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On distance matrices and Laplacians

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Abstract

We consider distance matrices of certain graphs and of points chosen in a rectangular grid. Formulae for the inverse and the determinant of the distance matrix of a weighted tree are obtained. Results concerning the inertia and the determinant of the distance matrix of an unweighted unicyclic graph are proved. If D is the distance matrix of a tree, then we obtain certain results for a perturbation of D^{-1}. As an example, it is shown that if $\widetilde{L}$ is the Laplacian matrix of an arbitrary connected graph, then $\left(D^{-1} - \widetilde{L}\right)^{-1}$ is an entrywise positive matrix. We consider the distance matrix of a subset of a rectangular grid of points in the plane. If we choose $m + k - 1$ points, not containing a closed path, in an $m \times k$ grid, then a formula for the determinant of the distance matrix of such points is obtained.

Keywords: Trees; Distance matrices; Laplacians; Determinants; Nonnegative matrices

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1. Introduction and background

A *graph* $G = (V, E)$ consists of a finite set of vertices V and a set of edges E. A *simple graph* has no loops or multiple edges and therefore its edge set consists of distinct pairs. A *weighted graph* is a graph in which each edge is assigned a *weight*, which is a positive number. An *unweighted graph*, or simply a *graph*, is thus a weighted graph with each of the edges bearing weight 1.

Let G be a connected, weighted graph on n vertices. The *distance between vertices* i and j is defined to be the minimum weight of all paths from i to j, where the *weight* of a path is just the sum of the weights of the edges on the path. The distance matrix D of G is an $n \times n$ matrix with zeros along the diagonal and with its (i, j)-entry equal to the distance between vertices i and j.

Distance matrices of graphs, particularly trees, have been investigated to a great extent in the literature. An early, remarkable result in this context concerns the determinant of the distance matrix of a tree: Graham and Pollack [3] showed that if T is a tree on n vertices with distance matrix D, then the determinant of D is $(-1)^{n-1}(n-1)2^{n-2}$, and thus is a function of only the number of vertices; that paper also discusses the inertia of D. (Recall that for symmetric matrix M, its *inertia* is the triple of integers $(n_+(M), n_0(M), n_-(M))$, where $n_+(M)$, $n_0(M)$, and $n_-(M)$ denote the number of positive eigenvalues of M, the multiplicity of 0 as an eigenvalue of M, and the number of negative eigenvalues of M, respectively.) In subsequent work, Graham and Lovasz [4] obtained a formula for D^{-1}, among other results. In Section 2 we extend Graham's and Lovasz's formula for D^{-1} to the case of a weighted tree. We also obtain an extension of the Graham and Pollack determinantal and inertial formulae to the weighted case. In Section 3 we further extend these results to distance matrices arising from unweighted unicyclic graphs.

Suppose that we have a weighted graph $G = (V, E)$ with n vertices and m edges, and that we assign an orientation to each edge of G. The associated (*vertex-edge*) *incidence matrix* Q of G is the $n \times m$ matrix defined as follows. The rows and the columns of Q are indexed by V and E respectively. The (i, j)-entry of Q is 0 if the ith vertex and the jth edge are not incident and it is $\sqrt{w(j)}$ (respectively, $-\sqrt{w(j)}$) if the ith vertex and the jth edge are incident, and the edge originates (respectively, terminates) at the ith vertex, where $w(j)$ denotes the weight of the jth edge. The *Laplacian matrix* L of G is defined as $L = QQ^{\mathrm{T}}$, and is independent of the orientation assigned to G. For basic properties of the Laplacian matrix see [1,7]. We note that in our results involving weighted trees, we will make use of the incidence matrix and the Laplacian matrix that arise by replacing each edge weight of the tree by its reciprocal.

In Section 4 we investigate a perturbation problem for distance matrices arising from weighted trees. Let D be a distance matrix arising from a weighted tree and let L be a Laplacian matrix of any weighted graph G. For $\epsilon > 0$, we consider perturbations of D^{-1} of the form $\epsilon D^{-1} - L$ and show that matrices of this form are invertible and have a nonnegative inverse.

Recall that if u and v are vectors in $\mathbb{R}^n$, then the ℓ_1-distance between u and v is defined as $\|u - v\|_1 = \sum_{i=1}^{n} |u_i - v_i|$. In Section 5 we obtain a formula for the determinant of the ℓ_1-distance matrix of a set of points in a rectangular grid. If $x_1, \ldots, x_n$ are distinct points in $\mathbb{R}^2$, then their ℓ_1-distance matrix $D = [d_{i,j}]$ is an $n \times n$ matrix with $d_{i,i} = 0$, $i = 1, 2, \ldots, n$, and $d_{i,j} = \|x_i - x_j\|_1$, if $i \neq j$. If $m + k - 1$ points are chosen from an $m \times k$ rectangular grid and if the points do not contain a closed path, then a formula for the determinant of D is obtained.

2. Distance matrix of a tree

In this section we extend some well known results on the distance matrix D of an unweighted tree T. The first result is due to Graham and Lovasz [4], who obtained a formula for D^{-1}. The latter two results are due to Graham and Pollack [3], who showed that if T has n vertices, then the determinant of D is $(-1)^{n-1}(n-1)2^{n-2}$, and that D has just one positive eigenvalue. In this section, we extend these results to the case of weighted trees.

Theorem 2.1. *Let T be a weighted tree on n vertices with edge weights $\alpha_1, \ldots, \alpha_{n-1}$ and let D be the corresponding distance matrix. Let L denote the Laplacian matrix for the weighting of T that arises by replacing each edge weight by its reciprocal. For each $i = 1, \ldots, n$, let d_i be the degree of the vertex i, let $\delta_i = 2 - d_i$, and set $\delta^{\mathrm{T}} = [\delta_1, \ldots, \delta_n]$. Then*

$$D^{-1} = -\frac{1}{2}L + \frac{1}{2\sum_{i=1}^{n-1}\alpha_i}\delta\delta^{\mathrm{T}}. \tag{2.1}$$

Proof. We use induction on n. For $n = 2$, we have $D = \begin{bmatrix} 0 & \alpha_1 \\ \alpha_1 & 0 \end{bmatrix}$, $L = \frac{1}{\alpha_1}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$, and $\delta = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, and the formula for D^{-1} follows readily. Now suppose we have a weighted tree on n vertices $1, 2, \ldots, n$, and form a new weighted tree $\overline{T}$ on vertices $1, \ldots, n+1$ by adding in a pendant vertex $n+1$, adjacent to vertex n with edge weight α_n. Let D, L, and δ be the appropriate quantities for T and let $\overline{D}$, $\overline{L}$, and $\bar{\delta}$ be the corresponding quantities for $\overline{T}$. Letting e_n be the nth standard unit basis vector in $\mathbb{R}^n$ and $\mathbf{1}$ be the all ones vector in $\mathbb{R}^n$, we have

$$\overline{L} = \left[\begin{array}{c|c} L + \frac{1}{\alpha_n}e_n e_n^{\mathrm{T}} & -\frac{1}{\alpha_n}e_n \\ \hline -\frac{1}{\alpha_n}e_n^{\mathrm{T}} & \frac{1}{\alpha_n} \end{array}\right], \quad \bar{\delta} = \left[\begin{array}{c} \delta - e_n \\ \hline 1 \end{array}\right],$$

and

$$\overline{D} = \left[\begin{array}{c|c} D & De_n + \alpha_n\mathbf{1} \\ \hline e_n^{\mathrm{T}}D + \alpha_n\mathbf{1}^{\mathrm{T}} & 0 \end{array}\right].$$

Let $\sigma_{n-1} = \sum_{i=1}^{n-1} \alpha_i$ and $\sigma_n = \sum_{i=1}^{n} \alpha_i$, and note that

$$
\begin{aligned}
-\frac{1}{2}\overline{L} + \frac{1}{2\sum_{i=1}^{n}\alpha_i}\bar{\delta}\bar{\delta}^{\mathrm{T}} &= -\frac{1}{2}\overline{L} + \frac{1}{2\sigma_n}\bar{\delta}\bar{\delta}^{\mathrm{T}} \\
&= \left[\begin{array}{c|c} -\frac{1}{2}L - \frac{1}{2\alpha_n}e_ne_n^{\mathrm{T}} + \frac{1}{2\sigma_n}\left(\delta\delta^{\mathrm{T}} - \delta e_n^{\mathrm{T}} - e_n\delta^{\mathrm{T}} + e_ne_n^{\mathrm{T}}\right) & \frac{1}{2\alpha_n}e_n + \frac{1}{2\sigma_n}(\delta - e_n) \\ \hline \frac{1}{2\alpha_n}e_n^{\mathrm{T}} + \frac{1}{2\sigma_n}\left(\delta^{\mathrm{T}} - e_n^{\mathrm{T}}\right) & -\frac{1}{2\alpha_n} + \frac{1}{2\sigma_n} \end{array}\right] \\
&= \left[\begin{array}{c|c} -\frac{1}{2}L + \frac{1}{2\sigma_n}\left(\delta\delta^{\mathrm{T}} - \delta e_n^{\mathrm{T}} - e_n\delta^{\mathrm{T}}\right) - \frac{\sigma_{n-1}}{2\alpha_n\sigma_n}e_ne_n^{\mathrm{T}} & \frac{1}{2\sigma_n}\delta + \frac{\sigma_{n-1}}{2\alpha_n\sigma_n}e_n \\ \hline \frac{1}{2\sigma_n}\delta^{\mathrm{T}} + \frac{\sigma_{n-1}}{2\alpha_n\sigma_n}e_n^{\mathrm{T}} & -\frac{\sigma_{n-1}}{2\alpha_n\sigma_n} \end{array}\right].
\end{aligned}
$$

From the induction hypothesis, $D^{-1}\mathbf{1} = \frac{1}{2\sigma_{n-1}}\delta^{\mathrm{T}}\mathbf{1}\delta = \frac{1}{\sigma_{n-1}}\delta$, so that $D\delta = \sigma_{n-1}\mathbf{1}$. Also from the induction hypothesis, $D^{-1} = -\frac{1}{2}L + \frac{1}{2\sigma_{n-1}}\delta\delta^{\mathrm{T}}$. We thus find that

$$
\begin{aligned}
&-\frac{1}{2}\overline{L} + \frac{1}{2\sigma_n}\bar{\delta}\bar{\delta}^{\mathrm{T}} \\
&\quad = \left[\begin{array}{c|c} D^{-1} - \frac{\alpha_n}{2\sigma_n\sigma_{n-1}}\delta\delta^{\mathrm{T}} - \frac{1}{2\sigma_n}\left(\delta e_n^{\mathrm{T}} + e_n\delta^{\mathrm{T}}\right) - \frac{\sigma_{n-1}}{2\alpha_n\sigma_n}e_ne_n^{\mathrm{T}} & \frac{1}{2\sigma_n}\delta + \frac{\sigma_{n-1}}{2\alpha_n\sigma_n}e_n \\ \hline \frac{1}{2\sigma_n}\delta^{\mathrm{T}} + \frac{\sigma_{n-1}}{2\alpha_n\sigma_n}e_n^{\mathrm{T}} & -\frac{\sigma_{n-1}}{2\alpha_n\sigma_n} \end{array}\right].
\end{aligned}
$$

Next, we note that

$$
\overline{D} = \left[\begin{array}{c|c} D & De_n + \alpha_n\mathbf{1} \\ \hline e_n^{\mathrm{T}}D + \alpha_n\mathbf{1}^{\mathrm{T}} & 0 \end{array}\right] = \left[\begin{array}{c|c} I & 0 \\ \hline e_n^{\mathrm{T}} & 1 \end{array}\right]\left[\begin{array}{c|c} D & \alpha_n\mathbf{1} \\ \hline \alpha_n\mathbf{1}^{\mathrm{T}} & -2\alpha_n \end{array}\right]\left[\begin{array}{c|c} I & e_n \\ \hline 0 & 1 \end{array}\right],
$$

so that

$$
\overline{D}^{-1} = \left[\begin{array}{c|c} I & -e_n \\ \hline 0 & 1 \end{array}\right]\left[\begin{array}{c|c} D & \alpha_n\mathbf{1} \\ \hline \alpha_n\mathbf{1}^{\mathrm{T}} & -2\alpha_n \end{array}\right]^{-1}\left[\begin{array}{c|c} I & 0 \\ \hline -e_n^{\mathrm{T}} & 1 \end{array}\right].
$$

Now a standard computation shows that

$$
\begin{aligned}
\left[\begin{array}{c|c} D & \alpha_n\mathbf{1} \\ \hline \alpha_n\mathbf{1}^{\mathrm{T}} & -2\alpha_n \end{array}\right]^{-1} &= \left[\begin{array}{c|c} \left(D + \frac{\alpha_n}{2}\mathbf{1}\mathbf{1}^{\mathrm{T}}\right)^{-1} & \frac{\sigma_{n-1}}{2\sigma_n}D^{-1}\mathbf{1} \\ \hline \frac{\sigma_{n-1}}{2\sigma_n}\mathbf{1}^{\mathrm{T}}D^{-1} & -\frac{\sigma_{n-1}}{2\alpha_n\sigma_n} \end{array}\right] \\
&= \left[\begin{array}{c|c} D^{-1} - \frac{\alpha_n}{2\sigma_n\sigma_{n-1}}\delta\delta^{\mathrm{T}} & \frac{1}{2\sigma_n}\delta \\ \hline \frac{1}{2\sigma_n}\delta^{\mathrm{T}} & -\frac{\sigma_{n-1}}{2\alpha_n\sigma_n} \end{array}\right],
\end{aligned}
$$

so that

$$
\overline{D}^{-1} = \left[\begin{array}{c|c} I & -e_n \\ \hline 0 & 1 \end{array}\right]\left[\begin{array}{c|c} D^{-1} - \frac{\alpha_n}{2\sigma_n\sigma_{n-1}}\delta\delta^{\mathrm{T}} & \frac{1}{2\sigma_n}\delta \\ \hline \frac{1}{2\sigma_n}\delta^{\mathrm{T}} & -\frac{\sigma_{n-1}}{2\alpha_n\sigma_n} \end{array}\right]\left[\begin{array}{c|c} I & 0 \\ \hline -e_n^{\mathrm{T}} & 1 \end{array}\right]
$$

$$= \left[\begin{array}{c|c} D^{-1} - \frac{\alpha_n}{2\sigma_n\sigma_{n-1}}\delta\delta^{\mathrm{T}} - \frac{1}{2\sigma_n}\left(\delta e_n^{\mathrm{T}} + e_n\delta^{\mathrm{T}}\right) - \frac{\sigma_{n-1}}{2\alpha_n\sigma_n}e_n e_n^{\mathrm{T}} & \frac{1}{2\sigma_n}\delta + \frac{\sigma_{n-1}}{2\alpha_n\sigma_n}e_n \\ \hline \frac{1}{2\sigma_n}\delta^{\mathrm{T}} + \frac{\sigma_{n-1}}{2\alpha_n\sigma_n}e_n^{\mathrm{T}} & -\frac{\sigma_{n-1}}{2\alpha_n\sigma_n} \end{array}\right]$$

as desired. □

Remark 2.2. For an unweighted tree on $n \geqslant 3$ vertices, the (i, j)-element of D^{-1} is zero if and only if $i \neq j$, one of the vertices i and j has degree 2, and i and j are not adjacent. To see this, observe that the (i, j)-element of D^{-1} is zero if and only if $(2 - d_i)(2 - d_j) = (n - 1)\ell_{ij}$, so that our conditions are clearly sufficient. Conversely, if i and j are adjacent, then $\ell_{ij} = -1$ so that $(n - 1)\ell_{ij} = -(n - 1)$. But $(2 - d_i)(2 - d_j) \geqslant 1[2 - (n - 1)] = -(n - 3)$ and the (i, j)-element of D^{-1} is nonzero. Finally, if $i = j$, we see that since $n \geqslant 3$, $(2 - d)^2 = (n - 1)d$ has no admissible positive integer solution for d.

In particular, each row of D^{-1} corresponding to a vertex of degree 2 has 3 nonzero entries: a -1 on the diagonal and $\frac{1}{2}$ in each spot corresponding to an adjacent vertex.

In order to discuss the determinant and inertia properties of distance matrices of weighted trees (and later, of unicyclic graphs), we begin by considering the following somewhat larger class of weighted graphs. Let G be a weighted graph, and suppose that we have a collection of weighted trees $B_1, \ldots, B_k$. We construct a new graph $\overline{G}$ from G and the trees $B_1, \ldots, B_k$ by adding, for each $i = 1, \ldots, k$, a weighted edge between some vertex of B_i and some vertex of G. We say that the new graph $\overline{G}$ *is constructed by adding the weighted branches* $B_1, \ldots, B_k$ *to* G. Evidently both the weighted trees and the unicyclic graphs can be constructed in this fashion.

Theorem 2.3. *Let G be a connected weighted graph on n vertices with distance matrix D, and suppose that $D\mathbf{1} = d\mathbf{1}$. Form $\overline{G}$ from G by adding weighted branches to G on a total of m new vertices, with positive weights $\alpha_1, \ldots, \alpha_m$ on the new edges. Let $\overline{D}$ be the distance matrix for $\overline{G}$. Then for each $x \in \mathbb{R}$, $\det(\overline{D} + xJ) = (-2)^m \det(D)\left(\prod_{i=1}^m \alpha_i\right)(1 + \frac{nx}{d} + \frac{n}{2d}\sum_{i=1}^m \alpha_i)$. Further, $n_0(D) = n_0(\overline{D})$ and if, in addition, D is nonsingular, then $n_+(D) = n_+(\overline{D})$.*

Proof. We prove the statement regarding $\det(\overline{D} + xJ)$ by induction on m. For the case that $m = 0$, note that the eigenvalues of D can be written as $d = \lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_n$, while the eigenvalues of $D + xJ$ are $d + nx$ and $\lambda_2, \ldots, \lambda_n$. It now follows that $\det(D + xJ) = d\left(1 + \frac{nx}{d}\right)\lambda_2 \cdots \lambda_n = \det(D)\left(1 + \frac{nx}{d}\right)$, as desired. Suppose now the statement holds for some $m \geqslant 0$ and that $\overline{G}$ is formed as described by adding branches on $m + 1$ new vertices to G. Without loss of generality, assume that vertex $n + m + 1$ is pendant, adjacent to vertex $n + m$, and that the weight of the corresponding pendant edge is α_{m+1}. Then

$$\overline{D}+xJ=\left[\begin{array}{c|c} I & 0 \\ \hline e_{n+m}^{\mathrm{T}} & 1 \end{array}\right]\left[\begin{array}{c|c} \widehat{D}+xJ & \alpha_{m+1}\mathbf{1} \\ \hline \alpha_{m+1}\mathbf{1}^{\mathrm{T}} & -2\alpha_{m+1} \end{array}\right]\left[\begin{array}{c|c} I & e_{n+m} \\ \hline 0 & 1 \end{array}\right],$$

where $\widehat{D}$ is the distance matrix for the weighted graph on $n+m$ vertices formed from $\overline{G}$ by deleting vertex $n+m+1$. Thus, $\det(\overline{D}+xJ)=(-2\alpha_{m+1})\det\left(\widehat{D}+xJ+\frac{\alpha_{m+1}}{2}J\right)=(-2\alpha_{m+1})(-2)^m\det(D)\left(\prod_{i=1}^m\alpha_i\right)\left(1+\frac{n}{d}\left(x+\frac{\alpha_{m+1}}{2}\right)+\frac{n}{2d}\sum_{i=1}^m\alpha_i\right)$, the first equality following from Schur's formula, and the second from an application of the induction hypothesis. We now find readily that $\det(\overline{D}+xJ)=(-2)^{m+1}\det(D)\left(\prod_{i=1}^{m+1}\alpha_i\right)\left(1+\frac{nx}{d}+\frac{n}{2d}\sum_{i=1}^{m+1}\alpha_i\right)$.

In order to deduce that $n_0(D)=n_0(\overline{D})$, we prove by induction on m that for any $x\geqslant 0$, $n_0(D)=n_0(\overline{D}+xJ)$. Note that the case $m=0$ is straightforward, since the eigenvalues of $\overline{D}+xJ$ consist of the Perron value $d+nx$, along with the remaining non-Perron eigenvalues of D. Next, suppose that the result holds for some $m\geqslant 0$, and that $\overline{G}$ is formed from G by adding branches on $m+1$ new vertices, with vertex $n+m+1$ pendant, and adjacent to vertex $n+m$ with an edge of weight α_{m+1} between them. As above we see that $\overline{D}+xJ$ is congruent to the matrix

$$M=\left[\begin{array}{c|c} \widehat{D}+xJ & \alpha_{m+1}\mathbf{1} \\ \hline \alpha_{m+1}\mathbf{1}^{\mathrm{T}} & -2\alpha_{m+1} \end{array}\right],$$

where $\widehat{D}$ is as described above. Evidently $n_0(M)=n_0(\overline{D}+xJ)$. Observe that the vector $\left[\begin{array}{c} v \\ \hline u \end{array}\right]$ (partitioned conformably with M) is a null vector for M if and only if v is a null vector for $\widehat{D}+(x+\alpha_{m+1}/2)J$ and $u=v^{\mathrm{T}}\mathbf{1}/2$. It follows then that the null vectors of M are in one to one correspondence with those of $\widehat{D}+(x+\alpha_{m+1}/2)J$. Applying the induction hypothesis, we find that $n_0\left(\widehat{D}+(x+\alpha_{m+1}/2)J\right)=n_0(D)$, from which we conclude that $n_0(\overline{D}+xJ)=n_0(M)=n_0(D)$.

Next suppose that D is nonsingular; we will prove the statement on $n_+(D)$ by induction on m, and note that the case $m=0$ is plain. Suppose that the statement holds for some $m\geqslant 0$, and that $\overline{G}$ is formed from G by adding branches on $m+1$ new vertices, with vertex $n+m+1$ pendant, and adjacent to vertex $n+m$ with an edge of weight α_{m+1} between them. As above we see that $\overline{D}$ is congruent to

$$\left[\begin{array}{c|c} \widehat{D} & \alpha_{m+1}\mathbf{1} \\ \hline \alpha_{m+1}\mathbf{1}^{\mathrm{T}} & -2\alpha_{m+1} \end{array}\right],$$

where $\widehat{D}$ is as described above. From interlacing, we find that $n_+\left(\widehat{D}\right)\leqslant n_+(\overline{D})$ and $n_-\left(\widehat{D}\right)\leqslant n_-(\overline{D})$; since $\det\left(\widehat{D}\right)$ and $\det(\overline{D})$ have opposite signs, it follows that in fact $n_+\left(\widehat{D}\right)=n_+(\overline{D})$. Applying the induction hypothesis, we find that $n_+(D)=n_+\left(\widehat{D}\right)=n_+(\overline{D})$. $\square$

Theorem 2.3 yields the following generalization of results of Graham and Pollack [3] for unweighted trees.

Theorem 2.4. *Let T be a weighted tree on n vertices with edge weights $\alpha_1, \ldots, \alpha_{n-1}$. Let D be the distance matrix of T. Then for any real number x,*

$$\det(D + xJ) = (-1)^{n-1} 2^{n-2} \left(\prod_{i=1}^{n-1} \alpha_i \right) \left(2x + \sum_{i=1}^{n-1} \alpha_i \right).$$

Further, the inertia of D is $(n_+(D), n_0(D), n_-(D)) = (1, 0, n-1)$.

Proof. Observe that we may construct any tree on n vertices with weights $\alpha_1, \ldots, \alpha_{n-1}$ by beginning with a single edge of weight α_1 (whose 2×2 distance matrix has constant row sum α_1, determinant $-\alpha_1{}^2$ and eigenvalues $\pm\alpha_1$) and then adding in branches on $n-2$ new vertices as described in Theorem 2.3. The results now follow from that theorem. $\square$

Corollary 2.5. *If D is as in Theorem 2.4, then*

$$\det(D) = (-1)^{n-1} 2^{n-2} \left(\prod_{i=1}^{n-1} \alpha_i \right) \left(\sum_{i=1}^{n-1} \alpha_i \right).$$

3. Distance matrix of a unicyclic graph

Recall that a graph is *unicyclic* if it is connected and has a single cycle. In this section we obtain results concerning the inertia and determinant for the distance matrix of an unweighted unicyclic graph. Many of our results in this section are stated separately for the cases that the length of the cycle is odd or even.

We begin by investigating the distance matrix for an unweighted cycle of odd length. Here we assume without loss of generality that the vertices of the cycle of length $2k+1$ are labelled so that for each $i = 1, \ldots, 2k+1$, vertex i is adjacent to vertices $i+1$ and $i-1$ (where these indices are taken modulo $2k+1$). As part of our investigation, we make use of the cyclic permutation matrix C (of order $2k+1$) having $C_{i,i+1} = 1$ for $i = 1, \ldots 2k+1$ (again taking indices modulo $2k+1$).

Theorem 3.1. *Let D be the distance matrix for the cycle on $2k+1$ vertices. Then*

$$D^{-1} = -2I - C^k - C^{k+1} + \frac{2k+1}{k(k+1)} J.$$

Proof. Since

$$e_1^{\mathrm{T}} D = [0, 1, 2, \ldots, k-1, k, k, k-1, \ldots, 2, 1],$$

$$e_{k+1}^{\mathrm{T}} D = [k, k-1, k-2, \dots, 1, 0, 1, 2, \dots, k-1, k],$$
$$e_{k+2}^{\mathrm{T}} D = [k, k, k-1, \dots, 2, 1, 0, 1, \dots, k-2, k-1],$$

we find that

$$\left(e_{k+1}^{\mathrm{T}} + e_{k+2}^{\mathrm{T}} + 2e_1^{\mathrm{T}}\right) D = [2k, 2k+1, \dots, 2k+1].$$

It follows that

$$\left(C^k + C^{k+1} + 2I\right) D = (2k+1)J - I,$$

and hence

$$\begin{aligned} I &= ((2k+1)J - I)^{-1} \left(C^k + C^{k+1} + 2I\right) D \\ &= \left(-I + \frac{2k+1}{4k(k+1)} J\right) \left(C^k + C^{k+1} + 2I\right) D \\ &= \left(-2I - C^k - C^{k+1} + \frac{2k+1}{k(k+1)} J\right) D. \qquad \square \end{aligned}$$

Corollary 3.2. *The distance matrix for a cycle on* $2k+1$ *vertices has just one positive eigenvalue.*

Proof. Evidently the eigenvalues for D^{-1} are $\frac{1}{k(k+1)}$ and

$$-2 - \left(\cos \frac{2kj}{2k+1}\pi + \cos \frac{2(k+1)j}{2k+1}\pi\right),$$

$j = 1, \dots, 2k$, and only the first eigenvalue is positive. $\square$

Remark 3.3. It is straightforward to see that $C^k + C^{k+1}$ is the adjacency matrix for a cycle of length $2k+1$, where for each $i = 1, \dots, 2k+1$, vertex i is adjacent to vertices $i+k$ and $i-k$ (taking those indices modulo $2k+1$). It follows readily that the matrix $2I + C^k + C^{k+1}$ is permutationally similar to $2I + C + C^{\mathrm{T}}$, so that $\det\left(2I + C^k + C^{k+1}\right) = \det\left(2I + C + C^{\mathrm{T}}\right)$. A simple proof by induction on k shows that $\det\left(2I + C + C^{\mathrm{T}}\right) = 4$.

Next we give formulae for the determinant and inertia for the distance matrix of an unweighted unicyclic graph having a cycle of odd length.

Theorem 3.4. *Let G be a unicyclic graph with $2k+1+m$ vertices and cycle length $2k+1$. Let $\overline{D}$ be the distance matrix of G. Then* $\det(\overline{D}) = (-2)^m\left[k(k+1) + \frac{2k+1}{2}m\right]$, *while the inertia of $\overline{D}$ is given by* $\left(n_+(\overline{D}), n_0(\overline{D}), n_-(\overline{D})\right) = (1, 0, 2k+m)$.

Proof. Let D be the distance matrix for the cycle on $2k+1$ vertices, and observe that $D\mathbf{1} = k(k+1)\mathbf{1}$. In particular, the hypothesis of Theorem 2.3 applies to D, and $\overline{D}$ is constructed from D as described in that theorem.

From Theorem 3.1 we find that the eigenvalues of D consist of the Perron value $k(k+1)$, along with the reciprocals of the eigenvalues of $-2I - C^k - C^{k+1}$ whose eigenvectors are orthogonal to $\mathbf{1}$. By Remark 3.3, $\det(-2I - C^k - C^{k+1}) = -4$ and clearly -4 is the eigenvalue of $-2I - C^k - C^{k+1}$ corresponding to $\mathbf{1}$, so that the remaining eigenvalues have product 1. Hence $\det(D) = k(k+1)$. Applying Theorem 2.3, it now follows readily that $\det(\overline{D}) = (-2)^m \left[k(k+1) + \frac{2k+1}{2}m\right]$.

From Corollary 3.2 we have $n_+(D) = 1$, and again applying Theorem 2.3, we find that $\left(n_+(\overline{D}), n_0(\overline{D}), n_-(\overline{D})\right) = (1, 0, 2k+m)$. $\square$

Next we develop parallel results for the distance matrix of an unweighted unicyclic graph having a cycle of even length. We begin by analyzing the distance matrix for an unweighted cycle of even length.

Remark 3.5. The distance matrix for the $2k$-cycle is the circulant

$$D = \mathrm{circ}([0, 1, 2, \dots, k, k-1, \dots, 2, 1]).$$

In particular, for any $x \geqslant 0$, $D + xJ$ has Perron value $k^2 + 2kx$ with Perron vector $\mathbf{1}$. Further, for each $j = 1, \dots, 2k-1$, consider the $2k$th root of unity $\chi = \mathrm{e}^{\pi \mathrm{i} j/k}$. It is straightforward to see that χ generates a non-Perron eigenvalue of $D + xJ$ as follows:

$$\begin{aligned}
&\sum_{t=1}^{k} t\chi^t + \sum_{t=1}^{k} (k-t)\chi^{k+t} \\
&\quad = k\sum_{t=1}^{k} \chi^{k+t} + (1-\chi^k)\sum_{t=1}^{k} t\chi^t \\
&\quad = k\chi^{k+1}\frac{1-\chi^k}{1-\chi} + \frac{(1-\chi^k)\chi}{(1-\chi)^2}\left[-(k+1)\chi^k + k\chi^{k+1} + 1\right].
\end{aligned}$$

If j is even, then $\chi^k = 1$, so we get a 0 eigenvalue. If j is odd, we get $\chi^k = -1$, so the eigenvalue becomes

$$\begin{aligned}
&2\left[-\frac{k\chi}{1-\chi} + \frac{\chi}{(1-\chi)^2}(k+2-k\chi)\right] \\
&\quad = \frac{2\chi}{(1-\chi)^2}(-k + k\chi + k + 2 - k\chi) \\
&\quad = \frac{4\chi}{(1-\chi)^2}
\end{aligned}$$

$$= \frac{4}{|1-\chi|^4}\chi\left(1-2\overline{\chi}+\overline{\chi}^2\right)$$
$$= \frac{4}{|1-\chi|^4}\left(\chi-2+\overline{\chi}\right)$$
$$= -\frac{8}{|1-\chi|^4}\left[1-\cos\frac{\pi j}{k}\right].$$

In particular, we see that for any $x \geqslant 0$, $D + xJ$ has one positive eigenvalue, and nullity $k-1$.

The following result will be useful in discussing the inertia of the distance matrix for a unicyclic graph.

Lemma 3.6. *Let G_0 be a connected graph with distance matrix D_0 and suppose that for all $x > 0$, $D_0 + xJ$ has a single positive eigenvalue (namely the Perron value). Form G_m from G_0 by adding in unweighted branches at various vertices of G_0, on a total of m new vertices. If D_m is the corresponding distance matrix, then $D_m + xJ$ has just one positive eigenvalue for any $x > 0$.*

Proof. We proceed by induction on m, and note that the case $m = 0$ is just the hypothesis on D_0. Note that for some i, we have that

$$D_{m+1} + xJ = \begin{bmatrix} D_m + xJ & D_m e_i + (x+1)\mathbf{1} \\ e_i^{\mathrm{T}} D_m + (x+1)\mathbf{1}^{\mathrm{T}} & x \end{bmatrix}.$$

Let $M = \begin{bmatrix} I & 0 \\ -e_i^{\mathrm{T}} & 1 \end{bmatrix}$. Then we see that

$$M(D_{m+1} + xJ)M^{\mathrm{T}} = \begin{bmatrix} D_m + xJ & \mathbf{1} \\ \mathbf{1}^{\mathrm{T}} & -2 \end{bmatrix} = A,$$

so that $D_{m+1} + xJ$ and A have the same inertia. Note also that A has a positive eigenvalue λ if and only if $D_m + \left(x + \frac{1}{2+\lambda}\right)J$ has λ as a positive eigenvalue, which, by the induction hypothesis, must be the Perron value. Note that, necessarily, such a λ is a simple eigenvalue of A since any Perron value is simple. Suppose that there are two positive eigenvalues λ_1 and λ_2 with $\lambda_1 > \lambda_2$ so that λ_i is the Perron value of $D_m + \left(x + \frac{1}{2+\lambda_i}\right)J$, $i = 1, 2$. Then, letting ρ denote the Perron value, we have

$$\lambda_1 = \rho\left(D_m + \left(x + \frac{1}{2+\lambda_1}\right)J\right) < \rho\left(D_m + \left(x + \frac{1}{2+\lambda_2}\right)J\right) = \lambda_2,$$

the inequality being strict since $\lambda_1 > \lambda_2$ which is a contradiction. Hence A can have only one positive eigenvalue so that $D_{m+1} + xJ$ has just one positive eigenvalue. $\square$

We now determine the inertia for the distance matrix of an unweighted unicyclic graph having a cycle of even length.

Theorem 3.7. *Let G be a unicyclic graph on $2k+m$ vertices with an even cycle of length $2k$. Let D be the distance matrix of G. Then the inertia of D is $(n_+(D), n_0(D), n_-(D)) = (1, k-1, k+m)$.*

Proof. By Lemma 3.6 and the Remark 3.5, we find that $n_+(D) = 1$. Also, applying Remark 3.5 and Theorem 2.3, we find that $n_0(D) = k-1$. Consequently, $n_-(D) = k+m$ and the result follows. $\square$

4. Inverse distance matrix perturbed by a Laplacian

We begin with a preliminary result.

Lemma 4.1. *Let T be a weighted tree with n vertices, and suppose that each edge of T has been assigned an orientation. Let D be the distance matrix of T, and let L and Q denote the Laplacian matrix and incidence matrix, respectively, for the weighting of T that arises by replacing each edge weight by its reciprocal. Denote the degree of vertex i by d_i, let $\delta_i = 2 - d_i$, $i = 1, \dots, n$, and let $\delta^{\mathrm{T}} = [\delta_1, \dots, \delta_n]^{\mathrm{T}}$. Then the following assertions hold*:

(i) $LD = \delta\mathbf{1}^{\mathrm{T}} - 2I$, $DL = \mathbf{1}\delta^{\mathrm{T}} - 2I$.
(ii) $Q^{\mathrm{T}}DQ = -2I$.
(iii) $(D^{-1} - L)^{-1} = \frac{1}{3}D + \frac{1}{3}\left(\sum_{i=1}^{n-1}\alpha_i\right)J$.

Proof. (i) As in Theorem 2.1, we denote the edge weights of T by $\alpha_1, \dots, \alpha_{n-1}$. It follows from Theorem 2.1 that $-\frac{1}{2}LD + \frac{1}{2\sum_{i=1}^{n-1}\alpha_i}\delta\delta^{\mathrm{T}}D = I$ and hence $LD = \frac{1}{\sum_{i=1}^{n-1}\alpha_i}\delta\delta^{\mathrm{T}}D - 2I$. Also, as seen in the proof of Theorem 2.1, $D\delta = \left(\sum_{i=1}^{n-1}\alpha_i\right)\mathbf{1}$ and thus $LD = \delta\mathbf{1}^{\mathrm{T}} - 2I$. The proof of the second part is similar.

(ii) By (i), $QQ^{\mathrm{T}}D = \delta\mathbf{1}^{\mathrm{T}} - 2I$ and, since Q has zero column sums, $QQ^{\mathrm{T}}DQ = -2Q$. The result follows since Q has full column rank and thus admits a left-inverse. We remark that the assertion made in (ii) is well-known in the unweighted case, see [6].

(iii) The result follows immediately from (i) and the fact (see Theorem 2.1) that $D^{-1}\mathbf{1} = \frac{1}{\sum_{i=1}^{n-1}\alpha_i}\delta$. $\square$

Motivated by the above results, we conducted certain numerical experiments concerning $(D^{-1} - L)^{-1}$, in which L was replaced by the Laplacian of an arbitrary connected graph, while D continued to be the distance matrix of a weighted tree. The results were interesting as well as unexpected and are presented in a sequence, culminating in Theorems 4.5 and 4.6.

Let D be the distance matrix of a weighted tree with at least two vertices. We will use the well-known fact that D, and any principal submatrix of D of order at least 2, has exactly one positive eigenvalue, while the remaining eigenvalues are negative. A square matrix is said to be an *N-matrix* if all its principal minors are negative. A *signature matrix* is a diagonal matrix with ± 1 on the diagonal.

Lemma 4.2. *Let S be an $n \times n$ symmetric, positive semidefinite matrix with $S\mathbf{1} = 0$. Let D be the distance matrix of a weighted tree on n vertices. Then $D^{-1} - S$ is nonsingular and has* 1 *positive and $n - 1$ negative eigenvalues.*

Proof. Using the notation and the conclusion of Theorem 2.1,

$$D^{-1} = -\frac{1}{2}L + \frac{1}{2\sum_{i=1}^{n-1}\alpha_i}\delta\delta^{\mathrm{T}}.$$

If $(D^{-1} - S)x = 0$ for some vector x, then, using the preceding equation,

$$\mathbf{1}^{\mathrm{T}}\left(-\frac{1}{2}L + \frac{1}{2\sum_{i=1}^{n-1}\alpha_i}\delta\delta^{\mathrm{T}} - S\right)x = 0$$

and thus

$$\mathbf{1}^{\mathrm{T}}\delta\delta^{\mathrm{T}}x = 0.$$

Since $\mathbf{1}^{\mathrm{T}}\delta = 2n - 2(n-1) = 2$, then $\delta^{\mathrm{T}}x = 0$. Thus $\left(-\frac{1}{2}L - S\right)x = 0$. Since L and S are positive semidefinite, it follows that $Lx = 0$ and hence x must be a scalar multiple of $\mathbf{1}$. Now since $\delta^{\mathrm{T}}x = 0$ and $\mathbf{1}^{\mathrm{T}}\delta \neq 0$, we have that $x = 0$. Thus the matrix $D^{-1} - S$ is nonsingular. Similarly, for any $t \geqslant 0$, $(D^{-1} - tS)$ is nonsingular and must have the same inertia for all $t \geqslant 0$. Since D^{-1} has 1 positive and $n - 1$ negative eigenvalues, it follows that $D^{-1} - S$ has 1 positive and $n - 1$ negative eigenvalues. $\square$

Lemma 4.3. *Let S be an $n \times n$ symmetric, positive semidefinite matrix with $S\mathbf{1} = 0$. Let D be the distance matrix of a weighted tree on n vertices. Then* (i) *For any $\alpha > 0$, $\alpha I - S^{\frac{1}{2}}DS^{\frac{1}{2}}$ is positive definite.* (ii) *$S^{\frac{1}{2}}DS^{\frac{1}{2}}$ is negative semidefinite.*

Proof. Evaluating the determinant of $\begin{bmatrix} D^{-1} & S^{\frac{1}{2}} \\ S^{\frac{1}{2}} & \alpha I \end{bmatrix}$ in two different ways, we see that

$$\det(D^{-1})\det\left(\alpha I - S^{\frac{1}{2}}DS^{\frac{1}{2}}\right) = \det(\alpha I)\det\left(D^{-1} - \frac{S}{\alpha}\right). \tag{4.1}$$

By Lemma 4.2, $D^{-1} - \frac{1}{\alpha}S$ has 1 positive and $n-1$ negative eigenvalues and thus $\det(D^{-1})$ and $\det\left(D^{-1} - \frac{S}{\alpha}\right)$ have the same sign. It follows from (4.1) that $\det\left(\alpha I - S^{\frac{1}{2}}DS^{\frac{1}{2}}\right) > 0$. Since D has inertia $(1, 0, n-1)$, $S^{\frac{1}{2}}DS^{\frac{1}{2}}$ has at most one positive eigenvalue. Thus $\alpha I - S^{\frac{1}{2}}DS^{\frac{1}{2}}$ has at most 1 negative eigenvalue. However, since $\det\left(\alpha I - S^{\frac{1}{2}}DS^{\frac{1}{2}}\right) > 0$, $\alpha I - S^{\frac{1}{2}}DS^{\frac{1}{2}}$ has no negative eigenvalue and hence it is positive definite. This proves (i). To prove (ii), first note that $S^{\frac{1}{2}}DS^{\frac{1}{2}}$ has at most 1 nonnegative eigenvalue. If $S^{\frac{1}{2}}DS^{\frac{1}{2}}$ has a positive eigenvalue, then we get a contradiction to (i) for small $\alpha > 0$. Thus $S^{\frac{1}{2}}DS^{\frac{1}{2}}$ has all eigenvalues nonpositive and hence is negative semidefinite. □

Lemma 4.4. *Let S be an $n \times n$ symmetric, positive semidefinite matrix with $S\mathbf{1} = 0$ and suppose $rank(S) = n-1$. Let D be the distance matrix of a weighted tree on n vertices and let $p \geqslant 1$ be an integer. Then any proper principal submatrix of $D^{-1}(I - DS)^p$ is negative definite.*

Proof. We have

$$\begin{aligned} D^{-1}(I - DS)^p &= D^{-1}\sum_{r=0}^{p}(-1)^r(DS)^r\binom{p}{r} \\ &= D^{-1} + \sum_{r=1}^{p}(-1)^r S(DS)^{r-1}\binom{p}{r} \\ &= D^{-1} - S^{\frac{1}{2}}ZS^{\frac{1}{2}}, \end{aligned}$$

where

$$Z = pI + \sum_{r=2}^{p}(-1)^{r-1}\left(S^{\frac{1}{2}}DS^{\frac{1}{2}}\right)^{r-1}\binom{p}{r}.$$

By Lemma 4.3, $S^{\frac{1}{2}}DS^{\frac{1}{2}}$ is negative semidefinite and hence Z is positive definite. Thus $S^{\frac{1}{2}}ZS^{\frac{1}{2}}$ is positive semidefinite. Note that $\operatorname{rank} S^{\frac{1}{2}}ZS^{\frac{1}{2}} = \operatorname{rank} S^{\frac{1}{2}}Z^{\frac{1}{2}}Z^{\frac{1}{2}}S^{\frac{1}{2}} = \operatorname{rank} S^{\frac{1}{2}}Z^{\frac{1}{2}} = \operatorname{rank} S^{\frac{1}{2}} = \operatorname{rank} S = n-1$. (Here we used the facts that $\operatorname{rank} XX' = \operatorname{rank} X$, for any X and that the rank is invariant under multiplication by a nonsingular matrix.) Since S is positive semidefinite and $S\mathbf{1} = 0$, it follows that $S^{\frac{1}{2}}\mathbf{1} = 0$, and thus $S^{\frac{1}{2}}ZS^{\frac{1}{2}}\mathbf{1} = 0$. Then all cofactors of $S^{\frac{1}{2}}ZS^{\frac{1}{2}}$ are equal, and, in particular, any principal submatrix of $S^{\frac{1}{2}}ZS^{\frac{1}{2}}$ of order $n-1$ is nonsingular. Denote by $A(i,i)$ the submatrix of A obtained by deleting row and column i. By the interlacing property, $D^{-1}(i,i)$ has at most one nonnegative eigenvalue. However, since $d_{ii} = 0$, $D^{-1}(i,i)$ is singular and thus it has only nonpositive eigenvalues. Thus $D^{-1}(i,i)$ is negative semidefinite. Since, in view of a preceding remark, $S^{\frac{1}{2}}ZS^{\frac{1}{2}}(i,i)$ is positive definite, it follows that $\left(D^{-1} - S^{\frac{1}{2}} Z S^{\frac{1}{2}}\right)(i,i) = D^{-1}(i,i) - S^{\frac{1}{2}} Z S^{\frac{1}{2}}(i,i)$ is negative definite and the proof is complete. □

Theorem 4.5. *Let S be an $n \times n$ symmetric, positive semidefinite matrix with $S\mathbf{1} = 0$ and suppose $rank(S) = n - 1$. Let D be the distance matrix of a weighted tree on n vertices and let $p \geqslant 1$ be an integer. Then $(I - tDS)^{-p} D > 0$, for all $t > 0$.*

Proof. As in the proof of Lemma 4.4 we may write $D^{-1}(I - tDS)^p = D^{-1} - tS^{\frac{1}{2}} Z S^{\frac{1}{2}}$, where Z is positive definite. It follows by Lemmas 4.2 and 4.4 that for $t > 0$, $U_t = -D^{-1}(I - tDS)^p$ has a negative determinant and any proper principal minor of U_t is positive. Thus U_t^{-1} is an N-matrix and there exists a signature matrix R_t (see Lemma 2 of [8]) such that $R_t U_t^{-1} R_t < 0$. By a continuity argument, R_t must in fact be the same for all $t > 0$. Since for sufficiently small $t > 0$, U_t^{-1}, being close to $-D$, is negative, it follows that $R_t = I$ for all $t > 0$. Thus $-(I - tDS)^{-p} D < 0$ and the proof is complete. □

We now prove an application of our results to Laplacians; the first part of the result is motivated by (iii), Lemma 4.1.

Theorem 4.6. *Let G be a weighted, connected, graph on n vertices and let $\widetilde{L}$ be the Laplacian of G. Let D be the distance matrix of a weighted tree on n vertices. Then* (i) $\left(D^{-1} - \widetilde{L}\right)^{-1}$ *is an entrywise positive matrix, and* (ii) *each entry of $F(\epsilon) = \left(\epsilon D^{-1} - \widetilde{L}\right)^{-1}$ is decreasing in $\epsilon > 0$.*

Proof. Since $\widetilde{L}$ is symmetric, positive semidefinite of rank $n - 1$ and satisfies $\widetilde{L}\mathbf{1} = 0$, (i) follows from the case $p = 1$ of Theorem 4.5. To prove (ii), note that the derivative of $F(\epsilon)$ with respect to ϵ is given by

$$
\begin{aligned}
F'(\epsilon) &= -\left(\epsilon D^{-1} - \widetilde{L}\right)^{-1} D^{-1} (\epsilon D^{-1} - L)^{-1} \\
&= -\left[\left(\epsilon D^{-1} - \widetilde{L}\right) D \left(\epsilon D^{-1} - \widetilde{L}\right)\right]^{-1} \\
&= -\epsilon^{-2}\left(D^{-1} - \widetilde{S}\right)^{-1},
\end{aligned}
$$

where $\widetilde{S} = 2(\widetilde{L}/\epsilon) - \left(\widetilde{L}/\epsilon\right) D \left(\widetilde{L}/\epsilon\right)$. We claim that $\widetilde{S}$ satisfies the hypotheses of Theorem 4.5. Note that

$$
\widetilde{S} = 2\left(\widetilde{L}/\epsilon\right) - \left(\widetilde{L}/\epsilon\right) D \left(\widetilde{L}/\epsilon\right) = \left(\widetilde{L}/\epsilon\right)^{\frac{1}{2}} \left(2I - \left(\widetilde{L}/\epsilon\right)^{\frac{1}{2}} D \left(\widetilde{L}/\epsilon\right)^{\frac{1}{2}}\right) \left(\widetilde{L}/\epsilon\right)^{\frac{1}{2}}.
$$

By (i) of Lemma 4.3, $2I - \left(\widetilde{L}/\epsilon\right)^{\frac{1}{2}} D \left(\widetilde{L}/\epsilon\right)^{\frac{1}{2}}$ is positive definite. Therefore, S is positive semidefinite. Furthermore, as in the proof of Lemma 4.4,

$$
\begin{aligned}
\operatorname{rank} \widetilde{S} &= \operatorname{rank}\left(2\left(\widetilde{L}/\epsilon\right) - \left(\widetilde{L}/\epsilon\right) D(\widetilde{L}/\epsilon)\right) \\
&= \operatorname{rank}\left(\widetilde{L}/\epsilon\right)^{\frac{1}{2}} \left(2I - \left(\widetilde{L}/\epsilon\right)^{\frac{1}{2}} D \left(\widetilde{L}/\epsilon\right)^{\frac{1}{2}}\right) \left(\widetilde{L}/\epsilon\right)^{\frac{1}{2}} \\
&= \operatorname{rank}\left(\widetilde{L}/\epsilon\right)^{\frac{1}{2}} \left(2I - \left(\widetilde{L}/\epsilon\right)^{\frac{1}{2}} D \left(\widetilde{L}/\epsilon\right)^{\frac{1}{2}}\right) \\
&= \operatorname{rank}\left(\widetilde{L}/\epsilon\right)^{\frac{1}{2}} \\
&= \operatorname{rank}\left(\widetilde{L}/\epsilon\right) \\
&= n - 1.
\end{aligned}
$$

Thus the claim is proved. Hence by Theorem 4.5, $F'(\epsilon) < 0$. Thus $F(\epsilon)$ is decreasing in $\epsilon > 0$ and the proof of (ii) is complete. □

5. Determinants of ℓ_1-distance matrices

We begin by recalling from the introduction that if $x_1, \ldots, x_n$ is a set of distinct points in $\mathbb{R}^2$, then the ℓ_1-*distance matrix* D for these n points is given by the $n \times n$ matrix $D = (d_{i,j})$ with $d_{i,i} = 0$; $i = 1, 2, \ldots, n$, and $d_{i,j} = \|x_i - x_j\|_1$, if $i \neq j$. Furthermore, it is convenient to keep in mind the rectangular grid to which the points $x_1, \ldots, x_n$ belong. For this purpose, we introduce the following notation. If $\sigma_1 < \cdots < \sigma_m$ and $\tau_1 < \cdots < \tau_k$, then we denote by $RG(\sigma_1, \ldots, \sigma_m; \tau_1, \ldots, \tau_k)$ the $m \times k$ rectangular grid $\{\sigma_1, \ldots, \sigma_m\} \times \{\tau_1, \ldots, \tau_k\}$. The notation RG will be used when the numbers are clear from the context.

Next, following Dyn, Light and Cheney [2], a path (this variation of a path is usually called a "lattice path") is a finite ordered set in RG, $[y_1, \ldots, y_r]$ such that the line segment joining consecutive points are of positive length and are alternately horizontal and vertical. Repetitions of points is permitted. (Strictly speaking, such a path should be called a "walk", but we continue to use the term "path" to keep the terminology consistent with [2].) The number r is then the length of the path. A path is said to be *closed* if r is even, if $y_r \neq y_1$ and if the line segment joining y_1 and y_r is perpendicular to the line segment joining y_r and y_{r-1}. We now prove a preliminary result.

Lemma 5.1. *Let $Z = X \cup Y$ be a subset of $RG = RG(\sigma_1, \ldots, \sigma_m; \tau_1, \ldots, \tau_k)$ containing $m + k$ points such that $Y = \{y_1, \ldots, y_r\}$ and $[y_1, \ldots, y_r]$ is a closed path. For $i = 1, 2, \ldots, r$; let $Z^i = Z \setminus \{y_i\}$, and let D^i be the ℓ_1-distance matrix of Z^i. Then for any $i, j \in \{1, \ldots, r\}$,* $\det(D^i) = \det(D^j)$.

Proof. It is easily verified that for any $u \in \mathbb{R}^2$, $\sum_{k=1}^{r}(-1)^{k-1}\|u - y_k\|_1 = 0$. Thus

$$\|u - y_1\|_1 = \sum_{k=2}^{r}(-1)^k\|u - y_k\|_1.$$

In D^1, add $(-1)^k$ times the column (respectively, row) corresponding to y_k to the column (respectively, row) corresponding to y_2, $k = 3, \dots, r$. The resulting matrix is clearly D^2. Since the determinant is unchanged by these operations, we conclude that $\det(D^1) = \det(D^2)$. We can similarly prove that $\det(D^i) = \det(D^{i+1})$, $i = 2, \dots, r-1$ and the proof is complete. □

A set of $m + k - 1$ points in RG not containing a closed path correspond to a basic feasible solution in a transportation problem, see, for example, [5]. We associate an $(m + k) \times mk$ matrix, denoted A_{RG}, with RG as follows. The columns of A_{RG} are indexed by $\{(i, j); i = 1, \dots, m; j = 1, \dots, k\}$. For each (i, j), the column corresponding to (i, j) has a 1 at the ith and the $(m + j)$-th places, and zeros elsewhere. Then it is well-known from the theory of the transportation problem, (see [5], Theorem 1, p. 477), that a set of $m + k - 1$ points in RG do not contain a closed path if and only if the corresponding columns of the matrix A_{RG} are linearly independent. This observation and elementary properties of independent subsets immediately lead to the following result.

Lemma 5.2. *Let $X = \{x_1, \dots, x_{m+k-1}\}$ and $Y = \{y_1, \dots, y_{m+k-1}\}$ be subsets of $RG = RG(\sigma_1, \dots, \sigma_m; \tau_1, \dots, \tau_k)$ that do not contain a closed path and let $y \in Y \setminus X$. Then there exists $x \in X \setminus Y$ such that $X \setminus \{x\} \cup \{y\}$ does not contain a closed path.*

The following is the main result of this section.

Theorem 5.3. *Let $x_1, \dots, x_{m+k-1}$ be a subset of $RG = RG(\sigma_1, \dots, \sigma_m; \tau_1, \dots, \tau_k)$ that does not contain a closed path and let D be the ℓ_1-distance matrix of $x_1, \dots, x_{m+k-1}$. Then*

$$\det(D) = (-1)^{m+k} 2^{m+k-3} (\sigma_m - \sigma_1 + \tau_k - \tau_1) \prod_{i=1}^{m-1} (\sigma_{i+1} - \sigma_i) \prod_{i=1}^{k-1} (\tau_{i+1} - \tau_i). \tag{5.1}$$

Proof. Let $X = \{x_1, \dots, x_{m+k-1}\}$ and $Y = \{y_1, \dots, y_{m+k-1}\}$ be subsets of $RG = RG(\sigma_1, \dots, \sigma_m; \tau_1, \dots, \tau_k)$ that do not contain a closed path and let D and $\widetilde{D}$ be their ℓ_1-distance matrices respectively. We claim that $\det(D) = \det\left(\widetilde{D}\right)$. Suppose $\det(D) \neq \det\left(\widetilde{D}\right)$ and we assume without loss of generality that, subject to this condition, $|X \cap Y|$ is the maximum possible. Let $y \in Y \setminus X$. Then by Lemma 5.2, there exists $x \in X \setminus Y$ such that $X_1 = X \setminus \{x\} \cup \{y\}$ does not contain a closed path. Let D' be the ℓ_1-distance matrix of X_1. By Lemma 5.1, $\det(D') = \det(D)$ and hence $\det(D') \neq \det\left(\widetilde{D}\right)$. However $|X_1 \cap Y| > |X \cap Y|$, which is a contradiction. Thus we conclude that $\det(D) = \det\left(\widetilde{D}\right)$ and the claim is proved. Now let Z be the set of $m + k - 1$ points $(\sigma_1, \tau_1), (\sigma_1, \tau_2), \dots, (\sigma_1, \tau_k), (\sigma_2, \tau_1), (\sigma_3, \tau_1), \dots, (\sigma_m, \tau_1)$ and let

D'' be the ℓ_1-distance matrix of Z. By the preceding conclusion, $\det(D) = \det(D'')$. Note that D'' is the distance matrix of a path with $m + k - 1$ vertices and with edge-weights $\sigma_2 - \sigma_1, \sigma_3 - \sigma_2, \ldots, \sigma_m - \sigma_{m-1}, \tau_2 - \tau_1, \tau_3 - \tau_2, \ldots, \tau_k - \tau_{k-1}$. By Corollary 2.5, $\det(D'')$ is given by (5.1) and the proof is complete. $\square$

For relevance of the ℓ_1-distance matrix considered in this section, in the context of numerical analysis, see [9,10].

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