The method of fundamental solutions for elliptic boundary value problems

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The aim of this paper is to describe the development of the method of fundamental solutions (MFS) and related methods over the last three decades. Several applications of MFS-type methods are presented. Techniques by which such methods are extended to certain classes of non-trivial problems and adapted for the solution of inhomogeneous problems are also outlined.

Keywords: elliptic boundary value problems, fundamental solutions, nonlinear least squares, boundary collocation

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1. Introduction

The method of fundamental solutions (MFS) is a technique for the numerical solution of certain elliptic boundary value problems which falls in the class of methods generally called boundary methods. Like the boundary element method (BEM), it is applicable when a fundamental solution of the differential equation in question is known, and it shares the same advantages as the BEM over domain discretization methods. Moreover, it has certain advantages over the BEM, which will be mentioned in the following.

In this paper, primary attention is devoted to the MFS solution of elliptic boundary value problems governed by equations of the form

$$\mathcal{L}u(P) = 0, \quad P \in \Omega,$$

where $\mathcal{L}$ is a linear elliptic partial differential operator and Ω is a bounded domain in $\mathcal{R}^2$ or $\mathcal{R}^3$. By a fundamental solution of the differential equation, we mean a function $K(P,Q)$ such that

$$\mathcal{L}K(P,Q) = -\delta(P,Q), \quad P,Q \in \mathcal{R}^n,\ n = 2 \text{ or } 3,$$

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where $\delta(P, Q)$ denotes the Dirac delta function. The function K is defined everywhere except when $P = Q$, where it is singular. Thus Q is called the singularity of the fundamental solution.

The basic ideas for the formulation of the MFS were first proposed by Kupradze and Aleksidze [1,80–83]. The approximate solution $u_N(P)$ is expressed as a linear combination of fundamental solutions

$$u_N(P) = \sum_{j=1}^{N} c_j K(P, Q_j), \quad P \in \overline{\Omega}, \tag{1.1}$$

with the singularities $\{Q_j\}_{j=1}^N$ placed outside the domain of the problem $\overline{\Omega} \equiv \Omega \cup \partial\Omega$. The locations of the singularities are either preassigned or determined along with the coefficients $\{c_j\}_{j=1}^N$ of the fundamental solutions so that the approximate solution satisfies the boundary conditions as well as possible. This is usually achieved by a least squares fit of the boundary data. If the locations of the singularities are to be determined, the resulting minimization problem is nonlinear and can be solved using readily available software.

The singularities in the MFS may be considered as lying on the boundary $\partial\Omega_a$ of a region Ω_a containing Ω. Boundary integral equation methods involving such an "auxiliary" boundary have been considered by several authors. In the method of Kupradze [81], the direct boundary integral equation formulation is tackled by choosing observation points on $\partial\Omega_a$, thus avoiding singularities in the boundary integral equations. The question of the uniqueness of the solution in Kupradze's method is addressed by Christiansen [26]. The idea of using a simple layer potential representation on an auxiliary boundary was introduced by Oliveira [95] and further investigated by Heise [45]. A similar approach is used in [70] to solve problems in elastostatics. In [15], a boundary integral equation approach is used for the solution of three-dimensional potential problems. Singularities in the resulting boundary integral equations are avoided by moving the observation points away from the boundary, which is equivalent to using an auxiliary boundary. This "desingularization technique" is also used to solve the inviscid flow about an arbitrary three-dimensional body [118], biharmonic problems [41,117,121], spherical shell problems [113], and elastostatics problems [46,98].

In the approach of Oliveira [95], the solution of the problem is expressed in terms of a simple layer potential representation on an auxiliary boundary $\partial\Omega_a$. For example, in the case of Laplace's equation,

$$\Delta u(P) = 0, \quad P \in \Omega,$$

we have

$$u(P) = \int_{\partial\Omega_a} \sigma_a(Q) \log r(P, Q)\, \mathrm{d}s_Q. \tag{1.2}$$

If we approximate the integral in (1.2) by a quadrature rule with nodes $\{Q_j\}_{j=1}^N$ and weights $\{\omega_j\}_{j=1}^N$, then $u(P)$ is approximated by

$$\widetilde{u}_N(P) = \sum_{j=1}^{N} \left[\omega_j \sigma_a(Q_j)\right] \log r(P, Q_j), \tag{1.3}$$

cf. (1.1), where $\{\sigma_a(Q_j)\}_{j=1}^N$ are determined by satisfying the boundary conditions on the boundary $\partial\Omega$ of Ω. Since the fundamental solution of Laplace's equation is a constant multiple of $\log r(P, Q)$, the MFS can be viewed as a discrete simple layer potential method [30]. This approach is useful in the development of MFS formulations for biharmonic problems [59–61] discussed in section 2.3.

The purpose of this paper is to review the development of the MFS over the last thirty years and to describe recent applications and extensions. Early uses of the method were for the solution of various linear potential problems in two and three space variables [49,89]. It has since been applied to a variety of more complicated problems such as plane potential problems involving nonlinear radiation-type boundary conditions, free boundary problems, biharmonic problems, elastostatics problems and wave scattering problems.

A brief outline of this paper is as follows. In section 2, the formulation and application of the MFS to potential, elastostatics, acoustics and biharmonic problems are described, and, in section 3, details concerning the implementation of the method are provided. In section 4, extensions of the MFS to certain classes of problems, namely problems with boundary singularities, free boundary problems, and problems with nonlinear boundary conditions, are presented, and in section 5, the application of the method to inhomogeneous problems is reviewed. Comments and conclusions are given in section 6.

2. Applications

2.1. Potential problems

The MFS with moving singularities was first proposed by Mathon and Johnston [89] for the solution of potential problems of the form

$$\begin{aligned} \Delta u(P) &= 0, \quad P \in \Omega, \\ Bu(P) &= 0, \quad P \in \partial\Omega, \end{aligned} \tag{2.1}$$

where Δ denotes the Laplace operator, u is the dependent variable, and Ω is a bounded domain in the plane with boundary $\partial\Omega$. The operator B specifies the boundary conditions (BCs) and has the form:

$$Bu(P) = \begin{cases} \alpha(P) + u(P), & P \in \partial\Omega_1 \quad \text{(Dirichlet BCs)}, \\ \alpha(P) + \dfrac{\partial u}{\partial n}(P), & P \in \partial\Omega_2 \quad \text{(Neumann BCs)}, \\ \alpha(P) + \beta(P)u(P) + \gamma(P)\dfrac{\partial u}{\partial n}(P), & P \in \partial\Omega_3 \quad \text{(Robin BCs)}, \end{cases} \tag{2.2}$$

where α, β and γ are prescribed functions, and $\partial\Omega = \partial\Omega_1 \cup \partial\Omega_2 \cup \partial\Omega_3$. The solution u is approximated by a function of the form

$$u_N(\mathbf{c}, \mathbf{Q}; P) = \sum_{j=1}^{N} c_j k_1(P, Q_j), \quad P \in \overline{\Omega}, \tag{2.3}$$

where $\mathbf{c} = (c_1, c_2, \ldots, c_N)$ and $\mathbf{Q}$ is a $2N$-vector containing the coordinates of the singularities Q_j, which lie outside $\overline{\Omega}$. The function $k_1(P, Q)$ is a fundamental solution of Laplace's equation given by

$$k_1(P, Q) = -\frac{1}{2\pi} \log r(P, Q), \tag{2.4}$$

with $r(P, Q)$ denoting the distance between the points P and Q. A set of observation points $\{P_i\}_{i=1}^{M}$ is selected on $\partial\Omega$ and the coefficients $\mathbf{c}$ and the locations of the singularities $\mathbf{Q}$ are determined by minimizing the functional

$$F(\mathbf{c}, \mathbf{Q}) = \sum_{i=1}^{M} \left| Bu_N(\mathbf{c}, \mathbf{Q}; P_i) \right|^2, \tag{2.5}$$

which is nonlinear in the coordinates of the Q_j. The minimization of this functional is done using a nonlinear least squares algorithm. The MFS can also be formulated in either a Galerkin or a variational approach [9,43,44]. In this paper, the emphasis is on the least squares approach.

The MFS formulation easily extends to other second-order linear elliptic problems in two and three space variables for which fundamental solutions are known. These include Laplace's equation in three space variables, for which

$$K(P, Q) = \frac{1}{4\pi r(P, Q)}$$

[49,89]; the Helmholtz equation, $(\Delta + \lambda^2)u(P) = 0$, for which

$$K(P, Q) = \begin{cases} -\dfrac{\mathrm{i}}{4} H_0^{(2)}\big(\lambda r(P, Q)\big) & \text{in } \mathcal{R}^2, \\ \dfrac{1}{4\pi r(P, Q)} \exp\big(-\mathrm{i}\lambda r(P, Q)\big) & \text{in } \mathcal{R}^3, \end{cases} \tag{2.6}$$

where $\mathrm{i} = \sqrt{-1}$ and $H_0^{(2)}$ is the Hankel function of the second kind of order zero [76]; the modified Helmholtz equation: $(\Delta - \lambda^2)u(P) = 0$, for which

$$K(P,Q) = \begin{cases} \dfrac{1}{2\pi} K_0\big(\lambda r(P,Q)\big) & \text{in } \mathcal{R}^2, \\ \dfrac{1}{4\pi r(P,Q)} \exp\big(-\lambda r(P,Q)\big) & \text{in } \mathcal{R}^3, \end{cases} \tag{2.7}$$

where K_0 is the modified Bessel function of the second kind of order zero [86]. The MFS can also be applied to exterior boundary value problems with equal ease.

The efficacy of the MFS for second order elliptic problems is demonstrated by numerical results presented in [30,49,53–56,86,89]. In particular, in [56], a rather difficult problem in the computation of dipole fields is solved. This problem involves a coupled system of second order equations in three space variables. Also, two examples are given in [30] to illustrate the adaptivity of the MFS in choosing appropriate locations for the singularities. In [43], Han and Olson use a method closely related to the MFS for the solution of Laplace's equation in two and three dimensions. The satisfaction of the boundary conditions is achieved by minimizing boundary residuals with Galerkin weighting functions.

The emphasis in this paper is on the MFS with moving singularities. Simpler MFS approaches involve fixed (preassigned) singularities with the solution being determined by linear least squares, or by collocating the boundary conditions at boundary points P_i, $i = 1, \ldots, N$, which gives rise to a square system of linear algebraic equations that is solved directly. Examples of these approaches for potential problems are given in [9,29,33,34,50,90,91,99]. Most of the papers in the engineering literature use the version of the MFS with fixed singularities and collocation. A comprehensive review of boundary collocation methods up to 1985, including some references on the MFS and related methods, is given in [74].

For the MFS method with collocation, the determination of the optimal choice of collocation points and singularities has been considered by several authors in the case of the Dirichlet problem for Laplace's equation in the plane with analytic boundary data. When the region is a disk of radius r, it is shown in [65,68] (see also [34]) that, when the collocation points and singularities are placed uniformly on the boundary of the disk and on a circle of radius R ($> r$), respectively, the error in the MFS approximation for N collocation points and N singularities satisfies $\sup_{P\in\Omega} |u(P) - u_N(P)| = \mathrm{O}((r/R)^N)$, that is, exponential convergence is achieved. In [66,67], this result is generalized to regions in the plane whose boundaries are analytic Jordan curves, but no practical rule is given for determining the collocation points and the singularities until the recent work of [69]. In [72,73], the stability of the MFS collocation method is examined and it is observed that, although the method can be highly ill-conditioned, this often does not affect the quality of the numerical solution. A Galerkin MFS-type method, called a delta-trigonometric Petrov–Galerkin method, is proposed by Cheng [25] for the solution of harmonic problems. It is shown that, for circular domains, this method also converges exponentially.

In (2.3), some authors [9,36,47] add a constant coefficient c_0 in order to avoid the non-uniqueness problems reported by Christiansen [26]. While it has been observed in the MFS with moving singularities that the inclusion of the constant term does not affect the quality of the numerical solution [9,100], it has been reported [36] that it does in the MFS with collocation; this is clearly an area for further study. The constant can be viewed as a normalizing factor in the expression for the fundamental solution, and such factors have been used in the MFS to accelerate the convergence of the method [43,49].

2.2. Elastostatics and acoustics problems

The MFS with fixed singularities has been applied to several problems in elastostatics: in [12,13,87,106,107] for problems in the plane, in [109] for axisymmetric problems, and in [108] for three-dimensional problems. Redekop and Cheung [108] reviewed similar methods applied to elastostatics.

Application of the method to steady-state wave propagation problems is more recent. Kress and Mohsen [79] use the MFS with fixed singularities to solve an exterior Dirichlet problem in acoustics. In [7,8], this method is applied to the analysis of acoustic wave scattering in fluids involving both rigid and fluid scatterers, and in [78,114,115] to acoustic radiation problems. It is also used in [77] to study the scattering/transmission of time-harmonic waves by an elastic obstacle embedded in a three-dimensional infinite elastic medium. In [76], acoustic wave scattering in fluids and wave scattering from rigid/vacuous obstacles in elastic regions are analyzed using the MFS with moving singularities and fixed singularities, respectively; see also [75,112]. Numerical experiments reported in [76,77] for the MFS solution of problems of this nature do not indicate the presence of the fictitious eigenfrequency problem encountered in the BEM.

As an example of acoustic scattering in fluids, consider a time-harmonic acoustic wave in an unbounded, homogeneous fluid domain Ω incident upon a rigid, fixed obstacle occupying the region $\Omega^c \in \mathcal{R}^3$, the complement of Ω, with boundary $\partial\Omega$. The fluid particle velocity at a point P is given by

$$\mathbf{u}(P,t) = -\nabla\phi(P,t),$$

where ϕ is the velocity potential. For a time harmonic wave,

$$\phi(P,t) = \Phi(P)\exp(\mathrm{i}\omega t),$$

where $\Phi(P)$ is the complex amplitude and ω is the circular frequency. The scattered wave is defined according to $\Phi^S = \Phi - \Phi^I$, where Φ^I represents the incident wave. The function Φ satisfies the Helmholtz equation

$$\Delta\Phi + k^2\Phi = 0,$$

in Ω, where $k = \omega/c$ is the wave number and c is the speed of propagation of the wave. On the boundary of the rigid scatterer,

$$\mathbf{u} \cdot \mathbf{n} = 0;$$

that is,

$$\frac{\partial \Phi}{\partial n} = 0,$$

where $\mathbf{n}$ is the unit normal to $\partial\Omega$. Rewriting the problem in terms of Φ^S gives

$$\begin{aligned}
\Delta\Phi^S(P) + k^2\Phi^S(P) = 0, \quad & P \in \Omega, \\
\frac{\partial \Phi^S}{\partial n}(P) + \frac{\partial \Phi^I}{\partial n}(P) = 0, \quad & P \in \partial\Omega.
\end{aligned}$$

Moreover, Φ^S vanishes at field points sufficiently far from the scatterer. Problems of this type are solved in [76] using the MFS with the appropriate fundamental solution from (2.6). In [17–19], an MFS-type method with fixed singularities for the Helmholtz equation is discussed.

Suppose now that Ω is an unbounded homogeneous and isotropic elastic solid instead of a fluid. The obstacle is either rigid and fixed, or is a cavity, and is impinged upon by a time-harmonic wave. The field variables of interest are the displacement vector $\mathbf{u}$ and the surface traction vector $\mathbf{t}$, which are related by Hooke's law. The governing system of differential equations in terms of the displacement vector is

$$\left(c_p^2 - c_s^2\right)\nabla(\nabla \cdot \mathbf{u}) + c_s^2\Delta\mathbf{u} + \omega^2\mathbf{u} = \mathbf{0},$$

where c_p and c_s are the dilatational and shear wave speeds, respectively, in Ω. The boundary conditions for the two types of obstacles are

$$\begin{aligned}
\mathbf{u} = \mathbf{0} \quad & \text{(rigid obstacle)}, \\
\mathbf{t} = \mathbf{0} \quad & \text{(cavity)}.
\end{aligned}$$

For this problem, the fundamental solution is the Stokes fundamental tensor [96].

2.3. Biharmonic problems

The approach of Oliveira [95] has been used to develop three MFS methods for biharmonic problems based on several simple layer potential representations of biharmonic functions; see [59–61]. To describe these methods, we first consider the biharmonic equation

$$\Delta^2 u(P) = 0, \quad P \in \Omega, \tag{2.8}$$

subject to the boundary conditions

$$B_1u(P) \equiv \alpha(P) + u(P) = 0, \quad B_2u(P) \equiv \beta(P) + \frac{\partial u}{\partial n}(P) = 0, \qquad P \in \partial\Omega,$$

or

$$B_1 u(P) \equiv \alpha(P) + u(P) = 0, \quad B_2 u(P) \equiv \beta(P) + \Delta u(P) = 0, \qquad P \in \partial\Omega,$$

where α and β are prescribed functions. The first biharmonic MFS formulation is based on the simple layer potential representation of biharmonic functions suggested in [88]; cf. [9]. In this formulation [59], the solution is approximated by a function of the form

$$u_N(\mathbf{c}, \mathbf{Q}; P) = \sum_{j=1}^{N} \left[c_j k_1(P, Q_j) + d_j k_2(P, Q_j) \right], \quad P \in \overline{\Omega}, \tag{2.9}$$

where k_1 is the fundamental solution of Laplace's equation given by (2.4) and k_2 is the fundamental solution of the biharmonic equation given by

$$k_2(P, Q) = -\frac{1}{8\pi} r^2(P, Q) \log r(P, Q). \tag{2.10}$$

Following the MFS approach in the case of Laplace's equation, the boundary conditions are imposed by minimizing the functional

$$F(\mathbf{c}, \mathbf{Q}) = \sum_{i=1}^{M} \left[\left| B_1 u_N(\mathbf{c}, \mathbf{Q}; P_i) \right|^2 + \left| B_2 u_N(\mathbf{c}, \mathbf{Q}; P_i) \right|^2 \right].$$

The second biharmonic MFS formulation is based on the Almansi representation of biharmonic functions [2,51]. Almansi showed that the general solution of the biharmonic equation is given by

$$u(P) = r^2(P)\phi^{(1)}(P) + \phi^{(2)}(P), \quad P \in \overline{\Omega},$$

where the functions $\phi^{(1)}$ and $\phi^{(2)}$ are harmonic in Ω, and $r(P)$ denotes the distance of the point P from the origin, which lies in Ω. If we replace both $\phi^{(1)}$ and $\phi^{(2)}$ by simple layer potential representations, viz.,

$$\phi^{(i)}(P) = \int_{\partial\Omega} \sigma^{(i)}(Q) \log r(P, Q) \, \mathrm{d}s_q, \quad P \in \overline{\Omega}, \ i = 1, 2,$$

then

$$u(P) = r^2(P) \int_{\partial\Omega} \sigma^{(1)}(Q) \log r(P, Q) \, \mathrm{d}s_q + \int_{\partial\Omega} \sigma^{(2)}(Q) \log r(P, Q) \, \mathrm{d}s_q, \quad P \in \overline{\Omega}. \tag{2.11}$$

This expression provides the motivation for the second biharmonic MFS formulation. In (2.11), the simple layer potential representation with respect to an auxiliary boundary $\partial\Omega_a$ is used and, following (1.2)–(1.3), $u(P)$ is approximated by a function of the form

$$u_N(\mathbf{c}, \mathbf{Q}; P) = \sum_{j=1}^{N} \left[c_j r^2(P) + d_j\right] k_1(P, Q_j), \quad P \in \overline{\Omega}, \tag{2.12}$$

where k_1 is given by (2.4).

Now consider the biharmonic equation subject to the boundary conditions

$$B_1 u(P) \equiv \alpha(P) + \frac{\partial u}{\partial x_P}(P) = 0, \quad B_2 u(P) \equiv \beta(P) + \frac{\partial u}{\partial y_P}(P) = 0, \qquad P \in \partial\Omega. \tag{2.13}$$

Problems of this type arise, for example, in fluid flow problems, where u denotes the stream function. They do not have unique solutions, but since the quantities of interest are usually derivatives of the solution, which give the velocity components, the nonuniqueness is inconsequential.

For the biharmonic problem with boundary conditions (2.13), Fichera [32] proposed the simple layer potential representation

$$u(P) = \int_{\partial\Omega} \left[\sigma(Q)\frac{\partial k_2}{\partial x_Q}(P, Q) + \mu(Q)\frac{\partial k_2}{\partial y_Q}(P, Q)\right] \mathrm{d}s_Q, \quad P \in \overline{\Omega}, \tag{2.14}$$

where $k_2(P, Q)$ is given by (2.10). It follows that

$$u(P) = \int_{\partial\Omega} \left[\sigma(Q)(P_1 - Q_1) + \mu(Q)(P_2 - Q_2)\right] \left[k_1(P, Q) + \frac{1}{4\pi}\right] \mathrm{d}s_Q, \quad P \in \overline{\Omega},$$

where $P = (P_1, P_2)$ and $Q = (Q_1, Q_2)$. This provides the motivation for the third biharmonic MFS formulation [61], in which the approximate solution has the form

$$u_N(\mathbf{c}, \mathbf{Q}; P) = \sum_{j=1}^{N} \left[c_j(P_1 - Q_{j_1}) + d_j(P_2 - Q_{j_2})\right] \left[k_1(P, Q_j) + \frac{1}{4\pi}\right], \quad P \in \overline{\Omega},$$

where $Q_j = (Q_{j_1}, Q_{j_2})$.

The results of extensive numerical experimentation involving each of these MFS approaches are presented in [59–61]. The effectiveness of each method is demonstrated by examining its performance on a variety of standard test problems as well as on problems of practical interest arising in elasticity and fluid flow. The MFS with fixed singularities has been considered for biharmonic problems arising in elasticity [52,92,93], and in Stokes fluid flow [11]. In [122], a biharmonic-type MFS with moving singularities in which the fundamental solutions are selected to satisfy certain boundary conditions of the problem is applied to the solution of Stokes flow past or due to motion of solid particles. The coefficients in the MFS expansion and the positions of the singularities are obtained by minimizing an appropriate functional.

This is achieved by alternately minimizing the functional with respect to the coefficients and with respect to the locations of the singularities. The iterative process is terminated when the difference between the values of both the MFS expansion coefficients and coordinates of the singularities at consecutive steps differ by less than a prescribed tolerance. The same approach but with fixed singularities, defined as the singularity method, is presented in [28] and [105] for similar problems. In some cases, this method is applied to Stokes flow problems with expansions involving spherical harmonic functions instead of fundamental solutions [10,71,119].

2.4. More general operators

The MFS is applicable when the fundamental solution of the differential equation governing the problem in question is available. Several authors have developed methods for constructing fundamental solutions of less commonly occurring elliptic equations. In [27], Clements considers second order linear elliptic partial differential equations of the general form

$$\frac{\partial}{\partial x}\left(a(x,y)\frac{\partial\phi}{\partial x}+b(x,y)\frac{\partial\phi}{\partial y}\right)+\frac{\partial}{\partial y}\left(c(x,y)\frac{\partial\phi}{\partial x}+d(x,y)\frac{\partial\phi}{\partial y}\right)+e(x,y)\phi=0,$$

and systems of differential equations of the same type, and gives various methods for obtaining fundamental solutions of such equations in special cases. For instance, in the case where $a(x,y)=d(x,y)=\mu(x,y)$ and $b(x,y)=c(x,y)\equiv 0$, one may take $\phi(x,y)=[\mu(x,y)]^{-1/2}u(x,y)$. The governing equation is then transformed into

$$\Delta u-\big[\mu(x,y)\big]^{-1/2}\Delta\big[\mu(x,y)\big]^{1/2}u=0.$$

In the special case where $\Delta\mu^{1/2}=0$, we have Laplace's equation whereas when $\Delta\mu^{1/2}=\kappa^2\mu^{1/2}$, where κ is a constant, we have the Helmholtz equation. Since the fundamental solutions of these equations are known, the fundamental solutions of the original equations may be easily recovered via the transformation $\phi(x,y)=\mu^{-1/2}u$. Other methods for obtaining fundamental solutions of such equations are given in [27]; see also references therein and [120].

3. Implementational considerations

In each of the formulations described in section 2, the MFS leads to a nonlinear least squares problem which is solved using readily available software. In early applications of the MFS [49,56], a slightly modified version of the Harwell subroutine VA07AD [48] was used. This method incorporates features from the Newton–Raphson, steepest descent and Marquardt methods. In [59], the MINPACK [35] routine LMDIF, which is a modified version of the Levenberg–Marquardt algorithm, is found to be more efficient than VA07AD through experimentation. In both VA07AD and LMDIF, the Jacobian is evaluated internally by finite differences.

In a recent study [101], the performance of the routines LMDIF and LMDER from MINPACK [35] and the routine E04UPF from NAG [94] is investigated. The routine LMDER is identical to LMDIF with the exception that the user has to provide the Jacobian. As expected, the use of LMDER leads to substantial savings in both storage and cost. The routine E04UPF uses a sequential quadratic programming algorithm and may be used for linear or nonlinear, constrained or unconstrained optimization. The choice of the least squares solver is very important as it may lead to substantial savings in terms of both memory and cost. A state-of-the-art routine such as E04UPF offers a variety of features, for example the use of constrained optimization which can be extremely useful for certain MFS applications.

The initial placement of the singularities can also be extremely important in the convergence of a least squares routine. Usually the singularities are distributed uniformly around the domain of the problem at a fixed distance from the boundary. The number of boundary points is chosen to be approximately three times the number of unknowns of the problem [49]. In order to decide how many singularities are needed to describe the problem satisfactorily, one may follow a process first suggested in [86] in which one starts the method with a small number of singularities and, after a certain number of iterations, the number of singularities is increased, with a concomitant increase in the number of boundary points. This is repeated until the required accuracy is obtained [62]. It has been observed that sometimes the singularities have a tendency to move to the interior of the region. This is rectified by an internal check and if a singularity is found inside the region it is moved outside the domain. Alternatively, if one uses a least squares minimization routine such as E04UPF, the condition that the singularities remain outside the domain may be imposed as a constraint.

The size of the least squares problem can be reduced considerably by exploiting any existing symmetries. If the problem is symmetric about one of the coordinate axes, only half the domain is discretized and if the problem is symmetric about two coordinate axes only a quarter of the domain is discretized, etc. Details of the implementation of this for Laplacian problems can be found in [49] and for biharmonic problems in [59–61].

4. Extensions

4.1. Problems with boundary singularities

Because of its adaptivity, the MFS gives reasonably accurate results when applied to problems with boundary singularities [59–61]. However, the local behavior of the solution in the neighborhood of the singularity can be easily incorporated into the MFS formulation leading to a more accurate representation of the solution and a more efficient solution procedure. This approach is applied to a number of harmonic and biharmonic problems in [57]. For example, consider the problem

$$\Delta u(P) = 0, \quad P \in \Omega,$$

subject to the boundary conditions

$$B_1u(P) = 0, \quad P \in \partial\Omega_1, \qquad B_2u(P) = 0, \quad P \in \partial\Omega_2,$$

where $\partial\Omega = \partial\Omega_1 \cup \partial\Omega_2$. Suppose that the change of boundary conditions occurs at the origin and leads to a boundary singularity. In polar coordinates, the asymptotic behavior is then given by

$$u_s(r,\theta) = \sum_{j=1}^{\infty} \alpha_j r^{\lambda_j} f_j(\theta), \quad (r,\theta) \in \Omega, \tag{4.1}$$

where the functions f_j and the singularity powers λ_j are known. The solution is now approximated by

$$u_N(\mathbf{c}, \boldsymbol{\alpha}, \mathbf{Q}; P) = \sum_{j=1}^{N} c_j k(P, Q_j) + \sum_{j=1}^{N_s} \overline{\alpha}_j r^{\lambda_j} f_j(\theta), \qquad P \in \overline{\Omega},$$

where N_s is the number of leading singular terms considered. The boundary conditions are satisfied by minimizing the functional (2.5), and the minimization yields the coefficients c_j, the coordinates of the singularities Q_j, and the coefficients $\overline{\alpha}_j$. The strategy for singular biharmonic problems is similar.

In [85], a two-domain MFS is applied to a variety of singular elliptic boundary value problems. A neighborhood of the singularity is one of the domains and the remainder of the problem domain the other. The local behavior of the singularity is only included in the MFS approximation in the first domain and the two MFS expansions are matched along with their normal derivatives on the subdomain interfaces. In [101], a modified MFS is applied to singular harmonic and biharmonic problems where the nature of the singularity is not assumed to be known *a priori*. This modified MFS predicts the nature of the leading singular term, in the case of harmonic problems, the term $r^{\lambda_1} f_1(\theta)$ in (4.1).

4.2. Free boundary problems

The MFS is an ideal candidate for the solution of free boundary problems as these combine the nonlinearity and boundary features which are characteristic of the method. The MFS is applied to free boundary problems governed by Laplace's equation in [58]. There, the MFS is used to solve several potential flow problems for each of which the method is modified accordingly in order to accommodate its particular difficulties. In a typical free boundary problem, we have

$$\Delta u(P) = 0, \quad P \in \Omega,$$

subject to the boundary conditions

$$Bu(P) = 0, \quad P \in \partial\Omega_{Fx},$$

and

$$B_1 u(P) = 0, \quad B_2 u(P) = 0, \qquad P \in \partial\Omega_{Fr},$$

where $\partial\Omega_{Fx}$ is the fixed part of the boundary, $\partial\Omega_{Fr}$ is the free part of the boundary and $\partial\Omega = \partial\Omega_{Fx} \cup \partial\Omega_{Fr}$. The location of the free boundary $\partial\Omega_{Fr}$ is unknown and must be found as part of the solution. This is compensated by the specification of a second boundary condition on $\partial\Omega_{Fr}$. The idea is to minimize a functional of the form

$$\begin{aligned} F\big(\mathbf{c}, \mathbf{Q}, \mathbf{P}^{Fr}\big) = {} & \sum_{i=1}^{M_{Fx}} \big|Bu_N(\mathbf{c}, \mathbf{Q}; P_i)\big|^2 \\ & + \sum_{i=M_{Fx}+1}^{M_{Fx}+M_{Fr}} \big[\big|B_1 u_N\big(\mathbf{c}, \mathbf{Q}; P_i^{Fr}\big)\big|^2 + \big|B_2 u_N\big(\mathbf{c}, \mathbf{Q}; P_i^{Fr}\big)\big|^2\big], \end{aligned} \tag{4.2}$$

where M_{Fx} and M_{Fr} are the numbers of observation points on $\partial\Omega_{Fx}$ and $\partial\Omega_{Fr}$, respectively, and the points P_i^{Fr} are unknown parameters representing the free boundary. Instead of taking both coordinates of each of the points P_i^{Fr} as unknown parameters, we may choose to take either their x- or y-coordinate to be unknown while specifying the other coordinate. These parameters, as well as the usual set of unknowns determining the solution, are found by minimizing the functional (4.2). One important advantage of the method is the ease with which it can incorporate the particular features of each free boundary problem. For instance, in one of the problems solved in [58], the exact shape of the free boundary is known in the neighborhood of a boundary singularity point. This known behavior of the free boundary is easily included in the MFS solution procedure.

In [102], the MFS is applied to Signorini problems, a class of free boundary problems arising in contact problems in electropainting, elasticity and fluid mechanics. Such problems are also solved in [9] using a fundamental solutions approximation in combination with a variational formulation of the problem. The main difficulty in Signorini problems is the fact that, on part of the boundary, two types of conditions alternate and the points where changes of type occur are unknown. In order to obtain the complete solution of the problem in this case, one needs to determine the solution u and the number and positions of the points where the boundary conditions change. The application of the MFS, in conjunction with a constrained least squares minimization routine, yields the position of these points as well as the solution u of the problem.

A typical Signorini problem is governed by Laplace's equation in a region Ω subject to the boundary conditions

$$Bu(P) = 0, \quad P \in \partial\Omega_1,$$

and

$$B_1 u(P) = 0, \quad B_2 u(P) < 0, \qquad P \in \partial\Omega_2,$$

or

$$B_2u(P) = 0, \quad B_1u(P) < 0, \qquad P \in \partial\Omega_2,$$

where $\partial\Omega_2$ is the part of the boundary where the conditions B_1 and B_2 alternate, and $\partial\Omega = \partial\Omega_1 \cup \partial\Omega_2$. The latter conditions can be combined to give

$$B_1u(P) \leqslant 0, \quad B_2u(P) \leqslant 0, \qquad P \in \partial\Omega_2,$$

and

$$B_1u(P)B_2u(P) = 0, \quad P \in \partial\Omega_2.$$

Then the functional

$$F(\mathbf{c}, \mathbf{Q}) = \sum_{i=1}^{M_1} \left|Bu_N(\mathbf{c}, \mathbf{Q}; P_i)\right|^2 + \sum_{i=M_1+1}^{M_1+M_2} \left|B_1u_N(\mathbf{c}, \mathbf{Q}; P_i)B_2u_N(\mathbf{c}, \mathbf{Q}; P_i)\right|^2$$

is minimized subject to the constraints

$$B_1u(P) \leqslant 0, \quad P \in \partial\Omega_2,$$

and

$$B_2u(P) \leqslant 0, \quad P \in \partial\Omega_2,$$

where M_1 and M_2 are the numbers of observation points on $\partial\Omega_1$ and $\partial\Omega_2$, respectively. This constrained minimization problem can be solved using a routine such as E04UPF mentioned in section 3.

Recently, in [103], a biharmonic MFS based on the simple layer potential representation of Fichera (2.14) was applied to a Stokes flow problem with a free surface, namely the Newtonian extrudate-swell problem. The MFS solution of such fourth order free boundary problems is similar to the MFS solution of second order free boundary problems described earlier in this section. However, in contrast with second order problems where two boundary conditions are prescribed on the free boundary, in fourth order problems, three boundary conditions are prescribed on this boundary. As a result, an additional term must be included in the functional (4.2). A three-dimensional free surface problem governed by Laplace's equation is solved in [84] by the MFS with fixed singularities.

4.3. Problems with nonlinear boundary conditions

Laplace's equation subject to nonlinear boundary conditions of the form

$$B\left[u(P)\right] = \alpha(P)u(P) + \beta(P)\frac{\partial u}{\partial n}(P) + \gamma(P)\left[u(P)\right]^4 + \delta(P), \quad P \in \partial\Omega,$$

where α, β, γ and δ are prescribed functions, can be easily handled by the MFS [62]. These nonlinear boundary conditions are included in the MFS formulation in exactly the same way that linear conditions are, by minimizing the functional

$$F(\mathbf{c}, \mathbf{Q}) = \sum_{i=1}^{M} \left| B\left[u_N(\mathbf{c}, \mathbf{Q}; P_i)\right] \right|^2.$$

4.4. *Axisymmetric problems*

The MFS has recently been applied to axisymmetric potential and acoustics problems [63,64]. In particular, the potential problems considered are of the type

$$\Delta u(P) = 0, \quad P \in \Omega', \tag{4.3}$$

subject to the boundary conditions

$$Bu(P) = f(P), \quad P \in \partial\Omega',$$

where the operator B is given by (2.2) and Ω' is a bounded domain in R^3 with boundary $\partial\Omega'$, which we shall assume to be piecewise smooth. Suppose that the region $\Omega' \in R^3$ is axisymmetric, that is, formed as a figure of revolution by rotating a plane region Ω with boundary $\partial\Omega$ about the z-axis. Then (4.3) reduces to

$$\frac{\partial^2 u(P)}{\partial r^2} + \frac{1}{r}\frac{\partial u(P)}{\partial r} + \frac{\partial^2 u(P)}{\partial z^2} = 0, \quad P \in \Omega. \tag{4.4}$$

When the boundary conditions are also axisymmetric, that is, on the boundary $\partial\Omega$, both u and its normal derivative are independent of θ, the three-dimensional problem reduces to solving the axisymmetric version of Laplace's equation (4.4) subject to the boundary conditions

$$Bu(P) = f(P), \quad P \in \partial\Omega.$$

If $P = (r_P, z_P)$ and $Q = (r_Q, z_Q)$ are two points in Ω and we set

$$R^2 = (r_Q + r_P)^2 + (z_Q - z_P)^2,$$

then the fundamental solution of (4.4) is given by (see [42,63] and references therein)

$$K(P, Q) = \frac{4\mathcal{K}(k)}{R},$$

where $\mathcal{K}(k)$ is the complete elliptic integral of the first kind,

$$\mathcal{K}(k) = \int_0^{\pi/2} \left[1 - k^2 \sin^2\theta(Q)\right]^{-1/2} \mathrm{d}\theta(Q),$$

and $k^2 = 4r_Q r_P/R^2$. Also, it can be shown that the normal derivative of K can be expressed in terms of $\mathcal{K}(k)$ and $\mathcal{E}(k)$, where $\mathcal{E}(k)$ is the complete elliptic integral of the second kind defined by

$$\mathcal{E}(k) = \int_0^{\pi/2} \left[1 - k^2 \sin^2 \theta(Q)\right]^{1/2} \mathrm{d}\theta(Q).$$

The complete elliptic integrals $\mathcal{K}(k)$ and $\mathcal{E}(k)$ may be evaluated using the NAG functions S21BBF and S21BCF [94], respectively. These functions compute the complete elliptic integrals from Carlson's algorithm [16].

When the body Ω' is axisymmetric but the boundary conditions are not, the solution u can be expanded in a Fourier series as

$$u(P) = \frac{u_0(r_P, z_P)}{2} + \sum_{m=1}^{\infty} \left[u_m^c(r_P, z_P)\cos(m\theta_P) + u_m^s(r_P, z_P)\sin(m\theta_P)\right], \qquad (4.5)$$

where each of the functions u_0, u_m^c and u_m^s satisfies a differential equation of the form

$$\frac{\partial^2 u(P)}{\partial r^2} + \frac{1}{r}\frac{\partial u(P)}{\partial r} + \frac{\partial^2 u(P)}{\partial z^2} - \left(\frac{m}{r}\right)^2 u = 0, \quad P \in \Omega,\ m = 0, 1, \ldots,$$

[42,63,110]. The idea is to expand the boundary conditions in terms of Fourier series and obtain the coefficient functions u_m^c and u_m^s, each of which satisfies a different boundary value problem. The solution is then obtained from the Fourier expansion (4.5). The fundamental solutions of the sequence of differential equations and their normal derivatives may again be expressed in terms of complete elliptic integrals [63]. In [63], the MFS is used to solve a number of examples involving both axisymmetric bodies with axisymmetric boundary conditions, and axisymmetric bodies with arbitrary boundary conditions. Axisymmetric problems in elastostatics using the MFS with fixed singularities are considered in [108].

In the case of axisymmetric problems in acoustics, the fundamental solution of the axisymmetric version of the Helmholtz equation and its normal derivative can also be expressed in terms of complete elliptic integrals [64,111]. Several examples from axisymmetric acoustic scattering and radiation are solved using the MFS in [64].

4.5. Bimaterial problems

Another recent application of the MFS is to problems of steady-state heat conduction in anisotropic bimaterials [5] of the form

$$\begin{aligned} k_{11}^I \frac{\partial^2 u^I(P)}{\partial x^2} + \left(k_{12}^I + k_{21}^I\right)\frac{\partial^2 u^I(P)}{\partial x \partial y} + k_{22}^I \frac{\partial^2 u^I(P)}{\partial y^2} &= 0, \qquad P \in \Omega^I, \\ k_{11}^{II} \frac{\partial^2 u^{II}(P)}{\partial x^2} + \left(k_{12}^{II} + k_{21}^{II}\right)\frac{\partial^2 u^{II}(P)}{\partial x \partial y} + k_{22}^{II} \frac{\partial^2 u^{II}(P)}{\partial y^2} &= 0, \qquad P \in \Omega^{II}, \end{aligned}$$

where Ω^I and Ω^{II} are domains in the plane with common boundary $\partial\Omega^C$ and the constants k_{ij}^I and k_{ij}^{II}, $i = 1, 2$, $j = 1, 2$, are the thermal conductivities of the materials in Ω^I and Ω^{II}, respectively. The boundary conditions are

$$\begin{aligned} B^I u^I(P) &= f^I(P), \qquad P \in \partial\Omega^1, \\ B^{II} u^{II}(P) &= f^{II}(P), \quad P \in \partial\Omega^2, \end{aligned}$$

and the interface continuity conditions are

$$B_1^I u^I(P) = B_1^{II} u^{II}(P), \quad B_2^I u^I(P) = B_2^{II} u^{II}(P), \qquad P \in \partial\Omega^C,$$

where $\partial\Omega^I = \partial\Omega^1 \cup \partial\Omega^C$, $\partial\Omega^{II} = \partial\Omega^2 \cup \partial\Omega^C$. For a perfectly conductive interface,

$$B_1^I = B_1^{II} = I, \quad B_2^I = k_{21}^I \frac{\partial}{\partial x} + k_{22}^I \frac{\partial}{\partial y} \quad \text{and} \quad B_2^{II} = k_{21}^{II} \frac{\partial}{\partial x} + k_{22}^{II} \frac{\partial}{\partial y}.$$

In [5], problems of this type are solved using two MFS approaches. The first uses a domain decomposition MFS, and the second, modified fundamental solutions which satisfy the interface conditions. These modified fundamental solutions are derived in [6].

5. Inhomogeneous problems

5.1. Poisson problems

Consider the Poisson equation

$$\Delta u(P) = f(P), \quad P \in \Omega, \tag{5.1}$$

subject to the boundary condition

$$u(P) = g(P), \quad P \in \partial\Omega.$$

If we construct a particular solution $\widehat{u}$ of (5.1) and set $v = u - \widehat{u}$ then

$$\Delta v(P) = 0, \quad P \in \Omega,$$

subject to the boundary condition

$$v(P) = g(P) - \widehat{u}(P), \quad P \in \partial\Omega,$$

which may be solved by the MFS. The main task is therefore to determine a particular solution of the Poisson equation.

As suggested in [3], a particular solution of (5.1) can be obtained by constructing the associated Newton potential

$$\widehat{u}(P) = \frac{1}{2\pi} \int\!\!\int_\Omega f(q) \log r(P, q) \, \mathrm{d}V(q), \quad P \in \Omega.$$

In [14], where the use of MFS-type methods for the solution of Poisson problems was first employed, this integral is evaluated directly using numerical quadrature. The use of quasi-Monte Carlo methods for the approximation of such integrals has also been investigated [22–24,123]. The evaluation of the (potentially complicated) domain integral can be circumvented in the following way [3]. Assuming that $\widehat{\Omega}$ is a domain which contains $\overline{\Omega}$ and that f can be smoothly extended to $\widehat{\Omega}$, the particular solution can be constructed by evaluating the integral

$$\hat{u}(P) = \frac{1}{2\pi}\int\int_{\widehat{\Omega}} f(q)\,\log r(P,q)\,\mathrm{d}V(q), \quad P \in \overline{\Omega}.$$

A judicious choice of $\widehat{\Omega}$, for example, an ellipse, makes the evaluation of this integral straightforward; see [3] for details. This technique was first used in the MFS for the solution of Poisson problems in [36]. In [100,104], this method is used with the MFS for the solution of Poisson problems with Dirichlet and mixed boundary conditions.

In [36,38], another approach for solving Poisson problems, based on the dual reciprocity method [97], is suggested. In this approach, the right hand side function f is approximated by a series of radial basis functions

$$\hat{f} = \sum_{i=1}^{n} a_i \hat{f}_i,$$

and a particular solution $\hat{u}$ is obtained by taking

$$\hat{u} = \sum_{i=1}^{n} a_i \hat{u}_i,$$

where each $\hat{u}_i$ satisfies

$$\Delta \hat{u}_i = \hat{f}_i.$$

The various approaches for determining particular solutions have been the subject of several recent studies involving MFS-type methods [20,22,23,37,39,40].

A class of Poisson problems of particular interest is that in which the right hand side f is harmonic. When f is an elementary harmonic function such as a constant or a polynomial, it is easy to construct a particular solution $\hat{u}$. The general case is addressed in [3], in which it is assumed that a particular solution is of the form $\hat{u}(x,y) = xH(x,y)/2$, with H harmonic. From the satisfaction of Poisson's equation, we obtain $\partial H/\partial x = f$. Integration yields

$$H(x,y) = \int_{x_0}^{x} f(s,y)\,\mathrm{d}s + h(y), \tag{5.2}$$

where the point x_0 and the function $h(y)$ are arbitrary. From the fact that H is harmonic, it follows that

$$h''(y) = -\frac{\partial f}{\partial x}(x_0, y).$$

Integrating this differential equation gives

$$h(y) = -\int_{y_0}^{y} (y-t)\frac{\partial f}{\partial x}(x_0,t)\,\mathrm{d}t,$$

from which, together with (5.2), one obtains the desired particular solution. This construction is considerably easier than the construction of a particular solution via the Newton potential as it only involves single integrals. Applications of this technique in combination with the MFS can be found in [100,104].

Alternatively, from the formulation of [31], one can apply the MFS directly to the Poisson problem with a harmonic right hand side f in the following way. Since

$$\Delta^2 u(P) = 0, \quad P \in \Omega,$$

$u(P)$ can be approximated by an expression of the form (2.9). Thus, since $\Delta u = f$, it follows that f can be approximated by

$$f_N(\mathbf{c},\mathbf{Q};P) = \Delta u_N(\mathbf{c},\mathbf{Q};P) = \sum_{j=1}^{N} d_j\left[k_1(P,Q_j) - \frac{1}{2\pi}\right].$$

In order to determine the coefficients c_j and d_j, we simply minimize the functional

$$F(\mathbf{c},\mathbf{Q}) = \sum_{i=1}^{M}\left[\left|u_N(\mathbf{c},\mathbf{Q};P_i) - g(P_i)\right|^2 + \left|\Delta u_N(\mathbf{c},\mathbf{Q};P_i) - f(P_i)\right|^2\right].$$

The efficacy of this approach is demonstrated in [100,104].

5.2. *More general inhomogeneous problems*

The idea of using particular solutions to remove the inhomogeneous terms can be extended to more general elliptic operators [3,39]. In the case of the linear inhomogeneous equation

$$\mathcal{L}u(P) = f(P), \quad P \in \Omega,$$

where $\mathcal{L}$ is a linear elliptic operator, a particular solution $\widehat{u}$ may be constructed by evaluating the integral

$$\widehat{u}(P) = -\int\int_{\widehat{\Omega}} f(q)\,K(P,q)\,\mathrm{d}V(q), \quad P \in \overline{\Omega}, \tag{5.3}$$

where the function K is the fundamental solution of $\mathcal{L}$ and $\widehat{\Omega}$ is a suitably chosen region containing $\overline{\Omega}$. In the particular case of the inhomogeneous biharmonic equation

$$\Delta^2 u(P) = f(P), \quad P \in \Omega,$$

a particular solution may be found by evaluating the integral (5.3) with $K(P,q) = k_2(P,q)$. As before, once a particular solution has been determined, the inhomogeneous problem can be transformed into a homogeneous one and solved using standard MFS methods. This approach is applied in [100,104] for the solution of inhomogeneous biharmonic problems.

5.3. *Nonlinear inhomogeneous problems*

The application of an MFS-type method to a boundary value problem governed by a nonlinear elliptic equation was first reported in [14] for Poisson problems. In [21], Chen considers the problem

$$\begin{aligned} &\Delta u(P) + \delta f\big(u(P)\big) = 0, \quad && P \in \Omega, \\ &u(P) = 0, && P \in \partial\Omega, \end{aligned}$$

where δ is the Frank–Kamenetskii parameter. The idea is to use a Picard-type iteration and solve the sequence of Poisson problems for $n = 1, 2, \ldots,$

$$\begin{aligned} &\Delta u_{n+1}(P) = -\delta f\big(u_n(P)\big), \quad && P \in \Omega, \\ &u_{n+1}(P) = 0, && P \in \partial\Omega. \end{aligned}$$

Each of these problems may be solved using the methods described in section 5.1. In [21], at each iteration, Chen uses a combination of the dual reciprocity method to evaluate a particular solution of the inhomogeneous Poisson problem and an MFS-type method for the solution of the homogeneous problem.

As in the case of Poisson problems, when applying MFS-type methods to two-dimensional elasticity problems with body forces, in order to calculate a particular solution of the inhomogeneous problem, it is necessary to evaluate an area integral. This point is addressed in [13] where it is shown that for polynomial body forces the evaluation of the area integral can be avoided by transforming it into a contour integral.

5.4. *Transient problems*

Traditional boundary element methods have been used for some time for the numerical solution of transient problems; see, for example, [4]. The time dependence is handled using one of three methods. Firstly, one may use Laplace transforms to remove the time variable. Secondly, finite differences in time may be used, or lastly one could use time-dependent fundamental solutions. This third approach was introduced by Kupradze [80] in the case of the heat equation. This approach is also used by Walker [116], who extends the discrete source superposition method of Fenner [33], which is the MFS with fixed singularities, to transient problems. In particular, the diffusion equation is solved by matching the potential generated by three sets of singularities. This method involves integrations in time but no integrations in space, and the first results reported in [116] are promising.

More recently, Chen and Golberg [24] (see also [124]) applied an MFS-type method for the solution of a time-dependent diffusion problem. The time variable is removed by taking Laplace transforms and the problem is transformed into one governed by the modified Helmholtz equation. In particular, Chen and Golberg considered the diffusion equation

$$\frac{1}{k}\frac{\partial u}{\partial t}(P,t) = \Delta u(P,t), \quad P \in \Omega,\ t > 0,$$

subject to the boundary conditions

$$\begin{aligned} u(P,t) &= f_1(P,t), \qquad P \in \partial\Omega_1,\ t > 0, \\ \frac{\partial u}{\partial n}(P,t) &= f_2(P,t), \quad P \in \partial\Omega_2,\ t > 0, \end{aligned}$$

where $\partial\Omega = \partial\Omega_1 \cup \partial\Omega_2$, and the initial condition

$$u(P,0) = u_0(P), \quad P \in \Omega.$$

By taking the Laplace transform

$$\mathcal{L}\big[u(P,t)\big] = U(P;s) = \int_0^\infty u(P,t)\,\mathrm{e}^{-st}\,\mathrm{d}t,$$

this diffusion problem becomes

$$\begin{aligned} &\Big(\Delta - \frac{s}{k}\Big)U(P;s) = -\frac{u_0(P)}{k}, && P \in \Omega, \\ &U(P;s) = F_1(P;s), && P \in \partial\Omega_1, \\ &\frac{\partial U}{\partial n}(P;s) = F_2(P;s), && P \in \partial\Omega_2, \end{aligned}$$

where F_1 and F_2 are the Laplace transforms of f_1 and f_2, respectively. This boundary value problem is subsequently solved by determining a particular solution of the inhomogeneous problem using the associated Newton potential and then solving the homogeneous problem using an MFS-type method with the appropriate fundamental solution from (2.7). The solution in the real-time space can be obtained by a numerical inverse Laplace transform scheme; see [24] for details.

6. Concluding remarks

Numerical experiments show that, in contrast with other boundary methods such as the boundary element method, the MFS requires relatively few boundary points and singularities to produce accurate results. Probably the most important feature of the MFS approach is its adaptivity. By permitting the singularities to move, the method is able to adapt the approximation automatically to reflect any bad behavior in the solution and produces a uniform distribution of the error on the boundary regardless of its shape. In particular, corners of the region, which can cause problems in boundary element methods, are not a specific source of inaccuracy. The price one pays for this adaptivity is that the least squares problem becomes nonlinear in the coordinates of the singularities. However, since good software for solving this problem is available, this is not a serious drawback in most situations. The MFS has other advantages over boundary element methods. For example, it does not require an elaborate discretization of the boundary, nor does it involve potentially troublesome and costly integrations over the boundary. Also, the determination of an approximation to the solution at a point in the interior of the domain of the problem only requires an evaluation of the approximate solution, whereas in the boundary element method a quadrature is required. The derivatives of the MFS approximation can also be evaluated directly. Finally, the MFS is easy to implement and requires little data preparation.

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