MAT 271: Applied & Computational Harmonic Analysis
Comments on Homework 1

Problem 1: This was an easy problem and everyone solved it correctly!

Problem 2: (c) You need to justify that $f(x) \to 0$ as $|x| \to \infty$ after integration by parts. This comes from the assumption that $f \in L^1$.

(d) You need to show that $f * g \in L^1(\mathbb{R})$ first, which is rather straightforward. Then, in the process of showing $\mathscr{F}\{f * g\}(\xi) = \widehat{f}(\xi) \cdot \widehat{g}(\xi)$, you need to justify why the order of integrations can be swapped. This should be done by Fubini's theorem. For engineering students who have not learned Fubini's theorem, please look at the Wikipedia entry: `http://en.wikipedia.org/wiki/Fubini's_theorem`, In essence, in order to change the integration order of a function $f(x, y)$, you need to check $f(\cdot, y) \in L^1$ as a function of the first variable for almost all y in the integration region and $f(x, \cdot) \in L^1$ as a function of the second variable for almost all x in the integration region. In that Wikipedia page, there is an interesting example demonstrating that changing the order of integrations gives you different answer, i.e.,

$$\int_0^1 \int_0^1 \frac{x^2 - y^2}{(x^2 + y^2)^2} \, \mathrm{d}x \, \mathrm{d}y = -\frac{\pi}{4}$$

whereas

$$\int_0^1 \int_0^1 \frac{x^2 - y^2}{(x^2 + y^2)^2} \, \mathrm{d}y \, \mathrm{d}x = \frac{\pi}{4}.$$

Another great reference for the change of integration order is the famous Fourier analysis book by T. Körner [1, Chap. 47, 48]. Anyone who has not looked at this book definitely should take a look at it. It contains all sorts of interesting history, facts, applications of the Fourier analysis, as well as its unexpected relationship to the other parts of mathematics and sciences.

Problem 3: There are several ways to derive the Fourier transform of the Gaussian. I believe the best way is the following.

Consider the derivative:

$$\begin{aligned}
&g'(x;\sigma) = -\frac{1}{\sqrt{2\pi}\sigma^3} x \mathrm{e}^{-x^2/2\sigma^2}. \\
&\Longrightarrow \sigma^2 g' = -xg \\
&\Longrightarrow \sigma^2 (2\pi \mathrm{i} \xi) \hat{g} = -\frac{\mathrm{i}}{2\pi} \frac{\mathrm{d}\hat{g}}{\mathrm{d}\xi} \\
&\Longrightarrow \frac{\mathrm{d}\hat{g}}{\mathrm{d}\xi} = -4\pi^2 \sigma^2 \xi \hat{g}
\end{aligned}$$

This is a simple ODE and we can get the solution:

$$\hat{g}(\xi;\sigma) = C\mathrm{e}^{-2\pi^2\sigma^2\xi^2}.$$

But $\hat{g}(0) = 1$ because this is the integral of the probability density function of the normal distribution with mean 0 and variance σ^2. Therefore, $C = 1$.

$$\hat{g}(\xi;\sigma) = \mathrm{e}^{-2\pi^2\sigma^2\xi^2}.$$

Note that several of you showed that

$$\int_{-\infty}^{\infty} \exp\left(-\frac{(x+2\pi\sigma^2\mathrm{i}\xi)^2}{2\sigma^2}\right)\mathrm{d}x = \int_{-\infty}^{\infty} \exp\left(-\frac{u^2}{2\sigma^2}\right)\mathrm{d}u.$$

This argument is too formal, and needs more precise explanation because the first integral contains the imaginary number. One can proceed from here if he/she is very careful, but it is better to avoid this argument. That's why the first proof using the ODE is preferable.

Problem 4: In order to prove sinc function is in L^2, several people stated that the integral:

$$\int_{-\infty}^{\infty} \frac{\mathrm{d}x}{\pi^2 x^2} < \infty,$$

which is not true! You need to split the integration range into two parts, say $|x| \le \delta$ and $|x| > \delta$ for some $\delta > 0$ as follows:

$$\begin{aligned}
\int_{-\infty}^{\infty} \frac{\sin^2(\pi x)}{\pi^2 x^2}\mathrm{d}x &= \int_{|x|\le\delta} \frac{\sin^2(\pi x)}{\pi^2 x^2}\mathrm{d}x + \int_{|x|>\delta} \frac{\sin^2(\pi x)}{\pi^2 x^2}\mathrm{d}x \\
&\le \int_{|x|\le\delta} 1\,\mathrm{d}x + \int_{|x|>\delta} \frac{\mathrm{d}x}{\pi^2 x^2} \\
&= 2\delta + \frac{2}{\pi^2\delta} < \infty.
\end{aligned}$$

It is not necessary to obtain the exact value of $\|\operatorname{sinc}(\cdot)\|_2$. The question only asks to show that $\|\operatorname{sinc}(\cdot)\|_2 < \infty$.

Problem 5: If you state the equality condition of the Cauchy-Schwarz inequality used in this uncertainty inequality, then it is automatically, "if and only if". The bottom line is the Cauchy-Schwarz inequality in this case becomes:

$$\|f\|^4 = 4\left(\mathrm{Re}\int x\overline{f(x)}f'(x)\,\mathrm{d}x\right)^2 \le \int x^2|f(x)|^2\,\mathrm{d}x \int |f'(x)|^2\,\mathrm{d}x,$$

and the equality holds *if and only if*

$$f'(x) = cxf(x), \quad \text{for some constant } c.$$

So, we can easily get the solution:

$$f(x) = a\mathrm{e}^{cx^2/2}, \quad \text{for some constants } a, c.$$

However, the function f must be in $L^2(\mathbb{R})$. So, we must have $c < 0$. Otherwise, this function cannot have a finite norm in $L^2(\mathbb{R})$. So, we can set $c = -1/\sigma^2$ for some $\sigma > 0$, and get the form:

$$f(x) = a\mathrm{e}^{-x^2/2\sigma^2}, \quad \text{for some constants } a \text{ and } \sigma > 0.$$

Problem 6: It is important to notice that this shah (or comb) function is not really a function, but a *generalized function* or also known as a *distribution*. So, *you cannot use the usual Fourier series/transform definitions.*

The best way to solve this problem is to split it into two subproblems:

(1) Show $\mathscr{F}\{\mathrm{III}_1\}(\xi) = \mathrm{III}_1(\xi)$;

(2) Apply the dilation operator $\boldsymbol{\delta}_A$ to $\mathrm{III}_1(x)$ and use the the Fourier transform formula: $\mathscr{F}\{\boldsymbol{\delta}_A f\}(\xi) = \boldsymbol{\delta}_{1/A}\mathscr{F}\{f\}(\xi)$.

As for (1), I stated this in my lecture when we discussed the sampling theorem, but this is what I really wanted you to show. The correct proof goes like this. Let us write $\mathrm{III}_1(x)$ as $\mathrm{III}(x)$. First of all, since the shah function is a generalized function, its Fourier transform is defined by pairing it with a very nice function ϕ in the Schwartz space $\mathcal{S}$ (a space of functions of infinitely many times differentiable and decaying faster than any polynomials in both space and frequency domain). With the slight abuse of notation of $\langle f, g\rangle = \int f(x)g(x)\,\mathrm{d}x$ instead of the usual $\int f(x)\overline{g(x)}\,\mathrm{d}x$, we have

$$\begin{aligned}
\langle \mathscr{F}\{\mathrm{III}\}, \phi\rangle &= \langle \mathrm{III}, \mathscr{F}\{\phi\}\rangle \\
&= \int \mathrm{III}(\xi)\widehat{\phi}(\xi)\,\mathrm{d}\xi \\
&= \sum_{k\in\mathbb{Z}} \widehat{\phi}(k) \\
&\overset{(a)}{=} \sum_{k\in\mathbb{Z}} \phi(k) \\
&= \int \mathrm{III}(\xi)\phi(\xi)\,\mathrm{d}\xi \\
&= \langle \mathrm{III}, \phi\rangle.
\end{aligned}$$

Since $\phi \in \mathcal{S}$ is arbitrary, we can conclude $\widehat{\mathrm{III}}(\xi) = \mathrm{III}(\xi)$. The equality (a) above is called the *Poisson summation formula*, and will be proved simply as follows: Consider a periodized version of $\phi(x)$, i.e.,

$$\sum_{k\in\mathbb{Z}} \phi(x-k).$$

This is clearly periodic with period 1. Thus, we can expand this into a Fourier series as:

$$\begin{aligned}
\sum_{k\in\mathbb{Z}} \phi(x-k) &= \sum_{n\in\mathbb{Z}} c_n \mathrm{e}^{2\pi\mathrm{i}nx} \\
&= \sum_{n\in\mathbb{Z}} \left(\int_{-1/2}^{1/2} \sum_{k\in\mathbb{Z}} \phi(y-k)\mathrm{e}^{-2\pi\mathrm{i}ny}\,\mathrm{d}y \right) \mathrm{e}^{2\pi\mathrm{i}nx} \\
&= \sum_{n\in\mathbb{Z}} \left(\sum_{k\in\mathbb{Z}} \int_{-1/2}^{1/2} \phi(y-k)\mathrm{e}^{-2\pi\mathrm{i}n(y-k)}\,\mathrm{d}y \right) \mathrm{e}^{2\pi\mathrm{i}nx} \\
&= \sum_{n\in\mathbb{Z}} \left(\int_{-\infty}^{\infty} \phi(y)\mathrm{e}^{-2\pi\mathrm{i}ny}\,\mathrm{d}y \right) \mathrm{e}^{2\pi\mathrm{i}nx} \\
&= \sum_{n\in\mathbb{Z}} \widehat{\phi}(n)\mathrm{e}^{2\pi\mathrm{i}nx}.
\end{aligned}$$

Setting $x = 0$ in both sides, we get the Poisson summation formula:

$$\sum_{k\in\mathbb{Z}} \phi(k) = \sum_{k\in\mathbb{Z}} \widehat{\phi}(k).$$

Once we establish (1), i.e., $\widehat{\mathrm{III}}_1(\xi) = \mathrm{III}_1(\xi)$, it is easy to get (2) by the dilation formula. First, it is important to notice that $\delta(ax) = (1/a)\delta(x)$ or $a\delta(ax) = \delta(x)$. Now,

$$\mathrm{III}_A(x) = \sum_{k\in\mathbb{Z}} \delta(x-kA) = \sum_{k\in\mathbb{Z}} \frac{1}{A}\delta(x/A-k) = \frac{1}{\sqrt{A}}\boldsymbol{\delta}_A \mathrm{III}(x).$$

$$\begin{aligned}
\mathscr{F}\{\mathrm{III}_A\}(\xi) &= \mathscr{F}\left\{ \frac{1}{\sqrt{A}}\boldsymbol{\delta}_A \mathrm{III} \right\}(\xi) \\
&= \frac{1}{\sqrt{A}}\boldsymbol{\delta}_{1/A}\mathrm{III}(\xi) \\
&= \mathrm{III}(A\xi) \\
&= \sum_{k\in\mathbb{Z}} \delta(A\xi-k) \\
&= \sum_{k\in\mathbb{Z}} \frac{1}{A}\delta(\xi-k/A) \\
&= \frac{1}{A}\mathrm{III}_{1/A}(\xi).
\end{aligned}$$

I noticed that some of you had tried to compute the Fourier transform of the exponential function $\exp(2\pi\mathrm{i}kx/A)$ in this problem, but this does not work because such an exponential function is neither in $L^1(\mathbb{R})$ nor $L^2(\mathbb{R})$.

References

[1] T. W. KÖRNER, *Fourier Analysis*, Cambridge Univ. Press, 1988.