MAT 280: Laplacian Eigenfunctions: Theory, Applications, and Computations
Lecture 4: Diffusions and Vibrations in 2D and 3D — I. Basics

Lecturer: Naoki Saito
Scribe: Brendan Farrell/Allen Xue

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The basic reference for this lecture is [1, Sec.10.1].

1 Wave Equation and Heat Equation

Consider a bounded domain $\Omega \subset \mathbb{R}^d, \quad d = 2, 3, \cdots$.

$$\begin{array}{cc} \underline{\text{wave equation}} & \underline{\text{heat equation}} \\ u_{tt} = c^2 \Delta u \quad \text{in } \Omega & u_t = k \Delta u \quad \text{in } \Omega \end{array} \tag{1}$$

with one of the three boundary conditions (BC) on $\partial\Omega$:

$$\begin{array}{llll} u = 0 & \text{(D)} & u = 0 & \text{(D)} \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{(N)} & \dfrac{\partial u}{\partial \nu} = 0 & \text{(N)} \\ \dfrac{\partial u}{\partial \nu} + au = 0 & \text{(R)} & \dfrac{\partial u}{\partial \nu} + au = 0 & \text{(R)} \end{array} \tag{2}$$

with initial conditions (IC):

$$\begin{array}{ll} u(\boldsymbol{x}, 0) = f(\boldsymbol{x}) & u(\boldsymbol{x}, 0) = f(\boldsymbol{x}) \\ u_t(\boldsymbol{x}, 0) = g(\boldsymbol{x}) & u_t(\boldsymbol{x}, 0) = g(\boldsymbol{x}). \end{array} \tag{3}$$

The abbreviations for the boundary conditions used here are: Dirichlet (D), Neumann (N), Robin (R). For the Robin BC, a is a constant.

We use the method of separation of variables and set $u(\boldsymbol{x},t) = T(t)v(\boldsymbol{x})$, which leads to the following equations

$$\begin{aligned} &\text{From wave equation:} \quad \frac{T''}{c^2T} = \frac{\Delta v}{v} = -\lambda. \\ &\text{From heat equation:} \quad \frac{T'}{kT} = \frac{\Delta v}{v} = -\lambda. \end{aligned} \tag{4}$$

Later in this lecture we will show that $\lambda \geq 0$, for at least either (D), (N), or (R) in (2) is satisfied.

Regardless of whether we consider the heat or the wave equation, we reach

$$\begin{gathered} -\Delta v = \lambda v \qquad \text{in } \ \Omega \\ \text{where } v \text{ satisfies either (D), (N), or (R).} \end{gathered} \tag{5}$$

Lots of mathematics are involved to prove that the set of λ satisfying (5) is discrete, i.e., $\lambda_1, \lambda_2, \cdots$, and there exist the corresponding eigenfunctions $\varphi_1, \varphi_2, \cdots$ that are mutually orthogonal. We'll cover those math later, but at this point, we assume the existence of $\lambda_1, \lambda_2, \cdots$ and $\varphi_1, \varphi_2, \cdots$. Once we have the eigenpairs $\{(\lambda_n, \varphi_n)\}_{n=1}^{\infty}$, we can write the solutions for (1) as

$$\begin{aligned} &\text{wave equation:} \quad u(\boldsymbol{x},t) = \textstyle\sum_{n=1}^{\infty} \left[A_n \cos(\sqrt{\lambda_n} ct) + B_n \sin(\sqrt{\lambda_n} ct)\right] \varphi_n(\boldsymbol{x}) \\ &\text{heat equation:} \quad u(\boldsymbol{x},t) = \textstyle\sum_{n=1}^{\infty} A_n \mathrm{e}^{-\lambda_n k t} \varphi_n(\boldsymbol{x}) \end{aligned} \tag{6}$$

where A_n and B_n are appropriate constants.

Preliminary: some important formulas used in the following sections:

- Divergence Theorem

$$\int_{\Omega} \nabla \cdot f \, \mathrm{d}\boldsymbol{x} = \int_{\partial\Omega} \boldsymbol{\nu} \cdot f \, \mathrm{d}S,$$

$\boldsymbol{\nu}$ is normal vector and $\mathrm{d}S$ is a surface measure on $\partial\Omega$.

- Green's first identity (G1): For $u, v \in C^2(\overline{\Omega})$,

$$\int_{\Omega} u \Delta v \, \mathrm{d}\boldsymbol{x} + \int_{\Omega} \nabla u \cdot \nabla v \, \mathrm{d}\boldsymbol{x} = \int_{\partial\Omega} u \frac{\partial v}{\partial \nu} \, \mathrm{d}S.$$

- Green's second identity (G2): For $u, v \in C^2(\overline{\Omega})$,

$$\int_\Omega (u\Delta v - v\Delta u)\,\mathrm{d}\boldsymbol{x} = \int_{\partial\Omega} \left(u\frac{\partial v}{\partial \nu} - v\frac{\partial u}{\partial \nu} \right)\,\mathrm{d}S.$$

- The definition of the directional derivative along $\boldsymbol{\nu}$:

$$\frac{\partial}{\partial \nu} \triangleq \boldsymbol{\nu} \cdot \nabla \tag{7}$$

2 Orthogonality of the Eigenfunctions

Define the inner-product

$$\langle f, g \rangle \triangleq \int_\Omega f(\boldsymbol{x})\overline{g(\boldsymbol{x})}\,\mathrm{d}\boldsymbol{x}, \quad \text{where } \Omega \in \mathbb{R}^d, \quad \mathrm{d}\boldsymbol{x} = \mathrm{d}x_1\,\mathrm{d}x_2 \ldots \mathrm{d}x_d.$$

Consider two functions $u, v \in C^2(\overline{\Omega})$, with $\overline{\Omega} = \Omega \bigcup \partial\Omega$, ($C^2$ condition can be weakened), we have

$$u\Delta v - (\Delta u)v = \nabla \cdot [u\nabla v - (\nabla u)v].$$

Then integrate both sides in Ω:

$$\begin{aligned} \int_\Omega (u\Delta v - (\Delta u)v)\,\mathrm{d}\boldsymbol{x} &= \int_\Omega \nabla \cdot [u\nabla v - (\nabla u)v]\,\mathrm{d}\boldsymbol{x} \\ &\overset{(a)}{=} \int_{\partial\Omega} \nu \cdot [u\nabla v - (\nabla u)v]\,\mathrm{d}S \\ &\overset{(b)}{=} \int_{\partial\Omega} \left(u\frac{\partial v}{\partial \nu} - v\frac{\partial u}{\partial \nu} \right)\,\mathrm{d}S, \end{aligned} \tag{8}$$

where (a) is derived by divergence theorem, and (b) is from the definition (7).

Now we can show that any $u, v \in C^2(\overline{\Omega})$ satisfying either (D), (N), or (R) also satisfy

$$\langle u, \Delta v \rangle = \langle \Delta u, v \rangle.$$

Proof. Equation (8) is equivalent to

$$\langle u, \Delta v \rangle - \langle \Delta u, v \rangle = \int_{\partial\Omega} \left(u\frac{\partial v}{\partial \nu} - v\frac{\partial u}{\partial \nu} \right)\,\mathrm{d}S.$$

If u and v satisfy (D) or (N), it is obvious that the above is equal to 0.
If u and v satisfy (R), we get

$$u\frac{\partial v}{\partial \nu} - v\frac{\partial u}{\partial \nu} = u(-av) - v(-au) = 0.$$

□

Therefore each of these three classical BC's is symmetric. Suppose both u, v are real eigenfunctions satisfying

$$-\Delta u = \lambda_1 u, \qquad -\Delta v = \lambda_2 v$$

and satisfying either (D), (N), or (R). Then λ_1, λ_2 are reals, and if $\lambda_1 \neq \lambda_2$, then $\langle u, v\rangle = 0$.

Proof.

$$\begin{aligned}
\lambda_1\langle u, u\rangle &= \langle \lambda_1 u, u\rangle \\
&= \langle -\Delta u, u\rangle \\
&= \langle u, -\Delta u\rangle \\
&= \langle u, \lambda_1 u\rangle \\
&= \overline{\lambda_1}\langle u, u\rangle,
\end{aligned}$$

which implies $(\lambda_1 - \overline{\lambda_1})\|u\|_2^2 = 0$. Since $\|u\|_2 \neq 0$, $\lambda_1 = \overline{\lambda_1} \Leftrightarrow \lambda_1 \in \mathbb{R}$. Similarly,

$$\begin{aligned}
\lambda_1\langle u, v\rangle - \lambda_2\langle u, v\rangle &= \langle \lambda_1 u, v\rangle - \langle u, \lambda_2 v\rangle \\
&= \langle -\Delta u, v\rangle - \langle u, -\Delta v\rangle \\
&= \langle u, -\Delta v\rangle - \langle u, -\Delta v\rangle \\
&= 0
\end{aligned}$$

which implies $(\lambda_1 - \lambda_2)\langle u, v\rangle = 0$. Since $\lambda_1 \neq \lambda_2$, $\langle u, v\rangle = 0$. □

We summarize the information above with the following theorem.

Theorem 2.1. *In the eigenvalue problem* (5)*, we have the following facts:*

- *all the eigenvalues are real*
- *the eigenfunctions can be chosen to be real-valued*
- *the eigenfunctions corresponding to distinct eigenvalues are necessarily orthogonal*
- *all the eigenfunctions can be chosen to be orthogonal, i.e., orthonormal.*

3 Multiplicity of the Eigenvalues

Definition 3.1. An eigenvalue λ has *multiplicity* m if it has m linearly independent eigenfunctions. The *eigenspace* E_λ is a linear space spanned by the set of eigenfunctions corresponding to λ. So, in this case $\dim(E_\lambda) = m$.

Notice: if $\dim(E_\lambda) = m$, and $E_\lambda = \text{span}\{w_1, ..., w_m\}$, but $\langle w_i, w_j\rangle \neq \delta_{ij}$, then we can use the Gram-Schmidt orthogonalization method to get

$$E_\lambda = \text{span}\{\varphi_1, ..., \varphi_m\}, \quad \text{with } \langle \varphi_i, \varphi_j\rangle = \delta_{ij}.$$

4 Generalized Fourier Series

Because of the Theorem 2.1, we have for $f \in L^2(\Omega)$

$$f(\boldsymbol{x}) = \sum_{n=1}^{\infty} f_n \varphi_n(\boldsymbol{x}), \quad f_n = \langle f, \varphi_n\rangle.$$

This is a generalization of the Fourier series, and we can discuss the decay of $\{f_n\}$, etc.

Theorem 4.1. *As in* (5)*, let λ_k be the Dirichlet-Laplacian eigenvalues, let ν_k be the Neumann-Laplacian eigenvalues, and let ρ_k be the Robin-Laplacian eigenvalues, where $k \in \mathbb{N}$. Then*

$$\lambda_k > 0, \quad \nu_k \geq 0, \quad \textit{and } \rho_k \geq 0, \textit{ if } a \geq 0.$$

Proof. Let u and v are corresponding eigenfunctions. Use Green's first identity (G1):

$$\int_\Omega u \Delta v \, \mathrm{d}\boldsymbol{x} + \int_\Omega \nabla u \cdot \nabla v \, \mathrm{d}\boldsymbol{x} = \int_{\partial\Omega} u \frac{\partial v}{\partial \nu} \, \mathrm{d}S,$$

Set $v = u$,

$$\int_\Omega u \Delta u \, \mathrm{d}\boldsymbol{x} + \int_\Omega |\nabla u|^2 \, \mathrm{d}\boldsymbol{x} = \int_{\partial\Omega} u \frac{\partial u}{\partial \nu} \, \mathrm{d}S,$$

For the boundary condition (D),

$$\int_\Omega u(-\lambda u) \, \mathrm{d}\boldsymbol{x} + \int_\Omega |\nabla u|^2 \, \mathrm{d}\boldsymbol{x} = 0 \;\Rightarrow\; \lambda = \frac{\int_\Omega |\nabla u|^2 \, \mathrm{d}\boldsymbol{x}}{\int_\Omega u^2 \, \mathrm{d}\boldsymbol{x}} \geq 0.$$

But $|\nabla u|^2 \neq 0$. Since if so, $u = \text{const}$, then $u \equiv 0$, which conflicts with the fact that u is eigenfunction. Therefore, $\lambda > 0$.

For the boundary condition (N),

$$\nu = \frac{\int_\Omega |\nabla u|^2 \,\mathrm{d}\boldsymbol{x}}{\int_\Omega u^2 \,\mathrm{d}\boldsymbol{x}} \geq 0.$$

Here $|\nabla u|^2 = 0$ is acceptable, i.e., $u \equiv \text{const} \neq 0$. Then $\nu \geq 0$, where $\nu = 0$ corresponds to the eigenfunction $\varphi_0(x) \equiv \text{const} \neq 0$.

For the boundary condition (R), we have

$$-\rho \int_\Omega |u|^2 \,\mathrm{d}\boldsymbol{x} + \int_\Omega |\nabla u|^2 \,\mathrm{d}\boldsymbol{x} = \int_{\partial\Omega} u(-au) \,\mathrm{d}S$$

$$\Rightarrow \rho = \frac{a \int_{\partial\Omega} |u|^2 \,\mathrm{d}S + \int_\Omega |\nabla u|^2 \,\mathrm{d}\boldsymbol{x}}{\int_\Omega |u|^2 \,\mathrm{d}\boldsymbol{x}} \geq 0, \qquad \text{if } a \geq 0.$$

□

5 Completeness of $\{\varphi_n\}_{n\in\mathbb{N}}$ in the L^2-sense

See [2, Sec.3.3-3.4] and [3, Chapter 4] for the elementary discussion on the completeness of a set of basis in $L^2(\Omega)$.

For all $f \in L^2(\Omega)$, we have

$$\left\| f - \sum_{n=1}^{N} f_n \varphi_n \right\|_{L^2} \to 0, \quad \text{as } N \to \infty.$$

Equivalently,

$$f = \sum_{n=1}^{\infty} f_n \varphi_n \qquad \text{in the } L^2 \text{ sense.}$$

This is important because if it were not the case, we could not represent arbitrary $L^2(\Omega)$ function in terms of $\{\varphi_n\}_{n\in\mathbb{N}}$. We will discuss more about the completeness later.

Example: Diffusion in a 3D cube.
Let $\Omega = Q = \{(x, y, z) \,|\, 0 < x < \pi,\ 0 < y < \pi,\ 0 < z < \pi\}$.

$$\begin{cases} \text{DE :} & u_t = k\varDelta u & \text{in } \Omega \\ \text{BC :} & u = 0 & \text{on } \partial\Omega \\ \text{IC :} & u = f(\boldsymbol{x}) & \boldsymbol{x} \in \Omega,\ \ t = 0 \end{cases}$$

Then by the separation of variables, let $u(\boldsymbol{x}, t) = T(t) \cdot v(\boldsymbol{x})$ as before, we have

$$-\varDelta v = \lambda v \quad \text{in } Q, \qquad \text{with } \ v = 0 \quad \text{on } \partial Q.$$

Because the sides of Q are parallel to the axes, one can do the separation of variables again.

$$\begin{aligned} & v(x, y, z) = X(x)Y(y)Z(z) \\ \Rightarrow \quad & \frac{X''}{X} + \frac{Y''}{Y} + \frac{Y''}{Y} = -\lambda. \end{aligned}$$

BC's are also separated as

$$X(0) = X(\pi) = Y(0) = Y(\pi) = Z(0) = Z(\pi) = 0.$$

Therefore, we can solve $v(x, y, z)$ for $(l, m, n) \in \mathbb{N}^3$,

$$\begin{aligned} v(x, y, z) &= \sin(lx)\sin(my)\sin(nz) \\ &= v_{l,m,n}(\boldsymbol{x}) \end{aligned}$$

whose orthonormal version is $(\frac{2}{\pi})^{3/2}\sin(lx)\sin(my)\sin(nz)$.
Then

$$\lambda = \lambda_{l,m,n} = l^2 + m^2 + n^2$$

Finally we get the solution

$$u(\boldsymbol{x}, t) = \sum_{l,m,n} A_{lmn} \mathrm{e}^{-(l^2+m^2+n^2)kt} \sin(lx)\sin(my)\sin(nz)$$

where $A_{l,m,n} = (\frac{2}{\pi})^3 \langle f, v_{l,m,n} \rangle$.

Here, different values for l, m, n can result in the same eigenvalue. For example, $\lambda = 27$. The valid values for (l, m, n) are $(5, 1, 1)$, $(1, 5, 1)$, $(1, 1, 5)$, and $(3, 3, 3)$. In other words, the multiplicity of $\lambda = 27$ is four.

References

[1] W. A. STRAUSS, *Partial Differential Equations: An Introduction*, John Wiley & Sons, 1992.

[2] G. B. FOLLAND, *Fourier Analysis and Its Applications*, Brooks/Cole, 1992.

[3] N. YOUNG, *An Introduction to Hilbert Space*, Cambridge Univ. Press, 1988.