MAT 271: Applied & Computational Harmonic Analysis
Homework 3: due Monday, 06/05/23

Problem 1: Read the the article on SVD by Gilbert Strang, which you can download from https://archive.siam.org/pdf/news/828.pdf.

Problem 2: An n-dimensional *spike process* simply generates the standard basis vectors $\{\boldsymbol{e}_j\}_{j=1}^n \subset \mathbb{R}^n$ in a random order with equal probability, where $\boldsymbol{e}_j$ has one at the jth entry and all the other entries are zero. One can view this process as a unit impulse located at a random position between 1 and n.

(a) Compute the covariance matrix of this process.

(b) Show that the Karhunen-Loève basis of this process is any orthonormal basis in $\mathbb{R}^n$ containing a "DC" basis vector $\frac{1}{\sqrt{n}}(1,1,\cdots,1)^\top$.

Problem 3: Consider the following stochastic process called the *ramp process*:

$$X(t) = t - H(t-\tau), \quad t \in [0,1), \tau \sim \text{unif}[0,1),$$

where $H(\cdot)$ is the Heaviside step function, i.e., $H(t) = 1$ if $t \geq 0$; and $H(t) = 0$ if $t < 0$.

(a) Show that the covariance function of this process is

$$\Gamma(s,t) = \min(s,t) - st, \quad 0 \leq s,t \leq 1.$$

(b) Show that the Karhunen-Loève basis functions of this process are of the following form:

$$\phi_k(t) = \sqrt{2}\sin \pi k t, \quad k = 1,2,\ldots$$

(c) Discretize this process as follows. Let our sampling points be $t_k = \frac{2k+1}{2n}$, $k = 0,\ldots,n-1$. Suppose the discontinuity $t = \tau$ does not happen exactly at the sampling points. Then all the realizations whose discontinuities are located anywhere in the open interval $(\frac{2k-1}{2n}, \frac{2k+1}{2n})$ have the same discretized version. Therefore, any realization now has the following form:

$$\boldsymbol{x}_j = (x_{0j},\ldots,x_{n-1,j})^\top, \quad x_{kj} = \begin{cases} \frac{2k+1}{2n}, & \text{for } k = 0,\ldots,j-1, \\ \frac{2k+1}{2n} - 1, & \text{for } k = j,\ldots,n-1, \end{cases}$$

where j is picked uniformly randomly from the set $\{0,1,\cdots,n-1\}$. (Note that the index of the vector components starts with 0 for convenience). Take $n = 256$, and generate 256 realizations of this process. (You only have to construct a data matrix X whose column vectors are $\boldsymbol{x}_j$, $j = 0,\ldots,255$. Then compute the covariance matrix, compute the eigenvectors (i.e., KL vectors) using Julia, and compare those eigenvectors with the sinusoids analytically obtained in (b).

Problem 4: We will work on *Rogue's Gallery* dataset for computing PCA/KLT. You can submit the print out of your Julia scripts with your comments and additional notes and figures.

- **(a)** Download the MATLAB file:
  https://www.math.ucdavis.edu/ saito/data/faces.mat
  on your computer, and load into your Julia session using the package MAT.jl.
- **(b)** Randomly split theses 143 faces into two groups of size 72 and 71. Let the training dataset be those 72 faces, and the test dataset be the remaining 71 faces. Compute the average face of the training dataset and display in Julia. Use the function `heatmap` with the option `ratio=:equal, colormap=grays` to display the image in the proper aspect ratio and with the grayscale colormap. Note that you need to install and use the Plots.jl package for this operation.
- **(c)** Subtract the average face from each face in the training and the test datasets and compute the *eigenfaces*, i.e., the Karhunen-Loève basis of the training dataset. [Hint: You need to use the SVD formulation we discussed in the class. Otherwise, your covariance matrix becomes too huge to handle, i.e., $128^2 \times 128^2$. Then display the top five KLB vectors as images in Julia corresponding to the five most significant eigenvalues. Display the five lowest frequency 2D DCT basis functions and compare them with the top five KLB. Note that the function `dct` is available in the package FFTW.jl. I strongly suggest that you check the documentation of the `dct` function.
- **(d)** Compute the KL expansion coefficients of both the training and the test datasets. Note that you must use the same KLB computed from the training dataset to compute these coefficients of both datasets. Compute also the DCT coefficients of the datasets.
- **(e)** Choose one face from the training dataset, and another face from the test dataset. Approximate these faces by the 72 KLB vectors and the 72 lowest frequency DCT vectors. Display the approximations as images in Julia using the `layout` option (i.e., arrange four images in one figure as 2×2 subfigures); see the Layouts documentation for the details. Compute the residual error of these approximations and display them similarly. Note that when you display and compare different images using `heatmap` commands, you should use the same value scaling by supplying the range of the pixel values via the option `clims` so that the same color/gray scale values corresponds to the same physical pixel values. Check the Plots documentation for the details.
- **(f)** Compute the average relative ℓ^2 errors of the training dataset by the KLB approximation as a function of the number of coefficients retained (starting from 0 retained coefficients up to all the coefficients and plot it. Compute the same by the DCT approximation and plot it on the same graph for comparison. Repeat this experiments for the test dataset. What conclusion can you obtain from these experiments?

Problem 5: Let $\Phi \in L^2(\mathbb{R}^2)$ and let $S : L^2(\mathbb{R}) \to L^2(\mathbb{R}^2)$ be a *windowed Fourier transform* with a window function g satisfying the appropriate conditions as a window function as was discussed in the lecture. Then prove the following statement:
There exists a function $f \in L^2(\mathbb{R})$ such that $\Phi(x,\xi) = Sf(x,\xi)$ if and only if

$$\Phi(x_0,\xi_0) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \Phi(x,\xi)\, K(x_0,x,\xi_0,\xi)\,\mathrm{d}x\,\mathrm{d}\xi,$$

where the so-called *reproducing* kernel is defined as

$$K(x_0,x,\xi_0,\xi) := \left\langle g_{x,\xi}, g_{x_0,\xi_0} \right\rangle.$$