MAT 271: Applied & Computational Harmonic Analysis Homework 1: due Monday, 04/24/23

Problem 1: Prove that the dilation operator:

$$\delta_s f(x) := \frac{1}{\sqrt{s}} f\left(\frac{x}{s}\right), \quad s > 0,$$

is an isometry (i.e., norm-preserving) in $L^2(\mathbb{R})$.

Problem 2: Suppose $f, g \in L^1$. Prove the following Fourier transform formulas:

- **(a)** $\mathscr{F}\{\tau_a f\}(\xi) = \mathrm{e}^{-2\pi \mathrm{i} a\xi} \widehat{f}(\xi)$, where $a \in \mathbb{R}$.
- **(b)** $\mathscr{F}\{\delta_s f\}(\xi) = \delta_{1/s} \widehat{f}(\xi) = \sqrt{s}\widehat{f}(s\xi)$, where $s > 0$.
- **(c)** If $f \in C^1(\mathbb{R})$ and $f'(x) \to 0$ as $|x| \to \infty$, then $\mathscr{F}\{f'\}(\xi) = (2\pi\mathrm{i}\xi)\widehat{f}(\xi)$.
- **(d)** $\mathscr{F}\{f * g\}(\xi) = \widehat{f}(\xi) \cdot \widehat{g}(\xi)$.

Problem 3: Compute the Fourier transform of the Gaussian function:

$$g(x;\sigma) := \frac{1}{\sqrt{2\pi}\sigma} \mathrm{e}^{-x^2/2\sigma^2}, \quad x \in \mathbb{R}, \sigma > 0.$$

Problem 4: Prove that the *sinc* function, $\operatorname{sinc}(x) := \dfrac{\sin \pi x}{\pi x}$, $x \in \mathbb{R}$, belongs to $L^2(\mathbb{R})$, but not to $L^1(\mathbb{R})$.

Problem 5: Prove that the *equality* in the Heisenberg inequality for $f \in L^2(\mathbb{R})$,

$$\Delta_{x_0} f \Delta_{\xi_0} \widehat{f} \geq \frac{1}{16\pi^2},$$

with $x_0 = \xi_0 = 0$ is satisfied if and only if $f(x) = a \exp(-x^2/2\sigma^2)$ for some constants $a, \sigma \in \mathbb{R}$. [Hint: Recall when the equality holds for the Cauchy-Schwarz inequality.]

Problem 6: Let $\mathrm{III}_A(x) := \sum_{k \in \mathbb{Z}} \delta(x - kA)$ be the *Shah* function with period A. Prove:

$$\mathscr{F}\{\mathrm{III}_A\}(\xi) = \frac{1}{A} \mathrm{III}_{1/A}(\xi) = \frac{1}{A} \sum_{k \in \mathbb{Z}} \delta(\xi - k/A),$$

where $\delta(\cdot)$ is the Dirac delta function.