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Fractals, trees and the Neumann Laplacian

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1 Introduction

In [3] the embedding map E of the Sobolev space $W^{1,p}(\Omega)$ into $L^p(\Omega)$ was studied for a new class of domains Ω in $\mathbf{R}^n$ with irregular boundaries. These were called *generalized ridged domains* and include 'rooms and passages', 'interlocking combs', finite and infinite spirals, trumpets and horns. A characteristic feature of these domains is that they possess what is called a *generalized ridge*, this being a Lipschitz curve which roughly forms a central axis of the domain. The idea was motivated by the ridge of a set in $\mathbf{R}^2$. Estimates were obtained for the quantity

$$\beta(E) := \inf\{\|E - P\| : P \in \mathscr{F}(W^{1,p}(\Omega), L^p(\Omega))\},$$

where $\mathscr{F}(W^{1,p}(\Omega), L^p(\Omega))$ denotes the set of linear maps from $W^{1,p}(\Omega)$ into $L^p(\Omega)$ which are bounded and have finite rank, and, in particular, these give a necessary and sufficient condition for E to be compact. In a subsequent paper [4] estimates for the approximation numbers of E were given; the case $p = 2$ is of special interest as then the approximation numbers of E are related to the eigenvalues of the Neumann Laplacian $-\Delta_{\Omega,N}$ of Ω.

There is a lot of current interest in domains with 'fractal' boundaries, especially in the problem of determining the asymptotic distribution of the eigenvalues of the Dirichlet and Neumann Laplacians on such domains (see [7, 8, 10]). The term fractal is taken to mean that the Hausdorff, or more appropriately it seems, the Minkowski dimension of the boundary takes a value in $(n-1, n]$. It is known that in some cases the second term in the asymptotic formula for the function $N(\lambda)$, which denotes the number of eigenvalues less than λ, depends on the Minkowski dimension and the Minkowski content of the boundary. In the case of the Neumann Laplacian it is important to know at the outset whether or not the spectrum of the operator is discrete. For examples like the Koch 'snowflake' this is a consequence of the fact that the domain is a quasi-disc and therefore has the $W^{1,p}$-extension property (see [11, Sect. 1.5.1]). The map E is then compact and hence the spectrum is discrete. As far as we know domains which do not possess the $W^{1,p}$-extension property lie outside existing theories and this is what partly motivates the present paper. We define a class of domains which have similar properties to the generalized ridged domains of [3] but now the Lipschitz curve which constitutes the generalized ridge in [3] is replaced by a tree. We show that the ideas in [3] can be extended to this problem and give as before estimates for the quantity $\beta(E)$; again a special case is a necessary and sufficient condition for E to be compact. As in [3] our objective was to define a class of domains which exhibit the pathological characteristics in a natural way and which are amenable to precise and manageable results. We contend that the example considered in Sect. 6 confirms that this has been achieved.

Our method for analysing the embedding $E: W^{1,p}(\Omega) \hookrightarrow L^p(\Omega)$ hinges on the result that the properties of E for the class of domains considered are related to those of an embedding map between weighted function spaces defined on a tree Γ. This problem on a tree is of independent interest and is discussed in some detail in Sect. 2. Our main requirement is an analogue of the celebrated result due to Muckenhaupt and others concerning weighted integral inequalities for operators of Hardy type; however serious difficulties are presented by the tree in general and we have been forced to make an assumption to obtain a two-sided inequality. The generalized ridged domains are defined in Sect. 3 and in Sect. 4 precise estimates are obtained which give necessary and sufficient conditions for E to be compact and also for $\beta(E) < 1$. The next step is to determine the asymptotic distribution of the approximation numbers and in Sect. 5 we go some way towards obtaining analogues of the Dirichlet–Neumann bracketing results which provide such a powerful method for analysing the distribution of eigenvalues in the case $p = 2$. In Sect. 6 we apply our abstract results to an example on a self-similar domain Ω in R^2 which does not have the $W^{1,p}$-extension property. We obtain asymptotic bounds for functions defined in Sect. 5 which determine the distribution of Dirichlet and Neumann approximation numbers for $p \in (1, \infty)$ and in the case $p = 2$ we also derive asymptotic formulae for the functions $\mathcal{N}(\lambda; -\Delta_{\Omega,D})$, $\mathcal{N}(\lambda; -\Delta_{\Omega,N})$ which count the eigenvalues of the negative Dirichlet and Neumann Laplacians respectively which are less than λ. Of particular interest is the Neumann problem when the boundary of Ω has outer Minkowski dimension $d_o \in (1, 2)$. We prove that as $\lambda \to \infty$,

$$\mathcal{N}(\lambda; -\Delta_{\Omega,N}) - (1/4\pi)|\Omega|\lambda \asymp \lambda^{d_o/2} ,$$

where $\asymp$ indicates that the quotient of the two sides is bounded above and below by positive constants. In this case the boundary of Ω has inner Minkowski dimension d_i satisfying $1 < d_i < d_o$ and as $\lambda \to \infty$,

$$\mathcal{N}(\lambda; -\Delta_{\Omega,D}) - (1/4\pi)|\Omega|\lambda = O(\lambda^{d_i/2}) .$$

This is also a consequence of Lapidus' result in [10, Corollary 2.1] since $\partial\Omega$ has finite d_i-dimensional upper Minkowski content relative to Ω.

An important contribution to the study of the spectral asymptotics of the Neumann problem for elliptic operators on domains with irregular boundaries was made by Métivier in [12], and this had a major influence on Lapidus' work in [10]. Another noteworthy reference for this problem is [6]. To see the extent of the degree of pathology possible in the spectrum of the Neumann Laplacian, one should consult the paper [9] by Hempel, Seco and Simon where it is proved that any closed subset of the non-negative real axis is the essential spectrum of the

Neumann Laplacian on some subdomain of the unit ball. Estimates in the literature for the approximation numbers of E for general $p \in (1, \infty)$ are usually for domains with smooth boundaries. For domains with a $W^{1,p}(\Omega)$ extension property, two-sided bounds were obtained by El Kolli in [2] for the Kolmogorov numbers of E: the Kolmogorov and approximation numbers are equal when $p = 2$ and relations exist between them in general; see [1, Sect. II.3] for details.

2 Analysis on trees

Let Γ be a tree i.e. a connected graph without loops or cycles where the edges are nondegenerate closed line segments whose end-points are vertices. We shall assume that Γ contains an infinite number of vertices and that each vertex is of finite degree, i.e. only a finite number of edges emanate from each vertex. For every $x, y \in \Gamma$ there is a unique polygonal path in Γ which joins x and y. The distance between x and y is defined to be the length of this polygonal path and in this way Γ is endowed with a metric topology.

Lemma 2.1. *Let $\tau(\Gamma)$ be the metric topology on Γ. Then*
(i) *a set $A \subset \Gamma$ is compact if and only if it is closed and meets only a finite number of edges;*
(ii) *$\tau(\Gamma)$ is locally compact;*
(iii) *Γ is the union of a countable number of vertices and edges. Thus if Γ is endowed with the natural 1-dimensional Lebesgue measure it is a σ-finite measure space.*

Proof. (i) Let A be compact and hence closed. Suppose A meets an infinite number of edges and choose a point t_k of A on each of these edges. A subsequence of (t_k) converges to a point t lying on some edge of Γ. But this would imply that in each neighbourhood of t there exists an infinity of edges, contradicting the assumption that only a finite number of edges meet at a vertex.

Conversely, let A be closed and meet only a finite number of edges. Then the intersection of A with each edge is compact and A is the union of a finite number of compact sets. It is therefore compact.
(ii) Any point a on Γ lies on only a finite number of edges. Take a closed neighbourhood of a with diameter less than the distance from a to the nearest vertex different from a. This is compact from (i).
(iii) Let A be the mid-point of an edge. The set of finite sequences of vertices $X_1, \ldots, X_k$ which lie on the path joining A to X_k is uniquely determined by X_k. Since each vertex has finite degree the result follows.

For $a \in \Gamma$ we define $t \succeq_a x$ (or equivalently $x \preceq_a t$) to mean that $x \in \Gamma$ lies on the path from a to $t \in \Gamma$. This is a partial ordering on Γ and the ordered graph so formed is referred to as the tree *rooted* at a and denoted by $\Gamma(a)$. If a is not a vertex of Γ we make it one by replacing the edge on which it lies by two. In this way every rooted tree $\Gamma(a)$ is the unique finite union of subtrees $\Gamma_i(a)$ $(i \in I_a)$ rooted at a, any two of which intersect only at a. The degree of a is

$$|I_a| := \# I_a < \infty \; ; \tag{2.1}$$

note that if a was not an original vertex of Γ then $|I_a| = 2$.

The path joining two points $x, y \in \Gamma$ may be parameterized by $s(t) := \text{dist}(x, t)$ and for $g \in L^1_{\text{loc}}(\Gamma)$ we have

$$\int_x^y g(t)dt = \int_{t \preceq_x y} g(t)dt = \int_0^{\text{dist}(x,y)} g(t(s))ds .$$

If F is locally Lipschitz on Γ and $G(s) = F(t(s))$, then G is Lipschitz on $[0, \text{dist}(x, y)]$ and

$$F(y) - F(x) = \int_0^{\text{dist}(x,y)} G'(s)ds ; \tag{2.2}$$

we may suppose that $G'(s)$ is defined everywhere by $G'(s) = \limsup_{n \to \infty} (n[G(s + n^{-1}) - G(s)])$. If μ is a measure defined on the path joining x, y which is absolutely continuous with respect to Lebesgue measure, it induces a measure on $[0, \text{dist}(x, y)]$ and we define

$$\int_x^y \phi\mu(dF) := \int_0^{\text{dist}(x,y)} G'(s)\phi(t(s))d\mu(t(s)) \tag{2.3}$$

provided ϕ is μ-integrable. Thus, if μ is a Borel measure on Γ with respect to which Lebesgue measure dt is absolutely continuous then

$$F(y) - F(x) = \int_x^y \frac{dt}{d\mu} \mu(dF) . \tag{2.4}$$

Note that $G'(s)$ depends only on $t(s)$ (except possibly at vertices of Γ) and the direction in which the edge in which it lies is described as s increases. Except possibly at the vertices of Γ, the modulus $|G'(s)|$ depends only on $t(s)$ and we denote it by $|F'(t)|$. Then

$$\int_0^{\text{dist}(x,y)} |G'(s)|ds = \int_x^y |F'(t)| \frac{dt}{d\mu} d\mu(t) . \tag{2.5}$$

We also note the following: if $s = \text{dist}(a, x)$ and $\theta \in L^1(\Gamma)$ then
(i) $\phi(s) = \int_a^x \theta(t)dt$ is an absolutely continuous function of s and $\phi'(s) = \theta(x(s))$ a.e.,
(ii) $\psi(s) = \int_{t \succeq_a x} \theta(t)dt$, the integral over the set $\{t : t \in \Gamma, x \preceq_a t\}$, is an absolutely continuous function on the interior of each edge and $\psi'(s) = -\theta(x(s))$ a.e.

Let $L^p(\Gamma, d\mu)$, $(1 \leqq p \leqq \infty)$ denote the set of complex-valued functions F on Γ which are measurable with respect to μ and for which $\|F\|_{p,\Gamma,d\mu} < \infty$, where

$$\|F\|_{p,\Gamma,d\mu} := \begin{cases} (\int_\Gamma |F(t)|^p d\mu(t))^{1/p} & \text{if } 1 \leqq p < \infty , \\ \mu - \sup_\Gamma |F(t)| & \text{if } p = \infty ; \end{cases}$$

when μ is Lebesgue measure we denote the space and norm by $L^p(\Gamma)$ and $\|\cdot\|_{p,\Gamma}$ respectively. Let $L_0^{1,p}(\Gamma, d\mu)$ denote the set of functions F which are locally Lipschitz on Γ and $|F'| \in L^p(\Gamma, d\mu)$. Then $\|F'\|_{p,\Gamma,d\mu}$ defines a pseudo-norm on $L_0^{1,p}(\Gamma, d\mu)$. Moreover, if $p \neq 1$ and $\psi := dt/d\mu \in L^{p'}_{\text{loc}}(\Gamma, d\mu)$, $p' = p/(p-1)$, we have

$$\begin{aligned} |F(y) - F(a)| &\leqq \int_0^{\text{dist}(x,a)} |G'(s)|ds = \int_a^y |F'(t)|\psi(t)d\mu(t) \\ &\leqq \left(\int_a^y |\psi(t)|^{p'} d\mu(t)\right)^{1/p'} \|F'\|_{p,\Gamma,d\mu} . \end{aligned} \tag{2.6}$$

Hence, if $\psi \in L^{p'}(\Gamma, d\mu)$, $|F(\cdot) - F(a)|$ is bounded on $L_0^{1,p}(\Gamma, d\mu)$ for any fixed $a \in \Gamma$. Furthermore, if $\mu(\Gamma) < \infty$ then $L_0^{1,p}(\Gamma, d\mu) \subset L^p(\Gamma, d\mu)$. In subsequent sections we shall be concerned with $L^{1,p}(\Gamma, d\mu) := L_0^{1,p}(\Gamma, d\mu) \cap L^p(\Gamma, d\mu)$. The above remarks show that $L^{1,p}(\Gamma, d\mu) = L_0^{1,p}(\Gamma, d\mu)$ if $\psi \in L^{p'}(\Gamma, d\mu)$ and $\mu(\Gamma) < \infty$.

Let $1 < p \leqq q < \infty$ and

$$(T_a f)(x) := v(x) \int_a^x u(t) f(t) dt \quad (x \in \Gamma(a)) . \tag{2.7}$$

Set

$$f(t) = |F'(t)| \psi(t)^{-1/p}, \quad u(t) = \psi(t)^{1/p}, \quad v(t) = \psi(t)^{-1/q} . \tag{2.8}$$

Then

$$|F(x) - F(a)| \psi(x)^{-1/q} \leqq (T_a f)(x) ,$$

$$\| F - F(a) \|_{q, \Gamma(a), d\mu} \leqq \| T_a f \|_{q, \Gamma(a)} \tag{2.9}$$

and

$$\| f \|_{p, \Gamma(a)} = \| F' \|_{p, \Gamma(a), d\mu} . \tag{2.10}$$

The efficacy of our method rests heavily on Proposition 2.4 below which, in view of (2.9) and (2.10), is related to the properties of T_a as a map from $L^p(\Gamma)$ into $L^q(\Gamma)$. The result we require is a consequence of the following extension to trees of a celebrated result due to Muckenhaupt; see [11, Sect. 1.3].

Proposition 2.2. *Let $\Gamma = \Gamma(a)$ be a tree rooted at a and suppose that*

$$0 \leqq u \in L_{\mathrm{loc}}^{p'}(\Gamma), \quad 0 \leqq v \in L^q(\Gamma), (1 \leqq p \leqq q \leqq \infty) , \tag{2.11}$$

where $p' = p/(p-1)$ if $p > 1$ and $p' = \infty$ otherwise. Define

$$J(T_a) := \sup_{x \in \Gamma} \left\{ \left(\int_a^x u^{p'}(t) dt \right)^{1/p'} \left(\int_{t \succeq_a x} v^q(t) dt \right)^{1/q} \right\} \tag{2.12}$$

for $1 < p \leqq q < \infty$ and with the usual interpretation for $p = 1$ and $q = \infty$. Suppose that there exists $c = c(q)$ such that for all $t \in \Gamma$

$$\int_{x \succeq_a t} v^q(x) \left[\int_{y \succeq_a x} v^q(y) dy \right]^{-1/q'} dx \leqq c \left(\int_{y \succeq_a t} v^q(y) dy \right)^{1/q} . \tag{2.13}$$

Then the operator T_a defined by (2.7) is a bounded linear map of $L^p(\Gamma)$ into $L^q(\Gamma)$ if and only if $J(T_a) < \infty$. Furthermore

$$1 \leqq \| T_a \| / J(T_a) \leqq c^{1/q} (q')^{1/p'} \quad (1 < p \leqq q < \infty) \tag{2.14}$$

$$1 \leqq \| T_a \| / J(T_a) \leqq \begin{cases} c & \text{if } p = 1 \\ 1 & \text{if } q = \infty \end{cases} . \tag{2.15}$$

Proof. The proof is similar to that in [11, Sect. 1.3.1] and we consider the case $1 < p \leqq q < \infty$ only.

Let

$$h(x) := \left(\int_a^x u(t)^{p'} dt \right)^{1/qp'} .$$

Then

$$\|T_a f\|_{q,\Gamma}^p = \left\{ \int_\Gamma \left| v(x) \int_a^x u(t) f(t) dt \right|^q dx \right\}^{p/q}$$

$$\leq \left\{ \int_\Gamma v^q(x) \left(\int_a^x |f(t)h(t)|^p dt \right)^{q/p} \left(\int_a^x |u(y)/h(y)|^{p'} dy \right)^{q/p'} dx \right\}^{p/q} \tag{2.16}$$

by Hölder's inequality. Next we use the following consequence of Minkowski's inequality for non-negative functions ϕ and ψ and $r \geqq 1$:

$$\left\{ \int_\Gamma \phi(x) \left(\int_a^x \psi(t) dt \right)^r dx \right\}^{1/r} \leqq \int_\Gamma \psi(t) \left(\int_{x \succeq_a t} \phi(x) dx \right)^{1/r} dt . \tag{2.17}$$

To establish this, note that the left-hand side is equal to

$$\left\{ \int_\Gamma \left(\int_\Gamma \phi(x)^{1/r} \psi(t) \chi(t; \Gamma(a, x)) dt \right)^r dx \right\}^{1/r} ,$$

where $\Gamma(a, x) = \{t \in \Gamma : t \preceq_a x\}$, which by Minkowski's inequality in the form

$$\left\| \int_\Gamma G(\cdot, t) dt \right\|_{r,\Gamma} \leqq \int_\Gamma \| G(\cdot, t) \|_{r,\Gamma} dt ,$$

is majorised by

$$\int_\Gamma \left(\int_\Gamma [\phi(x)^{1/r} \psi(t) \chi(t; \Gamma(a, x))]^r dx \right)^{1/r} dt$$

whence (2.17). With $r = q/p$, $\psi = (|f|h)^p$ and

$$\phi(x) = v(x)^q \left(\int_a^x |u(y)/h(y)|^{p'} dy \right)^{q/p'}$$

we obtain from (2.16) and (2.17) that

$$\| T_a f\|_{q,\Gamma}^p \leqq \int_\Gamma |f(t)h(t)|^p \left\{ \int_{x \succeq_a t} v^q(x) \left(\int_a^x |u(y)/h(y)|^{p'} dy \right)^{q/p'} dx \right\}^{p/q} dt . \tag{2.18}$$

From Remark (i) below (2.5) we see that

$$\int_a^x (u(y)/h(y))^{p'} dy = q' \left\{ \int_0^{\mathrm{dist}(a,x)} u^{p'}(t(r)) dr \right\}^{1/q'}$$

$$= q' \left\{ \int_a^x u^{p'}(y) dy \right\}^{1/q'} .$$

Hence, from (2.18) we obtain

$$\| T_a f \|_{q,\Gamma}^p \leqq (q')^{p/p'} \int_\Gamma |f(t)h(t)|^p \left\{ \int_{x \succeq_a t} v^q(x) \left[\int_a^x u^{p'}(y) dy \right]^{q/p'q'} dx \right\}^{p/q} dt$$

$$\leqq (q')^{p/p'} J(T_a)^{p/q'} \int_\Gamma |f(t)h(t)|^p \left\{ \int_{x \succeq_a t} v^q(x) \left[\int_{y \succeq_a x} v^q(y) dy \right]^{-1/q'} dx \right\}^{p/q} dt$$

$$\leqq c^{p/q} (q')^{p/p'} J(T_a)^{p/q'} \int_\Gamma |f(t)h(t)|^p \left[\int_{y \succeq_a t} v^q(y) dy \right]^{p/q^2} dt ,$$

by (2.13),

$$= c^{p/q}(q')^{p/p'} J(T_a)^{p/q'} \int_\Gamma |f(t)|^p \left\{ \left(\int_a^t u^{p'}(y)dy \right)^{1/p'} \left(\int_{y \gtrsim_a t} v^q(y)dy \right)^{1/q} \right\}^{p/q} dt$$

$$\leqq c^{p/q}(q')^{p/p'} J(T_a)^p \int_\Gamma |f(t)|^p dt .$$

Thus if $J(T_a) < \infty$, $T_a : L^p(\Gamma) \to L^q(\Gamma)$ is bounded and the second inequality in (2.14) follows.

Conversely, suppose that $T_a : L^p(\Gamma) \to L^q(\Gamma)$ is bounded, that is, for all $f \in L^p(\Gamma)$ there exists $K > 0$ such that

$$\left\{ \int_\Gamma \left| v(x) \int_a^x u(t) f(t) dt \right|^q dx \right\}^{1/q} \leqq K \left(\int_\Gamma |f(x|^p dx \right)^{1/p} . \tag{2.19}$$

For some $y \in \Gamma$ set $f(t) = u^{p'-1}(t)\chi(t; \Gamma(a, y))$. Then $f \in L^p(\Gamma)$ since $u \in L^{p'}(\Gamma(a, y))$ and (2.19) yields

$$\left\{ \int_{x \gtrsim_a y} v^q(x) \left[\int_a^y u^{p'}(t) dt \right]^q dx \right\}^{1/q} \leqq K \left(\int_a^y u^{p'}(x) dx \right)^{1/p}$$

and so

$$\left(\int_a^y u^{p'}(t) dt \right)^{1/p'} \left(\int_{x \gtrsim_a y} v^q(x) dx \right)^{1/q} \leqq K .$$

This gives $J(T_a) \leqq \infty$ and hence the first inequality in (2.14).

Remark 2.3. We have been unable to obtain a necessary and sufficient condition for the boundedness of T_a without the assumption (2.13). If Γ is a line segment the assumption is redundant and (2.13) holds with $c = q$ since $-v^q(x)$ is the derivative of the inner integral on the left-hand side. However for a tree Γ this inner integral is not even continuous at a vertex and hence the same simple argument does not apply. Note that (2.13) is only required to establish that $J(T_a) < \infty$ is a sufficient condition for T_a to be bounded and the right inequalities in (2.14) and (2.15). Without (2.13) it may be seen from the proof of Proposition 2.2 that a sufficient condition for T_a to be bounded is that

$$K(T_a) := \sup_{t \in \Gamma} \left\{ \left(\int_a^t u^{p'}(s) ds \right)^{1/p'} \int_{x \gtrsim_a t} v^q(x) \left(\int_a^x u^{p'}(y) dy \right)^{q/p'q'} dx \right\} < \infty \tag{2.20}$$

in which case

$$\| T_a \| / K(T_a)^{1/q} \leqq \begin{cases} (q')^{1/p'} & \text{if } 1 < p \leqq q < \infty , \\ 1 & \text{if } p = 1 \text{ or } q = \infty . \end{cases} \tag{2.21}$$

Note that $K(T_a) \geqq J(T_a)^q$.

Proposition 2.4. *Let dt be locally absolutely continuous with respect to the Borel measure μ on $\Gamma \equiv \Gamma(a)$, $1 < p \leqq q < \infty$, and suppose that $\psi := dt/d\mu \in L^{p'}_{\text{loc}}(\Gamma)$. Define*

$$J(\Gamma(a)) := \sup_{x \in \Gamma} \left\{ [\mu(x)]^{1/q} \left[\int_a^x \psi^{p'/p}(t) dt \right]^{1/p'} \right\} , \tag{2.22}$$

where $\mu(x) = \mu\{t : t \succeq_a x\}$, *and suppose that*

$$(1/\mu(t)) \int_{x \succeq_a t} [\mu(x)]^{-1/q'} d\mu(x) \leqq c\mu(t)^{-1/q'} . \tag{2.23}$$

Let Γ' *be any subtree of* Γ *whose complement in* Γ *is also a tree and which is such that* $\mu(\Gamma') \leqq \mu(\Gamma)/2$. *Then*

$$\sup\{\|F - F_\Gamma\|_{q,\Gamma,d\mu} : F \in L^{1,p}(\Gamma, d\mu),\ \|F'\|_{p,\Gamma,d\mu} = 1\}$$

$$\begin{cases} \leqq 2c^{1/q}(q')^{1/p'} J(\Gamma(a)) & (2.24) \\ \geqq (1 - 2^{-1/q'}) J(\Gamma'(a)) & (2.25) \end{cases}$$

where

$$F_\Gamma := \frac{1}{\mu(\Gamma)} \int_\Gamma F(t) d\mu(t) .$$

Proof. On applying Proposition 2.2 to (2.9) with T_a defined by (2.7) and (2.8) we have that

$$\|F - F(a)\|_{q,\Gamma,d\mu} \leqq c^{1/q}(q')^{1/p'} J(\Gamma(a)) \|F'\|_{p,\Gamma,d\mu} .$$

The inequality (2.24) follows on noting that

$$\begin{aligned} \|F - F_\Gamma\|_{q,\Gamma,d\mu} &\leqq \|F - F(a)\|_{q,\Gamma,d\mu} + \|(F - F(a))_\Gamma\|_{q,\Gamma,d\mu} \\ &\leqq \|F - F(a)\|_{q,\Gamma,d\mu} + |(F - F(a))_\Gamma| \mu(\Gamma(a))^{1/q} \\ &\leqq 2\|F - F(a)\|_{q,\Gamma,d\mu} . \end{aligned}$$

To prove (2.25) we first observe that by Proposition 2.2 applied to Γ', given $\varepsilon > 0$ there exists $g \in L^p(\Gamma')$ such that

$$\|G\|_{q,\Gamma',d\mu} \geqq [J(\Gamma'(a)) - \varepsilon] \|g\|_{p,\Gamma'} , \tag{2.26}$$

where $G(x) = \int_{t \preceq_a x, t \in \Gamma'} \psi^{1/p}(t) g(t) dt$ $(x \in \Gamma')$. Let $f(x) = g(x)$ for $x \in \Gamma'$ and $f(x) = 0$ otherwise. Then

$$F(x) := \int_a^x \psi^{1/p}(t) f(t) dt = \begin{cases} G(x) , & x \in \Gamma' \\ 0 , & x \notin \Gamma' \end{cases}$$

and $|F'(x)| = |\psi^{1/p}(x) f(x)|$. Hence

$$\begin{aligned} \|F - F_\Gamma\|_{q,\Gamma,d\mu} &\geqq \|F\|_{q,\Gamma,d\mu} - |F_\Gamma| \mu(\Gamma)^{1/q} \\ &= \|G\|_{q,\Gamma',d\mu} - \mu(\Gamma)^{-1/q'} \left| \int_{\Gamma'} G(t) d\mu(t) \right| \\ &\geqq \|G\|_{q,\Gamma',d\mu} - [\mu(\Gamma')/\mu(\Gamma)]^{1/q'} \|G\|_{q,\Gamma',d\mu} \\ &\geqq (1 - 2^{-1/q'}) \|G\|_{q,\Gamma',d\mu} \\ &\geqq (1 - 2^{-1/q'}) [J(\Gamma') - \varepsilon] \|g\|_{p,\Gamma'} \end{aligned}$$

by (2.26),

$$\begin{aligned} &= (1 - 2^{-1/q'}) [J(\Gamma'(a)) - \varepsilon] \|f\|_{p,\Gamma} \\ &= (1 - 2^{-1/q'}) [J(\Gamma'(a)) - \varepsilon] \|F'\|_{p,\Gamma,d\mu} . \end{aligned}$$

Since ε is arbitrary (2.25) is proved.

Remark 2.5. As in Remark 2.3, (2.23) is only required for (2.24). Without (2.23) the right-hand side of (2.24) becomes

$$2(q')^{1/p'} K(\Gamma(a))^{1/q}$$

where

$$K(\Gamma(a)) = \sup_{t\in\Gamma} \left\{ \left(\int_a^t \psi^{p'/p} \right)^{1/p'} \int_{x \succeq_a t} \left(\int_a^x \psi^{p'/p} \right)^{q/p'q'} d\mu(x) \right\}.$$

We shall see in the example in Sect. 6 that it is possible for $K(\Gamma(a))^{1/q} \asymp J(\Gamma(a))$ without (2.23) being satisfied.

Similarly we obtain from Proposition 2.2

Proposition 2.6. *Let the hypothesis of Proposition* 2.4 *be satisfied except that either* $p = 1$ *or* $q = \infty$. *Then*

$$(1 - 2^{-1/q'}) J(\Gamma'(a)) \leqq \sup\{ \|F - F_\Gamma\|_{q,\Gamma,d\mu} : F \in L^{1,p}(\Gamma, d\mu),\ \|F'\|_{p,\Gamma,d\mu} = 1\}$$
$$\leqq \begin{cases} 2cJ(\Gamma(a)) & \text{if } p = 1 , \\ 2J(\Gamma(a)) & \text{if } q = \infty . \end{cases}$$

3 Generalized ridged domains

Let Ω be a domain (i.e. an open connected set) in $\mathbf{R}^n (n \geqq 1)$ and denote by $W^{1,p}(\Omega) (1 < p < \infty)$ the Banach space of (equivalence classes of) complex-valued functions f in $L^p(\Omega)$ with weak first derivatives in $L^p(\Omega)$ and having the norm

$$\|f\|_{1,p,\Omega} := (\|\nabla f\|_{p,\Omega}^p + \|f\|_{p,\Omega}^p)^{1/p} , \tag{3.1}$$

where $\nabla f = (\partial f/\partial x_1, \ldots, \partial f/\partial x_n)$ is the gradient of f in $\mathbf{R}^n$ and $\|f\|_{p,\Omega}$ denotes the usual $L^p(\Omega)$ norm. We denote the natural embedding $W^{1,p}(\Omega) \hookrightarrow L^p(\Omega)$ by E and define

$$\beta(E) := \inf\{ \|E - P\| : P \in \mathscr{F}(W^{1,p}(\Omega), L^p(\Omega))\} , \tag{3.2}$$

where $\mathscr{F}(W^{1,p}(\Omega), L^p(\Omega))$ denotes the set of linear maps from $W^{1,p}(\Omega)$ into $L^p(\Omega)$ which are bounded and of finite rank. Since $L^p(\Omega)$ has the approximation property, E is compact if and only if $\beta(E) = 0$. It was proved in [3, Theorem 2.3] that $\beta(E) = 1$ if $|\Omega|$, the n-dimensional Lebesgue measure of Ω, is infinite. Hence, hereafter we shall assume that $|\Omega| < \infty$.

We shall denote by $B(x, r)$ the open ball $\{y : |y - x| < r\}$ in $\mathbf{R}^n$, where $|\cdot|$ is any norm on $\mathbf{R}^n$: different norms only affect the absolute constants involved in the estimates obtained. A tree Γ is defined as in Sect. 2, having an infinite number of vertices each of finite degree. Note that since Γ has an infinite number of edges its completion is not compact, by Lemma 2.1 (i).

We recall from Sect. 2 that the derivative of a function which is locally Lipschitz on Γ may be defined everywhere.

Definition 3.1. A domain Ω with $|\Omega| < \infty$ will be called a *generalized ridged domain* if there exist real-valued functions u, ρ, τ, a tree Γ and positive constants α, β, γ, δ such that the following conditions are satisfied:
(i) $u : \Gamma \to \Omega$, $\rho : \Gamma \to \mathbf{R}^+ \equiv (0, \infty)$ are Lipschitz;
(ii) $\tau : \Omega \to \Gamma$ is surjective and for each $x \in \Omega$ there exists a neighbourhood $V(x)$ such that for all $y \in V(x)$, $|\tau(x) - \tau(y)|_\Gamma \leqq \gamma |x - y|$, where $|\cdot|_\Gamma$ denotes the metric on Γ: thus τ is uniformly locally Lipschitz;

(iii) $|x - u \circ \tau(x)| \leqq \alpha\rho \circ \tau(x)$ for all $x \in \Omega$;
(iv) $|u'(t)| + |\rho'(t)| \leqq \beta$ for all $t \in \Gamma$;
(v) with $B_t := B(u(t), \rho(t))$ and $\mathscr{C}(x) := \{y : sy + (1-s)x \in \Omega$ for all $s \in [0, 1]\}$, we have that for all $x \in \Omega$, $B_{\tau(x)} \subset \Omega$ and $\mathscr{C}(x) \cap B_{\tau(x)}$ contains a ball $B(x)$ such that $|B(x)|/|B_{\tau(x)}| \geqq \delta > 0$.
The curve $t \mapsto u(t) : \Gamma \to \Omega$ will be called a *generalized ridge* of Ω.

Many domains which are defined by an iterative process are of the above type. In Sect. 6 we give a detailed description of such a domain. The Koch snowflake, the domain whose boundary is the Koch curve (see [5, Sect. 8.3]), with the excision of certain line segments, can similarly be described as a generalized ridged domain. One begins with a regular hexagon and constructs on the middle third of each of its sides another regular hexagon, connecting them by removing the common boundary. The process continues indefinitely unless blocked by the presence of an already constructed hexagon. The underlying tree consists of the centres of the hexagons and the line segments joining them. We do not include details of the analysis as it is similar to that of Sect. 6; also our method merely recovers results which are already known because the snowflake is a quasi-disc; see [10, Theorem 4.1] and [11, Sect. 1.5.1]. Another example of a generalized ridged domain is provided by a branching spiral domain. Take polar co-ordinates r, θ in the plane and let $\Omega = D \backslash S$, where D is the open unit disc punctured at the origin and S the union of a set of equiangular spirals defined by

$$S = \{(r, \theta) : 0 \leqq \theta < 2\pi,$$
$$r = \exp\{-\theta - 2\pi(n - p/2^n);\ p = 0, 1, \ldots, 2^n - 1;\ n = 0, 1, \ldots\}\ .$$

Thus Ω is a labyrinth of ever-narrowing passages spiralling inwards, each of which divides into two after circling the origin once. The underlying tree may be taken to consist of edges of unit length and vertices of degree two except for one vertex which has degree one. It is easy in this example to define suitable functions u, ρ, τ.

The above definition differs from that in [3] only in that the interval J in [3, Definition 4.1] is now replaced by the tree Γ.

As in [3], the map τ in Definition 3.1 defines a positive Borel measure μ on Γ as follows: for $F \in C_0(\Gamma)$

$$\int_\Gamma F(t)d\mu(t) := \int_\Omega (F \circ \tau)(x)dx\ .$$

For any open subset Γ_0 of Γ we have

$$\mu(\Gamma_0) = |\tau^{-1}(\Gamma_0)|\ .$$

The map $F \mapsto F \circ \tau : C_0(\Gamma) \to L^p(\Omega)$ extends by continuity to a map

$$T : L^p(\Gamma; d\mu) \to L^p(\Omega) \tag{3.3}$$

which satisfies $TF(x) = F \circ \tau(x)$ for a.e. $x \in \Omega$ and T is an isometry when $L^p(\Gamma, d\mu)$ is endowed with the norm $\|\cdot\|_{p,\Gamma,d\mu}$. Note that by [11, Sect. 1.2.4], if $\nabla\tau$ does not vanish on a set of positive measure,

$$\int_\Omega (F \circ \tau)(x)dx = \int_\Gamma F(t) \int_{\tau^{-1}(t)} |\nabla\tau(x)|^{-1} d\sigma(x)dt$$

where σ denotes $(n-1)$-dimensional Hausdorff measure. Hence μ is absolutely continuous with respect to Lebesgue measure and

$$\frac{d\mu}{dt} = \int_{\tau^{-1}(t)} |\nabla\tau(x)|^{-1} d\sigma(x) . \tag{3.4}$$

If $\tau^{-1}(t)$ is a rectifiable curve in Ω when $n = 2$, its 1-dimensional Hausdorff measure is equal to its length $l(t)$ (see [5, Lemma 3.2]) and hence, since $|\nabla\tau(x)| \leqq \gamma$ by Definition 3.1(ii), we have

$$\frac{d\mu}{dt} \geqq \gamma^{-1} l(t) .$$

Consequently, dt is absolutely continuous with respect to $d\mu$ on any compact subset of Γ on which $l(\cdot)$ is positive.

In the remainder of the paper we shall assume that $1 < p < \infty$, that Ω is a generalized ridged domain and that dt is absolutely continuous with respect to $d\mu$. Also we shall assume that Γ is ordered, i.e. $\Gamma = \Gamma(a)$ for some $a \in \Gamma$.

Let

$$(Mf)(t) := \frac{1}{|B_t|} \int_{B_t} f(x) dx \ (t \in \Gamma) , \tag{3.5}$$

where $B_t = B(u(t), \rho(t))$. The main feature of our technique for determining the number $\beta(E)$ is the reduction of the problem to an equivalent one on the tree Γ for which precise results can be obtained on using the estimates in Sect. 2. The maps T and M defined in (3.3) and (3.5) respectively are the tools we use to establish the equivalence and the following three lemmas provide the key to the analysis. They are analogous to the results in Lemmas 4.2–4.5 in [3] and we omit the proofs as they are similar.

Recall that $L^{1,p}(\Gamma, d\mu)$ denotes the set of functions F which are locally Lipschitz on Γ and are such that $F, F' \in L^p(\Gamma, d\mu)$. In what follows K will denote various constants which depend only on n and the constants α, β, δ in Definition 3.1.

Lemma 3.2. *The map T defined in* (3.3) *is a bounded linear map of $L^{1,p}(\Gamma, d\mu)$ into $W^{1,p}(\Omega)$ and for $F \in L^{1,p}(\Gamma, d\mu)$*

$$\|\nabla(TF)\|_{p,\Omega} \leqq \gamma \|F'\|_{p,\Gamma,d\mu} , \tag{3.6}$$

where γ is the constant in Definition 3.1(ii).

Lemma 3.3. *The map M defined in* (3.5) *is a bounded linear map of $W^{1,p}(\Omega)$ into $L^{1,p}(\Gamma, d\mu)$ and, for $f \in W^{1,p}(\Omega)$,*

$$\|Mf\|_{p,\Gamma,d\mu} \leqq K \|f\|_{p,\Omega}, \quad \|(Mf)'\|_{p,\Gamma,d\mu} \leqq K \|\nabla f\|_{p,\Omega} . \tag{3.7}$$

Lemma 3.4. *Let Ω_1 be a measurable subset of Ω and*

$$k(\Omega_1) := \sup_{\Omega_1} \{\rho \circ \tau(x)\} < \infty . \tag{3.8}$$

Then for all $f \in W^{1,p}(\Omega)$

$$\|f - TMf\|_{p,\Omega_1} \leqq K k(\Omega_1) \|\nabla f\|_{p,\Omega} . \tag{3.9}$$

From [3, Theorem 2.6], $\beta(E) < 1$ is equivalent to the Poincaré inequality on $W^{1,p}(\Omega)$, that is, there exists a constant K such that

$$\|f - f_\Omega\|_{p,\Omega} \leqq K \|\nabla f\|_{p,\Omega} \quad (f \in W^{1,p}(\Omega)) ,$$

where $f_\Omega := \frac{1}{|\Omega|}\int_\Omega f(x)dx$. It follows that when $\beta(E) < 1$,

$$\|f\|_{M,\Omega} := \|\nabla f\|_{p,\Omega}$$

defines a norm on

$$W_M^{1,p}(\Omega) := \{f \in W^{1,p}(\Omega) : f_\Omega = 0\} \tag{3.10}$$

which is equivalent to $\|\cdot\|_{1,p,\Omega}$. If $\mathscr{C}$ denotes the set of constant functions, we have the topological isomorphism

$$W^{1,p}(\Omega) = \mathscr{C} \oplus W_M^{1,p}(\Omega)$$

and $W_M^{1,p}(\Omega)$ is topologically isomorphic to the quotient space $W^{1,p}(\Omega)/\mathscr{C}$; see [3] for details. Let I denote the embedding

$$I : W_M^{1,p}(\Omega) \hookrightarrow L_M^p(\Omega) := \{f \in L^p(\Omega) : f_\Omega = 0\} \tag{3.11}$$

and define

$$\beta(I) := \inf\{\|I - P\| : P \in \mathscr{F}(W_M^{1,p}(\Omega), L^p(\Omega))\} .$$

Then $\beta(E)$ and $\beta(I)$ are related by

$$\beta(I)^p/(1 + \|I\|^p) \leqq \beta(E)^p \leqq \beta(I)^p/(1 + \beta(I)^p) ; \tag{3.12}$$

see [3, Theorem 2.10]. It follows that E and I are compact together.

4 Upper and lower bounds for $\beta(I)$

The first step is to define and analyse what we regard as the singular part of the boundary of Ω, bearing in mind that the domains we study may be singular in many (even all) directions. The boundary in the Stone-Cech compactification of Ω is what motivates the following.

Let

$$\mathscr{A}(\Gamma) := \{A \subset \Gamma : A \text{ is closed and } \overline{\Gamma \setminus A} \text{ is a compact subtree of } \Gamma\} , \tag{4.1}$$

$$\mathscr{A}(\Omega) := \{\tau^{-1}(A) : A \in \mathscr{A}(\Gamma)\} . \tag{4.2}$$

Note that the boundary of $\Gamma \setminus A$ is finite for any $A \in \mathscr{A}(\Gamma)$, by Lemma 2.1(i) and hence A is a finite union of closed, disjoint (and rooted) subtrees of Γ.

Lemma 4.1. *The set $\mathscr{A}(\Omega)$ in* (4.2) *is a filter base of relatively closed subsets of Ω which satisfy the following conditions*:
(i) *for each $A \in \mathscr{A}(\Omega)$ the embedding $W^{1,p}(\Omega) \hookrightarrow L^p(\Omega \setminus A) \subset L^p(\Omega)$ is compact,*
(ii) *$\mathscr{A}(\Omega)$ is finer than the filter base*

$$\mathscr{A}_0(\Omega) := \{A : A = \Omega \setminus \Omega', \Omega' \subset\subset \Omega\} .$$

Proof. Suppose $\emptyset \in \mathscr{A}(\Omega)$. Then for some $A \in \mathscr{A}(\Gamma)$, $\Gamma = \tau(\Omega) = \Gamma \setminus A$. Since Γ does not have compact completion we conclude that $\emptyset \notin \mathscr{A}(\Omega)$. Let $A_i = \tau^{-1}(A_i)$, $i = 1,2$ belong to $\mathscr{A}(\Omega)$. Define $\overline{\Gamma \setminus A}$ to be the union of $\overline{\Gamma \setminus A_1}$ and $\overline{\Gamma} \setminus A_2$ and a path connecting them. Then $A \in \mathscr{A}(\Gamma)$ and

$$\tau^{-1}(A) \subseteq \tau^{-1}(A_1) \cap \tau^{-1}(A_2) = A_1 \cap A_2 .$$

Consequently $\mathscr{A}(\Omega)$ is a filter base.

Let $A = \tau^{-1}(\Lambda) \in \mathscr{A}(\Omega)$. Then $\Omega \setminus A = \tau^{-1}(\Gamma \setminus \Lambda)$ and hence $\tau(\Omega \setminus A) \subseteq \overline{\Gamma \setminus \Lambda}$. It follows from Definition 3.1 that $\Omega \setminus A$ is bounded and on it $\rho \circ \tau$ is bounded away from zero. Thus by Definition 3.1(v), $\Omega \setminus A$ lies in a bounded open subset of Ω which satisfies the cone condition and consequently $W^{1,p}(\Omega \setminus A) \subsetneq L^p(\Omega \setminus A)$ is compact. Thus (i) is proved.

Finally, let $\Omega' \subset\subset \Omega$. Then $\tau(\bar{\Omega}')$ is compact and is contained in a compact subtree of Γ and hence $\tau(\bar{\Omega}') \subset (\Gamma')^\circ$, the interior of some compact subtree Γ' of Γ. Let $\Lambda = \Gamma/(\Gamma')^\circ = \overline{\Gamma \setminus \Gamma'}$. Then $\Lambda \in \mathscr{A}(\Gamma)$ and

$$\tau^{-1}(\Lambda) = \Omega \setminus \tau^{-1}[(\Gamma')^\circ] \subseteq \Omega \setminus \bar{\Omega}' \subseteq \Omega \setminus \Omega' .$$

The lemma is therefore proved.

Since $\mathscr{A}(\Omega)$ is a filter base it is directed by reverse inclusion, that is, by the order relation $\succ$ where $A_1 \succ A_2$ if $A_1 \subseteq A_2$. If $\{\psi_A\}$ is a family in $\mathbf{R}$ indexed by $\mathscr{A}(\Omega)$, the pair $(\{\psi_A\}, \succ)$ is a net in $\mathbf{R}$. It converges to a limit ψ in $\mathbf{R}$, written $\lim_{\mathscr{A}(\Omega)} \psi_A = \psi$, if for every neighbourhood U of ψ in $\mathbf{R}$ there is an $A_0 \in \mathscr{A}(\Omega)$ such that $\psi_A \in U$ for all $A \succ A_0$ in $\mathscr{A}(\Omega)$. Similarly $\mathscr{A}(\Gamma)$ is a filter base directed by reverse inclusion and $\lim_{\mathscr{A}(\Gamma)} \phi_\Lambda = \phi$ is defined.

Let $\Lambda \in \mathscr{A}(\Gamma)$ and $A = \tau^{-1}(\Lambda) \in \mathscr{A}(\Omega)$. As noted above, Λ is the finite union of closed disjoint (and rooted) subtrees of Γ, $\Lambda_i (i \in N_\Lambda)$ say. Set $A_i = \tau^{-1}(\Lambda_i)$, $i \in N_A \equiv N_\Lambda$, these being disjoint closed subsets of Ω and

$$A = \tau^{-1}(\Lambda) = \bigcup_{i \in N_A} A_i . \tag{4.3}$$

Let

$$H_\Lambda F := \sum_{i \in N_\Lambda} \chi(\Lambda_i) F_{\Lambda_i} \tag{4.4}$$

$$h_A f := \sum_{i \in N_A} \chi(A_i) f_{A_i} \tag{4.5, 4.6}$$

where χ denotes the characteristic function and

$$F_{\Lambda_i} := \frac{1}{\mu(\Lambda_i)} \int_{\Lambda_i} F(t) d\mu(t) \tag{4.7}$$

$$f_{A_i} := \frac{1}{|A_i|} \int_{A_i} f(x) dx . \tag{4.8}$$

Lemma 4.2. *Let $A \in \mathscr{A}(\Omega)$ and define*

$$\theta_A := \sup\{ \| f - h_A f \|_{p,A} : f \in W_M^{1,p}(\Omega), \| \nabla f \|_{p,\Omega} = 1 \} . \tag{4.9}$$

Then

$$\beta(I) \leq \inf_{\mathscr{A}(\Omega)} \theta_A \leq \limsup_{\mathscr{A}(\Omega)} \theta_A \leq 2\beta(I) . \tag{4.10}$$

Proof. The operator $h_A : W_M^{1,p}(\Omega) \to L^p(\Omega)$ is compact since it is bounded and of finite rank. It therefore follows from [3, Corollary 2.9] that for any fixed $A_o \in \mathscr{A}(\Omega)$ and

$$\theta_A^o := \sup\{ \| f - h_{A_o} f \|_{p,A} : f \in W_M^{1,p}(\Omega), \| \nabla f \|_{p,\Omega} = 1 \}$$

we have

$$\beta(I) = \inf_{A \in \mathscr{A}(\Omega)} \theta_A^o = \lim_{\mathscr{A}(\Omega)} \theta_A^o . \tag{4.11}$$

Hence $\beta(I) \leqq \theta^o_{A_o}$ for any $A_o \in \mathscr{A}(\Omega)$ which implies

$$\beta(I) \leqq \inf_{\mathscr{A}(\Omega)} \theta_A \leqq \limsup_{\mathscr{A}(\Omega)} \theta_A . \tag{4.12}$$

Furthermore

$$\| f - h_A f \|_{p,A} \leqq \| f - h_{A_o} f \|_{p,A} + \| h_{A_o} f - h_A f \|_{p,A}$$

and since $\sum_{i \in N_A} \chi(A_i) = 1$ on A,

$$\begin{aligned} \| h_{A_o} f - h_A f \|^p_{p,A} &= \Big\| \sum_{i \in N_A} \chi(A_i) [h_{A_o} f - f_{A_i}] \Big\|^p_{p,A} \\ &= \sum_{i \in N_A} \| h_{A_o} f - f_{A_i} \|^p_{p,A_i} \\ &= \sum_{i \in N_A} \| (h_{A_o} f - f)_{A_i} \|^p_{p,A_i} \\ &\leqq \sum_{i \in N_A} \| h_{A_o} f - f \|^p_{p,A_i} \end{aligned}$$

by Hölder's inequality,

$$= \| f - h_{A_o} f \|^p_{p,A} .$$

Hence

$$\| f - h_A f \|_{p,A} \leqq 2 \| f - h_{A_o} f \|_{p,A}$$

and by (4.11)

$$\limsup_{\mathscr{A}(\Omega)} \theta_A \leqq 2 \lim_{\mathscr{A}(\Omega)} \theta^o_A = 2\beta(I) .$$

This and (4.12) completes the proof.

The connection we seek between the problem on Ω and the analogous one on Γ is provided by the next theorem.

Theorem 4.3. *For $\Lambda \in \mathscr{A}(\Gamma)$ and $A = \tau^{-1}(\Lambda) \in \mathscr{A}(\Omega)$ define*

$$\phi_\Lambda := \sup \{ \| F - H_\Lambda F \|_{p,\Lambda,d\mu} : F \in L^{1,p}(\Gamma, d\mu), \| F' \|_{p,\Gamma,d\mu} = 1 \} , \tag{4.13}$$

where H_Λ is defined in (4.4), *and let θ_A be given by* (4.9). *Then there exists a positive constant K, depending on n and the constants α, β, δ in Definition* 3.1, *such that*

$$\gamma^{-1} \phi_\Lambda \leqq \theta_A \leqq K \{ k(A) + \phi_\Lambda \} , \tag{4.14}$$

where $k(A) = \sup_A \{ \rho \circ \tau(x) \}$.

Proof. For $F \in L^{1,p}(\Gamma, d\mu)$, we have that $TF = F \circ \tau \in W^{1,p}(\Omega)$ by Lemma 3.2 and note that, since $\mu(\Lambda_i) = |A_i|$,

$$TH_\Lambda F = h_A(TF) . \tag{4.15}$$

Furthermore, if $f \in W^{1,p}(\Omega)$, then $f - f_\Omega \in W^{1,p}_M(\Omega)$ and

$$\begin{aligned} \| f - h_A f \|_{p,A} &= \| (f - f_\Omega) - h_A (f - f_\Omega) \|_{p,A} \\ &\leqq \theta_A \| \nabla f \|_{p,\Omega} . \end{aligned}$$

Consequently

$$\begin{aligned}\|F - H_A F\|_{p,A,d\mu} &= \|TF - h_A(TF)\|_{p,A}\\ &\leqq \theta_A \|\nabla(TF)\|_{p,\Omega}\\ &\leqq \gamma\theta_A \|F'\|_{p,\Gamma,d\mu}\end{aligned}$$

by (3.6), whence $\phi_A \leqq \gamma\theta_A$.

For $f \in W^{1,p}(\Omega)$, $Mf \in L^{1,p}(\Gamma, d\mu)$ by Lemma 3.3 and

$$\begin{aligned}\|f - h_A f\|_{p,A} &\leqq \|f - TMf\|_{p,A} + \|TMf - h_A(TMf)\|_{p,A}\\ &\quad + \|h_A f - h_A(TMf)\|_{p,A}\\ &= \|f - TMf\|_{p,A} + \|Mf - H_A(Mf)\|_{p,A,d\mu}\\ &\quad + \left\{\sum_{i \in N_A} \|(f - TMf)_{A_i}\|^p_{p,A_i}\right\}^{1/p}\end{aligned}$$

by (4.15),

$$\begin{aligned}&\leqq 2\|f - TMf\|_{p,A} + \|Mf - H_A(Mf)\|_{p,A,d\mu}\\ &\leqq Kk(A)\|\nabla f\|_{p,\Omega} + \phi_A\|(Mf)'\|_{p,\Gamma,d\mu}\end{aligned}$$

by (3.9) and (4.13),

$$\leqq K\{k(A) + \phi_A\}\|\nabla f\|_{p,\Omega}$$

by (3.7). The proof is therefore complete.

Corollary 4.4. *Define*

$$\phi_+ := \limsup_{\mathscr{A}(\Gamma)} \phi_A, \quad \phi_- := \liminf_{\mathscr{A}(\Gamma)} \phi_A . \tag{4.16}$$

Then

$$\frac{1}{2\gamma}\phi_+ \leqq \beta(I) \leqq K\phi_- . \tag{4.17}$$

Proof. If $k_o := \lim_{\mathscr{A}(\Omega)} k(A) = 0$, we see that (4.17) is a consequence of (4.10) and (4.14). It is therefore sufficient to prove $\beta(I) = 0$ when $k_o \neq 0$ for (4.10) and (4.14) would then imply that $\phi_+ = \phi_- = 0$. The proof of the analogous result in [3, Corollary 4.8] remains valid, mutatis mutandis, and we give only a brief sketch of the argument.

Let

$$s(x, y) := \inf\{\text{length of } P(x, y) : P(x, y) \text{ a polygonal path in } \Omega \text{ joining } x \text{ and } y\}$$

and

$$D := \sup\{s(x, y) : x, y \in \Omega\} .$$

If $D < \infty$ it follows from Definition 3.1 that Γ is bounded and as ρ is Lipschitz on Γ, $k_o = \lim_{\mathscr{A}(\Gamma)} [\inf_A \rho(t)]$. Thus $k_o > 0$ implies that ρ is bounded away from zero on Γ and by Definition 3.1, Ω satisfies a cone condition. Since Ω is also bounded when $D < \infty$ we conclude that E and I are compact and so $\beta(I) = 0$.

If $D = \infty$ there exists a path π in Γ such that

$$\sup_{t_1, t_2 \in \pi} s(u(t_1), u(t_2)) = \infty .$$

From this fact and $k_o > 0$ we deduce that there exists a sequence of disjoint balls in Ω with radii bounded away from zero. But this contradicts the assumption $|\Omega| < \infty$ made throughout the paper. The corollary is therefore proved.

Corollary 4.5. *The following are equivalent:*
(i) *for some* $C(\Omega) > 0$

$$\| f - f_\Omega \|_{p,\Omega} \leqq C(\Omega) \| \nabla f \|_{p,\Omega} \quad (f \in W^{1,p}(\Omega)) ,$$

(ii) *for some* $c(\Gamma) > 0$

$$\| F - F_\Gamma \|_{p,\Gamma,d\mu} \leqq c(\Gamma) \| F' \|_{p,\Gamma,d\mu} \quad (F \in L^{1,p}(\Gamma, d\mu)) .$$

Moreover the least constants $C(\Omega)$, $c(\Gamma)$ *satisfy*

$$\gamma^{-1} c(\Gamma) \leqq C(\Omega) \leqq K \{ k(\Omega) + c(\Gamma) \}$$

for some $K > 0$.

Proof. The proof of Theorem 4.3 applies.

The next step is to estimate ϕ_+ and ϕ_- by means of Proposition 2.4. Recall that any $\Lambda \in \mathscr{A}(\Gamma)$ is a finite union of closed, disjoint and rooted subtrees of Γ. If $\{a_i : i \in N_\Lambda\}$ are the vertices of Γ which constitute the boundary of $\Gamma \backslash \Lambda$ then $\Lambda = \bigcup_{i \in N_\Lambda} \Lambda_i(a_i)$, where the subtrees $\Lambda_i(a_i)$ are rooted at a_i: note that the partial ordering on $\Lambda_i(a_i)$ is that induced by the partial ordering on Γ.

For each $i \in N_\Lambda$ let $\Lambda'_i \in \mathscr{A}(\Lambda_i(a_i))$ be given by

$$\Lambda'_i = \bigcup_{j \in N_{\Lambda'_i}} \Lambda'_{ij}(c_{ij}) , \quad \mu(\Lambda'_{ij}(c_{ij})) \leqq \frac{1}{2} \mu(\Lambda_i(a_i)) ,$$

where the $\Lambda'_{ij}(c_{ij}), j \in N_{\Lambda'_i}$, are closed, disjoint subtrees of $\Lambda_i(a_i)$, rooted at $c_{ij} \in \Lambda_i(a_i)$. Then if

$$\Lambda' = \bigcup_{i \in N_\Lambda} \Lambda'_i$$

we have $\Lambda' \in \mathscr{A}(\Gamma)$ and $\Lambda' \subset \Lambda$. On applying Proposition 2.4 to $\Gamma(a) = \Lambda_i(a_i)$ and with $q = p$, we have

$$\sup \{ \| F - F_{\Lambda_i(a_i)} \|_{p,\Lambda_i(a_i),d\mu} : F \in L^{1,p}(\Lambda_i(a_i), d\mu), \| F' \|_{p,\Lambda_i(a_i),d\mu} = 1 \} \left\{ \begin{array}{l} \leqq 2c^{1/p}(p')^{1/p'} J(\Lambda_i(a_i)) \\ \geqq (1 - 2^{-1/p'}) J(\Lambda'_i) \end{array} \right\} \tag{4.18}$$

where (2.23) is satisfied,

$$J(\Lambda_i(a_i)) = \sup_{x \in \Lambda_i(a_i)} \left\{ [\mu\{t : t \succeq_{a_i} x\}]^{1/p} \left[\int_{a_i}^{x} \psi^{p'/p}(t) dt \right]^{1/p'} \right\} \tag{4.19}$$

and

$$\begin{aligned} J(\Lambda'_i) &= \max_{j \in N_{\Lambda'_i}} J(\Lambda'_{ij}(c_{ij})) \\ &= \sup_{x \in \Lambda'_i} \left\{ [\mu\{t \in \Lambda'_i : t \succeq_a x\}]^{1/p} \left[\int_{t \preceq_a x, t \in \Lambda'_i} \psi^{p'/p}(t) dt \right]^{1/p'} \right\} . \end{aligned} \tag{4.20}$$

Our main result is

Theorem 4.6. *Let* (2.23) *be satisfied and let dt be absolutely continuous with respect to $d\mu$ on $\Gamma = \Gamma(a)$ and $\psi := dt/d\mu \in L^{p'}_{\mathrm{loc}}(\Gamma, d\mu)$, where $p' = p/(p-1)$, $1 < p < \infty$. Then*

$$\beta(I) \asymp \lim_{\mathscr{A}(\Gamma)} J(\Lambda) \tag{4.21}$$

where $\asymp$ indicates that the quotient of the two sides is bounded above and below by positive constants. In (4.21)

$$J(\Lambda) = \sup_{x\in\Lambda} \left\{ [\mu\{t\in\Lambda : t \succeq_a x\}]^{1/p} \left[\int_{t\preceq_a x, t\in\Lambda} \psi^{p'/p}(t)\,dt \right]^{1/p'} \right\}. \tag{4.22}$$

The same result holds if instead of (2.23) *we assume that $K(\Lambda_i(a_i))^{1/p} = O(J(\Lambda_i(a_i))$ for all $i \in N_\Lambda$, in the notation of Remark* 2.5.

Proof. From (4.18), for any $F \in L^{1,p}(\Lambda, d\mu)$, with $\Lambda = \bigcup_{i\in N_\Lambda} \Lambda_i(a_i) \in \mathscr{A}(\Gamma)$,

$$\begin{aligned}\|F - H_\Lambda F\|^p_{p,\Lambda,d\mu} &= \sum_{i\in N_\Lambda} \|F - F_{\Lambda_i(a_i)}\|^p_{p,\Lambda_i(a_i),d\mu}\\ &\leqq \left[2c^{1/p}(p')^{1/p'} \max_{i\in N_\Lambda} \{J(\Lambda_i(a_i))\}\right]^p \sum_{i\in N_\Lambda} \|F'\|^p_{p,\Lambda_i(a_i),d\mu}\\ &= [2c^{1/p}(p')^{1/p'} J(\Lambda)]^p \|F'\|^p_{p,\Lambda,d\mu}\,.\end{aligned}$$

Hence in (4.13),

$$\begin{aligned}\phi_\Lambda &\leqq \sup\{\|F - H_\Lambda F\|_{p,\Lambda,d\mu} : F \in L^{1,p}(\Lambda, d\mu),\ \|F'\|_{p,\Lambda,d\mu} = 1\}\\ &\leqq 2c^{1/p}(p')^{1/p'} J(\Lambda)\,.\end{aligned} \tag{4.23}$$

Given $\varepsilon > 0$, there exists $F \in L^{1,p}(\Gamma, d\mu)$ with support in $\Lambda_i(a_i)$ such that

$$\begin{aligned}\|F - H_\Lambda F\|_{p,\Lambda,d\mu} &= \|F - F_{\Lambda_i(a_i)}\|_{p,\Lambda_i(a_i),d\mu}\\ &\geqq [(1 - 2^{-1/p'})J(\Lambda_i') - \varepsilon]\, \|F'\|_{p,\Gamma,d\mu}\end{aligned}$$

by (4.18). Hence

$$\phi_\Lambda \geqq (1 - 2^{-1/p'}) J(\Lambda')\,. \tag{4.24}$$

The theorem follows from (4.17), (4.23) and (4.24); the final part is a consequence of Remark 2.5.

Corollary 4.7. *Under the hypothesis of Theorem* 4.6 *we have*
(i) $\beta(E) < 1$ *if and only if* $J(\Gamma) < \infty$,
(ii) *E and I are compact if and only if* $\lim_{\mathscr{A}(\Gamma)} J(\Lambda) = 0$.

Proof. (i) Choose $b \in \Gamma$ such that $\Gamma(b) = \bigcup_{i\in I_b} \Gamma_i(b)$ and $\mu(\Gamma_i(b)) \leqq (1/2)\mu(\Gamma)$. Then it follows as in the proof of Theorem 4.6 that for all $F \in L^{1,p}(\Gamma, d\mu)$ and $\|F'\|_{p,\Gamma,d\mu} = 1$.

$$\|F - F_\Gamma\|_{p,\Gamma,d\mu} \asymp J(\Gamma(b))\,.$$

Also, $J(\Gamma(b)) < \infty$ if and only if $J(\Gamma) < \infty$. By Corollary 4.5, we therefore infer that $J(\Gamma) < \infty$ is equivalent to the Poincaré inequality

$$\|f - f_\Omega\|_{p,\Omega} \leqq K\|\nabla f\|_{p,\Omega} \quad (f \in W^{1,p}(\Omega))$$

and this is equivalent to $\beta(E) < 1$ by [3, Theorem 2.6]. Thus (i) is proved.
(ii) This is an immediate consequence of (3.12) and (4.21).

5 Dirichlet-Neumann bracketing in L^p

Let Ω be an open subset of $\mathbf{R}^n$ and denote by $a_m(\Omega)$, $a_m^0(\Omega)$ the mth approximation numbers of the embedding maps $I: W_M^{1,p}(\Omega) \hookrightarrow L_M^p$, $I_0: W_0^{1,p}(\Omega) \hookrightarrow L^p(\Omega)$ respectively, that is

$$a_m(\Omega) := \inf\{\|I - P\| : P \in \mathscr{F}(W_M^{1,p}(\Omega), L_M^p(\Omega)), \text{rank } P < m\}$$

and similarly for $a_m^0(\Omega)$. Define

$$\nu_\Omega(\varepsilon) := \max\{m : a_m(\Omega) \geqq \varepsilon\}$$
$$\nu_\Omega^0(\varepsilon) := \max\{m : a_m^0(\Omega) \geqq \varepsilon\} . \tag{5.1}$$

Since the embedding I is injective it follows that if S is a finite dimensional subspace of $W_M^{1,p}(\Omega)$, the restriction of I^{-1} to $I(S)$ is a bounded linear operator with bound $\alpha(S) > 0$ say, that is

$$\alpha(S) := \sup_{x \in S}\{\|\nabla x\|_{p,\Omega} / \|Ix\|_{p,\Omega}\} . \tag{5.2}$$

Let $d(S)$ denote the dimension of S and define

$$\mu_\Omega(\varepsilon) := \max\{d(S) : \alpha(S) \leqq 1/\varepsilon\} . \tag{5.3}$$

Similarly we define $\mu_\Omega^0(\varepsilon)$ with respect to the embedding I_0.

We now study how the numbers $\nu_\Omega(\varepsilon)$, $\nu_\Omega^0(\varepsilon)$, $\mu_\Omega(\varepsilon)$ and $\mu_\Omega^0(\varepsilon)$ are related and their behaviour when Ω is a finite union of sets with disjoint interior. Our objective is to obtain a technique for determining the asymptotic behaviour of $\nu_\Omega(\varepsilon)$ and $\nu_\Omega^0(\varepsilon)$ as $\varepsilon \to 0$ which is analogous to that of Dirichlet-Neumann bracketing in the case $p = 2$ for determining the asymptotic distribution of the eigenvalues of the Dirichlet and Neumann Laplacians. If $p = 2$ and I, I_0 are compact we shall see that $\mu_\Omega(\varepsilon)$, $\mu_\Omega^0(\varepsilon)$ coincide with $\nu_\Omega(\varepsilon)$, $\nu_\Omega(\varepsilon)$ respectively.

Lemma 5.1. *We have*

$$\nu_\Omega^0(\varepsilon) \leqq \nu_\Omega(\varepsilon) + 1 \tag{5.4}$$
$$\mu_\Omega(\varepsilon) \leqq \nu_\Omega(\varepsilon) , \tag{5.5}$$
$$\mu_\Omega^0(\varepsilon) \leqq \nu_\Omega^0(\varepsilon) . \tag{5.6}$$

Proof. Let $P \in \mathscr{F}(W_M^{1,p}(\Omega), L_M^p(\Omega))$ have rank $r(P)$. Then P_0 defined by

$$P_0 f := f_\Omega + P(f - f_\Omega)$$

maps $W_0^{1,p}(\Omega)$ into L^p and has rank $r(P_0) \leqq r(P) + 1$. Suppose $r(P) + 2 \leqq \nu_\Omega^0(\varepsilon)$. Then $r(P_0) \leqq \nu_\Omega^0(\varepsilon) - 1$ and so

$$\|I_0 - P_0\| \geqq a_{\nu_\Omega^0(\varepsilon)}(\Omega) \geqq \varepsilon .$$

Hence, for any $\eta < \varepsilon$ there exists $g \in W_0^{1,p}(\Omega)$ such that

$$\|g - g_\Omega - P(g - g_\Omega)\|_{p,\Omega} \geqq \eta \|\nabla g\|_{p,\Omega} .$$

Since $g - g_\Omega \in W_M^{1,p}(\Omega)$, we infer that $\|I - P\| \geqq \eta$ and so $\|I - P\| \geqq \varepsilon$. As this is true for any P of rank $r(P)$ satisfying $r(P) + 1 \leqq \nu_\Omega^0(\varepsilon) - 1$, (5.4) follows.

Let S be a subspace of $W_M^{1,p}(\Omega)$ of dimension $d(S)$ and P a bounded linear map of $W_M^{1,p}(\Omega)$ into $L_M^p(\Omega)$ of rank $r(P) < d(S)$. Then if $e_1, \ldots, e_{d(S)}$ is a basis of

S there exist $\lambda_1, \ldots, \lambda_{d(S)}$, not all of which are zero, such that $P(\sum_{i=1}^{d(S)} \lambda_i e_i) = 0$. Hence, with $\psi = \sum_{i=1}^{d(S)} \lambda_i e_i$ we have

$$\|(I-P)\psi\|_{p,\Omega} = \|I\psi\|_{p,\Omega} \geqq \alpha(S)^{-1} \|\nabla\psi\|_{p,\Omega}$$

and consequently $\|I-P\| \geqq \alpha(S)^{-1}$. It follows that $a_{d(S)}(\Omega) \geqq \alpha(S)^{-1}$ and $a_{\mu_\Omega(\varepsilon)}(\Omega) \geqq \varepsilon$. Therefore (5.5) is proved. A similar proof holds for (5.6).

Lemma 5.2. *Let $p = 2$ and suppose that I and I_0 are compact. Then*

$$\mu_\Omega(\varepsilon) = \nu_\Omega(\varepsilon), \quad \mu_\Omega^0(\varepsilon) = \nu_\Omega^0(\varepsilon) .$$

Proof. From [1, Theorems II.5.7, II.5.10] we have that $a_m(\Omega) = \mu_m(I) = \mu_m(I^*)$, where $\mu_m(T)$ denotes the mth singular number of T, that is, the mth eigenvalue (the eigenvalues being arranged in decreasing order and repeated according to multiplicity) of the positive compact self-adjoint operator $|T| = (T^*T)^{1/2}$. Thus $a_m^2(\Omega) = \lambda_m(I^*I)$, the mth eigenvalue of I^*I, a compact self-adjoint operator in $W_M^{1,2}(\Omega)$. Consequently, by [1, Theorem II.5.6]

$$\begin{aligned} \nu_\Omega(\varepsilon) &= \sum_{a_m(\Omega) \geqq \varepsilon} 1 \\ &= \sum_{\lambda_m(I^*I) \geqq \varepsilon^2} 1 \\ &= \max\{\dim R : R \in \mathscr{K}(\varepsilon^2)\} , \end{aligned}$$

where $\mathscr{K}(\varepsilon^2)$ is the set of closed linear subspaces R of $W_M^{1,p}(\Omega)$ such that, for all $x \in R$,

$$\begin{aligned} \varepsilon^2 \|\nabla x\|_{2,\Omega}^2 &\leqq (I^*Ix, x)_{W_M^{1,2}(\Omega)} \\ &= \|Ix\|_{2,\Omega}^2 . \end{aligned}$$

Therefore, $\alpha(R) \leqq 1/\varepsilon$ and $\dim R \leqq \mu_\Omega(\varepsilon)$ for all $R \in \mathscr{K}(\varepsilon^2)$, whence $\nu_\Omega(\varepsilon) \leqq \mu_\Omega(\varepsilon)$. The reverse inequality has already been established in Lemma 5.1 and so $\nu_\Omega(\varepsilon) = \mu_\Omega(\varepsilon)$. The proof of $\nu_\Omega^0(\varepsilon) = \mu_\Omega^0(\varepsilon)$ is similar.

Lemma 5.3. *Let $\Omega = (\bigcup_{i=1}^q \Omega_i) \cup N$ where the Ω_i are disjoint open subsets of $\mathbf{R}^n$ and N is a null set. Then*

$$\nu_\Omega(\varepsilon) + 1 \leqq \sum_{i=1}^q [\nu_{\Omega_i}(\varepsilon) + 1] .$$

Proof. Since $a_{\nu_{\Omega_i}(\varepsilon)+1}(\Omega_i) < \varepsilon$, there exists a bounded linear operator $P_i \colon W_M^{1,p}(\Omega_i) \to L_M^p(\Omega_i)$ with $r(P_i) < \nu_{\Omega_i}(\varepsilon) + 1$ and such that

$$\|I_i - P_i\| =: \varepsilon_i < \varepsilon ;$$

here I_i is the embedding $W_M^{1,p}(\Omega_i) \hookrightarrow L_M^p(\Omega_i)$. Let

$$Pf = \sum_{i=1}^q \chi_{\Omega_i}\{f_{\Omega_i} + P_i(f - f_{\Omega_i})\} \quad (f \in W_M^{1,p}(\Omega)) .$$

Then

$$\|f - Pf\|_{p,\Omega}^p = \sum_{i=1}^q \|f - f_{\Omega_i} - P_i(f - f_{\Omega_i})\|_{p,\Omega_i}^p$$

$$\leqq (\max_{1 \leqq i \leqq q} \varepsilon_i)^p \sum_{i=1}^q \|\nabla f\|_{p,\Omega_i}^p$$

$$< \varepsilon^p \|\nabla f\|_{p,\Omega}^p .$$

Also, since $(\chi_{\Omega_1} - \chi_{\Omega_2}), \ldots, (\chi_{\Omega_{q-1}} - \chi_{\Omega_q})$ spans the range of the map $f \mapsto \sum_{i=1}^q f_{\Omega_i}\chi_{\Omega_i}: W_M^{1,p}(\Omega) \to L_M^p(\Omega)$, it follows that

$$r(P) \leqq \sum_{i=1}^q r(P_i) + q - 1 \leqq \sum_{i=1}^q \nu_{\Omega_i}(\varepsilon) + q - 1 .$$

Therefore

$$a_l(\Omega) < \varepsilon$$

where $l = \sum_{i=1}^q \nu_{\Omega_i}(\varepsilon) + q$, and so

$$\nu_\Omega(\varepsilon) \leqq \sum_{i=1}^q \nu_{\Omega_i}(\varepsilon) + q - 1 ,$$

whence the result.

Lemma 5.4. *Under the hypothesis of Lemma* 5.3,

$$\mu_\Omega^0(\varepsilon) \geqq \sum_{i=1}^q \mu_{\Omega_i}^0(\varepsilon) .$$

Proof. For each i there exists a subspace S_i of $W_0^{1,p}(\Omega_i)$ of dimension $\mu_{\Omega_i}^0(\varepsilon)$ and such that $\alpha(S_i) \leqq 1/\varepsilon$. The direct sum S of the S_i is a subspace of $W_0^{1,p}(\Omega)$ of dimension $\sum_{i=1}^q \mu_{\Omega_i}^0(\varepsilon)$ and $\alpha(S) \leqq 1/\varepsilon$, whence the result.

Lemma 5.5. *Let $\Omega = \Omega_1 \cup \Omega_2 \cup N$, where Ω_1 and Ω_2 are disjoint open subsets of $\mathbf{R}^n$ and N is a null set. Suppose that for all $f \in W_0^{1,p}(\Omega)$, $\|f\|_{p,\Omega_2} \leqq \varepsilon \|\nabla f\|_{p,\Omega_2}$. Then for all $\eta > \varepsilon$,*

$$\nu_\Omega^0(\eta) \leqq \nu_{\Omega_1}(\eta) + 1 .$$

Proof. Let $P \in \mathscr{F}(W_M^{1,p}(\Omega_1), L_M^p(\Omega_1))$ have rank $r(P)$ and define

$$Qf = \{f_{\Omega_1} + P(f - f_{\Omega_1})\}\chi_{\Omega_1} \quad (f \in W_0^{1,p}(\Omega)) .$$

Then rank $Q \leqq r(P) + 1$ and

$$\|(I_0 - Q)f\|_{p,\Omega}^p = \|(f - f_{\Omega_1}) - P(f - f_{\Omega_1})\|_{p,\Omega_1}^p + \|f\|_{p,\Omega_2}^p$$

$$\leqq \|I_{\Omega_1} - P\|^p \|\nabla f\|_{p,\Omega_1}^p + \varepsilon^p \|\nabla f\|_{p,\Omega_2}^p$$

where $I_{\Omega_1}: W_M^{1,p}(\Omega_1) \hookrightarrow L_M^p(\Omega_1)$,

$$\leqq \{\max(\|I_{\Omega_1} - P\|, \varepsilon)\|\nabla f\|_{p,\Omega}\}^p ,$$

whence

$$\|I_0 - Q\| \leqq \max(\|I_{\Omega_1} - P\|, \varepsilon) .$$

If $r(P) + 1 \leqq \nu_\Omega^0(\eta) - 1$, it follows that $\|I_0 - Q\| \geqq \eta$ and hence $\|I_{\Omega_1} - P\| \geqq \eta$. Since P is arbitrary, we infer that $r(P) + 1 \leqq \nu_{\Omega_1}(\eta)$, whence the result.

Lemma 5.6. *Let Ω_1 be the image of Ω under an affine transformation of $\mathbf{R}^n$ which magnifies distances by a factor λ. Then $\nu_{\Omega_1}(\varepsilon) = \nu_\Omega(\varepsilon/\lambda)$ and similarly for ν^0, μ^0 and μ.*

Proof. Since Ω_1 is obtained from Ω by a similarity transformation $t \mapsto a + \lambda t$ the result is straightforward.

We are now able to give our main results about the asymptotic behaviour of ν_Q and ν_Q^0.

Theorem 5.7. *Let Q be an open cube in $\mathbf{R}^n$. Then*

$$\lim_{\lambda \to \infty} \{\lambda^{-n} \nu_Q(1/\lambda)\} = \inf_{\lambda > 0} \{\lambda^{-n}[\nu_Q(1/\lambda) + 1]\} . \tag{5.7}$$

Proof. For simplicity we shall prove the result for the case when Q is a square of side 1 in $\mathbf{R}^2$, the general case being proved similarly.

Let R be a rectangle in $\mathbf{R}^2$. For the embedding $W_M^{1;p}(R) \hookrightarrow L^p(R)$ we have from the Poincaré inequality that $a_1(R) \leqq c \operatorname{diam} R$ for some absolute constant c. Hence $\nu_R(\varepsilon) = 0$ if $\operatorname{diam} R < \varepsilon/c$.

With $\lambda \geqq \lambda_0 \geqq 1$, we write $\lambda = [\lambda/\lambda_0]\lambda_0 + \theta\lambda_0$, where $[\cdot]$ denotes the integer part and $0 \leqq \theta < 1$. Then λQ can be expressed (modulo a null set) as a disjoint union of $[\lambda/\lambda_0]^2$ open squares congruent to $\lambda_0 Q$ together with an L-shaped region which can be cut up into $2[\lambda/\lambda_0] + 1$ rectangles each of diameter less than $\sqrt{2}\lambda_0$. Each of these rectangles R_j is the union of $\{[c\lambda_0\sqrt{2}\varepsilon^{-1}] + 1\}^2$ congruent rectangles of diameter less than ε/c and hence in view of the previous paragraph and Lemma 5.3,

$$\nu_{R_j}(\varepsilon) + 1 \leqq \{[c\lambda_0\sqrt{2}\varepsilon^{-1}] + 1\}^2 .$$

Therefore, by Lemma 5.3,

$$\nu_{\lambda Q}(\varepsilon) + 1 \leqq [\lambda/\lambda_0]^2(\nu_{\lambda_0 Q}(\varepsilon) + 1) + \{2[\lambda/\lambda_0] + 1\}\{[c\lambda_0\sqrt{2}\varepsilon^{-1}] + 1\}^2$$

and

$$\lambda^{-2}\{\nu_{\lambda Q}(\varepsilon) + 1\} \leqq \lambda_0^{-2}\{\nu_{\lambda_0 Q}(\varepsilon) + 1\} + \mathrm{O}(\lambda_0\lambda^{-1}[1 + \varepsilon^{-2}])$$

or, in view of Lemma 5.6 and with $\varepsilon = 1$,

$$\lambda^{-2}\{\nu_Q(1/\lambda) + 1\} \leqq \lambda_0^{-2}\{\nu_Q(1/\lambda_0) + 1\} + \mathrm{O}(\lambda_0/\lambda) . \tag{5.8}$$

Given $\delta > 0$, choose λ_0 such that

$$\lambda_0^{-2}\{\nu_Q(1/\lambda_0) + 1\} < \inf\{\lambda^{-2}[\nu_Q(1/\lambda) + 1]\} + \delta .$$

Then (5.8) yields

$$\limsup_{\lambda \to \infty} \{\lambda^{-2}[\nu_Q(1/\lambda) + 1]\} \leqq \inf_{\lambda > 0} \{\lambda^{-2}[\nu_Q(1/\lambda) + 1]\} + \delta$$

whence the result.

Theorem 5.8. *Let Q be an open cube in $\mathbf{R}^n$. Then*

$$\lim_{\lambda \to \infty} \{\lambda^{-n}\mu_Q^0(1/\lambda)\} = \sup_{\lambda > 0} \{\lambda^{-n}\mu_Q^0(1/\lambda)\} . \tag{5.9}$$

Proof. As in the proof of Theorem 5.7 above (with $n = 2$ and $\lambda \geqq \lambda_0 \geqq 1$) we express λQ as a union of $[\lambda/\lambda_0]^2$ open squares congruent to $\lambda_0 Q$ together with an L-shaped region. Then, by Lemma 5.4,

$$\mu_Q^0(1/\lambda) = \mu_{\lambda Q}^0(1) \geqq [\lambda/\lambda_0]^2 \mu_{\lambda_0 Q}^0(1)$$
$$\geqq \{(\lambda/\lambda_0)^2 - 1\} \mu_Q^0(1/\lambda_0)$$

and so

$$\lambda^{-2}\mu_Q^0(1/\lambda) \geqq \lambda_0^{-2}\mu_Q^0(1/\lambda_0) - \lambda^{-2}\mu_Q^0(1/\lambda_0) . \tag{5.10}$$

This implies

$$\liminf_{\lambda \to \infty} \{\lambda^{-2}\mu_Q^0(1/\lambda)\} \geqq \sup_{\lambda > 0} \{\lambda^{-2}\mu_Q^0(1/\lambda)\}$$

and hence (5.9).

Corollary 5.9. *If Q is an open cube in $\mathbf{R}^n$ then as $\lambda \to \infty$*

$$\mu_Q^0(1/\lambda), \nu_Q^0(1/\lambda), \nu_Q(1/\lambda) \asymp \lambda^n . \tag{5.11}$$

Proof. This follows from Theorems 5.7, 5.8 and

$$\mu_Q^0(1/\lambda) \leqq \nu_Q^0(1/\lambda) \leqq \nu_Q(1/\lambda) + 1 .$$

Similar proofs yield the same results in Theorems 5.7, 5.8 and Corollary 5.9 when the cubes are replaced by equilateral triangles and their n-dimensional analogues.

Suppose that $p = 2$. Then $a_m(Q)$ is the mth singular number of I^* and II^* is the inverse of the Neumann Laplacian $-\triangle_{Q,N}$ restricted to $L_M^2(Q)$. Consequently $a_m(Q) = \lambda_m^{-1/2}(Q, N)$, where $\lambda_m(Q, N)$ is the mth positive eigenvalue of $-\triangle_{Q,N}$ and $\nu_Q(1/\lambda) = \sum_{\lambda_m(Q,N) \leqq \lambda^2} 1$. Similarly $\mu_Q^0(1/\lambda) = \nu_Q^0(1/\lambda) = \sum_{\lambda_m(Q,D) \leqq \lambda^2} 1$, where $\lambda_m(Q, D)$ is the mth eigenvalue of the Dirichlet Laplacian $-\triangle_{Q,D}$. Therefore when $p = 2$ we have the well-known result

$$\lim_{\lambda \to \infty} \{\lambda^{-n}\nu_Q(1/\lambda)\} = \lim_{\lambda \to \infty} \{\lambda^{-n}\mu_Q^0(1/\lambda)\} = (2\pi)^{-n}\omega_n|Q| , \tag{5.12}$$

where ω_n is the n-dimensional Lebesgue measure of the unit ball in $\mathbf{R}^n$. For general p we are unable to prove whether or not the limits in (5.7) and (5.9) are equal.

6 A domain without the $W^{1,p}$ extension property

6.1 The domain

We construct a domain Ω in $\mathbf{R}^2$ from a succession of finite sets (generations) Θ_m of closed congruent rectangles Q_m of edge lengths $2\alpha_m \times 2\beta_m$ ($\alpha_m \leqq \beta_m$) and with disjoint interiors; see Fig. 1. The generation Θ_0 consists of a single rectangle as does Θ_1, a short edge of Q_1 being attached to the middle portion of a long edge of Q_0. For $m \geqq 1$, Θ_m contains 2^{m-1} rectangles and to each long edge of Q_m is attached a short edge of a rectangle Q_{m+1}, these 2^m rectangles Q_{m+1} being the members of Θ_{m+1}. The domain Ω is the interior of the connected set Θ constructed in this way:

$$\Omega = \Theta^\circ, \Theta = \bigcup_{m \in \mathbf{N}_0} \left(\bigcup \{Q_m : Q_m \in \Theta_m\}\right) . \tag{6.1}$$

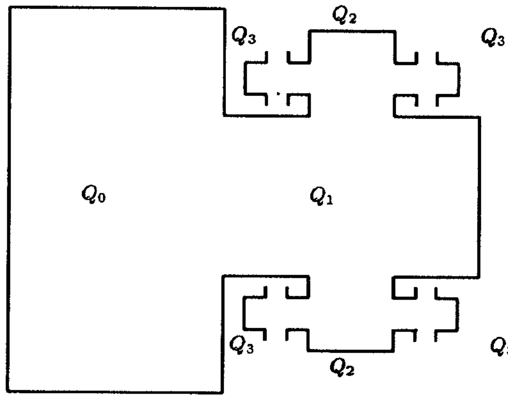

Fig. 1.

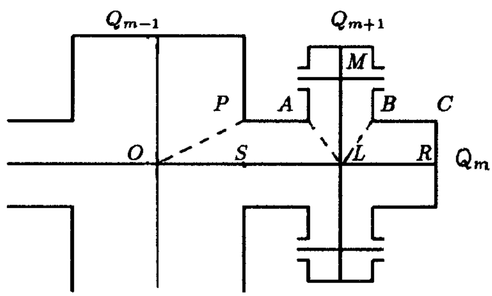

Fig. 2.

We shall assume that for all $m \in \mathbf{N}_0$,

$$\alpha_m \leqq \beta_m, 0 < \delta_1 \leqq \alpha_{m+1}/\alpha_m \leqq 1, \beta_{m+1}/\beta_m \leqq \delta_2 < 1, 2\beta_{m+2} < \beta_m - \alpha_{m+1} \, . \quad (6.2)$$

Note that in (6.2) the condition $2\beta_{m+2} < \beta_m - \alpha_{m+1}$ ensures there is no overlapping; also $|\Omega| < \infty$.

The major and minor axes of each rectangle make up a tree Γ of finite degree and this is taken to be the generalized ridge of Ω with $u: \Gamma \to \Omega$ the identification map. The portion of $u(\Gamma)$ in $Q_{m-1} \cup Q_m \cup Q_{m+1}$ is shown in Fig. 2. In $OPAL$ τ is defined to be the projection of PA onto OL and in $LBCR$ τ is the projection of BC onto LR. If t denotes the distance from O along Γ we have $u(t) = t$. Since Ω is covered by regions like $OPAL$ and $LBCR$ it is enough to analyse the properties of Ω vis-a-vis Definition 3.1 in these regions.

In Fig. 3 the co-ordinates are: $P(\alpha_{m-1}, \alpha_m)$, $A(\alpha_{m-1} + \beta_m - \alpha_{m+1}, \alpha_m)$, $L(\alpha_{m-1} + \beta_m, 0)$ and it is easy to see that X is the point $(\alpha_{m-1}\chi_m, \alpha_m\chi_m)$, where $\chi_m = (\alpha_{m-1} + \beta_m)/(\alpha_{m-1} + \alpha_{m+1})$. Thus in Fig. 3

$$\tan \phi = \alpha_m \chi_m/(\alpha_{m-1}\chi_m - t) \, . \quad (6.3)$$

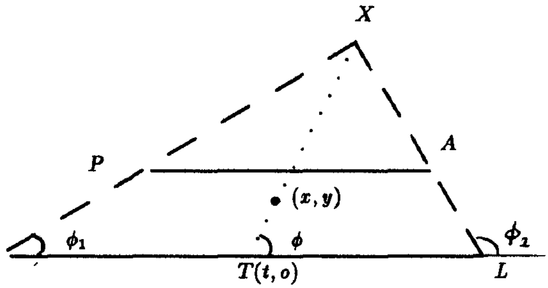

Fig. 3.

For any $(x, y) \in \Omega$ lying on the line XT we have $\tau(x, y) = t$ and so

$$t = x - y \cot \phi \,. \tag{6.4}$$

It follows that

$$\frac{\partial t}{\partial x} = 1 + y(\csc^2 \phi) \frac{\partial \phi}{\partial x} = 1 + y(\csc^2 \phi)\phi'(t) \frac{\partial t}{\partial x}$$

$$\frac{\partial t}{\partial y} = -\cot \phi + y(\csc^2 \phi)\phi'(t) \frac{\partial t}{\partial y}$$

and hence

$$[1 - y(\csc^2 \phi)\phi'] \nabla \tau(x, y) = (1, \; -\cot \phi) \,.$$

Since $(\csc^2 \phi)\phi' = 1/\alpha_m \chi_m$ from (6.3), we obtain

$$|\nabla \tau(x, y)| = |1 - y/\alpha_m \chi_m|^{-1} \csc \phi \,. \tag{6.5}$$

By (6.2),

$$1 \geqq 1 - y/\alpha_m \chi_m \geqq \frac{\beta_m - \alpha_{m+1}}{\alpha_{m-1} + \beta_m} \geqq \frac{1 - \delta_2}{1 + \delta_1^{-1}}$$

and so

$$|\nabla \tau(x, y)| \asymp \csc \phi \,. \tag{6.6}$$

Also in Fig. 3, $\phi_1 \leqq \phi \leqq \phi_2$ and hence

$$\begin{aligned} 1 \leqq \csc \phi &\leqq \max(\csc \phi_1, \csc \phi_2) \\ &= \max([1 + \alpha_{m-1}^2/\alpha_m^2]^{1/2}, [1 + \alpha_{m+1}^2/\alpha_m^2]^{1/2}) \\ &= [1 + \alpha_{m-1}^2/\alpha_m^2]^{1/2} \,. \end{aligned}$$

Therefore

$$|\nabla \tau(x, y)| \asymp 1 \,. \tag{6.7}$$

In $OPAL$ we define $\rho(t)$ to be the distance from $T(t, 0)$ to PA. It is straightforward to check that ρ is piecewise C^1 and $|\rho'(t)| \leqq 1$.

From (3.4) and (6.6)

$$\frac{d\mu}{dt} = \int_{\tau^{-1}(t)} |\nabla\tau(x, y)|^{-1}\, d\sigma$$
$$\asymp (\csc\phi)^{-1}[\alpha_m \csc\phi] = \alpha_m\,. \tag{6.8}$$

Thus dt is absolutely continuous with respect to $d\mu$ on OL. The other requirements of Definition 3.1 are easily seen to be satisfied on $OPAL$. Similar considerations apply to the region $LBCR$ and in particular (6.8) remains true.

We now estimate $J(\Lambda)$, $\Lambda \in \mathscr{A}(\Gamma)$. In the notation used in the proof of Theorem 4.6, $\Lambda = \bigcup_{i\in N_\lambda} \Lambda_i(a_i)$ where $a_i \preceq_a x$ for all $x \in \Lambda_i(a_i)$. Let $a_i \in Q_{k_i}$ and $r \in Q_s \cap \Lambda_i(a_i)$, $s > k_i$. Then by (6.8)

$$\int_{a_i}^{r} \psi^{p'/p}(t)\,dt \asymp \sum_{m=k_i}^{s} \alpha_m^{-p'/p}[\alpha_{m-1} + \beta_m]$$
$$\asymp \sum_{m=k_i}^{s} \beta_m \alpha_m^{-p'/p} \tag{6.9}$$

on using (6.2). Also

$$\mu\{t : t \succeq_{a_i} r\} = \int_{t\succeq_{a_i} r} (d\mu/dt)\,dt$$
$$\asymp \sum_{m=s}^{\infty} 2^{m-s}\alpha_m[\alpha_{m-1} + \beta_m]$$
$$\asymp \sum_{m=s}^{\infty} 2^{m-s}\alpha_m\beta_m\,. \tag{6.10}$$

Hence

$$J(\Lambda_i(a_i)) \asymp \sup_{k_i \leqq s < \infty} \left\{ \left(\sum_{m=s}^{\infty} 2^{m-s}\alpha_m\beta_m \right)^{1/p} \left(\sum_{m=k_i}^{s} \beta_m \alpha_m^{-p'/p} \right)^{1/p'} \right\}. \tag{6.11}$$

In Remark 2.5 we have

$$K(\Lambda_i(a_i)) \asymp \sup_{k_i \leqq s < \infty} \left\{ \left(\sum_{m=k_i}^{s} \beta_m \alpha_m^{-p'/p} \right)^{1/p'} \sum_{m=s}^{\infty} 2^{m-s} \left(\sum_{n=k_i}^{m} \beta_n \alpha_n^{-p'/p} \right)^{p/p'^2} \alpha_m\beta_m \right\}. \tag{6.12}$$

Also, in (2.23), we have if $t \in Q_r$

$$\mu(t) \asymp \sum_{m=r}^{\infty} 2^{m-r}\alpha_m\beta_m$$

and

$$\int_{x\succeq_a t} \mu(x)^{-1/p'}\,d\mu = \int_{x\succeq_a t} \mu(x)^{-1/p'}(d\mu/dx)\,dx$$
$$\asymp \sum_{s=r}^{\infty} 2^{s-r} \left(\sum_{m=s}^{\infty} 2^{m-s}\alpha_m\beta_m \right)^{-1/p'} \alpha_s[\alpha_s + \beta_s]$$
$$\asymp \sum_{s=r}^{\infty} 2^{s-r}\alpha_s\beta_s \left(\sum_{m=s}^{\infty} 2^{m-s}\alpha_m\beta_m \right)^{-1/p'}$$

Hence (2.23) is satisfied if

$$\sum_{s=r}^{\infty} 2^{s-r}\alpha_s\beta_s\left(\sum_{m=s}^{\infty} 2^{m-s}\alpha_m\beta_m\right)^{-1/p'} \asymp \left(\sum_{m=r}^{\infty} 2^{m-r}\alpha_m\beta_m\right)^{1/p}. \tag{6.13}$$

From Theorem 4.6 we therefore have

Theorem 6.1. *Let Ω be the domain* (6.1) *subject to* (6.2), *and suppose that either* (6.13) *is satisfied or* $K(\Lambda_i(a_i)) = O(J(\Lambda_i(a_i))^p)$ *for all* $i \in N_\Lambda$. *Then* $\beta(I) \asymp \lim_{\mathscr{A}(\Gamma)} J(\Lambda)$, *where* $J(\Lambda) = \max_{i\in N_\Lambda} J(\Lambda_i(a_i))$ *with* $\Lambda = \bigcup_{i \in N_\Lambda} \Lambda_i(a_i)$ *and* $J(\Lambda_i(a_i))$ *satisfying* (6.11).

6.2 Example

6.2.1 General estimates. Let $\alpha_m = c^{\alpha m}$, $\beta_m = c^m (m \in \mathbf{N}_0)$, where $\alpha \geqq 1$, $c^\alpha + 2c^2 < 1$ and $c^{1+\alpha/p+p/p'^2} < 1/2$ when $\alpha > p/p'$. Then (6.2) is satisfied. Also

$$\sum_{m=s}^{\infty} 2^{m-s}\alpha_m\beta_m = 2^{-s}\sum_{m=s}^{\infty}(2c^{1+\alpha})^m \asymp c^{(1+\alpha)s}$$

and

$$\sum_{m=k}^{s} \beta_m\alpha_m^{-p'/p} = \sum_{m=k}^{s} c^{(1-\alpha p'/p)m}$$

$$\asymp \begin{cases} c^{(1-\alpha p'/p)s} & \text{if } \alpha > p/p', \\ c^{(1-\alpha p'/p)k} & \text{if } \alpha < p/p', \\ s-k & \text{if } \alpha \doteq p/p'. \end{cases}$$

These estimates yield, as $s \to \infty$,

$$J_{k,s} := \left(\sum_{m=s}^{\infty} 2^{m-s}\alpha_m\beta_m\right)^{1/p}\left(\sum_{m=k}^{s}\beta_m\alpha_m^{-p'/p}\right)^{1/p'} \tag{6.14}$$

$$\asymp \begin{cases} c^s & \text{if } \alpha > p/p', \\ c^{k+(1+\alpha)(s-k)/p} & \text{if } \alpha < p/p', \\ (s-k)^{1/p'}c^{(1+\alpha)s/p} & \text{if } \alpha = p/p'. \end{cases}$$

Consequently $J(\Lambda_i(a_i)) \to 0$ as $k_i \to \infty$ and $\lim_{\mathscr{A}(\Gamma)} J(\Lambda) = 0$. It remains to analyse (6.12) and (6.13). Let

$$K_{k,s} := \left(\sum_{m=k}^{s} c^{(1-\alpha p'/p)m}\right)^{1/p'} \sum_{m=s}^{\infty} 2^{m-s}c^{(1+\alpha)m}\left(\sum_{n=k}^{m} c^{(1-\alpha p'/p)n}\right)^{p/p'^2}.$$

For $\alpha \leqq p/p'$, it is readily shown that $K_{k,s} \asymp J_{k,s}^p$ and this continues to be true for $\alpha > p/p'$ as long as $2c^{1+\alpha/p+p/p'^2} < 1$. In order for (6.13) to be satisfied we need $2^p c^{1+\alpha} < 1$: note that $c^{1+\alpha/p+p/p'^2} = c^{(1+\alpha)/p+p/p'} < c^{(1+\alpha)/p}$. This example shows that (2.23) is not necessary in Proposition 2.4 – see Remark 2.5. We have therefore proved that if $2c^{1+\alpha/p+p/p'^2} < 1$ when $\alpha > p/p'$ then I is compact.

If $\alpha = 1$, Ω can be shown to be a quasi-disc (see [11, Sect. 1.5.1, Example 1]) and hence has the $W^{1,p}$-extension property, which in turn implies that I is compact. However, if $\alpha > 1$, $\beta_m/\alpha_m \to \infty$ as $m \to \infty$ and so Ω is not a quasidisc. The compactness of I established above is not therefore attributable to the

$W^{1,p}$-extension property. It may well be that there exists a continuous extension of $W^{1,p}(\Omega)$ to a space $V(\mathbf{R}^2)$ which is compactly embedded in $L^p(\Omega)$ but this is far from obvious.

6.2.2 The inner and outer Minkowski dimensions of $\partial\Omega$. Let

$$(\partial\Omega)^i_\delta := \{x \in \Omega : \mathrm{dist}(x, \partial\Omega) < \delta\} ,$$

$$(\partial\Omega)^o_\delta := \{x \in \mathbf{R}^n \backslash \Omega : \mathrm{dist}(x, \partial\Omega) < \delta\} ,$$

$$\mathscr{M}^i_d(\partial\Omega) := \limsup_{\delta \to 0} \delta^{-(2-d)} |(\partial\Omega)^i_\delta| ,$$

$$d_i := \inf\{t : \mathscr{M}^i_t(\partial\Omega) < \infty\} ,$$

and define $\mathscr{M}^o_d(\partial\Omega)$, d_o similarly. Then $\mathscr{M}^i_d(\partial\Omega)$, $\mathscr{M}^o_d(\partial\Omega)$ are the upper Minkowski contents of $\partial\Omega$ relative to Ω and $\mathbf{R}^n\backslash\Omega$ respectively, and d_i, d_o are respectively the inner and outer Minkowski dimensions of $\partial\Omega$; see [10, Sect. 2]. Let integers M, N be such that $\alpha_{M+1} < \delta \leqq \alpha_M, \beta_{N+1} < \delta \leqq \beta_N$. Then, as $\delta \to 0$,

$$M = \frac{\log(1/\delta)}{\alpha \log(1/c)} + \mathrm{O}(1), \quad N = \frac{\log(1/\delta)}{\log(1/c)} + \mathrm{O}(1)$$

and

$$\begin{aligned} |(\partial\Omega)^i_\delta| &= 4\{(\alpha_0 + \beta_0) - 2\alpha_1\}\delta - \mathrm{O}(\delta^2) \\ &\quad + \sum_{k=1}^{M} 2^{k-1}\{[4(\beta_k - \alpha_{k+1}) + 2\alpha_k]\delta + \mathrm{O}(\delta^2)\} \\ &\quad + \sum_{k=M+1}^{N} 2^{k-1}\{4\alpha_k(\beta_k - \alpha_{k+1}) + \mathrm{O}(\delta\alpha_k)\} + \sum_{k=N+1}^{\infty} 2^{k-1}(4\alpha_k\beta_k) \\ &\asymp \delta \sum_{k=0}^{M} (2c)^k + \sum_{k=M+1}^{\infty} (2c^{1+\alpha})^k + \mathrm{O}\left(\delta^2 \sum_{k=1}^{M} 2^{k-1}\right). \end{aligned}$$

It follows that

$$|(\partial\Omega)^i_\delta| \asymp \begin{cases} \delta & \text{if } 2c < 1 , \\ \delta \log(1/\delta) & \text{if } 2c = 1 , \\ \delta^{1 - \log 2c/\alpha \log(1/c)} & \text{if } 2c > 1 . \end{cases}$$

Hence

$$d_i = \begin{cases} 1 & \text{if } = c \leqq 1/2 , \\ 1 + \log 2c/\alpha \log(1/c) & \text{if } = c > 1/2 . \end{cases} \tag{6.15}$$

Note also that $\mathscr{M}^i_{d_i}(\partial\Omega) = \infty$ if $2c = 1$ and is otherwise finite.

To calculate $\mathscr{M}^o_{d_o}(\partial\Omega)$ and d_o we choose N to be the largest integer such that $2\delta < \beta_{N+1} - \alpha_{N+2}$. Then $N = \log(1/\delta)/\log(1/c) + \mathrm{O}(1)$ and for all $k > N$ we have $\beta_k = \mathrm{O}(\delta)$. It follows that

$$||(\partial\Omega)^o_\delta| - S_N| \leqq K\delta^2 2^N ,$$

where, with $n_0 = 1$ and $n_k = 2^{k-1}$ for $k \geqq 1$,

$$S_N = \sum_{k=0}^{N} n_k\{4(\beta_k - \alpha_{k+1})\delta + 2\alpha_k\delta\}$$
$$\asymp \delta \sum_{k=0}^{N} (2c)^k .$$

This readily yields

$$|(\partial\Omega)^o_\delta| \asymp \begin{cases} \delta & \text{if } 2c < 1 , \\ \delta\log(1/\delta) & \text{if } 2c = 1 , \\ \delta^{1-\log(2c)/\log(1/c)} & \text{if } 2c > 1 . \end{cases}$$

Hence

$$d_o = \left\{ \begin{array}{ll} 1 & \text{if } 2c \leqq 1 , \\ 1 + \log(2c)/\log(1/c) & \text{if } 2c > 1 . \end{array} \right\} \tag{6.16}$$

Moreover, $\mathscr{M}^o_{d_o}(\partial\Omega) = \infty$ if $2c = 1$ and is otherwise finite. Note that if $2c > 1$, $1 < d_i < d_o < 2$ since $2c^2 < 1$ and the assumption $c^\alpha + 2c^2 < 1$ implies that $\alpha > 1$.

6.2.3 Asymptotics of $\nu_\Omega(\varepsilon)$ and $\mu^o_\Omega(\varepsilon)$ for $1 < p < \infty$. Let

$$\Omega_m := \bigcup_{Q_i \in \Theta_i,\, i \leqq m} Q_i, \quad \Omega\backslash\bar{\Omega}_m = \bigcup_{i=1}^{2^m} A_{i,m} . \tag{6.17}$$

Then, for each i, $A_{i,m}$ is a generalized ridged domain which is similar to Ω with generalized ridge $\Lambda_{i,m}$ say, which is a subtree of Γ rooted at the point S in Fig. 2. The map τ is now the projection of PA onto SL and not OL in Q_m but is otherwise the same as before: this modification does not affect the estimates from Sect. 6.2 used below.

By Theorem 4.3 and (4.18), for all $f \in W^{1,p}_M(A_{i,m})$,

$$\|f - f_{A_{i,m}}\|_{p,A_{i,m}} \leqq K[k(A_{i,m}) + \phi_{A_{i,m}}]\,\|\nabla f\|_{p,A_{i,m}}$$
$$\leqq K[k(A_{i,m}) + J(A_{i,m})]\,\|\nabla f\|_{p,A_{i,m}} \tag{6.18}$$

where $k(A_{i,m}) = \sup_{A_{i,m}}[(\rho\circ\tau)(x)]$. Hence

$$k(A_{i,m}) \asymp \alpha_{m+1} \asymp c^{\alpha m} \tag{6.19}$$

and from (6.11) and (6.14) it is easily seen that

$$J(A_{i,m}) \asymp c^m . \tag{6.20}$$

We now have that there exists $K_0 > 0$ such that

$$\|f - f_{A_{i,m}}\|_{p,A_{i,m}} \leqq K_0 c^m \|\nabla f\|_{p,A_{i,m}} \quad (f \in W^{1,p}_M(A_{i,m}))$$
$$< \varepsilon\|\nabla f\|_{p,A_{i,m}}$$

if $K_0 c^m < \varepsilon$, that is, $m > \log(K_0/\varepsilon)/\log(1/c)$. Therefore

$$\nu_{A_{i,m}}(\varepsilon) = 0 \quad \text{if } m = \left[\frac{\log(K_0/\varepsilon)}{\log(1/c)}\right] + 1 \tag{6.21}$$

and by Lemma 5.3,

$$\nu_{\Omega\setminus\bar{\Omega}_m}(\varepsilon) + 1 \leqq \sum_{i=1}^{2^m} 1 = 2^m$$
$$= O(\varepsilon^{-\log 2/\log(1/c)}) . \tag{6.22}$$

Furthermore, with $n_0 = 1$ and $n_i = 2^{i-1}$ for $i \geqq 1$,

$$\nu_{\Omega_m}(\varepsilon) + 1 \leqq \sum_{i=0}^{m} n_i(\nu_{Q_i}(\varepsilon) + 1) . \tag{6.23}$$

We now estimate $\nu_{Q_i}(\varepsilon)$ in (6.23). Let U denote the square $(0, 1)\times(0, 1)$ and set

$$L_U := \lim_{\lambda\to\infty} \lambda^{-2}\nu_U(1/\lambda) , \tag{6.24}$$

$$R(\lambda) := \lambda^{-2}[\nu_U(1/\lambda) + 1] - L_U ; \tag{6.25}$$

note that $R(\lambda) \geqq 0$ from Lemma 5.7. For $\mu > 0$ define $M = [2\lambda\beta_i/\mu]$, $N = [2\lambda\alpha_i/\mu]$ so that

$$2\lambda\beta_i = M\mu + \theta\mu, \quad 2\lambda\alpha_i = N\mu + \phi\mu \quad (\theta, \phi \in [0, 1)) ;$$

we allow both of M and N to be zero. In (6.23) with $\varepsilon = 1/\lambda$, $\lambda\beta_i \geqq \lambda\beta_m = \lambda c^m \geqq c/K_0$. We now subdivide λQ_i into MN squares of side μ and an L-shaped strip S which is the union of M rectangles of sides $\phi\mu\times\mu$, N rectangles of sides $\theta\mu\times\mu$ and a rectangle of sides $\theta\mu\times\phi\mu$. Further, we subdivide each rectangle in S into $k(\mu)$ say small rectangles T for which $\nu_T(1) = 0$. Hence, by Lemmas 5.3 and 5.6

$$\begin{aligned}\nu_{Q_i}(1/\lambda) + 1 = \nu_{\lambda Q_i}(1) + 1 &\leqq MN(\nu_{\mu U}(1) + 1) + (M + N + 1)k(\mu)\\ &\leqq \lambda^2|Q_i|(\mu^{-2}\{\nu_{\mu U}(1) + 1\}) + \{2\mu^{-1}\lambda(\alpha_i + \beta_i) + 1\}k(\mu)\\ &= \lambda^2|Q_i|(L_U + R(\mu)) + \{2\mu^{-1}\lambda(\alpha_i + \beta_i) + 1\}k(\mu) .\end{aligned} \tag{6.26}$$

It follows that

$$\limsup_{\lambda\to\infty}\{\lambda^{-2}\nu_{Q_i}(1/\lambda)\} \leqq |Q_i|(L_U + R(\mu))$$

and since $R(\mu)\to 0$ as $\mu\to\infty$,

$$\limsup_{\lambda\to\infty}\{\lambda^{-2}\nu_{Q_i}(1/\lambda)\} \leqq |Q_i|L_U . \tag{6.27}$$

From (6.22), (6.23) and (6.26) with $\varepsilon = 1/\lambda$,

$$\begin{aligned}\varepsilon^2[\nu_\Omega(\varepsilon) + 1] &\leqq \sum_{i=0}^{m} n_i\{|Q_i|(L_U + R(\mu)) + [2\mu^{-1}\varepsilon(\alpha_i + \beta_i) + \varepsilon^2]k(\mu)\}\\ &\quad + O(\varepsilon^{2-\frac{\log 2}{\log(1/c)}})\\ &\leqq (L_U + R(\mu))|\Omega| + 2\mu^{-1}\varepsilon k(\mu)\sum_{i=0}^{m}(2c)^i + O(\varepsilon^{2-\frac{\log 2}{\log(1/c)}}) .\end{aligned}$$

From (6.21)

$$\sum_{i=1}^{m-1}(2c)^i = \begin{cases} O(1) & \text{if } c < 1/2 ,\\ O(\log(1/\varepsilon)) & \text{if } c = 1/2 ,\\ O(\varepsilon^{1-\frac{\log 2}{\log(1/c)}}) & \text{if } c > 1/2 .\end{cases}$$

Since $2c^2 < 1$ we have $2 - \dfrac{\log 2}{\log(1/c)} > 0$ and hence

$$\limsup_{\varepsilon\to 0}\{\varepsilon^2[\nu_\Omega(\varepsilon)+1]\} \leqq (L_U + R(\mu))|\Omega| .$$

On allowing $\mu \to \infty$ we obtain

$$\limsup_{\varepsilon\to 0} \varepsilon^2 \nu_\Omega(\varepsilon) \leqq L_U|\Omega| . \tag{6.28}$$

To obtain the lower limit we argue in a similar way using Lemma 5.4. Using the same notation as above we now have

$$\mu_\Omega^0(\varepsilon) \geqq \sum_{i=0}^{m} n_i \mu_{Q_i}^0(\varepsilon)$$

and

$$\begin{aligned}\mu_{\lambda Q_i}^0(1) &\geqq MN\mu_{\mu U}^0(1)\\ &\geqq \{\mu^{-2}\lambda^2|Q_i| - 2\mu^{-1}\lambda(\alpha_i+\beta_i)\}\,\mu_{\mu U}^0(1)\\ &= \{\lambda^2|Q_i| - 2\lambda\mu(\alpha_i+\beta_i)\}\{L_U^0 + R^0(\mu)\} ,\end{aligned}$$

where

$$L_U^0 := \lim_{\lambda\to\infty} \lambda^{-2}\mu_U^0(1/\lambda) , \quad R^0(\mu) = \lambda^{-2}\mu_U^0(1/\lambda) - L_U^0 ; \tag{6.29}$$

note that $R^0(\mu) \leqq 0$ by Theorem 5.8. Arguing as before, we obtain as $\varepsilon \to 0$,

$$\varepsilon^2 \mu_\Omega^0(\varepsilon) \geqq (L_U^0 + R^0(\mu))(|\Omega| - \mu^o\dot{(1)}) - \mathrm{O}\big(\varepsilon^{2-\frac{\log 2c^{1+\alpha}}{\log(1/c)}}\big)$$

and

$$\liminf_{\varepsilon\to 0} \varepsilon^2 \mu_\Omega^0(\varepsilon) \geqq L_U^0|\Omega| . \tag{6.30}$$

We have therefore proved

Theorem 6.2. *Let* $\alpha_m = c^{\alpha m}$, $\beta_m = c^m$ *for* $m \geqq 0$, *where* $\alpha \geqq 1$, $c^\alpha + 2c^2 < 1$ *and* $c^{1+\alpha/p+p/p'^2} < 1/2$ *if* $\alpha > p/p'$. *Then*

(i) *I is compact,*

(ii) $\limsup_{\varepsilon\to 0}\varepsilon^2\nu_\Omega(\varepsilon) \leqq L_U|\Omega|$, *where* U *is the square* $(0,1)\times(0,1)$ *and* $L_U := \lim_{\lambda\to\infty}\lambda^{-2}\nu_U(1/\lambda)$,

(iii) $\liminf_{\varepsilon\to 0}\varepsilon^2\mu_\Omega^0(\varepsilon) \geqq L_U^0|\Omega|$, *where* $L_U^0 = \lim_{\lambda\to\infty}\lambda^{-2}\mu_U^0(1/\lambda)$.

6.2.4 Asymptotics of $\nu_\Omega(\varepsilon)$ *when* $p = 2$. When $p = 2$ we can obtain more precise information as follows. Let $\tau(A_{i,m}) = A_{i,m}$. Then from (6.10)

$$\begin{aligned}\mu(A_{i,m}) &\asymp \sum_{n=m}^{\infty} 2^{n-s}(c^{1+\alpha})^n\\ &\asymp c^{(1+\alpha)m} .\end{aligned}$$

Thus for l sufficiently large, and independent of m, $\mu(A_{i,m+l}) \leqq 1/2\mu(A_{i,m})$ and $\mu(A_{i,m+l}) \asymp c^{(1+\alpha)m}$. Hence, from Proposition 2.4 and Remark 2.5, and the fact that $K(A_{i,m}) \asymp J^p(A_{i,m}) \asymp c^{mp}$ by (6.20), there exists $F \in L^{1,p}(A_{i,m})$ and a constant $K_1 > 0$ such that

$$\|F - F_{A_{i,m}}\|_{2,A_{i,m},d\mu} \geqq 2K_1 c^m \|F'\|_{2,A_{i,m},d\mu} . \tag{6.31}$$

Since

$$\| F - F_{A_{i,m}} \|_{2, A_{i,m}, du} \leqq 2 \| F - F(0) \|_{2, A_{i,m}, d\mu}$$

we have with $G = F - F(0)$ that $G(0) = 0$ and

$$\| G \|_{2, A_{i,m}, d\mu} \geqq K_1 c^m \| G' \|_{2, A_{i,m}, d\mu} . \tag{6.32}$$

Let $f = G \circ \tau$. Then $f \in W^{1,2}(A_{i,m})$ and

$$\begin{aligned} \| f \|_{2, A_{i,m}} &= \| G \|_{2, A_{i,m}, d\mu} \\ &\geqq K_1 c^m \| G' \|_{2, A_{i,m}, d\mu} \\ &\geqq \gamma^{-1} k_1 c^m \| \nabla f \|_{2, A_{i,m}} \end{aligned} \tag{6.33}$$

on using (3.6). Moreover $f = 0$ on the *base* of $A_{i,m}$, i.e. the edge of $\bar{A}_{i,m}$ which meets $\bar{\Omega}_m$. Choose m to be the largest integer such that $\gamma^{-1} K_1 c^m \varepsilon^{-1} \geqq 1$; this gives

$$m = \frac{\log(1/\varepsilon)}{\log(1/c)} + \mathrm{O}(1) . \tag{6.34}$$

We have therefore established that for each $i \in \{1, 2, \ldots, 2^m\}$ there exists $f_i \in W^{1,2}(A_{i,m})$ such that

$$\| f_i \|_{2, A_{i,m}} \geqq \varepsilon \| \nabla f_i \|_{2, A_{i,m}} . \tag{6.35}$$

On setting $f_i = 0$ outside $A_{i,m}$ it follows that $f_i \in W^{1,2}(\Omega)$.

For $i \leqq m$

$$\nu^0_{Q_i}(\varepsilon) = \sharp \left\{ (p, q) : \frac{p^2}{\alpha_i^2} + \frac{q^2}{\beta_i^2} \leqq \frac{4}{\pi^2 \varepsilon^2}, p, q \in \mathbf{N} \right\}$$

$$\begin{cases} = 0 & \text{if} \left[\dfrac{2\alpha_i}{\pi\varepsilon} \right] < 1 \\ \geqq (1/4\pi\varepsilon^2)|Q_i| - \mathrm{O}((\alpha_i + \beta_i)\varepsilon^{-1}) & \text{otherwise} . \end{cases}$$

Hence, by Lemmas 5.2 and 5.4

$$\begin{aligned} \nu^0_{\Omega_m}(\varepsilon) &\geqq \sum_{i=0}^{m} n_i \nu^0_{Q_i}(\varepsilon) \\ &= \sum_{i=0}^{k} n_i \nu^0_{Q_i}(\varepsilon) , \end{aligned}$$

where k is the largest integer such that $\left[\dfrac{2\alpha_k}{\pi\varepsilon} \right] \geqq 1$. It follows that

$$k = \frac{\log(1/\varepsilon)}{\alpha \log(1/c)} + \mathrm{O}(1)$$

and

$$\begin{aligned} \nu^0_{\Omega_m}(\varepsilon) &\geqq (1/4\pi\varepsilon^2) \left\{ |\Omega| - \sum_{i=k+1}^{\infty} n_i |Q_i| \right\} - \mathrm{O}\left(\varepsilon^{-1} \sum_{i=0}^{k} (2c)^i \right) \\ &= (1/4\pi\varepsilon^2)|\Omega| - \mathrm{O}\left(\varepsilon^{-2} \sum_{i=k}^{\infty} (2c^{1+\alpha})^i \right) - \mathrm{O}\left(\varepsilon^{-1} \sum_{i=0}^{k} (2c)^i \right) \\ &= (1/4\pi\varepsilon^2)|\Omega| - \mathrm{O}(\varepsilon^{-2} R(\varepsilon)) \end{aligned} \tag{6.36}$$

where

$$R(\varepsilon) = \begin{cases} \varepsilon & \text{if } c < 1/2 \\ \varepsilon \log(1/\varepsilon) & \text{if } c = 1/2 \\ \varepsilon^{2-d_i} & \text{if } c > 1/2 \end{cases} \tag{6.37}$$

where d_i is given in (6.15).

Let $N = \nu^0_{\Omega_m}(\varepsilon) = \mu^0_{\Omega_m}(\varepsilon)$. Then there exists a subspace S of $W_0^{1,2}(\Omega_m)$ such that $\dim S = N$ and $\alpha(S) \leqq \varepsilon^{-1}$ in the notation of (5.2). Extend S to $W_0^{1,2}(\Omega)$ by setting its members to be zero outside Ω_m and denote $S \cup \operatorname{span}\{f_i : i = 1, 2, \ldots, 2^m\}$ by U. Then $\dim U = N + 2^m$, $U \subset W^{1,2}(\Omega)$ and for all $f \in U$

$$\|f\|_{2,\Omega} \geqq \varepsilon \|\nabla f\|_{2,\Omega}\,. \tag{6.38}$$

We claim that $\nu_\Omega(\varepsilon) \geqq N + 2^m - 1$. To prove this suppose that $P_n \in \mathscr{F}(W_M^{1,2}(\Omega), L_M(\Omega))$ has rank $n < N + 2^m - 1$ and set

$$Q_n f = f_\Omega + P_n(f - f_\Omega)$$

for $f \in W^{1,2}(\Omega)$. Then rank $Q_n \leqq n + 1$ and if $\{f_i : i = 1, 2, \ldots, N + 2^m\}$ is a basis of U, there exist constants $\lambda_i, i = 1, 2, \ldots, N + 2^m$, not all zero, such that $Q_n(\sum_{i=1}^{N+2^m} \lambda_i f_i) = 0$. Thus, with $f = \sum_{i=1}^{N+2^m} \lambda_i f_i$ we have in view of (6.38) that

$$\begin{aligned} \varepsilon \|\nabla f\|_{2,\Omega} &\leqq \|f\|_{2,\Omega} \\ &= \|f - Q_n f\|_{2,\Omega} \\ &= \|(f - f_\Omega) - P_n(f - f_\Omega)\|_{2,\Omega}\,. \end{aligned}$$

Consequently $a_{N+2^m-1}(\Omega) \geqq \varepsilon$ and $\nu_\Omega(\varepsilon) \geqq N + 2^m - 1$ as asserted. Hence, from (6.36),

$$\begin{aligned} \varepsilon^2 \nu_\Omega(\varepsilon) - (1/4\pi)|\Omega| &\geqq \varepsilon^2 2^m - \mathrm{O}(R(\varepsilon)) \\ &\geqq K_1 \varepsilon^{2 - \frac{\log 2}{\log(1/c)}} - K_2 R(\varepsilon)\,. \end{aligned} \tag{6.39}$$

We obtain an upper bound from (6.22) and (6.23) where we now have

$$\begin{aligned} \nu_{Q_i}(\varepsilon) &= \#\left\{(p, q) : \frac{p^2}{\alpha_i^2} + \frac{q^2}{\beta_i^2} \leqq \frac{4}{\pi^2 \varepsilon^2}, p, q \in \mathbf{N}_0, (p, q) \neq (0, 0)\right\} \\ &\leqq (1/4\pi\varepsilon^2)|Q_i| + 2(\alpha_i + \beta_i)\varepsilon^{-1}\,. \end{aligned}$$

Thus we have

$$\begin{aligned} \nu_\Omega(\varepsilon) + 1 &\leqq \sum_{i=0}^{m} n_i(\nu_{Q_i}(\varepsilon) + 1) + \mathrm{O}(\varepsilon^{-\frac{\log 2}{\log(1/c)}}) \\ &\leqq \left(\frac{1}{4\pi\varepsilon^2}\right)|\Omega| + 4\varepsilon^{-1} \sum_{i=0}^{m} (2c)^i + \mathrm{O}(\varepsilon^{-\frac{\log 2}{\log(1/c)}}) \end{aligned}$$

which yields

$$\varepsilon^2 \nu_\Omega(\varepsilon) - (1/4\pi)|\Omega| \leqq K_3 \varepsilon^{2 - \frac{\log 2}{\log(1/c)}} + K_4 R(\varepsilon)\,. \tag{6.40}$$

From (6.37), (6.39) and (6.40) we obtain

Theorem 6.3. *Let the hypothesis of Theorem 6.2 be satisfied. Then as* $\varepsilon \to 0$

$$\varepsilon^2 \nu_\Omega(\varepsilon) - (1/4\pi)|\Omega| \begin{cases} = \mathrm{O}(\varepsilon) & \text{if } c < 1/2\,, \\ = \mathrm{O}(\varepsilon \log(1/\varepsilon) & \text{if } c = 1/2\,, \\ \asymp \varepsilon^{2-d_o} & \text{if } c > 1/2\,, \end{cases}$$

where d_o *is given in* (6.16).

If $\mathcal{N}(\lambda;\, -\triangle_{\Omega,N}) = \sharp\{m : \lambda_m(\Omega, N) < \lambda\}$, we obtain, as $\lambda \to \infty$,

$$\mathcal{N}(\lambda;\, -\triangle_{\Omega,N}) - (1/4\pi)|\Omega|\lambda \begin{cases} = \mathrm{O}(\lambda^{1/2}) & \text{if } 2c < 1\,, \\ = \mathrm{O}(\lambda^{1/2}\log\lambda) & \text{if } 2c = 1\,, \\ \asymp \lambda^{d_o/2} & \text{if } 2c - 1\,. \end{cases} \tag{6.41}$$

The $c > 1/2$ case, when $\partial\Omega$ is fractal in the sense that the inner and outer Minkowski dimensions lie in (1, 2), is particularly interesting as we obtain the precise growth rate of the error term in (6.41). This improves on the general result in [10, Theorem 2.1] where the error is shown to be $\mathrm{O}(\lambda^{d_o/2})$. If $c \leqq 1/2$ the error in (6.41) is smaller than that in [10]: when $c < 1/2$ Lapidus' result implies the error $\mathrm{O}(\lambda^{1/2}\log\lambda)$ and when $c = 1/2$ he obtains $\mathrm{O}(\lambda^{s/2}\log\lambda)$ for any $s > 1$, since $\mathcal{M}^o_{d_o}(\partial\Omega) = \infty$ in this case.

6.2.5 Asymptotics of $\nu^o_\Omega(\varepsilon)$ *when* $p = 2$. From Lemmas 5.2 and 5.4 and (6.36) we obtain

$$\varepsilon^2 \nu^o_\Omega(\varepsilon) \geqq (1/4\pi)|\Omega| - \mathrm{O}(R(\varepsilon)) \tag{6.42}$$

where $R(\varepsilon)$ is given in (6.37).

To obtain an upper bound for $\nu^o_\Omega(\varepsilon)$ we use Lemma 5.5 with $\Omega_1 = \Omega_m$ and $\Omega_2 = \Omega \backslash \bar{\Omega}_m$. We first prove that there exists $K_1 > 0$ such that, for all $f \in W^{1,2}_0(\Omega)$

$$\|f\|_{2,\Omega_2} \leqq K_1 \alpha_m \|\nabla f\|_{2,\Omega_2}\,. \tag{6.43}$$

Consider the region $OPAL$ in Fig. 3, set $y = r\sin\phi$ and let $l(t)$ denote the length of $\tau^{-1}(t)$. We change co-ordinates to (t, r) where

$$x = t + r\cos\phi, \quad y = r\sin\phi \quad (\phi = \phi(t))\,.$$

Then

$$\begin{aligned} |f(x(t,r), y(t,r))| &= \left| \int_r^{l(t)} (\partial/\partial z) f(x(t,z), y(t,z))\, dz \right| \\ &= \left| \int_r^{l(t)} (f_1 \cos\phi + f_2 \sin\phi)\, dz \right| \\ &\leqq \left(\int_r^{l(t)} |\nabla f|^2\, dz \right)^{1/2} l(t)^{1/2}\,. \end{aligned}$$

Also, since $\phi' = (\sin^2\phi)/\alpha_m \chi_m$ in the notation of Sect. 6.1, we have

$$\begin{aligned} \frac{\partial(x,y)}{\partial(t,r)} &= \sin\phi - r\phi' \\ &= \alpha_m l(t)^{-1}(1 - r/l(t)\chi_m) \\ &\geqq (\alpha_m l^{-1})(1 - 1/\chi_m) \geqq (\alpha_m/l)\left(\frac{1-\delta_2}{1+\delta_1^{-1}}\right) \end{aligned}$$

by (6.2), where $l = \max_{0 \leqq t \leqq \beta_m} l(t) \asymp \alpha_m$. Hence

$$\alpha_m / l(t) \geqq \frac{\partial(x, y)}{\partial(t, r)} \geqq K > 0 ,$$

and

$$\iint_{OPAL} |f(x, y)|^2 \, dxdy \leqq \int_0^{\beta_m + \alpha_{m-1}} \int_0^{l(t)} l(t) \left(\int_r^{l(t)} |\nabla f|^2 dz \right) (\alpha_m / l(t)) \, drdt$$

$$\leqq K \alpha_m l \iint_{OPAL} |\nabla f|^2 \, dxdy$$

$$\leqq K \alpha_m^2 \iint_{OPAL} |\nabla f|^2 \, dxdy .$$

Since Ω_2 is made up of regions like *OPAL*, (6.43) follows.

Choose m to be the smallest integer such that $K_1 \alpha_m = K_1 c^{\alpha m} \leqq \varepsilon/2$. Then $m = \dfrac{\log(1/\varepsilon)}{\alpha \log(1/c)} + \mathrm{O}(1)$ and, by Lemma 5.5, $\nu_\Omega^0(\varepsilon) \leqq \nu_{\Omega_m}(\varepsilon) + 1$. Also, as in Sect. 6.2.4,

$$\nu_{\Omega_m}(\varepsilon) + 1 \leqq (1/4\pi\varepsilon^2)|\Omega| + 4\varepsilon^{-1} \sum_{i=0}^{m} (2c)^i + \sum_{i=0}^{m} 2^i$$

$$\leqq (1/4\pi\varepsilon^2)|\Omega| + \mathrm{O}(\varepsilon^{-2} R(\varepsilon)) .$$

We have therefore proved

Theorem 6.4. *Let the hypothesis of Theorem* 6.2 *be satisfied. Then, as* $\varepsilon \to 0$

$$\varepsilon^2 \nu_\Omega^0(\varepsilon) - (1/4\pi)|\Omega| = \begin{cases} \mathrm{O}(\varepsilon) & \text{if } 2c < 1 , \\ \mathrm{O}(\varepsilon \log(1/\varepsilon)) & \text{if } 2c = 1 , \\ \mathrm{O}(\varepsilon^{2-d_l}) & \text{if } 2c > 1 . \end{cases}.$$

If $\mathscr{N}(\lambda; -\triangle_{\Omega, D}) = \#\{m : \lambda_m(\Omega, D) < \lambda\}$ then our result yields

$$\mathscr{N}(\lambda; -\triangle_{\Omega, D}) - (1/4\pi)|\Omega|\lambda = \begin{cases} \mathrm{O}(\lambda^{1/2}) & \text{if } 2c < 1 , \\ \mathrm{O}(\lambda^{1/2} \log \lambda) & \text{if } 2c = 1 , \\ \mathrm{O}(\lambda^{d_l/2}) & \text{if } 2c > 1 . \end{cases} \tag{6.44}$$

In the case $c > 1/2$, (6.44) is included in [10, Corollary 2.1]. Similar remarks to those made for the Neumann problem at the end of the last subsection also apply to (6.44), namely that (6.44) gives better estimates than those of Lapidus in [10] when $c \leqq 1/2$.

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