Spectral Multidomain Technique with Local Fourier Basis *

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1 Introduction

The numerical solution of multidimensional non-linear evolution equations (and especially the particularly difficult case of the incompressible Navier-Stokes equations) is a heavy computational task. At the range of parameters, which is of interest in applications, the resolution requirements in time and space increase enormously. Consequently, a huge number of degrees of freedom must be computed, and memory and CPU requirements increase beyond the capacity of present day super-computers.

The emerging parallel machines of MIMD (message-passing and shared-memory) type are, in principle, able to grow in accordance with the requirements of the task. But such parallel machines can be effective only if efficient "parallelizing algorithms" can be supplied, otherwise, communication and synchronization overheads destroy completely the performance of the best serial methods. The numerical solution of Partial Differential Equations (PDE's) has the potential for efficient parallelization on local memory machines because partial derivatives describe a local behavior.

A wide class of parallel algorithms is based on Domain Decomposition techniques. The general approach consists of splitting a computational region into smaller subdomains, allocated to different processors with short

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communication links. The original problem is solved independently in each processor. Then some "patching" procedure is employed to match the overall solution, such as a direct influence matrix method or the iterative Schur method.

Domain Decomposition techniques can be successively combined with spectral (pseudospectral) methods, which are proven to be efficient for many fluid mechanics problems due to high-order accuracy and fast (exponential) convergence with the increasing of the number of basic functions (collocation points). Pseudospectral methods also allow fast transform techniques, such as well known FFT (Fast Fourier Transform).

The pseudospectral Fourier method is one which is the most simple for the evaluation of spatial derivatives, because the differential operator is represented in the transformed space by a diagonal matrix. The use of the trigonometric Fourier basis, however, is restricted to the case of periodic boundary conditions. For non-periodic problems, for example, viscous problem with no-slip walls, this basis is inappropriate, because periodically extended function would be not smooth at the boundaries. It is known, however, that the accuracy and the convergance rate of any spectral method are precisely dependent on the degree of smoothness- functions, which are not smooth enough, wold have a poore spectral approximation.

Domain Decomposition methods deal with non-periodic pieces of functions in each subdomain, even if they are periodic on a whole domain. For the non-periodic problems the polynomial Chebyshev or Legendre bases are mostly used. The disadvantages of these bases are non-diagonal matrices of the differential operators and non-uniform spatial grid. These bases are also associated with more severe restrictions for stability of time-dependent schemes.

Previously, attempts were made to use the spectral Fourier method for the solution of nonperiodic problems. In [1,2] the polynomial subtraction was employed to transform a function to a periodic form. This procedure, however, has a low order of accuracy, deteriorates for the short waves and is difficult to be generalized to two-dimensions.

Recently R. Coifman and Y. Meyer [3] constructed a Local Trigonometric Basis for the purposes of signal/image processing. We will show in this work, that a particular form of this basis is well compatible with multidomain techniques and can be efficiently used for solving of PDE's. This basis will enable us to blend distinct neighbouring domains of computation. The basis possesses the advantage of locality as well as the usual advantages of Fourier bases for diffirentiation, fast transform and spectral accuracy. We concentrate on elliptic equation of Helmholtz type, which contains as special cases the stationary Poisson's equation and implicit temporal discretization of time-dependent parabolic problems with diffusion-type linear operator. A direct matching procedure is developed, which exploits the exact homogeneous solutions of the original equation. Then the interface problem is reduced to a linear system with quasi-bidiagonal matrix, which provides explicit formulas for the matching parameters as functions of jumps at the interfaces.

In Section 2 a Local Fourier Basis is constructed and employed for solving nonperiodic differential equations. In Section 3 a direct pointwise matching procedure is described in one-dimensional case for different types of boundary conditions. The accuracy of multidomain algorithm with a Local Fourier Basis is examined and compared with the pseudospectral Chebyshev approximation in Section 4. In Section 5 the extention to two dimensions is discussed. In Section 6 the algorithm is applied for solving nonlinear time-dependent parabolic problems.

2 The Local Trigonometric Basis for Solution of Non-Periodic Problems

We start with the simplest one-dimensional boundary problem

$$u_{xx} = f(x), \quad x \in [a, b],$$
$$u(a) = u_a, \;\; u(b) = u_b, \tag{2.1}$$

where f is a C^∞ (not periodic) function on $[a, b]$. The direct application of the spectral Fourier method to this problem would give a poor resolution, because the periodic extention of $f(x)$ is not smooth. Nevertheless, being combined with a pertinent "smoothing" procedure it may become quite satisfactory. The efficient smoothing procedure was elaborated in [3] for the construction of the Orthogonal Local Trigonometric Basis, which appears as a "folding" of functions across the boundaries. Hereafter we describe a simplified version of that algorithm, which can be applied to the solution of differential equations.

Suppose, that $f(x)$ is known on an extended interval $[a_1, b_1] \supset [a, b]$. To define a "folding" procedure, we introduce an even non-negative function

$\psi(x)$ such that

$$\int_{-\infty}^{+\infty} \psi(x)dx = \frac{\pi}{2}, \quad \sup\psi \subset [-\epsilon, \epsilon], \quad \psi(x) = \psi(-x) \tag{2.2}$$

and related to it, the functions

$$\theta(x) = \int_{-\infty}^{x} \psi(\xi)d\xi, \quad \theta(x) + \theta(-x) = \frac{\pi}{2}; \tag{2.3}$$

$$s(x) = sin\theta(x), c(x) = \cos\theta(x); \; s^2(x) + c^2(x) = 1, \quad c(x) = s(-x). \tag{2.4}$$

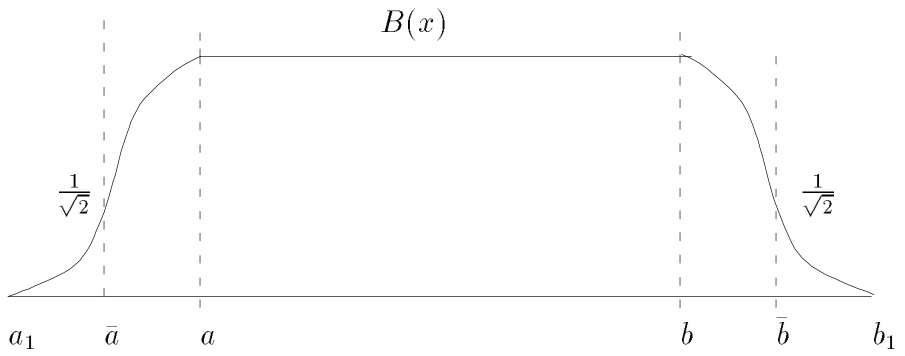

Figure 1: The bell function

Let us define the "bell" function $b(x)$

$$B(x) = \begin{cases} 0, & x < a_1, x > b_1 \\ s(x), & x \in [a_1, a] \\ 1, & x \in [a, b] \\ c(x), & x \in [b, b_1] \end{cases} \tag{2.5}$$

(Fig. 1). The "folding" of a function $f(x)$ across the boundary $x = \bar{x}$ is determined by the formula

$$\mathcal{F}_{\bar{x}} f(x) = B(x)f(x) - B(2\bar{x} - x)f(2\bar{x} - x) \tag{2.6}$$

The whole smoothing procedure, applied to $f(x)$ on the interval $[a_1, b_1]$ consists of two "foldings" across the lines $\bar{a} = (a + a_1)/2$ and $\bar{b} = (b + b_1)/2$

$$\tilde{f} \equiv B \cdot f = \mathcal{F}_{\bar{a}}\mathcal{F}_{\bar{b}} f(x) = B(x)f(x) - B(2\bar{a} - x)f(2\bar{a} - x) - B(2\bar{b} - x)f(2\bar{b} - x) \tag{2.7}$$

(the second term is "switched on" only on the interval $x \in [a_1, a]$ and respectively the third term- on the interval $x \in [b, b_1]$). It can be readily verified, that a "smoothed" function $\tilde{f}$ coincide with the original function f on $x \in [a, b]$, has a C^∞- continuation at the points x=a, x=b and all even derivatives $\tilde{f} = \tilde{f}'' = ... = \tilde{f}^{(2n)} = ... = 0$ at the points $x = \bar{a}, x = \bar{b}$ as in Fig. 2.

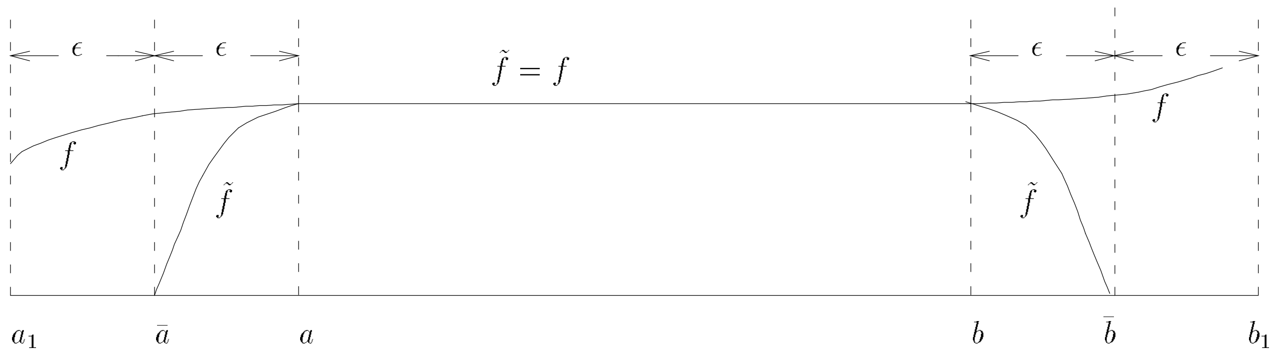

Figure 2: The folding operation

After the antisymmetric extension across the point $x = \bar{b}$ (or $x = \bar{a}$) we obtain the periodic function

$$F = \begin{cases} \tilde{f}(x) & x \in [\bar{a}, \bar{b}] \\ \tilde{f}(2\bar{b} - x) & x \in [\bar{b}, 2\bar{b} - \bar{a}] \end{cases} \tag{2.8}$$

Now the equation (2.1) with the function F in the RHS is easily solved by spectral Fourier method:

$$\begin{array}{lcrl} F(x) & \xrightarrow{\text{FT}} & & \hat{F}_k, \\ \hat{U}_k & = & - & \hat{F}_k/k^2, \\ \hat{U}_k & \xrightarrow{\text{IFT}} & & U(x) + c_0 + c_1 x \end{array} \tag{2.9}$$

where FT and IFT denote respectively Fourier and Inverse Fourier Transform and c_0 and c_1 are determined to satisfy the boundary conditions in (2.1). The resulting function $U(x)$ will obviously coincide with the solution $u(x)$ to the original problem (2.1) on the interval $x \in [a, b]$.

Several functions ψ (or θ), having the properties (2.2), (2.3), were tested for the smoothing procedure:

$$\theta(x) = \sqrt{\frac{\pi}{2a}}\left[1 + erf\left(\frac{x\sqrt{a}}{\sqrt{2}}\right)\right] \qquad (\psi(x) = e^{-ax^2}), \tag{2.10.1}$$

$$\theta(x) = tan^{-1}(e^{ax}), \tag{2.10.2}$$

$$\theta(x) = \frac{\pi}{4}(1 + \underbrace{sin(\frac{\pi}{2}\, sin(\frac{\pi}{2} .... sin\frac{\pi}{2}\frac{x}{\epsilon}}_{r}))). \tag{2.10.3}$$

The maximum numerical error $log(u - u^{exact})_{max}$ was examined for various intervals of smoothig ϵ, number N of collocation points on the interval $[a, b]$ and N_ϵ on the interval ϵ, and for several functions $f(x)$ in (2.1). The option (2.10.3) with $r = 3$ appeared to be the most successful one. Table 1 gives some details in the case $f(x) = sinkx,\ x \in [0, 2\pi]$.

			$\log(u - u^{exact})_{max}$			
N	N_ϵ	ϵ/π	$k=1$	$k=2$	$k=4$	$k=8$
16	8	1	-7.7	-7.0	-5.8	-4.1
32	8	1/2	-8.7	-7.7	-7.0	-5.8
64	8	1/4	-9.6	-8.7	-7.7	-7.0
128	8	1/8	-10.6	-9.6	-8.7	-7.7
32	16	1	-13.3	-12.8	-11.3	-9.6
64	16	1/2	-13.7	-13.3	-12.8	-11.3
128	16	1/4	-14.5	-13.7	-13.3	-12.8

Table 1: The maximum numerical error for the bell (2.10.3) with $r = 3$.

For a fixed interval of smoothing ϵ, the error is smaller, the bigger N. The enlargement of interval ϵ at a given N also decreases the error. If we compare two cases $N^{(2)} = nN^{(1)}$, $\epsilon^{(2)} = \epsilon^{(1)}/n$ $N_\epsilon^{(2)} = N_\epsilon^{(1)}$ (n is interger) with different wave numbers $k^{(2)} = nk^{(1)}$ such that a local distribution $f(x)$ near the boundaries is similar in both cases, then the error will be the same.

The following conclusion can be stated: as the resolution required increases the relative amount of overhead decreases as N_ϵ/N. It follows that for large enough problems the additional cost of smoothing will become insignificant.

3 Domain Decomposition Technique in One-Dimension

We consider the Helmholtz equation

$$u_{xx} - \lambda^2 u = f(x), \quad x \in [0, L], \tag{3.1}$$

with the different types boundary conditions:

$$u(0) = U_0, \quad u(L) = U_L \tag{3.2.1}$$

$$u_x(0) = D_0, \quad u_x(L) = D_L \tag{3.2.2}$$

$$u(0) = u(L). \tag{3.2.3}$$

Let's divide the whole region L into P equal (for the sake of simplicity) pieces $l = L/P$ and solve the equation on each interval by spectral collocation method with the Local Fourier Basis, as it is described in Section 2. The "bell" function is chosen to be equal $B = 1$ inside each subdomain $x \in [0, l]$ and decay outside of this range. This choice imply the overlaping of subdomains by the value 4ϵ (see Fig. 3).

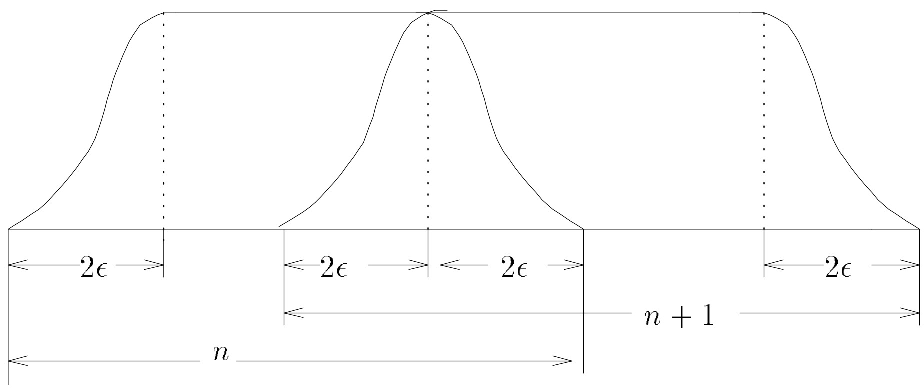

Figure 3: Sequence of overlapping bells

The unknown boundary conditions at the interfaces $\bar{x}_n = nl, \; n = 1, 2, ..., P-1$ are chosen arbitrarily at this stage. An overall particular solution $u_p =$

$\bigcup_{n=1}^{P} u_p^{(n)}$ and its first derivative will have jumps at the interfaces $\bar{x}_n$. To make the function $u(x)$ continuous, we exploit the homogeneous solutions of (3.1) in the form

$$u_h^{(n)} = A_n e^{\lambda(x-l)} + B_n e^{-\lambda x}, \quad x \in [0, l], \tag{3.3}$$

$$u = \bigcup_{n=1}^{P} u^{(n)}, \quad u^{(n)} = u_h^{(n)} + u_p^{(n)}. \tag{3.4}$$

The coefficients A_n and B_n are determined so that the continuity conditions on the interfaces

$$u^{(n)} = u^{(n+1)}, \quad u_x^{(n)} = u_x^{(n+1)}, \quad \bar{x}_n = l, 2l, ..., (P-1)l \tag{3.5}$$

are satisfied.

Combining (3.3)-(3.5), we obtain the following set of relations:

$$A_n = A_{n+1} e^{-\lambda l} - \alpha_n, \quad B_{n+1} = B_n e^{-\lambda l} - \beta_n$$

where

$$\alpha_n = \frac{1}{2}(\frac{\delta_n'}{\lambda} + \delta_n), \quad \beta_n = \frac{1}{2}(\frac{\delta_n'}{\lambda} - \delta_n),$$

$$\delta_n = u_p^{(n)}(l) - u_p^{(n+1)}(0), \quad \delta_n' = \frac{du_p^{(n)}(l)}{dx} - \frac{du_p^{(n+1)}(0)}{dx}, n = 1, 2, ..., P-1 \tag{3.6}$$

The corresponding relations for the two extreme processors depend on the global boundary conditions:

$$A_1 e^{-\lambda l} + B_1 = \alpha_0, \quad A_P + B_P e^{-\lambda l} = -\beta_P$$

$$\alpha_0 = U_0 - u_p^{(1)}(0), \quad \beta_P = u_p^{(P)}(l) - U_L \qquad \text{Dirichlet} \tag{3.7.1}$$

$$\alpha_0 = \frac{1}{\lambda}(D_0 - \frac{du_p^{(1)}(0)}{dx}), \quad \beta_P = \frac{1}{\lambda}(\frac{du_p^{(P)}(l)}{dx} - D_L) \qquad \text{Neumann} \tag{3.7.2}$$

$$A_P = A_1 e^{-\lambda l} - \alpha_0, \; B_1 = B_P e^{-\lambda l} - \beta_P$$

$$\alpha_0 = \frac{1}{2}(\frac{\delta_0'}{\lambda} + \delta_0), \; \beta_P = \frac{1}{2}(\frac{\delta_P'}{\lambda} - \delta_P),$$

$$\delta_0 = \delta_P = u_p^{(P)}(l) - u_p^{(1)}(0), \; \delta_0' = \delta_P' = \frac{du_p^{(P)}(l)}{dx} - \frac{du_p^{(1)}(0)}{dx} \qquad \text{periodic} \tag{3.7.3}$$

In the case of Dirichlet (Neumann) boundary conditions the interface problem (3.6), (3.7.1) (or (3.7.2)) is represented by a cyclic quasi-bidiagonal matrix (we write it down for $P = 4$):

$$\begin{pmatrix} E & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ -1 & E & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & E & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & E & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & -E & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & E & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & E & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & E \end{pmatrix} \begin{pmatrix} A_1 \\ A_2 \\ A_3 \\ A_4 \\ B_4 \\ B_3 \\ B_2 \\ B_1 \end{pmatrix} = \begin{pmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \beta_4 \\ \beta_3 \\ \beta_2 \\ \beta_1 \end{pmatrix} \tag{3.8}$$

where $E \equiv e^{-\lambda l}$.

This system can be easily resolved with respect to one of coefficients, say B_1:

$$B_1 = \frac{1}{1 - e^{-2P\lambda l}} \left[\sum_{n=0}^{P-1} (\alpha_n + \beta_{P-n} \exp^{-P\lambda l}) e^{-n\lambda l} \right].$$

Then the recursive relations (3.6), (3.7.1) give all other coefficients as functions of jumps δ_n and δ'_n across the interfaces.

For periodic boudary conditions both sets A_n and B_n are mutually uncoupled:

$$\begin{pmatrix} E & 0 & 0 & -1 \\ -1 & E & 0 & 0 \\ 0 & -1 & E & 0 \\ 0 & 0 & -1 & E \end{pmatrix} \begin{pmatrix} A_1 \\ A_2 \\ A_3 \\ A_4 \end{pmatrix} = \begin{pmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} \tag{3.9}$$

and similarly for the B_n. Since the explicit expression for A_P is determined

$$A_P = -\frac{1}{1 - e^{-P\lambda l}} \sum_{n=0}^{P-1} \alpha_n e^{-n\lambda l},$$

the other coefficients can be found by using the recurrence relations (3.6), (3.7.3). Thus, instead of the overall coupling, which is inherent in the spectral Fourier method, the communication among processors is reduced to exchanging of the interface data.

The use of the explicit analytical form of the homogeneous solution (3.3) enables us to obtain the rather general recurrence relations (3.6) for the parameters of the matching procedure, which do not depend on the function $f(x)$ in (3.1), but only on the structure of the equation itself.

At this point the matching algorithm described above can be viewed as a particular version of the influence matrix method. There are several technical differences which make our procedure more economical for the present implementation: 1) this procedure matches simultaneously the function and the derivative values; 2) we use analytically derived matching solutions; 3) in future papers we make use in an explicit way of exponentially small terms in the analytical form of the matching relations to minimize the communications between the processors.

A similar approach can be applied to other linear differential equations, such as the advection-diffusion equation

$$u_{xx} - a^2 u - 2bu_x = f(x).$$

The corresponding homogeneous solution has the form

$$u_h^{(n)} = A_n \exp^{\lambda_+(x-l)} + B_n \exp^{-\lambda_- x},$$

$$\lambda_{\pm} = b \quad \pm \ \sqrt{b^2 + a^2}$$

The matching relations can be resolved in the same way, as they are for the Helmholtz equation. Instead of relations (3.6) we will get

$$A_n = A_{n+1} e^{-\lambda_+ l} - \bar{\alpha}_n, \quad B_{n+1} = B_n e^{\lambda_- l} - \bar{\beta}_n$$

$$\bar{\alpha}_n = \frac{\delta_n' - \delta_n \lambda_-}{\lambda_+ - \lambda_-}, \quad \bar{\beta}_n = \frac{\delta_n' - \delta_n \lambda_+}{\lambda_+ - \lambda_-},$$

where δ_n and δ_n' are determined in (3.6).

4 Accuracy of the Local Fourier Method

The precision of our domain decomposition method with the Local Fourier Basis (LFB) is compared with the pseudospectral Chebyshev (PSC) discretization for the inner layer solution $u(x) = \frac{1}{2}(tanh 20x + 1)$, $x \in [-1, 1]$, which satisfies equation (3.1) with an appropriate $f(x)$ and boundary conditions (3.2.1). This function has a steep profile near the origin and it could confirm the accuracy of the method.

Table 2 illustrates the RMS error for various numbers of subdomains P and collocation points $N = N_1 + N_2 + N_n ... + N_P$; here $\lambda = 10$ and the interval of smoothing $\epsilon/l = 1/2$.

method	N	1-Domain	$2 - Domain^*$ $N_1 = N_2$	$3 - Domain^*$ $N_1 = N_2 = N_3$	$3 - Domain^{**}$
PSC [4]	34	2.22 (-2)	1.36 (-3)		3.49 (-3)
	74	4.94(-4)	1.12 (-7)		4.85 (-6)
LFB	32	1.21 (-3)	1.18 (-3)		
	30			1.58 (-3)	7.82 (-5)
	72	1.27 (-6)	1.30 (-6)	1.54 (-6)	5.84 (-8)

Table 2: The RMS error for inner layer solution ($^* N_n = N/P, l_n = L/P$; $^{**} N_1 = N_3 = H[\frac{1}{3}N], N_2 = N - 2N_1, l_1 = l_3 = 2l_2$).

One can see, that in 1-domain the LFB provides a 1-2 order better accuracy for the same number of collocation points N, than the PSC approximation. The multidomain technique, utilizing the LFB, practically does not affect the precision as compared with 1-domain (for the same N). In the 2-domain case, when the interface is located at the point of steepest profile, the PSC approximation has better accuracy for big enough $N (N = 74)$ due to condcentration of the Gauss-Lobatto collocation points towards the interface. In the 3-domain case with a constant grid spacing (case *- see legend of Table 2) the LFB method gives slightly smaller error in comparison with an adaptive grid PSC approximation (case **), and it is much more accurate on the adaptive mesh.

5 Two-Dimensional Elliptic Problems

The present approach can also be applied to multi-dimensional problems. For Domain Decomposition in the form of strips the problem can be easily reduced to a set of one-dimensional problems for the Fourier coefficients. Then the implementation of the one-dimensional routine is straightforward as follows.

In the two-dimensional case, for example, the Helmholtz equation takes the form

$$\Delta u - \lambda^2 u = f(x,y), \qquad \text{in } \Omega = (0, L_1) \times (0, L_2). \tag{5.1}$$

Let the rectangular Ω be decomposed into parallel strips $\Omega = \cup_{n=1}^{P} \omega_n$, $\omega_n = (\bar{x}_{n-1}, \bar{x}_n) \times (0, L_2)$, where $\bar{x}_n = nl, \quad n = 1, 2, ..., P$.

Suppose that $f(x,y)$ and $u(x,y)$ are periodic functions of y and expand them into Fourier series along the strips. Then we arrive at a set of one-dimensional equations for the Fourier coefficients $\hat{u}_k^{(n)}(x)$

$$\hat{u}_k^{(n)''} - (k^2 + \lambda^2)\hat{u}_k^{(n)} = \hat{f}_k^{(n)}(x) \qquad \text{in } \omega_n, \qquad k = -\frac{N_2}{2}, ..., \frac{N_2}{2}, \tag{5.2}$$

where N_2 is a number of collocation points in the y direction. The boundary conditions in the x-direction may be one of the (3.2.1)-(3.2.3) types.

The one dimensional equations (5.2) are solved by using the LFB with the exact matching procedure, as it is described in Section 3. Then the solution of the original problem (5.1) is obtained by the inverse Fourier transform in each strip ω_n.

If the function $f(x,y)$ is not periodic along the strips, as it was assumed before, it can be transformed into a periodic form by using the same "folding" procedure, as was described in Section 2. Then the previous situation is reproduced on the extended region $\bar{\Omega} = \in (0, L_1) \times (-\epsilon, L_2 + \epsilon)$. The solution of this modified problem will coincide with the original one on Ω.

6 Time-Dependent Problems

In this section we consider time-dependent parabolic problems with a linear operator of the diffusion type

$$\frac{\partial u}{\partial t} = \Delta u + b(u, x, t),$$

$$x = (x_1, x_2, ...x_d), \; x \in \Omega \subset R^d \tag{6.1}$$

where $b(u,x,t)$ contains all non-linear and inhomogeneous terms. An implicit (or semi-implicit) time discretization procedure for this problem leads to the elliptic equation (3.1) at each time step. Let us assume that we have the two-level scheme

$$\frac{u_{n+1} - u_n}{\tau} = \theta \Delta u_{n+1} + (1-\theta)\Delta u_n + b_n \tag{6.2}$$

where n refers to the times $t_n = n\tau$ and $\theta > 0$ (the particular cases $\theta = 1/2$ and $\theta = 1$ correspond to the Crank-Nikolson and the Euler-backward schemes respectively).

By setting $\lambda^2 = 1/\tau\theta$ we get

$$\Delta u_{n+1} - \lambda^2 u_{n+1} = \frac{\theta - 1}{\theta}\Delta u_n - \lambda^2 u_n - \frac{1}{\theta} b_n. \tag{6.3}$$

Equation (6.3) can be simplified by defining

$$v^* = \theta u_{n+1} + (1-\theta) u_n \tag{6.4}$$

then

$$\Delta v^* - \lambda^2 v^* = -\lambda^2 u_n - \frac{1}{\theta} b_n \tag{6.5}$$

This equation is solved for v^* in terms of the known u_n and b_n, then u_{n+1} is determined by using (6.4). This process can be repeated for each time step as many times as needed.

The folding procedure described in Section 2 operates with functions on an extended region. For time-dependent problems functions in the RHS of (6.3) or (6.5) are known only on the interior region, so that near the boundaries a different type of smoothing should be implemented (for example, a substraction procedure [2]). To avoid a special consideration on the boundaries, we presume the periodic boundary conditions what enables us to overlap the two extreme subdomains.

As an example, we consider the two-dimensional heat-transfer equation

$$\frac{\partial u}{\partial t} = \Delta u + f(x,y)$$

$$u(x,-\pi,t) = u(x,\pi,t), \quad u(-\pi,y,t) = u(\pi,y,t),$$

$$u(x,y,0) = 0$$

with the forcing term $f(x,y)$, which corresponds to the exact solution $u_{exact} = (1 - e^{-(k_x^2+k_y^2)t}) cos(k_x x) cos(k_y y)$. The succession of profiles across the strips is shown on the Fig.4 ($k_x = k_y = 1, P = 4$; the interfaces look as if they are straight lines). The numerical solution remains of high precision during the approach to steady state.

The capability of the method to resolve with good accuracy a very high gradient in a solution can be demonstrated of the viscous Burger's equation

$$\frac{\partial u}{\partial t} = -u\frac{\partial u}{\partial x} + \nu\frac{\partial^2 u}{\partial x^2}. \tag{6.6}$$

The exact periodic solution has the form [5]

$$u(x,t) = -2\nu\frac{\frac{\partial\phi}{\partial x}(x-ct,t+1)}{\phi(x-ct,t+1)}$$

$$\phi(x,t) = \sum_{n=-\infty}^{\infty} e^{-(x-(2n+1)\pi)^2/4\nu t}. \tag{6.7}$$

For ν small enough these are smoothed sawtooth waves with a steep profile in a narrow region of width $\nu(t+1)$ near the points $x = (2n+1)\pi + ct$. The waves propagate to the right with a constant speed c (corresponding to a mean profile value), and gradually diffuse to a uniform state.

We compare the accuracy of the present multidomain LFB technique with the "pure" pseudospectral Fourier method (PSF) in one domain. For time discretisation we use a semi-implicit stiffly stable scheme of 4-th order accuracy as in [6]; the time step is chosen to be small enough to reduce the time errors to be below the spatial ones. The solution starts with the exact distribution (6.7) at $t = 0$, then the error is computed at some specified time $t = \pi/8$. Table 4 gives the maximum error in the case $\nu = 0.2, c = 4$.

Comparison of the error at the same N testifies in favour of the PSF method. Because the number of collocation points (and respectively number of Fourier harmonics) in each subdomain is P times smaller than the total number N, the narrow high gradient region near $x = \pi + ct$ is difficult to resolve by the local basis. However, in the present case it can be seen, that if the local number of points N/P is equal to the number of points in one domain or even twice as small, then the multidomain approach becomes more accurate, than the ordinary one-domain PSF method.

spectral Fourier method [7] $P=1$		present LFB method $P=4$	
N	$(u-u^{ex})_{max}$	N	$(u-u^{ex})_{max}$
16	2.1 (-1)	16×4=64	2.9 (-3)
32	2.5 (-2)	32×4=128	2.8 (-6)
64	3.6 (-4)	64×4=256	7.5 (-10)
128	6.1 (-8)	128×4=512	6.0 (-11)

Table 3: Comparison of the numerical error for the one-domain PSF and multidomain LFB methods; P- number of subdomains, N- toal number of collocation points on a period.

7 Conclusions

The article presents a new parallel algorithm for the solving time dependent parabolic problems of the Navier-Stakes type and stationary elliptic problems of the Poissone's type in multidomain regions. On a first step the particular solutions to the local nonperiodic problems are constructed independently in each subdomain by using pseudospectral method with a Local Fourier Basis (LFB). This basis 1)supported on a constant grid spacing of collocation points; 2) diagonalizes the matrices of the differential operators in a transformed space; 3) ensure a high accuracy and spectral rate of convergance; 4) the relative amount of overhead due to overlapping of subdomains, needed for the implementation of this basis, is insignificant for the big enough number of collocation points.

On a second step a direct pointwise matching procedure is carryed out to provide the continuity of an overall solution by using the solutions to a corresponding homogeneous problem. The unknown coefficients are determined explicitly as functions of jumps between contiguous particular solutions on the interfaces. Thus the overall coupling, specific for the spectral Fourier method, is reduced to exchange only interface data.

The present parallel algorithm is applied to one- and two-dimensional rectangular regions, devided into strips. Nevertheless, this approach can be modified to cope with the domain decomposition in the form of rectangular boxes, that will be described elsewhere.

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