The Fourier Inversion Theorem

The Fourier transform $\mathcal{F}$ was defined initially on $L^1(\mathbb{R})$, a space of integrable functions, and $\mathcal{F} : L^1(\mathbb{R}) \to BC(\mathbb{R}) = C(\mathbb{R}) \cap L^\infty(\mathbb{R})$.

However, $\hat{f}$, the Fourier transform of $f \in L^1$, may not be in $L^1$.

An example: $f(x) = \chi(-\frac{1}{2}, \frac{1}{2})(x) \Rightarrow \hat{f}(\xi) = \text{sinc}(\xi) = \frac{\sin \pi \xi}{\pi \xi} \notin L^1$.

The Inverse Fourier Transform: For $f \in L^1$, $\hat{f}(x) = \int_{-\infty}^{\infty} f(\xi)e^{2\pi i \xi x} d\xi$.

[The Fourier Inversion Theorem] If both $f$ and $\hat{f}$ are in $L^1$, then $(\hat{f}) = (\hat{f}) = f$ almost everywhere.

There are many functions in $L^1$ whose Fourier transforms are also in $L^1$; one needs only a little smoothness of $f$ for necessary decay of $\hat{f}$ as $|\xi| \to \infty$.

An example: If $f \in C^2(\mathbb{R})$, $f'$ and $f''$ are both in $L^1$, then $\mathcal{F}(f'')(\xi) = -(2\pi \xi)^2 \hat{f}(\xi) \in BC(\mathbb{R})$.

This boundedness implies that $|\hat{f}(\xi)| \leq C/(1 + \xi^2)$. This, in turn, implies that $f \in L^1$.

The Fourier Transforms on $L^2$

The previous remark leads to the $L^2$ theory of the Fourier transforms. In general, simply assuming $f \in L^2$ is not enough: $\int_{-\infty}^{\infty} f(x)e^{-2\pi i \xi x} dx$ may not converge.

An example: $f(x) = \text{sinc}(x) = \frac{\sin \pi x}{\pi x} \in L^2$, but not in $L^1$.

We will overcome this problem as follows. Define a subspace of $L^1$, $\mathcal{X} \overset{\Delta}{=} \{ f \in L^1 | \hat{f} \in L^1 \}$. We first note that for such functions, we can have the Parseval equality: $\langle f, g \rangle = \langle \hat{f}, \hat{g} \rangle$ as well as the Plancherel equality. Also, for any $f \in \mathcal{X}$, $f, \hat{f} \in BC(\mathbb{R})$ as the remark after the Fourier inversion theorem. This implies that both $f$ and $\hat{f}$ are also in $L^2$; i.e., $\mathcal{X} \subset L^2$ (because $f \in L^1 \cap BC$ implies $f \in L^2$ thanks to the theorem: $L^p \cap L^r \subset L^q$ for $0 < p < q < r \leq \infty$, which in turn can be proved by Hölder’s inequality).

Now, the point is that $\mathcal{X}$ is also dense in $L^2$.

We can proceed as follows: for any $f \in L^2$, we can find a sequence $\{ f_n \} \subset \mathcal{X}$ such that $\| f_n - f \|_2 \to 0$ as $n \to \infty$, $\{ f_n \} \subset \mathcal{X}$. Now using the Plancherel equality to this sequence, we can see $\| \hat{f}_n - \hat{f}_m \|_2 = \| f_n - f_m \|_2 \to 0$ as $m, n \to \infty$. In other words, $\{ f_n \}$ is a Cauchy sequence in $L^2$. Since $L^2$ is complete, there exists the limit of $\hat{f}_n$ in $L^2$, and we define this limit as $\hat{f}$, the Fourier transform of $f \in L^2$.

[The Plancherel Theorem] For any $f, g \in L^2$, $\langle f, g \rangle = \langle \hat{f}, \hat{g} \rangle$ and $\| f \|_2 = \| \hat{f} \|_2$.

Finally, we can use all these facts for computing the Fourier transform of $L^2$ functions as follows: Suppose we set $\phi(x) = f(x)$ where $f \in L^2$. Then, $\phi(\xi) = f(-\xi)$. An example: $\phi(x) = \text{sinc}(x) \in L^2$. Then, $\phi(\xi) = \chi(-\frac{1}{2}, \frac{1}{2})(\xi)$.