MAT 271: Applied & Computational Harmonic Analysis
Homework 2: due Monday, 02/08/10

Problem 0: Familiarize yourself to the MATLAB environment using the MATLAB primers. (Only applicable for the people who do not have much MATLAB experience.) See "Useful Link" in my course web page to get MATLAB primers.

Problem 1: Let $\boldsymbol{w}_N^k := \frac{1}{\sqrt{N}}\left[\omega_N^0, \omega_N^k, \omega_N^{2k} \ldots, \omega_N^{k(N-1)}\right]^T \in \mathbb{C}^N$, where $\omega_N = \exp(2\pi \mathrm{i}/N)$. Prove

$$\left\langle \boldsymbol{w}_N^k, \boldsymbol{w}_N^\ell \right\rangle = \delta_{k,\ell},$$

where $\delta_{k,\ell}$ is Kronecker's delta and $0 \le k, \ell \le N-1$.

Problem 2: Consider a periodized versions of the function over $[-1/2, 1/2)$:

$$f(x) = ax, \quad -\frac{1}{2} \le x < \frac{1}{2}, \quad a > 0.$$

(a) Compute the Fourier coefficients c_k of this periodic function by hand.

(b) Using MATLAB, do the following:

1) Determine the value of a so that after the discretization of this function on a uniform grid of length 1024, the resulting vector has a unit ℓ^2-norm;

2) Apply MATLAB's `fft` to the input vector prepared in 1).

3) Display both the real and imaginary parts of the output vector computed in 2);

4) Plot the hand-computed Fourier coefficients in (a) with a computed in 1);

5) Do these two way of computing Fourier coefficients agree? What is your reasoning if they do not.

Problem 3: Consider a periodized version (with period 1) of the following function:

$$f(x) = ax^2, \quad -\frac{1}{2} \le x < \frac{1}{2}, \quad a > 0.$$

Repeat (a), (b) of Problem 2 for this function. In addition,

(c) Compare the speed of the decay of the Fourier coefficients of this function with that of Problem 2. Which decays faster? Why?

Problem 4: Consider a periodized version (with period 1) of the following function:

$$f(x) = a\mathrm{e}^{-x^2/2\sigma^2}, \quad -\frac{1}{2} \le x < \frac{1}{2}, \quad a, \sigma > 0.$$

Repeat (1)–(3) of (b) of Problem 3 with $\sigma = 1, 0.1, 0.01$. In addition,

4) Compare the speed of the decay of the Fourier coefficients of this function with these different values of σ;

5) Compare these decays with those of Problems 2 and 3. Which decays faster? Why?

6) Do they agree with the discretized version of the Fourier transform formula of Problem 3 of HW #1 (with appropriate multiplicative constants)? If not, state your interpretation/reasoning.

Problem 5: Let us use the definition of DFT as in my lecture and my supplementary note IV on DFT. Hence, given an input vector $\boldsymbol{f}$ of length N, the matrix-vector representation of the DFT applied to $\boldsymbol{f}$ is $\boldsymbol{F} = \widetilde{W}_N^* \boldsymbol{f}$.

(a) Let W_N be the DFT matrix defined in Note IV. Let D_N be the matrix representation of the MATLAB function `fft` so that the result of `fft` applied to the vector $\boldsymbol{f}$ of length N in MATLAB is $D_N \boldsymbol{f}$. Express D_N using W_N.

(b) Let S_N be the matrix representation of the MATLAB function `fftshift` as in Note IV. Then the MATLAB expression `fftshift(fft(f))` corresponds to the matrix-vector expression $S_N D_N \boldsymbol{f}$. show that $\widetilde{W}_N^* \boldsymbol{f} \neq S_N D_N \boldsymbol{f}$, and express $\widetilde{W}_N^*$ using S_N, D_N as well as the circulant-shift matrix T_N defined in Note IV.

(c) Using MATLAB, do the following exercise and submit the figures.

```
% Set up the x variable [-pi, pi].
N = 16;
x = ((-N/2+1):(N/2))*2*pi/N;

% Generate a simple example function f=cos(x).
f = cos(x);

% Do the fftshift(fft) using proper normalization.
F = fftshift(fft(f)/sqrt(N));

% Plot the real and imaginary parts of F.
figure(1)
stem(real(F)); hold on; stem(imag(F),'r*');
```

Print this figure and submit it. You may feel the result is counterintuitive!

(d) Using the result of (b), compute $\boldsymbol{F} = \widetilde{W}_N^* \boldsymbol{f}$ where $\boldsymbol{f}$ is the same $\cos$ function as in (c). Note that you need to either use `circshift` and `fftshift` functions or generate the matrices S_N and T_N and do matrix-vector multiplication to obtain $\boldsymbol{F}$. Then generate another window by `figure(2)`, and display the real and imaginary parts using the `stem` plot as before. What do you see here? You should see more intuitive results now. Submit this figure too.