Wavelets on Graphs

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May 30, 2012
Outline

1. Wavelets using Graph Laplacians
2. Haar-Like Wavelets on Graphs
3. Summary
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Conceptually, it is an adaptation of the *continuous wavelet transform* for graphs.

Let $G(V, E)$ be a weighted graph with $|V| = n$.

The **graph Fourier transform** of $f \in L^2(V)$ is defined as

$$
\hat{f}(\ell) = \langle f, \phi_\ell \rangle = \sum_{k=1}^{n} f(k) \phi^*_\ell(k), \quad \ell = 0, 1, \ldots, n - 1
$$

where $\phi_\ell := (\phi_\ell(1), \ldots, \phi_\ell(n))^T \in \mathbb{R}^n$ is the $\ell$th graph Laplacian eigenvector corresponding to the eigenvalue $\lambda_\ell$. The original vector $f$ can be reconstructed by

$$
f(k) = \sum_{\ell=0}^{n-1} \hat{f}(\ell) \phi_\ell(k), \quad k = 1, 2, \ldots, n.
$$
Let $T_g = g(L) : \mathcal{L}^2(V) \to \mathcal{L}^2(V)$ be defined as a Fourier multiplier as

$$\hat{T}_g f(\ell) = g(\lambda_\ell) \hat{f}(\ell),$$

where $g$ is a wavelet generating kernel, also called the spectral graph wavelet kernel (SGWT kernel). We now have:

$$(T_g f)(k) = \sum_{\ell=0}^{n-1} g(\lambda_\ell) \hat{f}(\ell) \phi_\ell(k).$$

The spectral graph wavelet operator at scale $s > 0$ is defined by $T_g^s = g(sL)$. Hence, the spectral wavelet function at scale $s$ and vertex $v_m$ is realized as $\psi_{s,m}(k) := T_g^s \delta_m(k)$ where $\delta_m$ is an impulse located at $v_m$.

$$\psi_{s,m}(k) = \sum_{\ell=0}^{n-1} g(s\lambda_\ell) \phi_\ell^*(m) \phi_\ell(k).$$
The wavelet coefficient of a given function $f \in L^2(V)$ is computed by

$$W_f(s, m) := \langle f, \psi_{s,m} \rangle = (T_g^s f)(m) = \sum_{\ell=0}^{n-1} g(s \lambda_\ell) \hat{f}(\ell) \phi_\ell(m).$$

**Lemma (H-V-G)**

*If the SGWT kernel $g$ satisfies the admissibility condition:*

$$\int_0^{\infty} \frac{g^2(x)}{x} \, dx := C_g < \infty, \text{ and } g(0) = 0, \text{ then}$$

$$\frac{1}{C_g} \sum_{m=1}^n \int_0^{\infty} W_f(s, m) \psi_{s,m}(k) \frac{ds}{s} =: f^\#(k)$$

*where $f^\# = f - \langle f, \phi_0 \rangle \phi_0$. In other words, $f$ can be reconstructed from the information $\{W_f(s, m)\}$ and the DC component.*
An example of $g(x)$:

$$g(x) = \begin{cases} 
(x/x_1)^\alpha & \text{for } 0 \leq x < x_1; \\
 s(x) & \text{for } x_1 \leq x \leq x_2; \\
(x_2/x)^\beta & \text{for } x > x_2.
\end{cases}$$

H-V-G used $x_1 = 1; x_2 = 2; \alpha = \beta = 2; \text{ and } s(x) = -5 + 11x - 6x^2 + x^3.$
The scaling function $\varphi$ can be defined as:

$$\varphi_m = \varphi_{1,m} := T_h \delta_m = h(L) \delta_m,$$

where $h : \mathbb{R}^+ \to \mathbb{R}^+$ acts as a low pass filter with $h(0) > 0$ and $h(x) \to 0$ as $x \to \infty$. An example is: $h(x) = \gamma \exp \left( -\left( x / 0.6\lambda_{\text{min}} \right)^4 \right)$ where $\gamma$ is set such that $h(0) = \max_{x \geq 0} g(x)$.

The scaling coefficient of a given function $f \in \mathcal{L}^2(V)$ is computed by

$$S_f(m) := \langle f, \varphi_m \rangle.$$

[Graph showing scaling function kernel $h(x)$, wavelet kernels $g(s, x)$, sum of squares $G$, and frame bounds.]

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For a discrete transform, sample the scale parameter $s$ in a logarithmically equispaced manner between $s_J = x_2/\lambda_{\text{max}}$ and $s_1 = x_2/\lambda_{\text{min}}$ where $\lambda_{\text{max}} \geq \lambda_{n-1}$, $\lambda_{\text{min}} = \lambda_{\text{max}}/K$ for some $K > 0$.

**Theorem (H-V-G)**

Given a set of scales $\{s_j\}_{1 \leq j \leq J}$, the set

$\mathcal{F} := \{\varphi_m\}_{1 \leq m \leq n} \cup \{\psi_{s_j,m}\}_{1 \leq j \leq J; 1 \leq m \leq n}$

constitutes a frame with bounds $A$, $B$ given by

$$A = \min_{\lambda \in [0, \lambda_{n-1}]} G(\lambda); \quad B = \max_{\lambda \in [0, \lambda_{n-1}]} G(\lambda)$$

where $G(\lambda) := h^2(\lambda) + \sum_{j=1}^{J} g^2(s_j \lambda)$.

Hence, for any $f \in \mathcal{L}^2(V)$, we have:

$$A \|f\|_2^2 \leq \sum_{m=1}^{n} \left( |S_f(m)|^2 + \sum_{j=1}^{J} |W_f(s_j, m)|^2 \right) \leq B \|f\|_2^2.$$
Work of Hammond-Vandergheynst-Gribonval . . .

- H-V-G proposed a fast transform instead of computing all of the graph Laplacian eigenvalues and eigenvectors that would require $O(n^3)$ operations.
- The fast algorithm requires $O(C \cdot |E| + C' \cdot Jn)$, where $C$, $C' > 0$ are some constants.
- The fast algorithm is based on the Chebyshev polynomial approximation $p(s_jx)$ to the function $g(s_jx)$ and fully utilizes the Chebyshev recurrence relation.
- Hence $W_f(s_j, m) \approx \delta_m^* p(s_jL) f$, i.e., done by matrix-vector products.
- It is especially effective if the graph (hence $L$) is sparse.
- As for the inverse transform, a stable algorithm exists because $\mathcal{F}$ forms a frame. As the frame theory indicates, it involves the pseudo inverse. Hence, it is not super fast even if one uses the conjugate gradient method.
- Software demo now!
The eigenvalue axis is not the same as the frequency axes particularly if a given graph comes from data from a topologically skewed shape or a narrow strip. In those cases, the eigenvalue orders are not intuitive.

The inverse transform is still slow even if one uses the CG method.
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The following slides are through the courtesy of R. R. Coifman, M. Gavish, and B. Nadler.
Haar-like bases
Haar-like bases
Haar-like bases
Work of Coifman-Gavish-Nadler

Haar-like bases

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Haar-like bases
With B. Nadler, Weizmann

\[ x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9 \]

\[ X \]
Haar-like bases
With B. Nadler, Weizmann
Haar-like bases
With B. Nadler, Weizmann
Haar-like bases

With B. Nadler, Weizmann
Haar-like bases
With B. Nadler, Weizmann
Haar-like bases
With B. Nadler, Weizmann

\[ \ell = 1 \]
\[ \ell = 2 \]
\[ \ell = 3 \]
Haar-like bases
With B. Nadler, Weizmann

\begin{figure}
\centering
\includegraphics[width=\textwidth]{haar-like_bases.png}
\end{figure}
Haar-like bases
With B. Nadler, Weizmann

\( \ell = 1 \)

\( \ell = 2 \)

\( \ell = 3 \)
Haar-like bases
With B. Nadler, Weizmann

\[ \ell = 1 \]
\[ \ell = 2 \]
\[ \ell = 3 \]

\[ \psi_1 \]
Haar-like bases

With B. Nadler, Weizmann
Haar-like bases
With B. Nadler, Weizmann

\[ \ell = 1 \]
\[ \ell = 2 \]
\[ \ell = 3 \]

\[ \psi_{2,2} \]
Haar-like bases
With B. Nadler, Weizmann

\[ \psi_{3,1} \]

\[ \ell = 1 \]
\[ \ell = 2 \]
\[ \ell = 3 \]
Haar-like bases
With B. Nadler, Weizmann
Haar-like bases
With B. Nadler, Weizmann
Haar-like bases
With B. Nadler, Weizmann
Haar-like bases
With B. Nadler, Weizmann

\[ \ell = 1 \]
\[ \ell = 2 \]
\[ \ell = 3 \]

[Diagram showing wavelet functions with levels \( \ell = 1, 2, 3 \) and function \( \psi_{3,5} \).]
Haar-like bases
With B. Nadler, Weizmann

\[ \ell = 1 \]
\[ \ell = 2 \]
\[ \ell = 3 \]
\[ \psi_{3,6} \]
Haar-like bases
With B. Nadler, Weizmann
Tensor product of Haar-like bases
Tensor product of Haar-like bases
Tensor product of Haar-like bases
Tensor product of Haar-like bases
Tensor product of Haar-like bases
Work of Coifman-Gavish-Nadler

Tensor product of Haar-like bases
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\[ = \sum_{i,j} a_{ij} \psi_i \otimes \varphi_j \]
Tensor Haar-like basis function
Is there any automatic algorithm to build a multiscale tree of graphs (i.e., multiscale graph coarsening) of a given original graph?

Can one develop smoother wavelets instead of Haar-like basis on a graph?

If so, when such smoother wavelets become critically useful? 

Maybe some network flow data, e.g.:
- flow measurements in river systems or drainage systems;
- biological system (e.g., pulse propagation on nervous systems)
- ...
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Another very interesting and recent idea is to adapt Donoho’s average interpolating wavelets to graph setting proposed by R. M. Rustamov: ArXiv:1110.2227v1 [math.FA], which should be studied very closely.

Very important to transfer harmonic and wavelet analysis techniques originally developed on the usual Euclidean spaces to graphs.

Discrete harmonic, analytic, Green’s functions on graphs have been developed (L. Lovasz, F. Chung, S.-T. Yau, . . .)

Cannot avoid tight interactions with mathematicians in different disciplines (discrete math, graph theory, optimization, numerical linear algebra, geometry & topology, PDEs, probability and statistics, . . .) and with domain experts (biology, sociology, electrical engineering, computer science, geology, geophysics, . . .)

Many interesting mathematics will come out from this endeavor!

There are many more proposals to construct wavelets on graphs that I could not cover today, e.g., diffusion wavelets of Coifman & Maggioni, an application of the lifting scheme to graphs by Jansen, Nason, & Silverman, . . .

Analysis of directed graphs needs more attention.

How about models based on quantum graphs?