A Brief History of the Convergence of the Fourier Series

**Theorem 1** (Dirichlet, 1829) Suppose $f$ is 1-periodic, piecewise smooth on $\mathbb{R}$. Then, $n$th partial sum, $S_n[f](x) := \sum_{n=-n}^{n} c_k e^{2\pi i k x}$, satisfies

$$\lim_{n \to \infty} S_n[f](x) = \frac{1}{2} \left[ f(x+) + f(x-) \right].$$

In particular, if $x$ is a point of continuity, then $\lim_{n \to \infty} S_n[f](x) = f(x)$.

**Theorem 2** (du Bois Reymond, 1876) There exists $f \in C(I)$ such that $\{S_n[f](0)\}$ diverges, where $I$ is an interval of unit length.

**Theorem 3** (A weak version of Fejér's Theorem) If $f$ is 1-periodic, continuous, and piecewise smooth on $\mathbb{R}$, then the Fourier series of $f$ converges to $f$ absolutely and uniformly.

**Definition:** Suppose a series of functions $\sum_{n=1}^{\infty} g_n(x)$ converges to $g(x)$ on a set $x \in I$. Then, the convergence is called *absolute* if $\sum_{n=1}^{\infty} |g_n(x)|$ also converges for $x \in I$.

If we have $\sup_{x \in I} \left| g(x) - \sum_{n=1}^{N} g_n(x) \right| \to 0$ as $N \to \infty$, then we call this a *uniform* convergence.

**Theorem 4** (Fejér 1904) If $f \in C(I)$, then the Cesàro means of $S_n[f]$ converge uniformly to $f$.

**Definition:** The $m$th Cesàro mean of partial sums is the mean of the first $m+1$ partial sums, i.e.,

$$\sigma_m[f](x) := \frac{1}{m+1} \sum_{n=0}^{m} S_n[f](x).$$

**Theorem 5** (Size of the Fourier coefficients and the smoothness of the functions) Suppose $f$ is 1-periodic. If $f \in C^{k-1}(\mathbb{R})$ and $f^{(k-1)}$ is piecewise smooth (i.e., $f^{(k)}$ exists and piecewise continuous), then the Fourier coefficients of $f$, $c_n$, satisfy $\sum_{n=1}^{\infty} n^k c_n^2 < \infty$. In particular, $n^k c_n \to 0$. On the other hand, suppose $c_n, n \neq 0$, satisfy $|c_n| \leq C |n|^{-(k+\gamma)}$ for some $C > 0$ and $\gamma > 1$. Then $f \in C^k(\mathbb{R})$.

**Theorem 6** (Kolmogorov, 1926) There exists $f \in L^1(I)$ such that $\{S_n[f](x)\}$ diverges for every $x$.

**Theorem 7** (Carleson, 1966) If $f \in L^2(I)$, then $S_n[f](x)$ converges to $f(x)$ almost everywhere.

**Theorem 8** (Hunt, 1967) If $f \in L^p(I)$, $p > 1$, then $S_n[f](x)$ converges to $f(x)$ almost everywhere.

Mathematicians are still trying to simplify the proof of the Carlson-Hunt theorem as of today. For the details of the above facts, see [1, Chap. 1,2], [2, Chap. 1], [3, Part 1], and [4, Chap. 1]. [5, Chap. 1].

**References**


