Properties of Graph Laplacian Eigenvalues

Algebraic Connectivity $a(G) := \lambda_1(G)$

Wiener Index
1. Properties of Graph Laplacian Eigenvalues

2. Algebraic Connectivity $a(G) := \lambda_1(G)$

3. Wiener Index
In this lecture, we only consider **undirected** and **unweighted** graphs and their **unnormalized** Laplacians \( L(G) = D(G) - A(G) \).

It is your exercise to see how the statements change for the normalized or symmetrically-normalized graph Laplacians.

Let \( |V(G)| = n, |E(G)| = m \), and assign each edge an arbitrary orientation to turn \( G \) into a directed graph temporarily. Then let us define the **directed incidence matrix** \( R = R(G) = (r_{ij}) \in \mathbb{R}^{n \times m} \) of \( G \) by

\[
  r_{ij} = \begin{cases} 
    1 & \text{if } e_j = [v_i', v_i] \text{ for some } i'; \\
    -1 & \text{if } e_j = [v_i, v_i'] \text{ for some } i'; \\
    0 & \text{otherwise.}
  \end{cases}
\]

Then, we can show that \( L(G) = R(G)R(G)^\top \); hence it is **positive semi-definite**. Note that \( L(G) \) is orientation independent.
In this lecture, we only consider **undirected** and **unweighted** graphs and their **unnormalized** Laplacians $L(G) = D(G) - A(G)$.

It is your exercise to see how the statements change for the normalized or symmetrically-normalized graph Laplacians.

Let $|V(G)| = n$, $|E(G)| = m$, and assign each edge an arbitrary orientation to turn $G$ into a directed graph temporarily. Then let us define the directed incidence matrix $R = R(G) = (r_{ij}) \in \mathbb{R}^{n \times m}$ of $G$ by

$$r_{ij} = \begin{cases} 
1 & \text{if } e_j = [v_{i'}, v_i] \text{ for some } i'; \\
-1 & \text{if } e_j = [v_i, v_{i'}] \text{ for some } i'; \\
0 & \text{otherwise}.
\end{cases}$$

Then, we can show that $L(G) = R(G)R(G)^T$; hence it is **positive semi-definite**. Note that $L(G)$ is orientation independent.
In this lecture, we only consider **undirected** and **unweighted** graphs and their **unnormalized** Laplacians \( L(G) = D(G) - A(G) \).

It is your exercise to see how the statements change for the normalized or symmetrically-normalized graph Laplacians.

Let \(|V(G)| = n, |E(G)| = m\), and assign each edge an arbitrary orientation to turn \( G \) into a directed graph temporarily. Then let us define the **directed incidence matrix** \( R = R(G) = (r_{ij}) \in \mathbb{R}^{n \times m} \) of \( G \) by

\[
    r_{ij} = \begin{cases} 
        1 & \text{if } e_j = [v_i', v_i] \text{ for some } i' ; \\
        -1 & \text{if } e_j = [v_i, v_i'] \text{ for some } i' ; \\
        0 & \text{otherwise} .
    \end{cases}
\]

Then, we can show that \( L(G) = R(G)R(G)^T \); hence it is **positive semi-definite**. Note that \( L(G) \) is orientation independent.
In this lecture, we only consider **undirected** and **unweighted** graphs and their **unnormalized** Laplacians \( L(G) = D(G) - A(G) \).

It is your exercise to see how the statements change for the normalized or symmetrically-normalized graph Laplacians.

Let \(|V(G)| = n\), \(|E(G)| = m\), and assign each edge an arbitrary orientation to turn \( G \) into a directed graph temporarily. Then let us define the **directed incidence matrix** \( R = R(G) = (r_{ij}) \in \mathbb{R}^{n \times m} \) of \( G \) by

\[
 r_{ij} = \begin{cases} 
 1 & \text{if } e_j = [v_{i'}, v_i] \text{ for some } i'; \\
 -1 & \text{if } e_j = [v_i, v_{i'}] \text{ for some } i'; \\
 0 & \text{otherwise.}
\end{cases}
\]

Then, we can show that \( L(G) = R(G)R(G)^T \); hence it is **positive semi-definite**. Note that \( L(G) \) is orientation independent.
Hence, we can sort the eigenvalues of $L(G)$ as $0 = \lambda_0(G) \leq \lambda_1(G) \leq \cdots \leq \lambda_{n-1}(G)$ and denote the set of these eigenvalue by $\Lambda(G)$.

- $m_G(\lambda) :=$ the multiplicity of $\lambda$.
- Let $I \subset \mathbb{R}$ be an interval of the real line. Then define $m_G(I) := \#\{\lambda_k(G) \in I\}$. 
Hence, we can sort the eigenvalues of $L(G)$ as $0 = \lambda_0(G) \leq \lambda_1(G) \leq \cdots \leq \lambda_{n-1}(G)$ and denote the set of these eigenvalue by $\Lambda(G)$.

$m_G(\lambda) :=$ the multiplicity of $\lambda$.

Let $I \subset \mathbb{R}$ be an interval of the real line. Then define $m_G(I) := \#\{\lambda_k(G) \in I\}$. 
Hence, we can sort the eigenvalues of $L(G)$ as $0 = \lambda_0(G) \leq \lambda_1(G) \leq \cdots \leq \lambda_{n-1}(G)$ and denote the set of these eigenvalue by $\Lambda(G)$.

$m_G(\lambda) :=$ the multiplicity of $\lambda$.

Let $I \subset \mathbb{R}$ be an interval of the real line. Then define $m_G(I) := \#\{\lambda_k(G) \in I\}$. 
Graph Laplacian matrices of the same graph are permutation-similar. In fact, graphs $G_1$ and $G_2$ are isomorphic iff there exists a permutation matrix $P$ such that

$$L(G_2) = P^T L(G_1) P.$$ 

rank $L(G) = n - m_G(0)$ where $m_G(0)$ turns out to be the number of connected components of $G$. Easy to check that $L(G)$ becomes $m_G(0)$ diagonal blocks, and the eigenspace corresponding to the zero eigenvalues is spanned by the indicator vectors of each connected component.

In particular, $\lambda_1 \neq 0$ iff $G$ is connected.

This led M. Fiedler (1973) to define the algebraic connectivity of $G$ by $a(G) := \lambda_1(G)$, viewing it as a quantitative measure of connectivity.
Graph Laplacian matrices of the same graph are permutation-similar. In fact, graphs $G_1$ and $G_2$ are isomorphic iff there exists a permutation matrix $P$ such that

$$L(G_2) = P^T L(G_1) P.$$ 

$\text{rank } L(G) = n - m_G(0)$ where $m_G(0)$ turns out to be the number of connected components of $G$. Easy to check that $L(G)$ becomes $m_G(0)$ diagonal blocks, and the eigenspace corresponding to the zero eigenvalues is spanned by the indicator vectors of each connected component.

In particular, $\lambda_1 \neq 0$ iff $G$ is connected.

This led M. Fiedler (1973) to define the algebraic connectivity of $G$ by $a(G) := \lambda_1(G)$, viewing it as a quantitative measure of connectivity.
Graph Laplacian matrices of the same graph are permutation-similar. In fact, graphs $G_1$ and $G_2$ are isomorphic iff there exists a permutation matrix $P$ such that

$$L(G_2) = P^T L(G_1) P.$$ 

$\text{rank } L(G) = n - m_G(0)$ where $m_G(0)$ turns out to be the number of connected components of $G$. Easy to check that $L(G)$ becomes $m_G(0)$ diagonal blocks, and the eigenspace corresponding to the zero eigenvalues is spanned by the indicator vectors of each connected component.

In particular, $\lambda_1 \neq 0$ iff $G$ is connected.

This led M. Fiedler (1973) to define the algebraic connectivity of $G$ by $a(G) := \lambda_1(G)$, viewing it as a quantitative measure of connectivity.
Graph Laplacian matrices of the same graph are permutation-similar. In fact, graphs $G_1$ and $G_2$ are isomorphic iff there exists a permutation matrix $P$ such that

$$L(G_2) = P^T L(G_1) P.$$ 

$\text{rank } L(G) = n - m_G(0)$ where $m_G(0)$ turns out to be the number of connected components of $G$. Easy to check that $L(G)$ becomes $m_G(0)$ diagonal blocks, and the eigenspace corresponding to the zero eigenvalues is spanned by the indicator vectors of each connected component.

In particular, $\lambda_1 \neq 0$ iff $G$ is connected.

This led M. Fiedler (1973) to define the algebraic connectivity of $G$ by $a(G) := \lambda_1(G)$, viewing it as a quantitative measure of connectivity.
Denote the complement of $G$ (in $K_n$) by $G^c$. Then, we have

$$L(G) + L(G^c) = L(K_n) = nI_n - J_n,$$

where $J_n$ is the $n \times n$ matrix whose entries are all 1.

We also have:

$$\Lambda(G^c) = \{0, n - \lambda_{n-1}(G), n - \lambda_{n-2}(G), \ldots, n - \lambda_1(G)\}.$$
Denote the complement of $G$ (in $K_n$) by $G^c$.

The Petersen graph and its complement in $K_{10}$ (from Wikipedia)

Then, we have

$$L(G) + L(G^c) = L(K_n) = nl_n - J_n,$$

where $J_n$ is the $n \times n$ matrix whose entries are all 1.

We also have:

$$\Lambda(G^c) = \{0, n - \lambda_{n-1}(G), n - \lambda_{n-2}(G), \ldots, n - \lambda_1(G)\}.$$
Denote the complement of $G$ (in $K_n$) by $G^c$.

The Petersen graph and its complement in $K_{10}$ (from Wikipedia)

Then, we have

$$L(G) + L(G^c) = L(K_n) = nl_n - J_n,$$

where $J_n$ is the $n \times n$ matrix whose entries are all 1.

We also have:

$$\Lambda(G^c) = \{0, n - \lambda_{n-1}(G), n - \lambda_{n-2}(G), \ldots, n - \lambda_1(G)\}.$$
From the above, we can see that

$$\lambda_{\text{max}}(G) = \lambda_{n-1}(G) \leq n,$$

and $$m_G(n) = m_{G^c}(0) - 1.$$ 

On the other hand, Grone and Merris showed in 1994

$$\lambda_{\text{max}}(G) = \lambda_{n-1}(G) \geq \max_{1 \leq j \leq n} d_j + 1.$$ 

Let $$G$$ be a connected graph and suppose $$L(G)$$ has exactly $$k$$ distinct eigenvalues. Then

$$\text{diam}(G) \leq k - 1.$$
From the above, we can see that

\[ \lambda_{\text{max}}(G) = \lambda_{n-1}(G) \leq n, \]

and \( m_G(n) = m_{G^c}(0) - 1 \).

On the other hand, Grone and Merris showed in 1994

\[ \lambda_{\text{max}}(G) = \lambda_{n-1}(G) \geq \max_{1 \leq j \leq n} d_j + 1. \]

Let \( G \) be a connected graph and suppose \( L(G) \) has exactly \( k \) distinct eigenvalues. Then

\[ \text{diam}(G) \leq k - 1. \]
From the above, we can see that

\[ \lambda_{\text{max}}(G) = \lambda_{n-1}(G) \leq n, \]

and \( m_{G}(n) = m_{G^c}(0) - 1 \).

On the other hand, Grone and Merris showed in 1994

\[ \lambda_{\text{max}}(G) = \lambda_{n-1}(G) \geq \max_{1 \leq j \leq n} d_j + 1. \]

Let \( G \) be a connected graph and suppose \( L(G) \) has exactly \( k \) distinct eigenvalues. Then

\[ \text{diam}(G) \leq k - 1. \]
Now define a **cut vertex** by any vertex that increases the number of connected components of $G$ when removed.

The vertices with mixed color are the cut vertices here (from Wikipedia).

Let $u$ be a cut vertex of the connected graph $G$. If the largest component of $G \setminus \{u\}$ contains $k$ vertices, then $\lambda_{n-2}(G) \leq k + 1$. 
Now define a cut vertex by any vertex that increases the number of connected components of $G$ when removed.

Let $u$ be a cut vertex of the connected graph $G$. If the largest component of $G \setminus \{u\}$ contains $k$ vertices, then $\lambda_{n-2}(G) \leq k + 1$. 

The vertices with mixed color are the cut vertices here (from Wikipedia)
A vertex of degree 1 is called a **pendant** vertex; a vertex adjacent to a pendant vertex is called **pendant neighbor**.

Let $p(G)$ and $q(G)$ be the number of pendant vertices and that of pendant neighbors, respectively.

The number of pendant neighbors of $G$ is bounded as:

$$p(G) - m_G(1) \leq q(G) \leq m_G(2, n],$$

where the second inequality holds if $G$ is connected and satisfies $2q(G) < n$. 
A vertex of degree 1 is called a **pendant** vertex; a vertex adjacent to a pendant vertex is called **pendant neighbor**.

Let $p(G)$ and $q(G)$ be the number of pendant vertices and that of pendant neighbors, respectively.

The number of pendant neighbors of $G$ is bounded as:

$$p(G) - m_G(1) \leq q(G) \leq m_G(2, n],$$

where the second inequality holds if $G$ is connected and satisfies $2q(G) < n$. 
A vertex of degree 1 is called a **pendant** vertex; a vertex adjacent to a pendant vertex is called **pendant neighbor**.

Let $p(G)$ and $q(G)$ be the number of pendant vertices and that of pendant neighbors, respectively.

The number of pendant neighbors of $G$ is bounded as:

$$p(G) - m_G(1) \leq q(G) \leq m_G(2, n],$$

where the second inequality holds if $G$ is connected and satisfies $2q(G) < n$. 
1. Properties of Graph Laplacian Eigenvalues

2. Algebraic Connectivity $a(G) := \lambda_1(G)$

3. Wiener Index
Some Graph Operations

- $G$ is said to be $k$-vertex-connected if $k$ is the size of the smallest subset of vertices such that the graph becomes disconnected if they are deleted.
- A 1-vertex-connected graph is called connected while a 2-vertex-connected graph is said to be biconnected.
- The vertex-connectivity $\kappa(G)$ of $G$ is the largest $k$ for which $G$ is $k$-vertex-connected.
- Similarly, we can define the $k$-edge-connectedness and the edge-connectivity $\epsilon(G)$. 
Some Graph Operations

- $G$ is said to be **$k$-vertex-connected** if $k$ is the size of the smallest subset of vertices such that the graph becomes disconnected if they are deleted.

- A 1-vertex-connected graph is called **connected** while a 2-vertex-connected graph is said to be **biconnected**.

- The vertex-connectivity $\kappa(G)$ of $G$ is the largest $k$ for which $G$ is $k$-vertex-connected.

- Similarly we can define the **$k$-edge-connectedness** and the **edge-connectivity** $\epsilon(G)$.
$G$ is said to be $k$-vertex-connected if $k$ is the size of the smallest subset of vertices such that the graph becomes disconnected if they are deleted.

A 1-vertex-connected graph is called connected while a 2-vertex-connected graph is said to be biconnected.

The vertex-connectivity $\kappa(G)$ of $G$ is the largest $k$ for which $G$ is $k$-vertex-connected.

Similarly we can define the $k$-edge-connectedness and the edge-connectivity $\epsilon(G)$. 
Some Graph Operations

- $G$ is said to be $k$-vertex-connected if $k$ is the size of the smallest subset of vertices such that the graph becomes disconnected if they are deleted.
- A 1-vertex-connected graph is called connected while a 2-vertex-connected graph is said to be biconnected.
- The vertex-connectivity $\kappa(G)$ of $G$ is the largest $k$ for which $G$ is $k$-vertex-connected.
- Similarly we can define the $k$-edge-connectedness and the edge-connectivity $\epsilon(G)$. 

saito@math.ucdavis.edu (UC Davis)
Some Graph Operations . . .

- The **Edge-union** \( G(V, E) \) of \( G_1(V, E_1) \) and \( G_2(V, E_2) \) is defined as \( E = E_1 \cup E_2 \) and \( V \) is common among \( G \), \( G_1 \), and \( G_2 \).

- The **Cartesian product** \( G = G_1 \times G_2 \) (or also written as \( G = G_1 \Box G_2 \)):

- \( G_1(V_1, E_1) \) and \( G_2(V_2, E_2) \) are said to be obtained from a **vertex decomposition** of \( G(V, E) \) if \( V = V_1 \cup V_2 \) and \( V_1 \cap V_2 = \emptyset \).

- If \( L(G) = \begin{bmatrix} L(G_1) & O \\ O & L(G_2) \end{bmatrix} \), then \( G \) is said to be the **direct sum** of \( G_1 \) and \( G_2 \) and written as \( G = G_1 \oplus G_2 \).
Some Graph Operations . . .

- The **Edge-union** $G(V, E)$ of $G_1(V, E_1)$ and $G_2(V, E_2)$ is defined as $E = E_1 \cup E_2$ and $V$ is common among $G$, $G_1$, and $G_2$.

- The **Cartesian product** $G = G_1 \times G_2$ (or also written as $G = G_1 \square G_2$):

![Cartesian product diagram](image)

The Cartesian product of two graphs (from Wikipedia)

- $G_1(V_1, E_1)$ and $G_2(V_2, E_2)$ are said to be obtained from a **vertex decomposition** of $G(V, E)$ if $V = V_1 \cup V_2$ and $V_1 \cap V_2 = \emptyset$.

- If $L(G) = \begin{bmatrix} L(G_1) & O \\ O & L(G_2) \end{bmatrix}$, then $G$ is said to be the **direct sum** of $G_1$ and $G_2$ and written as $G = G_1 \oplus G_2$. 

saito@math.ucdavis.edu (UC Davis)  |  Graph Laplacian Eigenvalues  |  April 16, 2012  |  13 / 20
Some Graph Operations . . .

- The **Edge-union** $G(V, E)$ of $G_1(V, E_1)$ and $G_2(V, E_2)$ is defined as $E = E_1 \cup E_2$ and $V$ is common among $G$, $G_1$, and $G_2$.

- The **Cartesian product** $G = G_1 \times G_2$ (or also written as $G = G_1 \Box G_2$):

  

The Cartesian product of two graphs (from Wikipedia)

- $G_1(V_1, E_1)$ and $G_2(V_2, E_2)$ are said to be obtained from a **vertex decomposition** of $G(V, E)$ if $V = V_1 \cup V_2$ and $V_1 \cap V_2 = \emptyset$.

- If $L(G) = \begin{bmatrix} L(G_1) & O \\ O & L(G_2) \end{bmatrix}$, then $G$ is said to be the **direct sum** of $G_1$ and $G_2$ and written as $G = G_1 \oplus G_2$. 

saito@math.ucdavis.edu  (UC Davis)  Graph Laplacian Eigenvalues  April 16, 2012  13 / 20
The Edge-union $G(V, E)$ of $G_1(V, E_1)$ and $G_2(V, E_2)$ is defined as $E = E_1 \cup E_2$ and $V$ is common among $G$, $G_1$, and $G_2$.

The Cartesian product $G = G_1 \times G_2$ (or also written as $G = G_1 \square G_2$):

$G_1(V_1, E_1)$ and $G_2(V_2, E_2)$ are said to be obtained from a vertex decomposition of $G(V, E)$ if $V = V_1 \cup V_2$ and $V_1 \cap V_2 = \emptyset$.

If $L(G) = \begin{bmatrix} L(G_1) & O \\ O & L(G_2) \end{bmatrix}$, then $G$ is said to be the direct sum of $G_1$ and $G_2$ and written as $G = G_1 \oplus G_2$. 
<table>
<thead>
<tr>
<th>Operations</th>
<th>Relations of $a(G)$, $a(G_i)$, $i = 1, 2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$G^c$</td>
<td>$a(G^c) = n - \lambda_{n-1}$</td>
</tr>
<tr>
<td>$G_1 = G \setminus {e}$</td>
<td>$a(G_1) \leq a(G)$</td>
</tr>
<tr>
<td>$G_1 = G \setminus {v_{i_1}, \ldots, v_{i_k}}$</td>
<td>$a(G) \leq a(G_1) + k$</td>
</tr>
<tr>
<td>$G_1 = G \cup {e}$</td>
<td>$a(G) \leq a(G_1) \leq a(G) + 2$</td>
</tr>
<tr>
<td>$G$: edge-union of $G_1, G_2$</td>
<td>$a(G_1) + a(G_2) = a(G)$</td>
</tr>
<tr>
<td>$G = G_1 \times G_2$</td>
<td>$a(G) = \min{a(G_1), a(G_2)}$</td>
</tr>
<tr>
<td>$G_1, G_2$: vertex decomposition of $G$</td>
<td>$a(G) \leq \min{a(G_1) +</td>
</tr>
<tr>
<td>$G = G_1 \oplus G_2$</td>
<td>$a(G_1) + a(G_2) \leq a(G_1 \oplus G_2)$</td>
</tr>
</tbody>
</table>
## Algebraic Connectivities of Specific Graphs (de Abreu, 2007)

<table>
<thead>
<tr>
<th>Graph $G$</th>
<th>Algebraic Connectivity $a(G)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Complete graph $K_n$</td>
<td>$a(K_n) = n$</td>
</tr>
<tr>
<td>Path $P_n$</td>
<td>$a(P_n) = 2 \left(1 - \cos \frac{\pi}{n}\right)$</td>
</tr>
<tr>
<td>Cycle $C_n$</td>
<td>$a(C_n) = 2 \left(1 - \cos \frac{2\pi}{n}\right)$</td>
</tr>
<tr>
<td>Bipartite complete graph $K_{p,q}$</td>
<td>$a(K_{p,q}) = \min{p, q}$</td>
</tr>
<tr>
<td>Star $K_{1,q}$</td>
<td>$a(K_{1,q}) = 1$</td>
</tr>
<tr>
<td>Cube $m$-dimension $Cb_m$</td>
<td>$a(Cb_m) = 2$</td>
</tr>
<tr>
<td>Petersen Graph $P$</td>
<td>$a(P) = 2$</td>
</tr>
</tbody>
</table>
Fiedler showed in 1973 the following bounds to $a(G)$:

- For $G \neq K_n$, $a(G) \leq n - 2$;
- $2 \min_j d_j - n + 2 \leq a(G) \leq \frac{n}{n-1} \min_j d_j$;
- $a(G) \leq \kappa(G) \leq \epsilon(G) \leq \min_j d_j$;
- $2\epsilon(G) \left(1 - \cos \frac{\pi}{n}\right) \leq a(G)$;
- $2 \left(\cos \frac{\pi}{n} - \cos \frac{2\pi}{n}\right) \kappa(G) - 2 \cos \frac{\pi}{n} \left(1 - \cos \frac{\pi}{n}\right) \max_j d_j \leq a(G)$. 
Fiedler showed in 1973 the following bounds to \( a(G) \):

- For \( G \neq K_n \), \( a(G) \leq n - 2 \);
- \( 2 \min_j d_j - n + 2 \leq a(G) \leq \frac{n}{n-1} \min_j d_j \);
- \( a(G) \leq \kappa(G) \leq \epsilon(G) \leq \min_j d_j \);
- \( 2\epsilon(G) \left( 1 - \cos \frac{\pi}{n} \right) \leq a(G) \);
- \( 2 \left( \cos \frac{\pi}{n} - \cos \frac{2\pi}{n} \right) \kappa(G) - 2 \cos \frac{\pi}{n} \left( 1 - \cos \frac{\pi}{n} \right) \max_j d_j \leq a(G) \).
Fiedler showed in 1973 the following bounds to $a(G)$:

- For $G \neq K_n$, $a(G) \leq n - 2$;
- $2 \min_j d_j - n + 2 \leq a(G) \leq \frac{n}{n-1} \min_j d_j$;
- $a(G) \leq \kappa(G) \leq \epsilon(G) \leq \min_j d_j$;
- $2\epsilon(G) \left(1 - \cos \frac{\pi}{n}\right) \leq a(G)$;
- $2 \left(\cos \frac{\pi}{n} - \cos \frac{2\pi}{n}\right) \kappa(G) - 2 \cos \frac{\pi}{n} \left(1 - \cos \frac{\pi}{n}\right) \max_j d_j \leq a(G)$. 
Fiedler showed in 1973 the following bounds to $a(G)$:

- For $G \neq K_n$, $a(G) \leq n - 2$;
- $2 \min_j d_j - n + 2 \leq a(G) \leq \frac{n}{n-1} \min_j d_j$;
- $a(G) \leq \kappa(G) \leq \epsilon(G) \leq \min_j d_j$;
- $2\epsilon(G) \left(1 - \cos \frac{\pi}{n}\right) \leq a(G)$;
- $2 \left(\cos \frac{\pi}{n} - \cos \frac{2\pi}{n}\right) \kappa(G) - 2 \cos \frac{\pi}{n} (1 - \cos \frac{\pi}{n}) \max_j d_j \leq a(G)$. 
Fiedler showed in 1973 the following bounds to $a(G)$:

- For $G \neq K_n$, $a(G) \leq n - 2$;
- $2 \min_j d_j - n + 2 \leq a(G) \leq \frac{n}{n-1} \min_j d_j$;
- $a(G) \leq \kappa(G) \leq \epsilon(G) \leq \min_j d_j$;
- $2\epsilon(G) \left(1 - \cos \frac{\pi}{n}\right) \leq a(G)$;
- $2 \left(\cos \frac{\pi}{n} - \cos \frac{2\pi}{n}\right) \kappa(G) - 2 \cos \frac{\pi}{n} \left(1 - \cos \frac{\pi}{n}\right) \max_j d_j \leq a(G)$.
Fiedler showed in 1973 the following bounds to $a(G)$:

- For $G \neq K_n$, $a(G) \leq n - 2$;
- $2 \min_j d_j - n + 2 \leq a(G) \leq \frac{n}{n-1} \min_j d_j$;
- $a(G) \leq \kappa(G) \leq \epsilon(G) \leq \min_j d_j$;
- $2\epsilon(G) \left(1 - \cos \frac{\pi}{n}\right) \leq a(G)$;
- $2 \left(\cos \frac{\pi}{n} - \cos \frac{2\pi}{n}\right) \kappa(G) - 2 \cos \frac{\pi}{n} \left(1 - \cos \frac{\pi}{n}\right) \max_j d_j \leq a(G)$.
A cycle is a connected graph where every vertex has exactly two neighbors.

A tree $T$ is a connected graph without cycles.

Grone, Merris, and Sunder showed in 1990:

$$a(T) \leq 2 \left(1 - \cos \left(\frac{\pi}{\text{diam}(T) + 1}\right)\right).$$

They also showed: if $T \neq K_{1,n-1}$ with $n \geq 6$, then $a(T) < 0.49.$
Algebraic Connectivity of Trees

- **A cycle** is a connected graph where every vertex has exactly two neighbors.
- **A tree** $T$ is a connected graph without cycles.
- Grone, Merris, and Sunder showed in 1990:
  
  \[ a(T) \leq 2 \left( 1 - \cos \left( \frac{\pi}{\text{diam}(T) + 1} \right) \right). \]

- They also showed: if $T \neq K_{1,n-1}$ with $n \geq 6$, then $a(T) < 0.49$. 

saito@math.ucdavis.edu (UC Davis)

Graph Laplacian Eigenvalues

April 16, 2012 17 / 20
A cycle is a connected graph where every vertex has exactly two neighbors.

A tree $T$ is a connected graph without cycles.

Grone, Merris, and Sunder showed in 1990:

$$a(T) \leq 2 \left( 1 - \cos \left( \frac{\pi}{\text{diam}(T) + 1} \right) \right).$$

They also showed: if $T \neq K_{1,n-1}$ with $n \geq 6$, then $a(T) < 0.49$. 
A cycle is a connected graph where every vertex has exactly two neighbors.

A tree $T$ is a connected graph without cycles.

Grone, Merris, and Sunder showed in 1990:

$$a(T) \leq 2 \left( 1 - \cos \left( \frac{\pi}{\text{diam}(T) + 1} \right) \right).$$

They also showed: if $T \neq K_{1,n-1}$ with $n \geq 6$, then $a(T) < 0.49$. 
**Isoperimetric Number**

- Let $S \subset V(G)$ be a nonempty subset of vertices of $G$.
- $\partial S := \{ e = (u,v) \in E(G) \mid u \in S, v \notin S \}$, which is called the boundary of $S$.
- The isoperimetric number of $G$ is defined as
  $$i(G) := \inf \left\{ \frac{|\partial S|}{|S|} \mid \emptyset \neq S \subset V, |S| \leq \frac{n}{2} \right\},$$
  which is closely related to the conductance of a graph, i.e., how fast a random walk on $G$ converges to a stationary distribution.
- For $n \geq 4$, the isoperimetric number $i(G)$ satisfies
  $$i(G) < \sqrt{\left(2 \max_{v \in V(G)} d_v - a(G)\right) a(G)}.$$
Let $S \subset V(G)$ be a nonempty subset of vertices of $G$.

$\partial S := \{ e = (u, v) \in E(G) \mid u \in S, v \notin S \}$, which is called the boundary of $S$.

The isoperimetric number of $G$ is defined as

\[
i(G) := \inf \left\{ \frac{|\partial S|}{|S|} \mid \emptyset \neq S \subset V, |S| \leq \frac{n}{2} \right\},
\]

which is closely related to the conductance of a graph, i.e., how fast a random walk on $G$ converges to a stationary distribution.

For $n \geq 4$, the isoperimetric number $i(G)$ satisfies

\[
i(G) < \sqrt{\left(2 \max_{v \in V(G)} d_v - a(G)\right) a(G)}.
\]
Isoperimetric Number

- Let $S \subseteq V(G)$ be a nonempty subset of vertices of $G$.
- $\partial S := \{e = (u, v) \in E(G) | u \in S, v \notin S\}$, which is called the boundary of $S$.
- The isoperimetric number of $G$ is defined as

$$i(G) := \inf \left\{ \frac{|\partial S|}{|S|} \left| \emptyset \neq S \subset V, |S| \leq \frac{n}{2} \right. \right\},$$

which is closely related to the conductance of a graph, i.e., how fast a random walk on $G$ converges to a stationary distribution.

- For $n \geq 4$, the isoperimetric number $i(G)$ satisfies

$$i(G) < \sqrt{\left( 2 \max_{v \in V(G)} d_v - a(G) \right)} a(G).$$
Isoperimetric Number

- Let $S \subset V(G)$ be a nonempty subset of vertices of $G$.
- $\partial S := \{ e = (u, v) \in E(G) \mid u \in S, v \notin S \}$, which is called the boundary of $S$.
- The isoperimetric number of $G$ is defined as

$$i(G) := \inf \left\{ \frac{|\partial S|}{|S|} \left| \emptyset \neq S \subset V, |S| \leq \frac{n}{2} \right. \right\},$$

which is closely related to the conductance of a graph, i.e., how fast a random walk on $G$ converges to a stationary distribution.
- For $n \geq 4$, the isoperimetric number $i(G)$ satisfies

$$i(G) < \sqrt{\left( 2 \max_{v \in V(G)} d_v - a(G) \right) a(G)}.$$
Outline

1. Properties of Graph Laplacian Eigenvalues

2. Algebraic Connectivity \( a(G) := \lambda_1(G) \)

3. Wiener Index
The distance matrix $\Delta(G)$ of $G$ represents “distances” among the vertices, i.e., $\Delta(G)_{i,j} = d(v_i, v_j)$ is the length (or cost) of the shortest path from vertex $v_i$ to vertex $v_j$.

The Wiener index\(^1\) $W(G)$ of a graph $G$ is the sum of the entries in the upper triangular part of the distance matrix $\Delta(G)$.

The Wiener index of a molecular graph has been used in chemical applications because it may exhibit a good correlation with physical and chemical properties (e.g., the boiling point, density, viscosity, surface tension, . . . ) of the corresponding molecule/material.

Let $G$ be a tree. Then

$$W(G) = \sum_{k=1}^{n-1} \frac{n}{\lambda_k}.$$ 

\(^1\)proposed by Harry Wiener of Brooklyn College in 1947
The distance matrix $\Delta(G)$ of $G$ represents “distances” among the vertices, i.e., $\Delta(G)_{i,j} = d(v_i, v_j)$ is the length (or cost) of the shortest path from vertex $v_i$ to vertex $v_j$.

The Wiener index\(^1\) $W(G)$ of a graph $G$ is the sum of the entries in the upper triangular part of the distance matrix $\Delta(G)$.

The Wiener index of a molecular graph has been used in chemical applications because it may exhibit a good correlation with physical and chemical properties (e.g., the boiling point, density, viscosity, surface tension, ... ) of the corresponding molecule/material.

Let $G$ be a tree. Then

$$W(G) = \sum_{k=1}^{n-1} \frac{n}{\lambda_k}.$$ 

\(^1\) proposed by Harry Wiener of Brooklyn College in 1947
Wiener Index

- The distance matrix $\Delta(G)$ of $G$ represents “distances” among the vertices, i.e., $\Delta(G)_{i,j} = d(v_i, v_j)$ is the length (or cost) of the shortest path from vertex $v_i$ to vertex $v_j$.

- The Wiener index $W(G)$ of a graph $G$ is the sum of the entries in the upper triangular part of the distance matrix $\Delta(G)$.

- The Wiener index of a molecular graph has been used in chemical applications because it may exhibit a good correlation with physical and chemical properties (e.g., the boiling point, density, viscosity, surface tension, ...) of the corresponding molecule/material.

- Let $G$ be a tree. Then

$$W(G) = \sum_{k=1}^{n-1} \frac{n}{\lambda_k}.$$  

1 proposed by Harry Wiener of Brooklyn College in 1947
Wiener Index

- The distance matrix $\Delta(G)$ of $G$ represents “distances” among the vertices, i.e., $\Delta(G)_{i,j} = d(v_i, v_j)$ is the length (or cost) of the shortest path from vertex $v_i$ to vertex $v_j$.

- The Wiener index$^1$ $W(G)$ of a graph $G$ is the sum of the entries in the upper triangular part of the distance matrix $\Delta(G)$.

- The Wiener index of a molecular graph has been used in chemical applications because it may exhibit a good correlation with physical and chemical properties (e.g., the boiling point, density, viscosity, surface tension, . . . ) of the corresponding molecule/material.

- Let $G$ be a tree. Then

$$W(G) = \sum_{k=1}^{n-1} \frac{n}{\lambda_k}.$$ 

---

$^1$ proposed by Harry Wiener of Brooklyn College in 1947