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Old and new results on algebraic connectivity of graphs

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Abstract

This paper is a survey of the second smallest eigenvalue of the Laplacian of a graph G, best-known as *the algebraic connectivity of* G, denoted $a(G)$. Emphasis is given on classifications of bounds to algebraic connectivity as a function of other graph invariants, as well as the applications of *Fiedler vectors* (eigenvectors related to $a(G)$) on trees, on hard problems in graphs and also on the combinatorial optimization problems. Besides, limit points to $a(G)$ and characterizations of extremal graphs to $a(G)$ are described, especially those for which the algebraic connectivity is equal to the vertex connectivity.

AMS classification: 05C50

Keywords: Laplacian of graph; Algebraic connectivity; Vertex and edge connectivities; Bounds for the algebraic connectivity; Fiedler vectors; Limit points; Extremal graphs; Laplacian integral graphs

1. Introduction

The Laplacian matrix of a graph and its eigenvalues can be used in various areas of mathematics, mainly discrete mathematics and combinatorial optimization, with interpretation in several physical and chemical problems. The adjacency matrix and its eigenvalues have been investigated more than the Laplacian matrix, see [6,14,17,18], and according to Mohar [64] the Laplacian eigenvalues are more intuitive and much more important than the spectrum of the adjacency matrix.

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Among all eigenvalues of the Laplacian of a graph, one of the most popular is the second smallest, called by Fiedler [25], the algebraic connectivity of a graph. Its importance is due to the fact that it is a good parameter to measure, to a certain extent, how well a graph is connected. For example, it is well-known that a graph is connected if and only if its algebraic connectivity is different from zero.

Recently, the algebraic connectivity has received much more attention, see [11,25,30,34,59–61,64,65] for surveys and books; [31,33,48,55,58,68,71] for application on trees; [5,26,27,21,24,34,63,64,67] for applications on hard problems in graph theory: the expanding properties of graphs, weighted graphs, absolute algebraic connectivity, isoperimetric number, genus and other invariants of a graph; [41,72] for the study of the asymptotic behavior of algebraic connectivity for random graphs; [4,13,26,28,44,54,63,64] for applications on combinatorial optimization problems: the problem of certain flowing process, the maximum cut problem and the traveling salesman problem.

Besides, the algebraic connectivity is of relevance to the theory of elasticity [72], to the correspondence between continuous and discrete mathematics (an inequality for continuous analogue of the algebraic connectivity in compact Riemannian manifolds was obtained by Cheeger [10]), and to an investigation of a bandwidth-type problem using the spectral parameter by Mohar [64]. Also, Ghosh and Boyd [29] describe a method to maximize the second smallest eigenvalue over the convex hull of the Laplacian of graphs in a particular family which is a convex optimization problem. The eigenvectors corresponding to the algebraic connectivity, called Fiedler vectors, are also of interest. Motivated by Fiedler [26], these eigenvectors have received a lot of attention recently, see [21,48,49]. According to Barnad et al., as cited by Merris [62], Fiedler vectors are used in algorithms for distributed memory parallel processors, as well the algebraic connectivity and Fiedler vectors for trees have been studied extensively [31,33,48].

More recently, the limit points of Laplacian spectra, and so of the algebraic connectivity, have been a subject of interest [37,45,69]. Finally, at the beginning of this millenium, the characterization of extremal graphs that satisfy certain maximal and minimal invariants seems to place a great emphasis on graph theory research. Hence, this could not be different with respect to the algebraic connectivity, as described in [4,22,47,49].

In this paper, new results are treated with greater attention and results are considered as *new* if they do not appear either in the classic article by Fiedler [25] or in one the following surveys: Mohar [65] and Merris [61].

This survey is developed through the following sections: Section 2 relates two classic theorems, Matrix-tree and Courant-Fisher, from which have come the concept and properties of $a(G)$. Section 3 describes expressions for $a(G)$ where G is a graph obtained from graph operations, and the algebraic connectivity of graphs in special families. In Section 4, a classification of bounds for $a(G)$ in terms of other invariants is given; Section 5 approaches applications of algebraic connectivity to ordering trees, to expanding graphs and $a(G)$-variations. Also, it describes applications of Fiedler vectors to combinatorial optimization problems; Section 6 relates the limit points and extremal graphs to $a(G)$. Characterizations of graphs for which the algebraic connectivity is equal to the vertex connectivity are the subject of Section 7.

2. Algebraic connectivity of simple and weighted graphs

The Laplacian matrix $L(G)$ is frequently used to enumerate spanning trees of a graph G, according to one of the oldest theorems in Graph Theory, Theorem 2.1, whose proof can be found in Biggs, [6].

Theorem 2.1 (Matrix-Tree Theorem). *Let u and v be vertices of a graph G, and let $L(G)(u, v)$ be the submatrix obtained from $L(G)$ by deleting row u and column v. Let $\mu(T)$ be the number of spanning trees of G. Then,* $|\det L(G)_{(u,v)}| = \mu(T)$.

The spectrum of the Laplacian of G is the sequence of its eigenvalues $S(L(G)) = (\lambda_1, \ldots, \lambda_{n-1}, \lambda_n)$, given in non-increasing order, $\lambda_1 \geqslant \cdots \geqslant \lambda_{n-1} \geqslant \lambda_n$. Since $L(G)$ is a semidefinite positive singular matrix, then $\lambda_n = 0$ and as an immediate consequence of the Matrix-Tree Theorem $\lambda_{n-1} = 0$ if and only if G is a disconnected graph. It is possible to obtain the same result by the well known Perron–Frobenius Theorem, [17,30], applied to the matrix $(n-1)I - L(G)$, where I is the identity matrix. Because of that, Fiedler [25] called λ_{n-1} the algebraic connectivity of G, denoted $a(G) = \lambda_{n-1}$. It is related to several important graph invariants and it has been extensively investigated. Most of the results related to $a(G)$ with other invariants of graphs are consequences of the well-known Courant-Fisher Theorem [25].

Theorem 2.2. *Let* **0** *be the null vector and* **1** *be the vector such that each of its coordinates is equal to* 1. *Since* **1** *is an eigenvector for* $\lambda_n = 0$, *then,*

$$a(G) = \min_{v \neq \mathbf{0}, v \perp \mathbf{1}} \frac{\langle L(G)v, v\rangle}{\langle v, v\rangle}.$$

Fiedler [26] extended this result to a weighted graph with non-negative values on the edges. The generalized Laplacian matrix $A_C(G)$ of a weighted graph G, where C is its cost matrix, for which $c_{ij} = c_{ji}$ is the weight (or the cost) of the edge (v_i, v_j), is

$$A_C(G) = \begin{cases} -c_{ij} & i \neq j, (i,j) \in E, \\ 0 & i \neq j, (i,j) \notin E, \\ \sum_{(i,j)\in E} c_{ij} & i \neq j. \end{cases}$$

Another way to describe $A_C(G)$ is by means of its quadratic form:

$$\langle A_C(G)X, X\rangle = \sum_{(i,j)\in E} c_{ij}(x_i - x_j)^2,$$

where $X = (x_1, x_2, \ldots, x_n)$ and the sum is over the pairs $i < j$ for which $\{v_i, v_j\}$ is an edge of G. See, [60].

The second smallest eigenvalue $a_C(G)$ will be analogously called algebraic connectivity of G, because of the following proposition:

Proposition 2.1. *Let* $G = (V, E)$ *be a connected graph valuated with positive weights* c_{ij}. *Then the algebraic connectivity of G is positive and equal to the minimum of the function*

$$\varphi(x) = n\frac{\sum_{(i,j)\in E} c_{ij}(x_i - x_j)^2}{\sum_{(i,j)\in E, i<j} (x_i - x_j)^2},$$

over all non-constant n-tuples $x = (x_i)$.

In Fiedler [27] we can find the definition of the *absolute algebraic connectivity* of a weighted graph $G = (V, E)$ as the maximum of algebraic connectivities for all non-negative valuations of G whose average value on the edges of G is one. The *absolute algebraic connectivity* $\hat{a}(G)$ is formally defined as

$$\hat{a}(G) = \max_{C \in \mathscr{C}(G)} a(G_C), \tag{2.1}$$

over the set $\mathscr{C}(G)$ of all non-negative valuations of G such that

$$\sum_{(i,k)\in E} c_{ik} = |E|. \tag{2.2}$$

Since the (0, 1)-valuation satisfies (2.2) and thus belongs to $\mathscr{C}(G)$, we have

$$a(G) \leqslant \hat{a}(G). \tag{2.3}$$

In his paper [27], Fiedler introduced the notion of the *absolute center of gravity* of G_m, when G_m is a metric space assigned to G as every point M in G_m for which a function

$$S(X) = \sum_{k\in V} d^2(X, k) \tag{2.4}$$

attains to minimum. The distance $d(X, k)$, where X is a point on the edge (i, k) can be determined by its distances x_1 and x_2 from i and k which $x_1 > 0$, $x_2 > 0$, $x_1 + x_2 = 1$. The distance between two vertices in V is defined as usual.

For variance of G, we call the following number:

$$\nu(G) = \frac{1}{n-1} \min_{X\in G_m} S(X), \tag{2.5}$$

where n is the number of vertices of G.

Based on the notions, Fiedler, in the same paper, solves completely the problem of finding $\hat{a}(T)$ where T is a tree. He proves that the absolute algebraic connectivity of a tree T is the reciprocal of the variance of T:

$$\hat{a}(T) = \frac{1}{\nu(T)}. \tag{2.6}$$

Since of the introduction of the absolute algebraic connectivity and its characterization for trees, the only one result found in the literature is due to Kirkland and Pati [50]. They present an upper bound on $\hat{a}(G)$ as a function of n and the vertex connectivity of G. See [50] for more details.

3. Algebraic connectivity of graphs obtained from operations

The use of operations on graphs in order to determine invariants has been a useful technique. The same technique is applied to the algebraic connectivity. The unary and binary, or more general, k-ary operations which are considered here are: the *complement* of G; the *removal of an edge* from G; the *removal of some vertices* from G and the *addition of an edge* to G as unary operations, and an *edge-union* of graphs; the *cartesian product* of graphs; the *decomposition of vertex set* and the *directed sum* of graphs as binary or k-ary operations. The union and join operations are defined later. All these operations are well-known and their definitions, as used here, can be found in [6,30,39,36]. Table 3.1 displays, in the left column, the unary and binary operations of graphs G, G_1 and G_2 and, the relations between their algebraic connectivities are in the right column.

The are several popular graphs for which their algebraic connectivity is known. The first one, in Theorem 3.1, gives the algebraic connectivity when G is a k-regular graph, [6]. Since $a(G)$ is determined as a function of the second smallest eigenvalue of the adjacency matrix of G, Theorem 3.1 provides a bridge between the eigenvalues of the Laplacian of G and those of its adjacency matrix, known as the *eigenvalues of graph.*

Theorem 3.1. *Let G be a k-regular graph and $\theta_{(n-1)}$ be the second smallest eigenvalue of G. Then $a(G) = k - \theta_{(n-1)}$.*

Table 3.1
Relations between $a(G)$ and $a(G_i)$, where G_i is a component of some operation

Operations	Relations of $a(G)$ and $a(G_i), i = 1, 2$
The complementary graph $\overline{G}$ of G	$a(\overline{G}) = n - \lambda_1$
G_1 obtained by removal of an edge from G	$a(G_1) \leqslant a(G)$
G_1 obtained by removal of k vertices from G	$a(G) \leqslant a(G_1) + k$
G_1 obtained by adding an edge of G	$a(G) \leqslant a(G_1) \leqslant a(G) + 2$
Edge-union of $G = G_1 \cup G_2$	$a(G_1) + a(G_2) = a(G)$
Cartesian product $G = G_1 \times G_2$	$a(G) = \min\{a(G_1); a(G_2)\}$
G_1 and G_2 obtained from $V(G)$-decomposition	$a(G) \leqslant \min\{a(G_1) + \|V_2\|; a(G_2) + \|V_1\|\}$
Direct sum $G = G_1 \bigoplus G_2$	$a(G_1) + a(G_2) \leqslant a(G_1 \bigoplus G_2)$

Table 3.2
Graphs for which $a(G)$ are known

Graph G	Algebraic connectivity, $a(G)$
Complete graph	$a(K_n) = n$
Path	$a(P_n) = 2(1 - \cos \frac{\pi}{n})$
Cycle	$a(C_n) = 2(1 - \cos \frac{2\pi}{n})$
Bipartite complete graph	$a(K_{p,q}) = \min(p, q)$
Star $K_{1,q}, q > 1$	$a(K_{1,q}) = 1$
Cube m-dimension	$a(Cb_m) = 2$
Petersen Graph	$a(P) = 2$

Using the theorem above we can obtain the exact expressions for the algebraic connectivity of complete graphs, cycles, bipartite complete graphs, cube graphs, Petersen graph and others. Some of these values can be seen in Table 3.2.

Concerning trees we can give a result due to Grone et al. [34].

Theorem 3.2. *If $T \neq K_{1,n-1}$ is a tree on $n \geqslant 6$ vertices then $a(T) < 0.49$.*

From this result, we conclude that, for every tree T, except for K_2 where $a(K_2) = 2$, $a(T)$ belonging to interval (0, 1]. Besides, there is a large gap between the algebraic connectivity of a star equal to 1 and that of any other tree on $n \geqslant 6$ vertices which is unable to reach 0.49, see Table 3.2 and Theorem 3.2.

4. Bounds to algebraic connectivity

There are several bounds to the algebraic connectivity related to other parameters of a graph. So, we group them in distinct classes: The first one $B_1 = \{n, m, \delta(G), \Delta(G), \kappa(G), e(G)\}$ contains the following basic invariants of G: the vertex number n, the edge number m; the minimal degree $\delta(G)$, the maximal degree $\Delta(G)$, the vertex connectivity $\kappa(G)$ and the edge connectivity $e(G)$; the second class $B_2 = \{\text{diam}(G), \alpha(G), g(G)\}$ contains the diameter $\text{diam}(G)$, the maximal independence number $\alpha(G)$, and the genus $g(G)$ of G. Finally, some non-usual parameters, which will be defined later, belong to the last class $B_3 = \{m(G), \mu(v), \rho(S)\}$. They are, respectively, $m(G)$, the mean distance of G; $\mu(v)$, the average of the degrees of all vertices of G adjacent to $v \in V$ and, $\rho(S)$, the edge density of subsets $S \subseteq V$. Tables 4.1–4.3 display the bounds of $a(G)$

Table 4.1
Bounds to $a(G)$ related with basic invariants in B_1

Authors	Bounds of $a(G)$
(i) Fiedler [25]	For $G \neq K_n$, $a(G) \leqslant n-2$
(ii) Fiedler [25]	$2\delta(G) - n + 2 \leqslant a(G) \leqslant \frac{n}{n-1}\delta(G)$
(iii) Fiedler [25]	$a(G) \leqslant \kappa(G) \leqslant e(G) \leqslant \delta(G)$
(iv) Fiedler [25]	$2e(G)(1-\cos\frac{\pi}{n}) \leqslant a(G)$
(v) Fiedler [25]	$2(\cos(\frac{\pi}{n} - \cos 2\frac{\pi}{n}))\kappa(G) - 2\cos\frac{\pi}{n}(1-\cos\frac{\pi}{n})\Delta(G) \leqslant a(G)$
(vi) Belhaiza et al. [4]	For $G \neq K_n$, $a(G) \leqslant \lfloor -1 + \sqrt{1+2m} \rfloor$

as functions of invariants in B_1, B_2 and B_3, respectively. In each table, the references are listed in the left column while their expressions are shown in the right.

In Table 4.1, while we owe to Fiedler [25] all the first bounds to algebraic connectivity, (i)–(v), which are related to basic invariants, the last bound, (vi), was obtained by Belhaiza et al. [4] with the help of a computer system. It is worth pointing out that computers are increasingly being used in Graph Theory in order to: (i) determine the numerical value of graph invariants; (ii) enumerate graphs, taking into account several constraints and exploiting symmetry, see McKay [56,57]; (iii) generate graphs, display, modify and study many parameters through interactive approach. The pioneer system of graphs in the last category was developed by Cvetković and Kraus [16], Cvetković and Simić [19] and was followed by *Grafitti* by Fajtlowicz and DeLaVina, see Larson [51]. A new generation of systems is *NewGraph System* developed by Stevanović et al. [73] and *AutoGraphiX* (*AGX*) *system* developed at GERAD, Montréal since 1997 [8,9,4]. This was the system used by Belhaiza et al. [4] to get the last bound on Table 4.1 which is sharp for all $m \geqslant 2$.

Table 4.2 displays the bounds on algebraic connectivity related to *diameter*, *maximal independence number* and *genus* of G. For definitions, see [20,30]. In this table, the lower-bound (i) and the upper-bound (v) are some of the most classic for $a(G)$. This last one was obtained by McKay and proved by Mohar [64] who also improved the bound (ii) obtaining the result (vii). The Boshier's bound leads us to see that large graphs of bounded genus and with bounded maximal degree have small algebraic connectivity. In order to prove (x), Molitierno [67] used the concept of the isoperimetric number. He also applies a result concerning the determination of graphs G_1 and G_2 such that G is a join of them, $G = G_1 \vee G_2$, given by Kirkiland et al. [47]. The join operation and the isoperimetric number will be defined later.

The next theorem due to Grone et al. [34] give a special lower-bound for a tree as a function of its diameter.

Theorem 4.1. *If T is a tree with diameter* $\mathrm{diam}(T)$ *then*

$$a(T) \leqslant 2\left(1 - \cos\left(\frac{\pi}{\mathrm{diam}(T)+1}\right)\right).$$

A rooted Bethe tree $\mathscr{B}_{d,k}$ is an unweighted rooted tree of k levels in which the vertex root has degree d, the vertices in level 2 to level $(k-1)$ have degree $(d+1)$ and the vertices in level k are pendant. If $d = 2$, $\mathscr{B}_{2,k}$ is a balanced binary tree of k levels. Molitierno et al. [68] obtained

Table 4.2
Bounds on $a(G)$ related to invariants in B_2

Authors	Bounds of $a(G)$
(i) Fiedler [25]	$a(G) \leqslant n - \alpha(G)$
(ii) Alon and Milman [2]	$\mathrm{diam}(G) \leqslant \sqrt{\frac{2\Delta(G)}{a(G)}} \log_2 n^2$
(iii) Boshier [7]	If g is a genus of G such that $n > 18(g+2)2$ then $a(G) \leqslant \frac{(6g+2)\Delta(G)}{\sqrt{\frac{n}{2}-3(g+2)}}$
(iv) Chung et al. [12]	$\mathrm{diam}(G) \leqslant \left\lceil \frac{\log(n-1)}{\log \frac{\lambda_1+a(G)}{\lambda_1-a(G)}} \ln(n-1) \right\rceil$
(v) Mohar [64]	$\frac{4}{\mathrm{diam}(G)n} \leqslant a(G)$
(vi) Mohar [64]	$\mathrm{diam}(G) \leqslant \lceil \frac{\Delta(G)+a(G)}{4a(G)} \ln(n-1) \rceil$
(vii) Mohar [64]	If $\beta \in \mathbb{R}$ and $\beta > 1$, $\mathrm{diam}(G) \leqslant \left\lceil \sqrt{\frac{\lambda_1}{a(G)}} \cdot \sqrt{\frac{\beta^2-1}{4\beta}} + 1 \right\rceil \cdot \left\lceil \log_\beta \frac{n}{2} \right\rceil$
(viii) Chang [11]	$a(G) \leqslant 1 - \frac{\sqrt{\Delta(G)-1}}{\Delta(G)}\left(1 - \frac{2}{\mathrm{diam}(G)}\right) + \frac{2}{\mathrm{diam}(G)}$
(ix) Molitierno [67]	Let $k \geqslant 1$ be the genus of G and let $p_k = \left\lfloor \frac{7+\sqrt{1+48k}}{2} \right\rfloor$ If $G = K_t$ such that t is the largest number of vertices that can be embedded on surface S_k, then $a(G) = p_k$ Otherwise, $a(G) \leqslant p_k - 1$
(x) Molitierno [67]	If G is a planar graph (the genus $k = 0$), then $a(G) \leqslant 4$ Moreover, equality holds if and only if $G = K_4$ or $G = K_{2,2,2}$
(xi) Molitierno [67]	For some positive integer c, if any of the equalities holds: $k = c(12c-1)$; $k = c(12c+1)$; $k = (4c-1)(3c-1)$ and $k = (4c+1)(3c+1)$, then $a(G) < p_k$ for all non-complete graphs G of genus k

quite tight upper and lower bounds on the algebraic connectivity of $\mathscr{B}_{2,k}$. These bounds are in (4.1) and (4.2) as follows:

$$a(\mathscr{B}_{2,k}) \leqslant \frac{1}{(2^k - 2k + 3) - \frac{2k-2}{2^{k-1}-1}} \tag{4.1}$$

and

$$a(\mathscr{B}_{2,k}) \geqslant \frac{1}{(2^k - 2k + 2) - \frac{2k-\sqrt{2}(2k-1-2^{k-1})}{2^k-1-\sqrt{2(2^{k-1}-1)}} + \frac{1}{3-2\sqrt{2}\cos(\frac{\pi}{2k-1})}}. \tag{4.2}$$

Rojo and Medina [71] obtain quite tight upper and lower bounds on the algebraic connectivity of the general case of Bethe tree according to the respective inequations (4.3) and (4.4):

$$a(\mathscr{B}_{d,k}) \leqslant \frac{(d-1)^2}{d^k - (2k-2)d + (2k-1) - \frac{(2k-2)(d-1)}{d^{k-1}-1}} \tag{4.3}$$

Table 4.3
Bounds to $a(G)$ related to invariants in B_3

Authors	Bounds of $a(G)$
(i) Mohar [63]	$\frac{2}{a(G)} + \frac{n-2}{2} \leqslant (n-1)\mu(G) \leqslant \frac{n}{n-1}\lceil\frac{\Delta(G)+a(G)}{4a(G)\ln(n-1)}\rceil$
(ii) Mohar [65]	Let $\rho(X)$ be the density of $X \subseteq V$ and E_X the edge cut-set related to X Then $a(G) \leqslant \min_{X\subseteq V} \rho(X)$, where E_X is an edge-cut set of G related to X
(iii) Merris [62]	Let $b(G) = \max_{v\in V}\{m(G) + d(v)\}$. Then $n - b(\overline{G}) \leqslant a(G)$

and

$$a(\mathscr{B}_{d,k}) \geqslant \frac{1}{\frac{1}{(d-1)^2}(d^k - (2k-1)(d-1) - \sqrt{d} + \frac{(d-1)(2k-1)}{d^{k-\frac{1}{2}}+1}) + \frac{1}{(d+1)-2\sqrt{d}\cos\frac{\pi}{2k-1}}}. \tag{4.4}$$

The tree $\mathscr{B}_{d,k_1,k_2}$ is obtained by the union of two Bethe trees $\mathscr{B}_{d,k_1}$ and $\mathscr{B}_{d,k_2}$ having a common vertex root. The last authors prove that $a(\mathscr{B}_{d,k_1,k_2}) = a(\mathscr{B}_{d,k})$. Consequently, the bounds given by (4.3) and (4.4) also are bounds to $a(\mathscr{B}_{d,k_1,k_2})$.

Let us consider two conjectures generated by Grafitti system: *Wow Conjecture* 584 and *Wow Conjecture* 636. Both of them are proved by Zhang [74]. The first one gives an upper-bound for the algebraic connectivity of the complement of a tree as a function of its maximal independence number and the other refers to those parameters where G is a non-trivial graph.

Theorem 4.2. *Let T be a tree of order n and $\alpha(T)$ its independence number. If $\overline{T}$ is the complement of T, then $a(\overline{T}) > n - 2(\alpha(T))$.*

Theorem 4.3. *Let G be a non-trivial graph of order n and $\overline{G}$ be its complement. Then $\alpha(\overline{G}) \geqslant \frac{n}{n-a(\overline{G})}$ with equality if and only if α is a factor of n and G has α components equal to $K_{\frac{n}{\alpha}}$.*

Table 4.3 displays the bounds of $a(G)$ related to invariants in B_3. The *mean distance*, $m(G)$, is the average of all distances between distinct vertices of G; $\mu(v)$ is the average of the degrees of the adjacent vertices to v, for each $v \in V$ and, $\rho(S)$ is the edge density of S, for $S \subseteq V$. Finally, for S a subset of vertices and E_S the edge cut-set of G related to S, the *edge density* of S is given by

$$\rho(S) = \frac{|V||E_S|}{|S||V-S|}.$$

Since each expression in Table 4.3 depends on the determination of vertex subsets of a graph, then it is hard to calculate.

Closing this section, it is interesting to mention an upper bound on the algebraic connectivity of graphs with many cutpoints. Let G be a graph with k cutpoints. Kirkland [43] builds an upper bound on $a(G)$ for the case that $1 \leqslant k \leqslant \frac{n}{2}$ and characterizes graphs which attain that bound. An year later, he [44] completes his investigation with a tight upper bound on $a(G)$ in terms of n and k where $k > \frac{n}{2}$. The technique used in his more recent paper relies on the analysis of the various connected components which arise from the deletion of a cutpoint v and all edges incident in it. For more details see [43,44].

5. Some applications of algebraic connectivity

There are several problems where we can apply $a(G)$. Among them we describe something about ordering trees, isoperimetric or conductance number, expanding graphs, the use of Fiedler vectors on combinatorial optimization problems and $a(G)$-variations.

Ordering trees: Cvetković et al. [15] outlined some approaches to ordering graphs by their spectra. Grone and Merris [31,33,58] found several results defining distinct ordering of trees. We chose one to show the technique they used in order to compare a pair of trees according to $a(G)$. Let $1 \leqslant s \leqslant t \leqslant n-2$. A tree is called a $T(s,t)$-tree when it has $n = s+t+2$ vertices and exactly two adjacent and non-pendant vertices u and v such that they are connected to s and t pendant vertices, respectively. Proposition 5.1, proved by Grone and Merris [33], leads to a partial order in the class of $T(s,t)$-trees.

Proposition 5.1. *Let $n = s+t+2$ be fixed and $L(T(s,t))$ be the Laplacian matrix of a $T(s,t)$-tree with algebraic connectivity $a(T(s,t))$. The unique eigenvalue of $L(T(s,t))$ less than unity is exactly $a(T(s,t))$. Furthermore, for every s, $1 \leqslant s \leqslant \frac{n-2}{2}$, $a(T(s,t))$ is a strictly decreasing function of s.*

Figs. 1 and 2 display two $T(s,t)$-trees. The first one has $s = 3$ and the second, $s = 5$. According to Proposition 5.1, $a(T(3,8)) > a(T(5,6))$. In fact, we have $a(T(3,8)) = 0.27706$ and $a(T(5,6)) = 0.25361$, [33].

Although several results obtained by Grone and Merris [33] provided distinct partial orders by $a(T)$ on different subclasses of trees, their results are not enough to give us a total ordering on the set of all trees. For example, the tree T_7, Fig. 3, has $a(T_7) = 0.296$ and another, T_8, Fig. 4, has $a(T_8) = 0.268$. Since the results from Grone and Merris are not enough to justify the inequality $a(T_8) < a(T_7)$, finding a total ordering of trees by $a(G)$ is still an open problem.

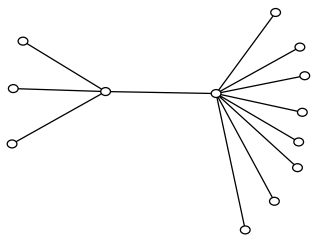

Fig. 1. $T(3,8)$ is the $T(s,t)$-tree, when $s = 3$ and $t = 8$.

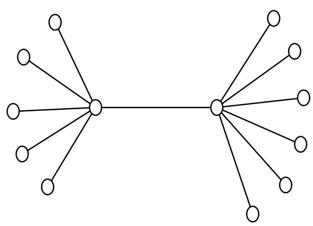

Fig. 2. $T(5,6)$ is the $T(s,t)$-tree, when $s = 5$ and $t = 6$.

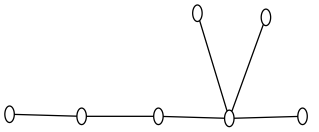

Fig. 3. T_7 with $a(T) = 0.296$.

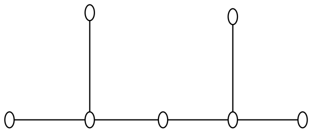

Fig. 4. Tree T_8 with $\mathrm{diam}(T_8) = 4$ and $a(T_8) = 0.268$.

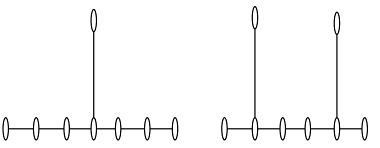

Fig. 5. T with $\mathrm{diam}(T) = 6$ and $a(T) = 0.198$, on the left, versus T' with $\mathrm{diam}(T') = 5$ and $a(T') = 0.186$, on the right.

Until 1990, it was thought that $a(T)$ *decreases as the diameter increases*. Grone and Merris [33] gave some counter-examples for this conjecture in their paper [33]. We chose one of them to display here. See T and T' on Fig. 5.

The isoperimetric number, edge-cutset and max cut problem: Let $G = (V, E)$ and $S \subseteq V$. We have $\delta S = \{(u, v) \in E/u \in S$ and $v \in V - S\}$ as an *edge-cutset* of G. This set can be useful to a network problem when the network is modelled by a graph. In this case, it is necessary to find a minimal δS. The parameter *isoperimetric number*, also known as the *conductance number*, is based on the cardinalities of S and δS:

$$i(G) = \min_{S \subseteq V} \frac{|\delta S|}{|S|}, \quad |S \leqslant 2.$$

Since the idea of a link between all vertices of a graph is shared with the algebraic connectivity and the isoperimetric number we can hope that there are some bounds related to them. Table 5.1 shows several inequalities which involve these invariants.

To close this part, we present an upper bound to the algebraic connectivity related to the cardinality of the largest component of a subgraph of G obtained from the removal a vertex cutset of G. This result was proved by Merris and Grone [32].

Theorem 5.1. *Let S be a vertex cutset of G such that the largest component of $G - S$ has order k. Then, $a(G) \leqslant k + 1$.*

The Max-Cut Problem, MCP, is a well-known NP-hard problem belonging to combinatorial optimization and, so, it is very important to have bounds to it. As a decision problem MCP can be defined as: *Given a graph $G = (V, E)$ and an integer k, is there a partition of V into V_1 and V_2 such that there are at least k edges in E between V_1 and V_2*. The algebraic connectivity was

Table 5.1
Bounds to $a(G)$ related to conductance number

Authors	Bounds of $a(G)$
(i) Mohar [63]	$\frac{a(G)}{2} \leqslant i(G)$
(ii) Mohar [65]	$i(G) \leqslant \sqrt{a(G)(2\Delta(G) - a(G))}$
(iii) Berman and Zhang [5]	$a(G) \leqslant \Delta(G) - \sqrt{\Delta(G)^2 - i(G)^2}$
(iv) Alon and Milan [2]	$a(G) \leqslant \frac{n\|\delta S\|}{(n-\|S\|)\|S\|}$

used by Mohar and Poljak [66], in order to obtain an upper-bound to MCP according to Theorem 5.2. See [65,70].

Theorem 5.2. *Let $MC(G)$ be the number of edges in maximal cut in G and $\overline{G}$ be the complement of G. Then, $MC(G) \leqslant \frac{n^2 - na(\overline{G})}{4}$.*

Fiedler vectors, edge-cutset and bandwith problem: Fiedler vectors, also known as *characteristic valuations*, are eigenvectors of $L(G)$ with respect to $a(G)$. Let Y be a Fiedler vector and u a vertex of G. We say that u is a *characteristic vertex* of G if and only if $Y(u) = 0$ and if there is a $w \in V$, w adjacent to u such that $Y(w) \neq 0$. An edge $\{u, w\}$ is a *characteristic edge* of G when $Y(u) \cdot Y(w) < 0$. The collection of all characteristic vertices and characteristic edges of G at Y is called the *characteristic set of G related to Y* and denoted S_Y. A *Perron-branch at S_Y* is a connected component of $G - S_Y$ with the smallest eigenvalue of the corresponding principal submatrix of $L(G)$ at most $a(G)$. G_Y is the *valuated connected graph G by Y* where the coordinates of a Fiedler vector Y are assigned to edges of G. Fiedler [26] shows that the characteristic valuations are able to determine particular edge-cutsets of a graph constituted by only two connected components. See Theorem 5.3.

Theorem 5.3. *Let G_Y be a valuated connected graph by Y where $Y = (y_i)$, $1 \leqslant i \leqslant n$, is a Fiedler vector of G with $1, 2, \ldots, n$ vertices. If $y_i \neq 0$, $1 \leqslant i \leqslant n$, then the set of all edges (i, j) for which $y_i \cdot y_k < 0$ forms a cut of G with only two blocks.*

The weights of vertices in a graph G in Fig. 6 are the coordinates of a Fiedler vector $Y = (-0.205, -0.761, -0.001, -0.081, 0.354, 0.287, 0.407)$. Each coordinate y_i is related to the vertex i in the graph above and on the left. Each edge of the edge cutset, which is below and on the left, has a negative value. According to Theorem 5.3, the graph on the right obtained by removing the edges of the cutset has just two blocks.

Juvan and Mohar [42] discussed how Fiedler vectors can be used in the optimal labelling problems known as the bandwidth problems. It is required to arrange the vertices of a graph G in a linear order $v_1, v_2, \ldots, v_n$ in a such a way that the edges will not have too long jumps, that is, if (v_i, v_j) is an edge of G then $|i - j|$ should be small. A reasonably good ordering is obtained by ordering the vertices of G related to the coordinates of a Fiedler vector Y:

$$v \leqslant u \iff y_v \leqslant y_u, \quad u, v \in V.$$

The algebraic connectivity is useful for this problem since the average square of jumps for any linear ordering of vertices of G obeys the inequality given by the Proposition 5.2 by Juvan and Mohar [42].

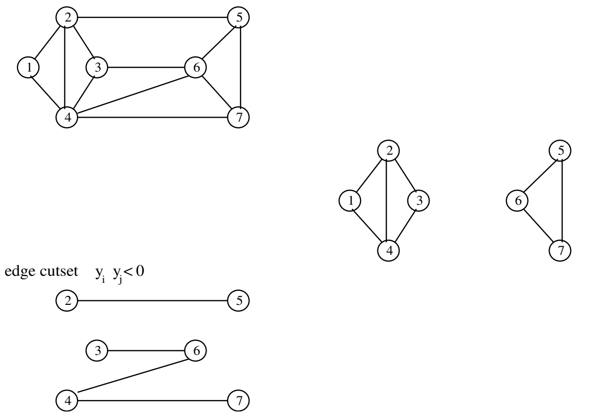

Fig. 6. An instance for Theorem 4.3.

Proposition 5.2. *Let G be a graph and $e = (v_i, v_j) \in E$ an edge of G. If* $\text{jump}(e) = |i - j|$ *then* $\sum_{e\in E} (\text{jump}(e))^2 \geqslant a(G)\frac{n(n^2-1)}{12}$.

Algebraic connectivity variations: Kirkland [45] characterizes all graphs in which spectral integral variation occurs in two places by adding an edge. Also, he discusses the case where one of the changed eigenvalues is just $a(G)$. Barik and Pati [3] posed the problem of characterizing graphs in which spectral variation occurs in one place by adding an edge where the changed eigenvalue is exactly $a(G)$ which increases by 2. Their results are based on the Fiedler vector of G and Perron-branches of G at S, the characteristic set with respect to Y.

The next theorem, see [3], characterizes a small family of graphs for which the multiplicity of $a(G)$ is equal to one and the spectral variation occurs in only one place.

Theorem 5.4. *Let G be a connected graph and i and j be two non-adjacent vertices in G. Let $a(G)$ have multiplicity one. The spectral variation occurs in one place by adding the edge (i, j) where the changed eigenvalue is $a(G)$ if and only if $G = K_n - \{(i, j)\}$.*

For example, let $G = K_4 - e$ be a complete graph without an edge e. Its spectrum is $S(G) = \{4, 4, 2, 0\}$, where the multiplicity of $a(G) = 2$ is one. It is well-known that the spectrum of K_4 is $\{4, 4, 4, 0\}$ and, consequently, $a(G) = 4$ and only one eigenvalue was changed.

Barik and Pati [3] also considered the case where the multiplicity of $a(G)$ is equal 2 and spectral integral variation of G occurs in one place by adding an edge between i and j where again the changed eigenvalue is algebraic connectivity.

Theorem 5.5. *Let G be a connected graph and i and j be two non-adjacent vertices in G. The spectral integral variation of G occurs in one place by adding the edge (i, j) where $a(G)$ is the changed eigenvalue if and only if $G = G^* \cup (G_1 + \{i\} + \{j\})$, where G^* is a graph of order k, $1 \leqslant k \leqslant n - 2$, while $a(G^*) = (2 - k - n)$ and G_1 is any graph on $(n - k - 2)$ vertices.*

Expansion properties of graphs: According to Alon [1] the Laplacian spectrum of a graph, in particular the algebraic connectivity, appears more naturally in the study of expanding properties of graphs than eigenvalues of adjacency matrix. In general,*expanders* are graphs with certain high connectivity properties and they can be constructed from graphs known as *c-magnifiers,* which are highly connected. They are used in the construction of networks. For more details, see [1]. The proposition below gives a necessary condition for a graph to be a c-magnifier [1]. For this, the algebraic connectivity of graph has an upper-bound as a function of c. On the other hand, when c is given a specific function of $a(G)$ then G is a c-magnifier.

Proposition 5.3. *Let G be a graph. If G is a c-magnifier then* $a(G) \geqslant \frac{2c^2}{4+2c^2}$. *Conversely, if* $c = \frac{2a(G)}{\Delta(G)+2a(G)}$ *then G is a c-magnifier.*

6. Limit points and extremal graphs to $a(G)$

In this section we present two directions of research. One concerns the limits to certain parameters of graphs sequenced by the cardinalities of their vertex sets. The other concerns the extremal graphs which satisfy certain properties or constraints with respect to graph invariants.

The limit points for $a(G)$: The study of the limit points of the eigenvalues of graphs was initiated by Hoffman [40], followed by Kirkland [45], and then Guo [37] who gives a formal definition to it with respect to $a(G)$: *A real number r is a limit point for the algebraic connectivity if there is a sequence of graphs* $(G_n)_{n\in\mathbb{N}}$ *such that the sequence of their respective algebraic connectivities* $(a(G_n))_{n\in\mathbb{N}}$ *converges to r, and* $a(G_n) \neq a(G_m)$, *when* $n \neq m$. He found the two largest limit points for algebraic connectivity of trees as guess in Theorem 6.1.

Theorem 6.1. *Let* $(T_n)_{n\in\mathbb{N}}$ *be a sequence of trees. The largest limit point of* $a\,((T_n))_{n\in\mathbb{N}}$ *is* $\frac{3-\sqrt{5}}{2}$, *while the second largest limit point of* $a\,((T_n))_{n\in\mathbb{N}}$ *is* $2-\sqrt{3}$.

Kirkland, in his paper [45], proved that each non-negative real number is a limit point for some sequence of algebraic connectivities. Also, he characterized the limit points for $(a(T_n)_{n\in\mathbb{N}})$, where $(T_n)_{n\in\mathbb{N}}$ is a sequence of trees. In the same paper, he applied a different technique from Guo [37] to find the k-largest limit point of $(a(T_n))_{n\in\mathbb{N}}$.

For $a, b \in \frac{\mathbb{N}}{2}$, Lima et al. [52] defined a new class of graphs called *(a,b)-linear*, denoted $L(a, b)$. Let n be the number of vertices and m be the number of edges of a graph G. We say that G is an (a, b)-linear graph if and only if $m = an - b$ and $n > 2b$. There is important characteristic for this class: the union of $L(a, b)$ taken over all $a, b \in \frac{\mathbb{N}}{2}$, constitutes the category of all simple graphs. Also, there is an equivalent definition for an (a, b)-linear graph as a function of the average degree of G, $d(G) = \frac{\sum_{1\leqslant i\leqslant n} d(v_i)}{n}$. The ceiling of $d(G)$ is equal to $2a$ while $2b$ is given as a multiple of the difference between $d(G)$ and its ceiling. For more details consult [52,69]. Oliveira et al. [69] built sequences of (a, b)-linear connected graphs $(G_n)_{n\in\mathbb{N}}$, where $\Delta(G_n)$ is bounded by a constant, for every $n \in \mathbb{N}$. In their paper [69] the authors show that the correspondent sequence of diameters $(d(G_n))_{n\in\mathbb{N}}$ diverges while its correspondent sequence of algebraic connectivities $(a(G_n))_n \in \mathbb{N}$ converges to zero.

Theorem 6.2. *Let* $(G_n)_n \in \mathbb{N}$ *be an increasing sequence of* (a, b)*-linear connected graphs such that there is* $k \in \mathbb{N}$, *for all* $n \in \mathbb{N}$, $1 < \Delta(G_n) \leqslant k$. *If the sequence of diameters* $(d(G_n))_{n\in\mathbb{N}}$ *is a monotonic non-decreasing sequence then the sequence of algebraic connectivities* $(a(G_n))_{n\in\mathbb{N}}$ *converges to zero.*

The extremal graphs for $a(G)$: According to Godsil and Royle [30] *graphs with small values of $a(G)$ tend to be elongated graphs of large diameter with bridges, whereas graphs with larger values of $a(G)$ tend to be rounder with smaller diameter and large girth and connectivity.*

We approach the extremal graphs through minimizing and maximizing the algebraic connectivity.

Minimizing graphs: For certain families of graphs, certain of these graphs are known to be extremal with respect to $\min a(G)$. Some of these cases are described below.

Let $T(k, \ell, d)$ be the tree on n vertices built by taking a path on vertices $1, 2, \ldots, d$, and adding k pendant vertices adjacent to vertex 1 and ℓ pendant vertices adjacent to vertex d. Fallat and Kirkland [21] gave an extremal tree among all trees $T(k, \ell, d)$ with respect to $a(G)$.

Theorem 6.3. *Among all trees on n vertices with fixed diameter $d + 1$, the minimum algebraic connectivity is attained by $T(k, k, d)$, where $k = \lceil \frac{n-d}{2} \rceil$. In this case, $T(k, k, d)$ is unique up to isomorphism.*

We can find the extremal cubic graphs with minimum $a(G)$ in Godsil and Royle [30]. They consider two cases: (i) when $n \geqslant 10$ and $n \equiv 2 \bmod 4$, and (ii) when $n \geqslant 12$ and $n \equiv 0 \bmod 4$. In Fig. 7 an instance for each case is given.

Fallat et al. [22] called the *lollipop graph*, denoted $C_{n,s}$, a connected unicyclic graph on n vertices such that its cycle has length s and satisfies the following property: *there are at most* 2 *connected components at each vertex on the cycle, and the component not including the vertices on the cycle (if it exists) is a path with length $n - s$.*

The *girth of a graph* is the length of its shortest cycle. An acyclic graph has infinite girth. Fallat and Kirkland [21] show in Theorem 6.4 that the algebraic connectivity of lollipop graphs minimizes the algebraic connectivity among all graphs with given girth.

Theorem 6.4. *Among all connected graphs on n vertices with fixed girth s, the algebraic connectivity is minimized by the lollipop graph $C_{n,s}$.*

Fallat and Kirkland [21] posed the following conjecture: *Each minimizer for graphs on n vertices with girths s is isomorphic to a lollipop graph $C_{n,s}$*. In the same paper, they proved this conjecture for $s = 3$. Some years after, Fallat et al. [22] extended the proof for $n \geqslant 3s - 1$. In both cases $a(G) \geqslant a(C_{n,s})$ and the equality holds if and only if G is isomorphic to $C_{n,s}$. The conjecture for $n < 3s - 1$ is apparently still opened.

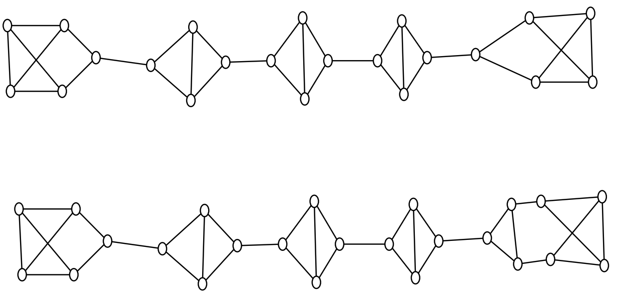

Fig. 7. The graph above refers to case (i) and, the other refers to case (ii).

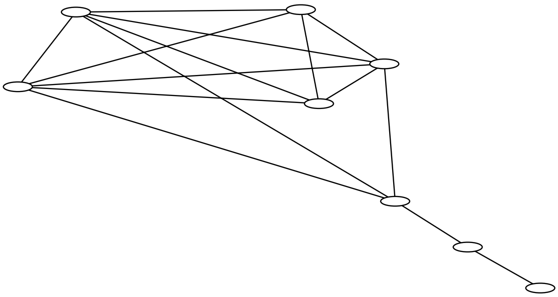

Fig. 8. A (9, 3, 4)-path-complete graph.

Belhaiza et al. [4] used the AGX-system in order to find connected graphs $G \neq K_n$ with minimum algebraic connectivity. All graphs found by AGX can be defined as follows: Let $n, m, t, p \in \mathbb{N}$, with $1 \leqslant t \leqslant (n-2)$ and $1 \leqslant p \leqslant n-t-1$. A graph with n vertices and m edges such that

$$\frac{(n-t)(n-t-1)}{2} + t \leqslant m \leqslant \frac{(n-t)(n-t-1)}{2} + (n-2)$$

is called an (n, p, t)-*path-complete graph*, denoted $PC_{n,p,t}$ if and only if the three conditions hold: (i) the maximal clique is K_{n-t}; (ii) it has a path $P_{t+1} = [v_0, v_1, \ldots, v_t]$ such that $v_0 \in K_{n-t} \cap P_{t+1}$ and v_1 is joined to K_{n-t} by p edges; (iii) there are no other edges.

These graphs constitute an almost unknown family described by Soltés [4]. Also they were considered by Harary [39] who proved that (n, p, t)-path-complete graphs are (non-unique) connected graphs with maximum diameter among all graphs with n vertices and m edges.

Fig. 8 shows an instance to $PC_{n,p,t}$, when $n = 9$; $p = 3$ and $t = 4$.

The AGX-system raises the following conjecture: *The connected graphs $G \neq K_n$ with minimum algebraic connectivity are all $PC_{n,p,t}$ graphs*. Belhaiza et al. [4] did not prove this conjecture, although they proved some partial results.

For n and $t \in \mathbb{N}$, $1 \leqslant t \leqslant n-2$, let $\mathbf{G}(n, m_t(r))$ be a family of connected graphs with n vertices and $m_t(r)$ edges, where for $t \leqslant r \leqslant n-2$, $G \in \mathbf{G}(n, m_t(r))$ has, exactly, $m_t(r) = \frac{(n-t)(n-t-1)}{2} + r$ edges. Propositions 6.1–6.3 describe these partial results.

Proposition 6.1. *Among all $G \in \mathbf{G}(n, m_1)$ with $\Delta(G) = n-1$, $a(G) \geqslant a(PC_{n,1,1})$.*

Proposition 6.2. *For every $G \in \mathbf{G}(n, m_1)$ with $\delta(G) \geqslant \frac{n-2}{2} + \frac{p}{2}$ and, $1 \leqslant p \leqslant n-2$, then $a(G) \geqslant a(PC_{n,p,1}) = p$.*

Proposition 6.3. *For every $G \in \mathbf{G}(n, m_2)$ such that $\delta(G) \geqslant \frac{n-1}{2}$ then $a(G) \geqslant 1 \geqslant a(PC_{n,p,2})$.*

Maximizing graphs: Take d to be even and let $P_{l,d}$ be the tree obtained from a path on vertices $1, 2, \ldots, d+2$ by adding ℓ pendant vertices to vertex $\frac{d+2}{2}$ and $n-\ell-d-2$ pendant vertices to vertex $\frac{d+4}{2}$.

Fallat and Kirkland [21] proved that trees like $P_{l,d}$ have the maximal algebraic connectivity among all trees with a certain diameter.

Theorem 6.5. *Among all trees on n vertices with fixed diameter $d+1$, the maximum $a(G)$ is attained by $P_{\ell,d}$, where $d = n-d-2$. In this case, $P_{\ell,n-d-2}$ is unique up to isomorphism.*

In the next theorem Fallat and Kirkland [21] gave the graph which maximize $a(G)$ when G is an unicycle graph.

Theorem 6.6. *The unique unicycle graph on n vertices with girth* 3 *of maximum algebraic connectivity is the graph* $K_{1,n-1} + \{e\}$ *for which* $a(K_{1,n-1} + \{e\}) = 1$.

Fallat et al. [23] consider the class of unicycle graphs on n vertices with girth g. Over that class, for a fixed g, they show that there is a natural number N such that for each $n > N$, the maximizing graph consists of a cycle with length g with $n - g$ pendant vertices adjacent to a common vertex on the cycle.

In order to find extremal graphs for the maximum of $a(G)$, Belhaiza et al. [4] used AGX. They obtained graphs with many edges whose structures are easy to understand if we consider their complements, see Theorem 6.7.

Theorem 6.7. *Let* $m \geqslant 2$. *There is a graph* $G \neq K_n$ *with m edges and maximum* $a(G)$, *whose complement* $\overline{G}$ *is the disjointed union of* K_3, P_3, K_2 *and* K_1.

The authors proved Theorem 6.7 based on the classic result from Merris [60] related to the complement of a graph G [4].

7. Characterizing graphs for which $a(G) = \kappa(G)$

It is well-known that, for any graph the algebraic connectivity is at most equal to the vertex connectivity (item (iii) on Table 4.1). So, it is natural to investigate graphs for which $a(G) = \kappa(G)$. First we review the definitions of the *union* and *join of graphs*.

If $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are two graphs on disjoint sets of vertices, their *union* is $G_1 + G_2 = (V_1 \cup V_2, E_1 \cup E_2)$. The *join* $G_1 \vee G_2$ is a graph G obtained from $G_1 + G_2$ by adding new edges from each vertex in G_1 to every vertex of G_2.

Theorem 7.1 due to Kirkland et al. [47] characterizes graphs for which $a(G) = \kappa(G)$, based on the join operation.

Theorem 7.1. *Let be a connected non-complete graph with n vertices. Then* $a(G) = \kappa(G)$ *if and only if G can be written as* $G_1 \vee G_2$, *where* G_1 *is a disconnected graph on* $n - \kappa(G)$ *vertices and* G_2 *is a graph on* $\kappa(G)$ *vertices and* $a(G_2) \geqslant 2\kappa(G) - n$.

From a graph G, Kirkland et al. [47] give practical conditions to determine graphs G_1 and G_2 (if it is possible) such that $G = G_1 \vee G_2$:

- Let $\overline{G}$ be the complement of G. If $\overline{G}$ is connected, it is impossible to have G_1 and G_2 such that $G = G_1 \vee G_2$, and $a(G) = \kappa(G)$;
- If $\overline{G}$ is a non-connected graph, let S_i, $1 \leqslant i \leqslant t$, be its connected component. It can be verified that the only possibility for G_1 is one of these S_i.
- If each $\overline{S_i}$ is connected, then G does not satisfy $G = G_1 \vee G_2$. Otherwise, there is j, $1 \leqslant j \leqslant t$, such that $\overline{S_j}$ is disconnected and has the largest number of vertices among all S_i for which $\overline{S_i}$ is disconnected. Do $G_1 = \overline{S_j}$ and G_2 the induced subgraph on the vertices of $G - G_1$. Then, $G = G_1 \vee G_2$.

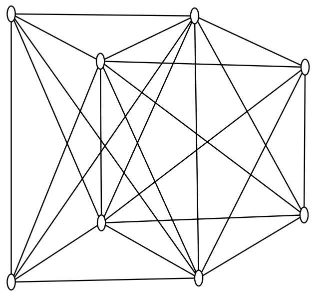

Fig. 9. G is not a Hakimi graph.

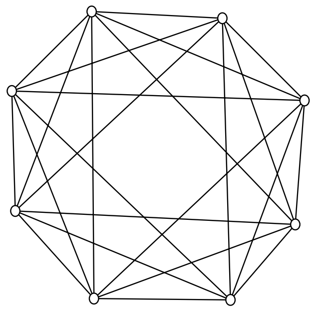

Fig. 10. H_8 is a Hakimi-graph.

For given natural numbers n and m, Harary [39] defines the set of graphs $\mathbb{G}_{n,m} = \{G$ is a connected graph with n vertices and m edges: $n \leqslant m < \frac{n(n-1)}{2}\}$. He shows that, for $n \geqslant 3$, there is a subclass $\mathbb{H}_{n,m} \subseteq \mathbb{G}_{n,m}$ such that if $H \in \mathbb{H}_{n,m}$, then $\kappa(H) = e(H) = \max_{G\in\mathbb{G}_{n,m}} \kappa(G) = \max_{G\in\mathbf{G}_{n,m}} e(G)$. Besides, given n and k, he makes a construction of a k-connected graph, called a (k,n)-*Harary graph* that belongs to $\mathbb{H}_{n,m}$ and has the fewest possible edges. For more details, see Gross and Yellen [35].

The algorithm $\mathscr{HA}$ of Hakimi [38], based on Harary's construction, builds graphs $H \in \mathbb{H}_{n,m}$, when n and m are given. We call each graph obtained from $\mathscr{HA}$ a *Hakimi graph.* The set of all Hakimi graphs is $\mathbb{HH}_{n,m}$.

$\mathscr{HA}$ begins with n and $m \in \mathbb{N}$ satisfying $n \leqslant m < \frac{n(n-1)}{2}$ and a trivial graph G ($E = \emptyset$). The goal is to share the edges between all n vertices in order to have a graph as *well-balanced as possible*. See [38] for details.

For a fixed n, let us define $\mathbb{HH}_n = \cup \mathbb{HH}_{n,m}$, such that $n \leqslant m < \frac{n(n-1)}{2}$. Of course, if $H \in \mathbb{HH}_n$ the equality $\kappa(H) = e(H)$ holds.

Although Theorem 7.1 gives necessary and sufficient conditions for a graph G have $a(G) = \kappa(G)$, it does not consider the condition that $\kappa(G) = e(G)$. So, this result is not enough to guarantee the maximality of $a(G)$. Lima et al. [53] characterize extremal Hakimi graphs with respect to the minimum number of edges among all graphs in $\mathbb{HH}_n$, such that $a(H) = \kappa(H) = e(H)$. Fig. 9 displays a graph G which is not a Hakimi graph, but has $a(G) = \kappa(G) = 4$. However, $e(G) = 5$. In this case, $a(G)$ is not the maximal algebraic connectivity among all algebraic connectivities of graphs in $G_{8,24}$. On the other hand, the 6-regular graph in Fig. 10 is a Hakimi graph which belongs to $\mathbb{HH}_8$ and has $a(H) = \kappa(H) = e(H) = 6$.

Table 7.1
Algebraic and vertex connectivities in $\mathbb{HH}_7$

m	7	8	9	10	11	12	13
$\kappa(G)$	2	2	2	2	3	3	3
$a(G)$	0.753	0.753	1.000	1.586	2.139	2.325	2.340
m	14	15	16	**17**	**18**	**19**	**20**
$\kappa(G)$	4	4	4	**4**	**5**	**5**	**5**
$a(G)$	3.198	3.198	3.382	**4.000**	**5.000**	**5.000**	**5.000**

Table 7.2
Algebraic and vertex connectivities in $\mathbb{HH}_8$

m	8	9	10	11	12	13	14	15	16	17
$\kappa(G)$	2	2	2	2	3	3	3	3	4	4
$a(G)$	0.586	0.586	0.764	1.268	2.000	2.000	2.104	2.244	2.586	2.586
m	18	19	20	21	22	23	**24**	**25**	**26**	**27**
$\kappa(G)$	4	4	5	5	5	5	**6**	**6**	**6**	**6**
$a(G)$	2.764	3.268	4.000	4.152	4.198	4.586	**6.000**	**6.000**	**6.000**	**6.000**

Lima et al. [53] proved Lemma 7.1 and Theorem 7.2. These results give extremal graphs in $\mathbb{H}_{n,m}$ which can also be obtained through $\mathscr{HA}$. So, they are also Hakimi graphs.

Lemma 7.1. *Let n be given and let $H \in \mathbb{HH}_n$ with $\lfloor\frac{n(n-2)}{2}\rfloor$ edges. If n is even, $\kappa(H) = n-2$ and, if n is odd, $\kappa(H) = n-3$. For each x, there is only one graph in $\mathbb{HH}_n$ for which $\kappa(H)$ takes these values.*

Theorem 7.2. *Let $H^* \in \mathbb{HH}_n$ with $\lfloor\frac{n(n-2)}{2}\rfloor$ edges. Then, $a(H^*) = \kappa(H^*) = e(H^*) = \delta(H^*)$ and for every $H \in \mathbb{HH}_n$ with $m = \lfloor\frac{n(n-2)}{2}\rfloor - 1$ then $a(H) < a(H^*)$.*

A graph which satisfies Theorem 7.2 is called an *extremal Hakimi graph*, denoted $H_{e,n}$. It has the minimal number of edges among all graphs in $\mathbb{HH}_n$ such that $a(H) = \kappa(H)$.

For a fixed n, a Hakimi graph in $\mathbb{HH}_n$ has n vertices and m edges such that $n \leqslant m \leqslant \frac{n(n-1)}{2} - 1$. Tables 7.1 and 7.2 list values of algebraic connectivity and vertex connectivity for all graphs in $\mathbb{HH}_7$ and $\mathbb{HH}_8$, respectively.

In Table 7.1, we observe that from $m \geqslant 17$, $a(H)$ achieves the maximum value $a(H)$ and its value is equal to the vertex connectivity for every $H \in \mathbb{HH}_7$. In Table 7.2, for $H \in \mathbb{HH}_8$, it same happens from $m \geqslant 24$. Through these examples, we can see that $a(H)$ is equal to $\kappa(H)$ for $m \geqslant \lfloor\frac{n(n-2)}{2}\rfloor$, according to Lima et al. [53]. Moreover, these authors show that, for each n, $H_{e,n}$ is a *Laplacian integral* that is, it has its Laplacian spectrum as an integer subset. They also proved that if x edges are inserted on $H_{e,n}$, $0 < x \leqslant \frac{n}{2}$, the resultant graph $H_{x,n} \in \mathbb{HH}_n$ is an *integrally completable Laplacian graph*, that is, each graph $H_{x,n}$ also has its Laplacian spectrum as an integer subset. Finally, according to Kirkland [46] and also proved by Lima et al. [53], the connectivity parameters of these graphs satisfy $a(H_{x,n}) = \kappa(H_{x,n}) = e(H_{x,n})$.

This survey ends with another important characterization of graphs for which $a(G) = \kappa(G)$, given by Kirkland et al. [47]. Their result is based on the notion of group inverse and it is in Theorem 7.3.

$$\mathscr{X}(L_G^\sharp) \doteq \frac{1}{2} \max_{1\leqslant i,j\leqslant n} \sum_{1\leqslant s\leqslant n} |\ell^\sharp{}_{i,s} - \ell^\sharp{}_{j,s}, |,$$

where $L_G^\sharp$ is the group inverse of the Laplacian matrix of G, $L(G)$.

Theorem 7.3. *Let $G \neq K_n$ be a connected graph on n vertices with $n \geqslant (\kappa(G)^2$. Then, $a(G) = \kappa(G)$ if and only if $\kappa(G) = \frac{1}{\mathscr{X}(L_G^\sharp)}$.*

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