MAT 280: Laplacian Eigenfunctions: Theory, Applications, and Computations
Lecture 7: Nodal Sets

Lecturer: Naoki Saito
Scribe: Bradley Marchand/Allen Xue

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Consider the eigenfunctions $v(\boldsymbol{x})$ of the Laplacian,

$$-\Delta v = \lambda v \ \text{ in } \Omega, \tag{1}$$

with Dirichlet, Neumann, or Robin boundary conditions. We define the nodal set, $\mathcal{N}$, as follows:

Definition 1.1. The nodal set, $\mathcal{N}$, is the set of points in Ω such that the eigenfunctions of (1) are zero. Thus,

$$\mathcal{N} \stackrel{\Delta}{=} \{\boldsymbol{x} \in \Omega \,|\, v(\boldsymbol{x}) = 0\}$$

Note that since Ω is open, no point on the boundary of Ω is in $\mathcal{N}$ (i.e., $\partial\Omega \not\subset \mathcal{N}$).

Nodal sets are important because they allow us to visualize the sets where $v(\boldsymbol{x}) > 0$ and $v(\boldsymbol{x}) < 0$. They mark a natural division of the domain into regions. It is this property that interested a German physicist named Ernst Chladni [6]. Chladni's most well known work was developing techniques for demonstrating the various modes of vibration in a surface. By drawing a bow over a piece of metal lightly covered in sand, Chladni was able to visually show the nodal sets for varying frequencies. Similar techniques are particularly important in making musical instruments, since the symmetries in the nodal lines can be interpreted as a measure of "quality."

To gain a better understanding of nodal sets let us consider a few examples.

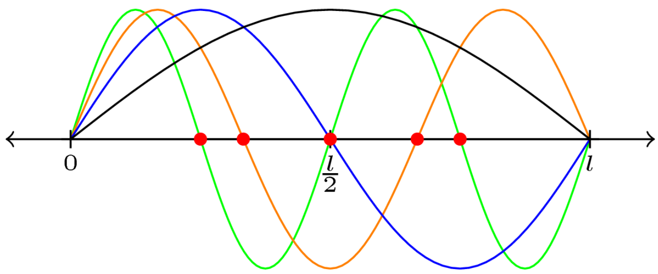

Figure 1: First four eigenfunctions for 1D Dirichlet-Laplacian eigenvalue problem in the domain of $\Omega = (0, \ell)$. Red points are Nodal nodes.

Example 1.2. *1D Vibrating String*

Earlier we showed that the eigenfunctions for the 1D Dirichlet-Laplacian eigenvalue problem in the domain of $\Omega = (0, \ell)$ are

$$v(x) = \sqrt{\frac{2}{\ell}} \sin\left(\frac{n\pi}{\ell} x\right) = \varphi_n(x), \quad n \in \mathbb{N}. \tag{2}$$

From the eigenfunctions we can see that

$$\mathcal{N}_n = \text{ Nodal set for } \varphi_n \ = \left\{ x = \frac{k\ell}{n} \,\middle|\, k = 1, 2, \dots, n-1 \right\}.$$

Note that $k = 0$ and $k = n$ are not in $\mathcal{N}$ since they are on the boundary.

The nodes in the nodal set can also be given a physical interpretation. Let us consider a wave equation with the Dirichlet boundary condition. We already know that

$$u(\boldsymbol{x}, t) = (A \cos \sqrt{\lambda} ct + B \sin \sqrt{\lambda} ct) v(\boldsymbol{x}) \tag{3}$$

is a solution for the wave equation (no initial condition is fixed at this point). Here $v(\boldsymbol{x})$ is an eigenfunction of the Dirichlet-Laplacian.

Then for all $\boldsymbol{x} \in \mathcal{N}$, $u(\boldsymbol{x}, t) \equiv 0$ for all $t \geq 0$. Thus, all the nodes in the nodal set are stationary points for all time. This has physical implications if we consider a guitar string. If the guitar player holds his finger on the middle of the string then the only tones that have nodes at $x = \frac{\ell}{2}$ survive (see Figure 2), which means

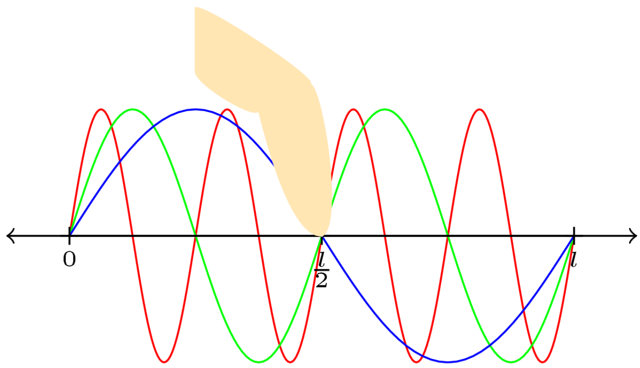

Figure 2: If a guitar string is held in the middle, only eigenfunctions with nodal points in the middle are possible. Thereby, limiting the possible sounds that the string can produce.

that the frequencies $\frac{n\pi c}{\ell}$ with even n survive and that the frequencies with odd n cannot happen.

Example 1.3. *The Square in 2D*

Let us consider the eigenfunctions of Dirichlet-Laplacian on a 2D square, i.e., $\Omega = \{(x, y) \,|\, 0 < x < \pi,\ 0 < y < \pi\}$. Previously, we found that the eigenfunctions and eigenvalues are

$$\begin{cases} \varphi(x,y) &= A \sin nx \sin my \\ \lambda_{nm} &= n^2 + m^2 \end{cases} \qquad \text{with } n, m \in \mathbb{N}. \tag{4}$$

Since we know what the eigenvalues and functions are, we can tabulate them in order of increasing eigenvalues. For eigenvalues with multiple eigenfunctions (i.e. eigenvalues with multiplicity) we consider a linear combination of the eigenfunctions.

For example, there are two corresponding eigenfunctions φ_{12} and φ_{21} for $\lambda = 5$. Thus, we write $\varphi(x, y) = A\varphi_{12}(x, y) + B\varphi_{21}(x, y)$ as the eigenfunction for $\lambda = 5$. Table 3 contains a few of the tabulated eigenvalues and functions. Note that by ordering the eigenvalues in this fashion we are able to index the eigenvalues and eigenfunctions with one index instead of two. See the Table 3.

There are some interesting questions we would like to consider. First, how many

λ_n	$\varphi_n(x, y)$
2	$A \sin x \sin y$
5	$A \sin 2x \sin y + B \sin x \sin 2y$
8	$A \sin 2x \sin 2y$
10	$A \sin 3x \sin y + B \sin x \sin 3y$

Figure 3: Table of eigenvalues and eigenfunctions for Dirichlet-Laplacian problem on the 2D square whose side length is π, where A, B are appropriate coefficients.

ways can a given integer λ be written as the sum of two squares of integers? Secondly, we want to ask "what do the nodal sets corresponding to different eigenvalues look like?"

The first question is answered by Number Theory. Specifically, we know the following:

Theorem 1.4. *A positive integer n can be written as the sum of two squares if and only if no prime congruent to 3 modulo 4 appears an odd number of times in the factorization of n into primes.*

See [2, Chap. 7.2] for details.

Let us investigate a bit about the second question. First we consider eigenvalues of single multiplicity. Since the eigenvalue is of single multiplicity we know that $\varphi(x, y) = A \sin nx \sin my$ for some n, m in $\mathbb{N}$. Thus, we can easily see that the nodal set will be the set $\{x : x = \pi/p\} \cup \{y : y = \pi/q\}$ where $p, q \in \mathbb{N}$ such that $1 < p < n$ divides n and $1 < q < m$ divides m. Those interested in the number of divisors of n and m can once again look to Number Theory for the answer [2, Chap. 3]. In the case where $\lambda = 2$, we get the lines $x = \pi/2$ and $y = \pi/2$.

For eigenvalues of double multiplicity the nodal sets can be much more exotic since we have a linear combination of two eigenfunctions. When A or B is zero, we have the solutions that were discribed above. For different values of A and B we get different looking nodal sets. Images of nodal sets for $\lambda = 10$ can be seen in Figure 4. More images of nodal sets on the square can be seen in [3, Sec. 10.4]. Trefethen [5, Chap. 11] provides insight into the programming involved in solving certain differential equations numerically.

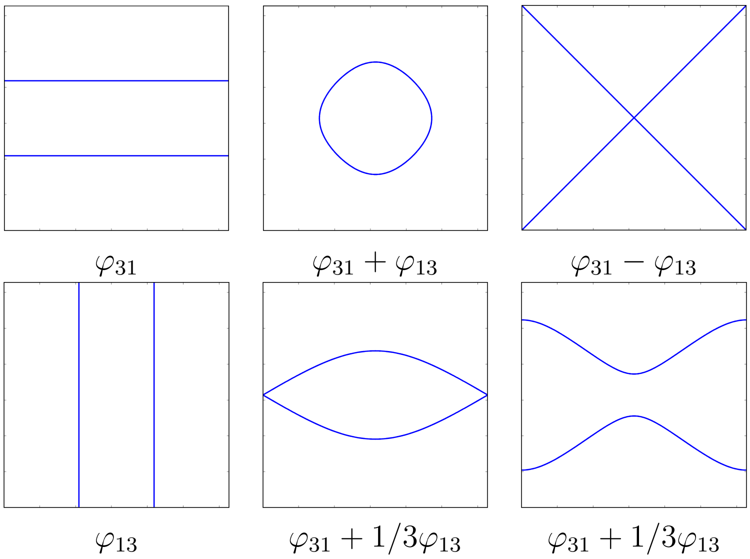

Figure 4: Some nodal sets for $\lambda = 10$ of the Dirichlet-Laplacian on the square.

Example 1.5. *The 3D Ball*

Recall that the eigenfunctions for the Dirichlet-Laplacian on a 3D ball, i.e., $\Omega = \{(x, y, z) \,|\, x^2 + y^2 + z^2 < a^2\}$ are

$$\varphi_{\ell mj}(r, \theta, \phi) = \frac{J_{\ell+1/2}(\sqrt{\lambda_{\ell j}}r)}{\sqrt{r}} P_\ell^m(\cos\theta)(A\cos m\phi + B\sin m\phi). \tag{5}$$

where we use the spherical coordinates, which were introduced in Lecture 6.

The nodal set for $\varphi_{\ell mj}$ is a union of the following kinds of surfaces:

(i) Spheres inside Ω, (corresponding to the zeros of the Bessel functions);

(ii) Vertical planes, i.e., $\phi = \text{const}$; and

(iii) Horizontal planes, i.e., $\theta = \text{const}$,

which in fact contains $j-1$ spheres, m vertical planes and $\ell - m$ horizontal planes.

So, how many regions can the nodal set (of the Dirichlet-Laplacian) divide a general domain Ω into (assuming Ω is connected)? The following theorem limits the possibilities.

Theorem 1.6 (Courant Nodal Domain Theorem)**.**

(i) *The first eigenfunction, $\varphi_1(x)$ corresponding to the smallest eigenvalue, λ_1, cannot have any nodes.*

(ii) *For $n \geq 2$, $\varphi_n(x)$ corresponding to the nth eigenvalue counting multiplicity, divides the domain Ω into at least 2 and at most n pieces.*

Discussions on nodal sets and the "Courant Nodal Domain Theorem" can be found in [1, Vol. I, Sec. V.5, VI.6] and [3, Sec. 10.4]. We will prove this later in this course when we deal with the general eigenvalue problem using the calculus of variations.

At this point it is easy to show that $\varphi_n(\boldsymbol{x})$ must divide Ω into at least 2 pieces if $n \geq 2$ if we assume (*i*) is correct.

Proof. We know that φ_1 is perpendicular to φ_n for $n \geq 2$. So,

$$\int_\Omega \varphi_1(\boldsymbol{x})\varphi_n(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} = 0$$

and from (*i*) we know that $\varphi_1(\boldsymbol{x}) > 0$ or $\varphi_1(\boldsymbol{x}) < 0$ in Ω. Thus, $\varphi_n(\boldsymbol{x})$ must change its sign in Ω. So there exist zeros of $\varphi_n(\boldsymbol{x})$ in Ω by the continuity of φ_n. These zeros form a nodal set. $\square$

REMARK:

In 1950, Szegő conjectured a similar theorem to the Courant Nodal Domain Theorem for the biharmonic eigenvalue problem,

$$\begin{cases} \Delta^2 u = \lambda u & \text{in } \Omega, \\ u = \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega. \end{cases} \tag{6}$$

Szegő's conjecture was the following:

Conjecture 1.7 (Szegő, 1950)**.** *If $\Omega \subset \mathbb{R}^2$ is a "nice" domain (i.e., $\partial\Omega$ is an analytic curve), then φ_1 for* (6) *does not change its sign.*

However, surprisingly, the conjecture is not even true for the first eigenfunction. For the details and its history including Szegő's conjecture see [4].

References

[1] R. COURANT AND D. HILBERT, *Methods of Mathematical Physics*, Wiley-Interscience, 1953.

[2] C. V. EYNDEN, *Elementary Number Theory*, McGraw-Hill, 2001.

[3] W. A. STRAUSS, *Partial Differential Equations: An Introduction*, John Wiley & Sons, 1992.

[4] G. SWEERS, "When is the first eigenfunction for the clamped plate equation of fixed sign?", *Election. J. Diff. Eqns.*, Conf. 06, 2001, pp. 285-296.

[5] L. N. TREFETHEN, *Spectral Methods in MATLAB*, SIAM, 2000.

[6] To know about Ernst Chladni, search on http://en.wikipedia.org/