MAT 271: Applied & Computational Harmonic Analysis Comments on Homework 3

Problem 1: Everyone got Part (a) correct. However, to show the claim of Part (b), what you really need to show is:

1. Compute the eigenvalues of the covariance matrix Γ, which is 0 and $1/n$ with geometric multiplicity 1 and $n-1$, respectively.
2. Γ is a real symmetric matrix; thus it can be *unitarily* diagonalizable, i.e., there exists an orthonormal basis diagonalizing Γ.
3. The above two also means that the eigenspace corresponding to the eigenvalue 0 and the one corresponding to $1/n$ are orthogonal.

For more information and the origin of the interest of this process, please read my own paper [1], [8], [7], and references therein.

Problem 2: Everyone got Part (a) correct. However, many people did Part (b) unsatisfactorily. Many people simply substituted $\phi_k(t) = \sqrt{2}\sin(k\pi t)$ into the integral equation, computing the eigenvalues, and claimed they are the eigenfunctions. With this argument, you cannot be sure whether there exists other eigenfunctions. The correct argument is to derive the eigenvalue problem in the ordinary differential equation from the integral equation, that is:

$$
\begin{aligned}
\lambda\phi(t) &= \int_0^1 \Gamma(t,s)\phi(s)\,\mathrm{d}s \\
&= \int_0^1 (\min(t,s) - ts)\phi(s)\,\mathrm{d}s \\
&= \int_0^t (s - ts)\phi(s)\,\mathrm{d}s + \int_t^1 (t - ts)\phi(s)\,\mathrm{d}s \\
&= \int_0^t s\phi(s)\,\mathrm{d}s - t\int_0^t s\phi(s)\,\mathrm{d}s + t\int_t^1 (1-s)\phi(s)\,\mathrm{d}s
\end{aligned}
$$

Now, differentiating both sides with respect to t leads to the following ODE:

$$\phi''(t) = -\frac{1}{\lambda}\phi(t)$$

The boundary condition can be derived by setting $t = 0$ and $t = 1$ in the above integral equation. It turns out to be the *Dirichlet* boundary condition:

$$\phi(0) = \phi(1) = 0.$$

From these, we can derive the desired solution.

For Part (c), some people used SVD instead of EIG and claimed that SVD gave them better or closer eigenvectors to the analytical ones compared to EIG. OK, why does this happen? It's a good exercise to think about it!

For more information about this process, please read the following papers [2], [3], [5, p. 19], [9].

Problem 3: Here, I would like to point out the two major mistakes several people made.

- If you use the function DCTMTX, then specifying the lowest 72 frequency DCT coefficients are trickier than using DCT2. Several people used $72^2 = 5184$ coefficients. That's why the DCT reconstructions were so good for some of you.
- It is very important to know that the MATLAB EIG function sorts the eigenvalues and eigenvectors in the *increasing order*, i.e., from the smallest to the largest. Thus, to use the top k KLB vectors means that you need to use the last k KLB vectors in the KLB matrix if you do not reorder it immediately after getting it from EIG. That is why the relative ℓ^2 curves for KLT were worse than those of DCT for some of you. For those of you made that mistake, I would strongly suggest that you recompute the error curve and plot against those of the DCT!

The other thing I want to point out is that *one should use the inverse transform routines to compute the basis functions*. Note that if the input signal is one of the basis functions/vectors, then the output is one of the standard basis vector. This means that if you apply the inverse transform to the identity matrix, you get all the basis functions. Thus, use IDCT2 to compute the DCT basis vectors! That's much faster and nicer than the code segments some of you wrote.

For more information about this dataset, please read the following papers [4], [6].

References

[1] B. BÉNICHOU AND N. SAITO, *Sparsity vs. statistical independence in adaptive signal representations: A case study of the spike process*, in Beyond Wavelets, G. V. Welland, ed., vol. 10 of Studies in Computational Mathematics, Academic Press, San Diego, CA, 2003, ch. 9, pp. 225–257.

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