MAT 271: Applied & Computational Harmonic Analysis
Comments on Homework 1

Problem 1: Some of you used the following definition of the L^2-norm:

$$\|f\|_{L^2}^2 = \int_{-\infty}^{\infty} f(x)^2 \, \mathrm{d}x,$$

which is only true if $f(x)$ is real-valued. In general, $f(x)$ could be complex-valued, and it is the best practice to use the absolute value before squaring, i.e.,

$$\|f\|_{L^2}^2 = \int_{-\infty}^{\infty} |f(x)|^2 \, \mathrm{d}x,$$

Problem 2: (c) You need to justify that $f(x) \to 0$ as $|x| \to \infty$ after integration by parts. This comes from the assumption that $f \in L^1$.

(d) You need to show that $f * g \in L^1(\mathbb{R})$ first, which is rather straightforward. Then, in the process of showing $\mathscr{F}\{f * g\}(\xi) = \widehat{f}(\xi) \cdot \widehat{g}(\xi)$, you need to justify why the order of integrations can be swapped. This should be done by Fubini's theorem. For engineering students who have not learned Fubini's theorem, please look at the Wikipedia entry: `http://en.wikipedia.org/wiki/Fubini's_theorem`, In essence, in order to change the integration order of a function $f(x, y)$, you need to check $f(\cdot, y) \in L^1$ as a function of the first variable for almost all y in the integration region and $f(x, \cdot) \in L^1$ as a function of the second variable for almost all x in the integration region. In that Wikipedia page, there is an interesting example demonstrating that changing the order of integrations gives you different answer, i.e.,

$$\int_0^1 \int_0^1 \frac{x^2 - y^2}{(x^2 + y^2)^2} \, \mathrm{d}x \, \mathrm{d}y = -\frac{\pi}{4}$$

whereas

$$\int_0^1 \int_0^1 \frac{x^2 - y^2}{(x^2 + y^2)^2} \, \mathrm{d}y \, \mathrm{d}x = \frac{\pi}{4}.$$

Another great reference for the change of integration order is the famous Fourier analysis book by T. Körner [1, Chap. 47, 48]. Anyone who has not looked at this book definitely should take a look at it. It contains all sorts of interesting history, facts, applications of the Fourier analysis, as well as its unexpected relationship to the other parts of mathematics and sciences.

Problem 3: There are several ways to derive the Fourier transform of the Gaussian. I believe the best way is the following.

Consider the derivative:

$$
\begin{aligned}
& g'(x;\sigma) = -\frac{1}{\sqrt{2\pi}\sigma^3} x \mathrm{e}^{-x^2/2\sigma^2}. \\
& \Longrightarrow \sigma^2 g' = -xg \\
& \Longrightarrow \sigma^2 (2\pi \mathrm{i}\xi)\hat{g} = -\frac{\mathrm{i}}{2\pi}\frac{\mathrm{d}\hat{g}}{\mathrm{d}\xi} \\
& \Longrightarrow \frac{\mathrm{d}\hat{g}}{\mathrm{d}\xi} = -4\pi^2\sigma^2\xi\hat{g}
\end{aligned}
$$

This is a simple ODE and we can get the solution:

$$\hat{g}(\xi;\sigma) = C\mathrm{e}^{-2\pi^2\sigma^2\xi^2}.$$

But $\hat{g}(0) = 1$ because this is the integral of the probability density function of the normal distribution with mean 0 and variance σ^2. Therefore, $C = 1$.

$$\hat{g}(\xi;\sigma) = \mathrm{e}^{-2\pi^2\sigma^2\xi^2}.$$

Note that several of you showed that

$$\int_{-\infty}^{\infty} \exp\left(-\frac{(x+2\pi\sigma^2\mathrm{i}\xi)^2}{2\sigma^2}\right)\mathrm{d}x = \int_{-\infty}^{\infty} \exp\left(-\frac{u^2}{2\sigma^2}\right)\mathrm{d}u.$$

This argument is too formal, and needs more precise explanation because the first integral contains the imaginary number. One can proceed from here if he/she is very careful, but it is better to avoid this argument. That's why the first proof using the ODE is preferable.

Problem 4: Again, you need to justify the change of the integration order. Also, a couple of people used the Dirac delta function in the proof. The better proof not involving any delta functions is the following.

$$
\begin{aligned}
\langle f, g\rangle &= \int f(x)\overline{g(x)}\,\mathrm{d}x \\
&= \int\int f(x)\overline{\mathrm{e}^{2\pi\mathrm{i}\xi x}\hat{g}(\xi)}\,\mathrm{d}\xi\,\mathrm{d}x \quad \text{via the inverse Fourier transform} \\
&= \int\int f(x)\mathrm{e}^{-2\pi\mathrm{i}\xi x}\overline{\hat{g}(\xi)}\,\mathrm{d}\xi\,\mathrm{d}x \\
&= \int\int f(x)\mathrm{e}^{-2\pi\mathrm{i}\xi x}\overline{\hat{g}(\xi)}\,\mathrm{d}x\,\mathrm{d}\xi \quad \text{via Fubini's theorem} \\
&= \int \hat{f}(\xi)\overline{\hat{g}(\xi)}\,\mathrm{d}\xi \\
&= \langle \hat{f}, \hat{g}\rangle.
\end{aligned}
$$

Problem 5: In order to prove sinc function is in L^2, several people stated that the integral:

$$\int_{-\infty}^{\infty} \frac{\mathrm{d}x}{\pi^2 x^2} < \infty,$$

which is not true! You need to split the integration range into two parts, say $|x| \le \delta$ and $|x| > \delta$ for some $\delta > 0$ as follows:

$$\begin{aligned}\int_{-\infty}^{\infty} \frac{\sin^2(\pi x)}{\pi^2 x^2}\,\mathrm{d}x &= \int_{|x|\le\delta} \frac{\sin^2(\pi x)}{\pi^2 x^2}\,\mathrm{d}x + \int_{|x|>\delta} \frac{\sin^2(\pi x)}{\pi^2 x^2}\,\mathrm{d}x \\ &\le \int_{|x|\le\delta} 1\,\mathrm{d}x + \int_{|x|>\delta} \frac{\mathrm{d}x}{\pi^2 x^2} \\ &= 2\delta + \frac{2}{\pi^2\delta} < \infty.\end{aligned}$$

It is not necessary to obtain the exact value of $\|\operatorname{sinc}(\cdot)\|_2$. The question only asks to show that $\|\operatorname{sinc}(\cdot)\|_2 < \infty$.

Problem 6: If you state the equality condition of the Cauchy-Schwarz inequality used in this uncertainty inequality, then it is automatically, "if and only if". The bottom line is the Cauchy-Schwarz inequality in this case becomes:

$$\|f\|^4 = 4\left(\operatorname{Re}\int x\overline{f(x)}f'(x)\,\mathrm{d}x\right)^2 \le \int x^2|f(x)|^2\,\mathrm{d}x \int |f'(x)|^2\,\mathrm{d}x,$$

and the equality holds *if and only if*

$$f'(x) = cxf(x), \quad \text{for some constant } c.$$

So, we can easily get the solution:

$$f(x) = a\mathrm{e}^{cx^2/2}, \quad \text{for some constants } a, c.$$

However, the function f must be in $L^2(\mathbb{R})$. So, we must have $c < 0$. Otherwise, this function cannot have a finite norm in $L^2(\mathbb{R})$. So, we can set $c = -1/\sigma^2$ for some $\sigma > 0$, and get the form:

$$f(x) = a\mathrm{e}^{-x^2/2\sigma^2}, \quad \text{for some constants } a \text{ and } \sigma > 0.$$

References

[1] T. W. KÖRNER, *Fourier Analysis*, Cambridge Univ. Press, 1988.