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LINEAR ALGEBRA AND ITS APPLICATIONS

Laplacian graph eigenvectors

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Dedicated to the memory of David L. Ross

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Abstract

If G is a graph, its Laplacian is the difference of the diagonal matrix of its vertex degrees and its adjacency matrix. The main thrust of the present article is to prove several Laplacian eigenvector "principles" which in certain cases can be used to deduce the effect on the spectrum of contracting, adding or deleting edges and/or of coalescing vertices. One application is the construction of two isospectral graphs on 11 vertices having different degree sequences, only one of which is bipartite, and only one of which is decomposable.

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1. Introduction

Let $G = (V, E)$ be a graph [1] with vertex set $V = \{1, 2, \ldots, n\}$ and edge set E of cardinality $o(E) = m$. Denote the degree of vertex i by $d_G(i)$ and let $D(G) = \text{diag}\,(d_G(1), d_G(2), \ldots, d_G(n))$ be the diagonal matrix of vertex degrees. The *Laplacian matrix* is $L(G) = D(G) - A(G)$, where $A(G)$ is the (0,1)-adjacency

[1] All graphs in this article are finite and undirected with no loops or multiple edges. A good recent reference for graph theoretic notions is [20].

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matrix. It follows from Geršgorin's Theorem that $L(G)$ is positive semi-definite and, because its rows sum to 0, $e_nL(G) = 0$, where e_n is the row n-tuple each of whose entries is 1. We will call e_n the *trivial eigenvector* of $L(G)$. The *spectrum* of G is

$$S(G) = (\lambda_1(G), \lambda_2(G), \dots, \lambda_n(G)),$$

where $\lambda_1(G) \geqslant \lambda_2(G) \geqslant \cdots \geqslant \lambda_n(G)$ are the eigenvalues of $L(G)$ arranged in nonincreasing order, and $\lambda_n(G) = 0$ is the eigenvalue afforded by e_n.

The degree sequence $d(G) = (d_1, d_2, \dots, d_n)$ consists of the vertex degrees of G, arranged in nonincreasing order. (We are not necessarily assuming that $d_i = d_G(i)$.) It is an old result of Schur that $S(G)$ *majorizes* $d(G)$, i.e.,

$$\sum_{i=1}^{t} \lambda_i(G) \geqslant \sum_{i=1}^{t} d_i, \quad 1 \leqslant t \leqslant n,$$

with equality when $t = n$.

We will abuse the language by referring to the eigenvalues and eigenvectors of $L(G)$ as the eigenvalues and eigenvectors of G. Thus, $x = (x_1, x_2, \dots, x_n)$ is an eigenvector of G affording λ if and only if $x \neq 0$ and

$$(d_G(i) - \lambda)x_i = \Sigma_i(G), \quad 1 \leqslant i \leqslant n, \tag{1}$$

where

$$\Sigma_i(G) = \sum_{\substack{j \in V \\ \{i,j\} \in E}} x_j$$

is the sum, over the vertices j adjacent in G to i, of x_j.

Among the earliest results relating $S(G)$ to other graph invariants emerges the Matrix-Tree Theorem, one statement of which is that every entry of the classical adjoint of $L(G)$ is equal to the spanning tree number, $t(G)$. Hence, by looking in two different ways at the coefficient of x in the characteristic polynomial of $L(G)$, we see that

$$nt(G) = \prod_{i=1}^{n-1} \lambda_i(G). \tag{2}$$

In particular, the *algebraic connectivity* $a(G) = \lambda_{n-1}(G) > 0$ if and only if G is connected [8]. Now commonly called "Fiedler vectors", the eigenvectors of G corresponding to $a(G)$ have been found useful, for example, in algorithms for distributed memory parallel processors [2].

One of the difficulties obscuring a better understanding of $S(G)$ and its relation to other graph invariants involves the still poorly understood effect on the spectrum of adding (or deleting) edges. To be more precise, suppose $G = (V, E)$ is a graph. Let $i, j \in V$, $i \neq j$. If $e = \{i, j\} \in E$, then the *edge deleted subgraph* $G - e = (V, E \setminus \{e\})$ is obtained from G by removing edge e (and leaving vertices i and j alone). If $e \notin E(G)$, then $G + e = (V, E \cup \{e\})$ is the graph obtained from G by adding an edge joining vertex i to vertex j. It is well known (see, e.g., [10], Theorem 4.1) that

$$\lambda_i(G + e) \geqslant \lambda_i(G) \geqslant 0, \quad 1 \leqslant i \leqslant n \tag{3}$$

(with equality throughout when $i = n$), and

$$\sum_{i=1}^{n} \lambda_i(G + e) = 2 + \sum_{i=1}^{n} \lambda_i(G). \tag{4}$$

Still, as Figs. 1 and 2 show, many possibilities exist within these constraints. Fig. 2(c), for example, illustrates the effect on the spectrum of the graph in Fig. 1(a) of adding an edge between its two "bottom" vertices. Informally, one might describe this effect by saying that one eigenvalue has gone up by 2. It seems just as valid, on the other hand, to claim that two eigenvalues have gone up by 1. The main thrust of the present article is to address these and other questions by means of eigenvectors. (We shall return to Figs. 1(a) and 2(c) in the discussion immediately preceding Theorem 3.10.)

It will be convenient to associate with eigenvector x a labeling of G in which vertex i is labeled x_i. Such labelings are sometimes called "valuations" [9]. Formally, the (vertex) *valuation* afforded by eigenvector x is the function

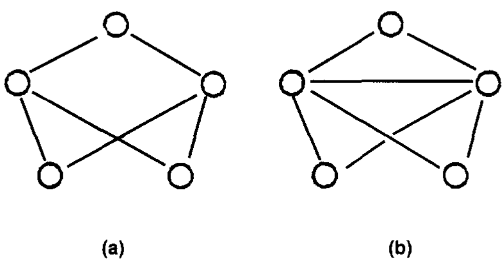

Fig. 1. (a) $S(G) = (5, 3, 2, 2, 0)$. (b) $S(G + e) = (5, 5, 2, 2, 0)$.

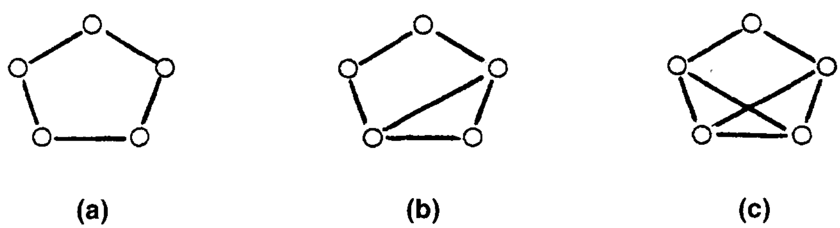

Fig. 2. $\phi = (1 + \sqrt{5})/2$. (a) $S(G - e_1) = (2 + \phi, 2 + \phi, 3 - \phi, 3 - \phi, 0)$. (b) $S(G) = (3 + \phi, 2 + \phi, 4 - \phi, 3 + \phi, 0)$. (c) $S(G + e_2) = (5, 4, 3, 2, 0)$.

$x: V \to \mathbb{R}$ defined by $x(i) = x_i$, $1 \leqslant i \leqslant n$. As the notation indicates, we will feel free to confuse the eigenvector with its associated valuation.

Because the coefficients of the characteristic polynomial of $L(G)$ are integers, any nonzero rational eigenvalue of G is an integer divisor of $nt(G)$. If λ is an irrational eigenvalue of multiplicity s, then each of its conjugates, λ^*, is also an eigenvalue of G of multiplicity s. Indeed, if K is the splitting field for det $(xI_n - L(G))$ over the rationals, there is an automorphism α of K (fixing Q) such that $\alpha(\lambda) = \lambda^*$. If x is an eigenvector of G corresponding to λ then (apply α to Eq. (1)) $\alpha(x) = (\alpha(x_1), \alpha(x_2), \ldots, \alpha(x_n))$ is an eigenvector of G affording λ^*. In particular, if just one eigenvalue of G "goes up by 2" as the result of adding an edge, it must be an integer eigenvalue.

Example 1.1. Eigenvectors (valuations) affording each of the nontrivial eigenvalues of the path, P_4, are illustrated in Fig. 3. In this case, $K = Q(\sqrt{2})$, $\alpha(a + b\sqrt{2}) = a - b\sqrt{2}$, and

$$\alpha(\tau, -1, 1, -\tau) = -\tau^{-1}(1, \tau, -\tau, -1).$$

2. Preliminaries

The *complement* of $G = (V, E)$, is $G^c = (V, E^c)$, where $e \in E(G^c) = E^c$ if and only if $e \notin E$. Observe that $L(G) + L(G^c) = nI_n - J_n$, where J_n, is the $n \times n$ matrix each of whose entries is 1. Because $L(G)J_n = 0 = J_nL(G)$, $L(G)$ commutes with $L(G^c)$. It follows that

$$\lambda_{n-i}(G^c) = n - \lambda_i(G), \quad 1 \leqslant i < n. \tag{5}$$

Therefore, $\lambda_1(G) \leqslant n$ with equality if and only if $a(G^c) = 0$, if and only if G^c is disconnected. Moreover, x is a nontrivial eigenvector of G affording λ if and only if it is a nontrivial eigenvector of G^c affording $n - \lambda$ (so a graph and its complement have the same eigenvectors). In particular, a Fiedler vector of G^c is an eigenvector of G affording $\lambda_1(G)$. For the purposes of this article we shall extend the definition of a *Fiedler vector* of G to be an eigenvector affording either $a(G)$ or $a(G^c)$, that is, either $\lambda_1(G)$ or $\lambda_{n-1}(G)$.

If $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are graphs on disjoint sets of r and s vertices, respectively, their *union* is the graph $G_1 + G_2 = (V_1 \cup V_2, E_1 \cup E_2)$, and their *join* is $G_1 \vee G_2 = (G_1^c + G_2^c)^c$, the graph on $n = r + s$ vertices obtained from $G_1 + G_2$ by inserting new edges from each vertex of G_1 to every vertex of G_2. Both union and join are associative, commutative binary operations.

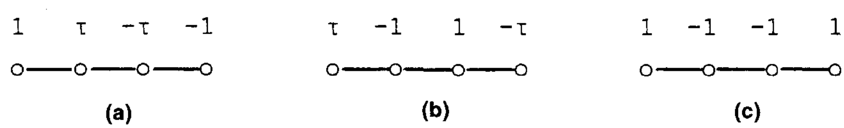

Fig. 3. $\tau = -1 + \sqrt{2}$: (a) $\lambda = 2 - \sqrt{2}$. (b) $\lambda = 2 + \sqrt{2}$. (c) $\lambda = 2$.

Suppose G is a graph on n vertices. If G^c has (exactly) q connected components then [11] the multiplicity of $\lambda = n$ as an eigenvalue of G is $q - 1$. In particular, G is the join of two graphs if and only if G^c is disconnected, if and only if $\lambda_1(G) = n$.

Theorem 2.1. *Let G_1 and G_2 be graphs on disjoint sets of r and s vertices, respectively. If $S(G_1) = (\mu_1, \dots, \mu_r)$ and $S(G_2) = (\nu_1, \dots, \nu_s)$, then the eigenvalues of $G_1 \vee G_2$ are $n = r + s$; $\mu_1 + s, \dots, \mu_{r-1} + s$; $\nu_1 + r, \dots, \nu_{s-1} + r$; and 0. Suppose y is an eigenvector of G_1 that is orthogonal to e_r. Extend y to a valuation of $G_1 \vee G_2$ by defining it to be zero on $V(G_2)$. If y affords the eigenvalue μ, the extension of y is an eigenvector of $G_1 \vee G_2$ affording $\mu + s$. Similarly, an eigenvector of G_2 affording ν extends to an eigenvector of $G_1 \vee G_2$ affording $\nu + r$. The eigenvalue $\lambda = r + s$ corresponds to an eigenvector whose value is $-s$ on each of the r vertices of G_1 and r on each of the s vertices of G_2. Finally, the trivial eigenvalue is afforded by e_n.*

A part of the Laplacian "folklore", this result is an immediate consequence of the previous definitions and observations.

Example 2.2. Let $G_1 = K_2^c = K_1 + K_1$ be the graph consisting of two isolated vertices. Then $x = (1, -1)$ and $e_2 = (1, 1)$ afford $\lambda_1(G_1) = 0$ and $\lambda_2(G_1) = 0$. If $G_2 = K_3^c$, then $y = (0, 1, -1), z = (1, 0, -1)$, and e_3 afford its spectrum. The join of these two graphs is $G_1 \vee G_2 = K_{2,3}$, the complete bipartite graph. Eigenvectors for each of its nontrivial eigenvalues (multiplicities included) are illustrated in Fig. 4.

Corollary 2.3. *Let G be a graph on n vertices. If $0 \neq \mu < n$ is an eigenvalue of G, then any eigenvector affording μ takes the value 0 on every vertex of degree $n - 1$.*

Proof. Suppose G has exactly q vertices of degree $n - 1$. If $q = 0$, there is nothing to prove. Otherwise, we may assume $d_G(i) = n - 1$, $1 \leqslant i \leqslant q$. Let $x_{[i]}$, $1 \leqslant i \leqslant q$, be the n-tuple whose ith coordinate is $1 - n$ and whose remaining coordinates all equal 1. Then (Theorem 2.1) $x_{[i]}$ is an eigenvector of G affording $\lambda = n$. Let $z = (z_1, z_2, \dots, z_n)$ be an eigenvector of G affording μ. Because z is orthogonal to $x_{[i]}$, $1 \leqslant i \leqslant q$, and to e_n, it is orthogonal to $x_{[i]} - e_n$, $1 \leqslant i \leqslant q$. Hence, $z_i = 0$, $1 \leqslant i \leqslant q$.

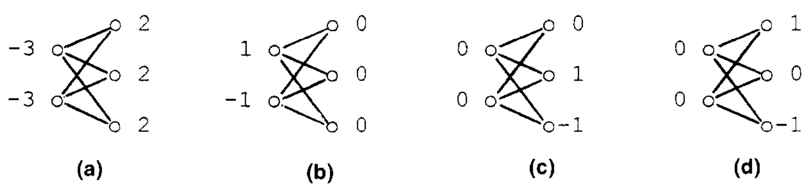

Fig. 4. (a) $\lambda = 5$. (b) $\lambda = 3$. (c) $\lambda = 2$. (d) $\lambda = 2$.

Example 2.4. Suppose $S(G) = (7,5,5,5,4,2,0)$. Then G is a graph on $n = 7$ vertices. Because $\lambda_1(G) = n$, G is a join. Given that $x = (2,2,-5,2,2,-5,2)$ is a Fiedler vector of G affording $\lambda_1(G) = 7$, it must be that $G = G_1 \vee G_2$ where $V(G_1) = \{3,6\}$ and $V(G_2) = \{1,2,4,5,7\}$. Thus, $r = \mathrm{o}(V(G_1)) = 2$ and $s = \mathrm{o}(V(G_2)) = 5$. If $G_1 \cong K_2$, then $\lambda_1(G_1) = 2$, and $\lambda = 2 + s = 7$ would be an eigenvalue of G afforded by an eigenvector that is zero on $V(G_2)$ and, hence, not a multiple of x. This would contradict the fact that the multiplicity of $\lambda = 7$ in $S(G)$ is 1. Thus $G_1 = K_2^c$, and (from $S(G)$ and Theorem 2.1) $S(G_2) = (3,3,2,0,0)$. So, G_2 is disconnected. Given that $y = (2,2,0,-3,-3,0,2)$ is a Fiedler vector of G affording $a(G) = \lambda_6(G) = 2$, it follows that $y' = (2,2,-3,-3,2)$ is a nontrivial eigenvector of G_2 affording $\lambda_4(G_2) = 0$. Hence, $G_2 = H_1 + H_2$, where $V(H_1) = \{4,5\}$ and $V(H_2) = \{1,2,7\}$. Because $\lambda_1(H_1) \leqslant 2$, $S(H_2) = (3,3,0)$ and $S(H_1) = (2,0)$. Thus, both H_1 and H_2 are joins. Evidently, $H_1 = K_2 = K_1 \vee K_1$ while $H_2 = K_2 \vee K_1 = (K_1 \vee K_1) \vee K_1 = K_3$. Thus, $G = K_2^c \vee (K_2 + K_3)$, which can be written as

$$G = (K_1 + K_1) \vee [(K_1 \vee K_1) + ([K_1 \vee K_1] \vee K_1)].$$

While any graph can be reconstructed from its spectrum and a corresponding basis of eigenvectors, we were able in this instance to recover G from $S(G)$ and substantially fewer than a complete set of its eigenvectors.

A *decomposable* graph is one that can be "constructed" from isolated vertices by joins and unions. [2] It follows from Theorem 2.1 that the spectrum of a decomposable graph is *Laplacian integral*, meaning that it consists entirely of integers. Moreover, as illustrated in Example 2.4, a blueprint for constructing the decomposable graph G typically can be read from $S(G)$ and a handful of its eigenvectors. If S is a nonempty subset of $V = V(G)$, the *induced subgraph* $G[S] = (S,F)$, where F is the subset of $E(G)$ consisting of those edges $e = \{i,j\}$ such that $i,j \in S$.

Theorem 2.5. *A graph is decomposable if and only if it does not have an induced subgraph isomorphic to* P_4.

This result is stated but not proved in [12]. Because an induced subgraph of $G = G_1 \vee G_2$ isomorphic to P_4 must be an induced subgraph of G_1 or G_2, decomposable graphs are "P_4-free". Conversely, if G contains no induced subgraph isomorphic to P_4 then (because $P_4^c \cong P_4$) every induced subgraph of G and of G^c is P_4-free. Therefore, the result is a consequence of the following lemma.

[2] Evidently, connected decomposable graphs have diameter 2, a property shared by "almost all" connected graphs [18].

Lemma 2.6. *Let G be a graph on $n \geqslant 2$ vertices. If G and G^c are both connected, then G has an induced subgraph isomorphic to P_4.*

The author is grateful to an anonymous referee for pointing out that Lemma 2.6 is implicit in [21], Sections 3 and 4, and for the following straight-forward proof by induction on n: If $2 \leqslant n \leqslant 3$, the results is vacuously true. If $n = 4$ then $G \cong P_4$. So, suppose G is a graph on $n > 4$ vertices such that G and G^c are both connected. Let $u \in V(G)$. If G contains no induced P_4, then neither does $G - u$. By the induction hypothesis, either $G - u$ or its complement is disconnected. Without loss of generality, we may assume it is $G - u$. Because u is not an isolated vertex in G or G^C, there exist vertices $v, w \in V(G)$ such that $\{u, v\}$ and $\{v, w\}$ are edges of G, but $\{u, w\}$ is not. If x is a vertex of a component of $G - u$ different from the one that contains v and w, then $G[\{x, u, v, w\}] \cong P_4$.

A *threshold graph* is one whose degree sequence is maximal with respect to majorization. An expository account of these graphs appears in [15]. In particular, G is a threshold graph if and only if it does not contain an induced subgraph isomorphic to one of the three forbidden graphs, $K_2 + K_2$, its complement C_4, or P_4. Thus, every threshold graph is decomposable.

Another construction of interest to us here is the *graph product*. If $G = (V, E)$ and $H = (W, F)$ are graphs, then $V(G \times H) = V \times W$, the cartesian product of $V(G)$ and $V(H)$. Vertices (i_1, k_1) and (i_2, k_2) are adjacent in $G \times H$ if $i_1 = i_2$ and $\{k_1, k_2\} \in F$ or if $\{i_1, i_2\} \in E$ and $k_1 = k_2$. One may view $G \times H$ as the graph obtained from G by replacing each of its vertices with a copy of H and each of its edges with $o(W)$ edges joining corresponding vertices of H in the two copies. Fig. 5 contains two illustrations of $K_3 \times K_2 = C_3 \times P_2$ and one of $P_3 \times P_2$.

If $A = (a_{ij})$ and $B = (b_{ij})$ are $r \times r$ and $s \times s$ matrices, their *Kronecker product* is the $rs \times rs$ partitioned matrix

$$A \otimes B = \begin{pmatrix} a_{11}B & a_{12}B & \dots & a_{1r}B \\ a_{21}B & a_{22}B & \dots & a_{2r}B \\ & & \dots & \\ a_{r1}B & a_{r2}B & \dots & a_{rr}B \end{pmatrix}.$$

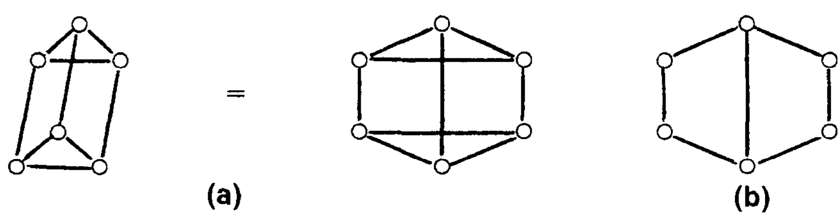

Fig. 5. (a) $C_3 \times P_2$. (b) $P_3 \times P_2$.

It is a routine exercise in the theory of Kronecker products (see, e.g., [5], Section 2.5, [6], Section 2.3, or [16], Ch. 5) that

$$L(G \times H) = L(G) \otimes I_s + I_r \otimes L(H). \tag{6}$$

It follows from Eq. (6) that the multiset of eigenvalues of $G \times H$ is $\{\lambda_i(G) + \lambda_k(H): 1 \leqslant i \leqslant r,\ 1 \leqslant k \leqslant s\}$. Moreover, if x is an eigenvector of G affording μ and y an eigenvector of H affording v, then $x \otimes y$ is an eigenvector of $G \times H$ affording $\mu + v$.

The graph product is a commutative, associative binary operation on graphs. If K is a third graph, on t vertices, then

$$L(G \times H \times K) = L(G) \otimes I_s \otimes I_t + I_r \otimes L(H) \otimes I_t + I_r \otimes I_s \otimes L(K).$$

Example 2.7. Let $H_1 = C_3 \times P_2$. Because $C_3 = K_3$ and $P_2 = K_2$ we have from Example 2.4 that $S(C_3) = (3, 3, 0)$ and $S(P_2) = (2, 0)$. Therefore, $S(H_1) = (5, 5, 3, 3, 2, 0)$. Because $C_6 = H_1^c$, we have from Eq. (5) that $S(C_6) = (4, 3, 3, 1, 1, 0)$. If $G_1 = H_1 \vee H_1^c = (H_1^c + H_1)^c$, then

$$S(G_1) = (12, 11, 11, 10, 9, 9, 9, 9, 8, 7, 7, 0).$$

Suppose $H_2 = P_3 \times P_2$. Because $P_3 = K_1 \vee K_2^c$, $S(P_3) = (3, 1, 0)$. Thus $S(H_2) = (5, 3, 3, 2, 1, 0)$ and $S(H_2^c) = (5, 4, 3, 3, 1, 0)$. If $G_2 = H_2 \vee H_2^c$, then

$$S(G_2) = (12, 11, 11, 10, 9, 9, 9, 9, 8, 7, 7, 0).$$

Two graphs which share the same Laplacian spectrum are said to be *isospectral*. This particular isospectral pair of nonisomorphic graphs is Laplacian integral. We shall have more to say about isospectral graphs in Section 4.

3. Main results

Theorem 3.1 (Edge Principle). *Let λ be an eigenvalue of G afforded by eigenvector x. If $x_i = x_j$, then λ is an eigenvalue of G' afforded by x, where G' is the graph obtained from G by deleting or adding $e = \{i, j\}$ depending on whether or not it is an edge of G.*

Proof. Suppose that vertices i and j are not adjacent in G. Because x is an eigenvector of G, $(d - \lambda)x_i = \Sigma_i(G)$, where $d = d_G(i)$. Hence, $((d + 1) - \lambda)x_i = x_i + \Sigma_i(G)$, which is the condition that must be met at vertex i for x to be an eigenvector of G' affording λ. The eigenvector condition at vertex j is confirmed similarly, and the conditions at the other vertices are the same for G' as they are for G. Reversing the argument, one establishes the case in which i and j are adjacent in G. □

Suppose $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are graphs on disjoint sets of vertices having eigenvectors y and z that afford (the same eigenvalue) λ. Then the valuation $x: V_1 \cup V_2 \to \mathbb{R}$ defined by

$$x(k) = \begin{cases} y_k & \text{if } k \in V_1, \\ z_k & \text{if } k \in V_2, \end{cases}$$

is an eigenvector of $G = G_1 + G_2$ that affords λ. If $y_i = z_j$ then, by the Edge Principle, x is an eigenvector of the graph G' obtained from $G_1 + G_2$ by adding an edge (sometimes called a "bridge") joining vertex i of G_1 and vertex j of G_2. (If $y_i \neq z_j$, and neither of them is zero, then $w = (y_i/z_j)z$ is an eigenvector of G_2 for which $y_i = w_j$.)

Example 3.2. Eigenvectors affording the three nontrivial eigenvalues of $G_1 = P_4 = G_2$ appear in Fig. 3. From these and the Edge Principle, we obtain the eigenvectors of C_8 illustrated in Fig. 6. A 45° rotation of any one of these octagons produces a linearly independent eigenvector corresponding to the same eigenvalue. Thus, each of these three eigenvalues has multiplicity (at least) 2. Together with $\lambda = 0$, we have accounted for all but one of the eigenvalues of C_8. Because it has 8 vertices and 8 spanning trees, Eq. (2) applied to C_8 yields

$$64 = \lambda[2(2-\sqrt{2})(2+\sqrt{2})]^2 = 16\lambda,$$

where $\lambda = 4$ is the missing eigenvalue.

Suppose r and s are positive integers. Let $G = (V, E)$. If V is the disjoint union of two nonempty sets V_r and V_s such that every vertex in V_r has degree r and every vertex in V_s degree s, then G is (r, s)-*semiregular*.

Theorem 3.3. *Suppose $G = (V, E)$ is an (r, s)-semiregular, bipartite graph in which V_r and V_s are the parts of the bipartition. Then the valuation defined by*

$$u(i) = \begin{cases} -r & \text{if } i \in V_r, \\ s & \text{if } i \in V_s, \end{cases} \tag{7}$$

is an eigenvector of G affording $\lambda = r + s$.

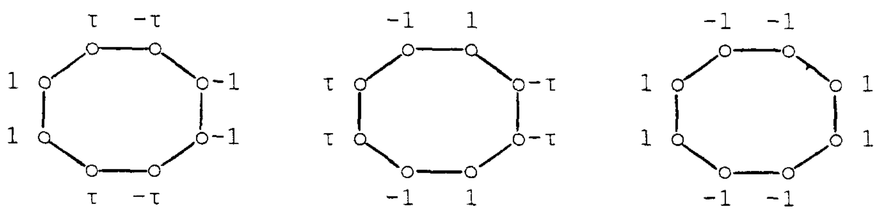

Fig. 6. $\tau = -1 + \sqrt{2}$: (a) $\lambda = 2 - \sqrt{2}$. (b) $\lambda = 2 + \sqrt{2}$. (c) $\lambda = 2$.

Note that P_4 is semiregular and bipartite but does not satisfy the hypothesis of Theorem 3.3. On the other hand, if G does satisfy the hypothesis of Theorem 3.3 then ([1], Theorem 2) $r + s = \lambda_1(G)$, the largest eigenvalue of G.

Proof. The eigenvector condition at a vertex of V_r is $-r(r - \lambda) = rs$; at a vertex of V_s it is $s(s - \lambda) = -rs$. Because $r \neq 0 \neq s$, each of these conditions is equivalent to $\lambda = r + s$. □

Example 3.4. Let G be an (r, s)-semiregular graph. If $r = s$, then G is regular. If in addition G is bipartite then, rescaling the vector u in Eq. (7), we obtain the *regular bipartite eigenvector* $u' = u/r$, consisting of alternating ± 1's. If G is an even cycle, the regular bipartite eigenvector corresponds to $\lambda_1(G) = 4$. In particular, the "missing" eigenvalue of Example 3.2 is accounted for by the regular bipartite eigenvector illustrated in Fig. 7(a). Figs. 7(b) and 7(c) show how the Edge Principle can be used to extend a regular bipartite eigenvector to regular graphs that are not bipartite and to graphs that are neither semiregular nor bipartite.

Let x be an eigenvector of G_1 affording λ. Suppose $x_i = x(i) = 0$. Extend the valuation x to an eigenvector of $G = G_1 + G_2$ by defining $x(k) = 0$ for all vertices k of G_2. By the Edge Principle, this extension is an eigenvector of any graph G' obtained from G by adding edges that join vertex i of G_1 to any number of vertices of G_2. (If $x_i = 0 = x_j$ for two vertices of G_1, then edges may be added to G' joining vertex j to vertices of G_2 as well.)

Suppose two or more linearly independent eigenvectors of G_1 afford λ. If each of them is zero at vertex i, their extensions to G' are linearly independent. If λ is an eigenvalue of G_1 of multiplicity $q > 1$, and i is an arbitrary vertex of G_1, then there exist at least $q - 1$ linearly independent eigenvectors for λ that are zero at vertex i. Thus, λ is an eigenvalue of G' of multiplicity *at least* $q - 1$. (These observations lead to new proofs of Theorems 2.5 and 3.1 of [10].)

Example 3.5. Fig. 8 illustrates the sum of two linearly independent eigenvectors of $G_1 = C_8$ corresponding to $\lambda = 2$. A rescaled sum appears in Fig. 9(a). From the previous remarks, $\lambda = 2$ is an eigenvalue of the graphs in Figs. 9(b) and 9(c).

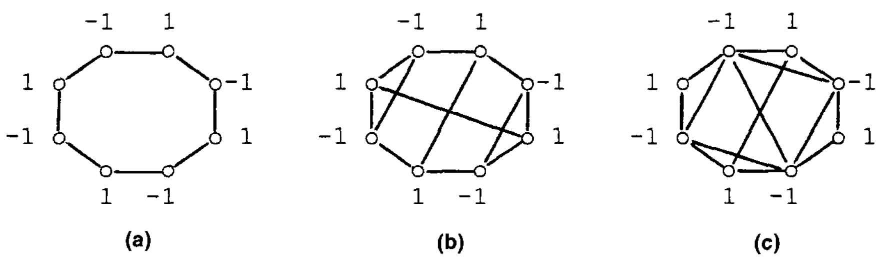

Fig. 7. (a) $\lambda = 4$. (b) $\lambda = 4$. (c) $\lambda = 4$.

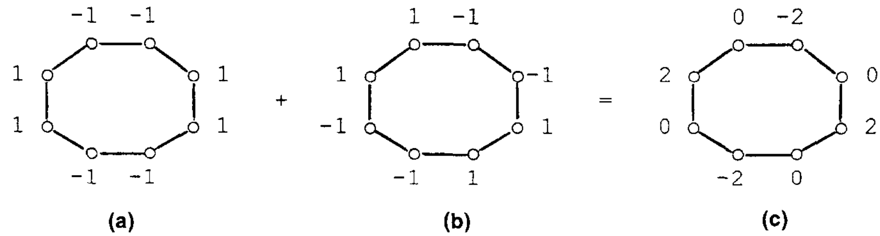

Fig. 8. (a) $\lambda = 2$. (b) $\lambda = 2$. (c) $\lambda = 2$.

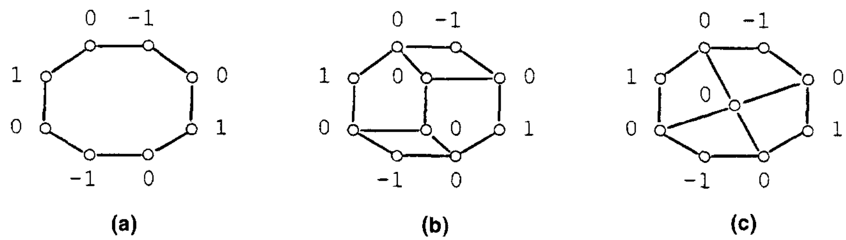

Fig. 9. (a) $\lambda = 2$. (b) $\lambda = 2$. (c) $\lambda = 2$.

Theorem 3.6 (Principle of Reduction and Extension). *Let $G = (V, E)$ be a graph. Fix a nonempty subset W of V. Delete all the vertices in $V \backslash W$ that are adjacent in G to no vertex of W (with the understanding that when a vertex is deleted from a graph, all edges incident with it are deleted as well). Remove any remaining edges that are incident with no vertex of W. Call the resulting* reduced graph *$G\{W\}$ (not to be confused with the induced graph $G[W]$). Suppose x is an eigenvector of $G\{W\}$ that affords λ and is* supported *by W in the sense that if $x(i) \neq 0$, then $i \in W$. Then x extends to an eigenvector of G affording λ.*

Proof. The valuation $x : V(G\{W\}) \to \mathbb{R}$ may be extended to a valuation $x : V \to \mathbb{R}$ by defining $x(k) = 0$ for all $k \in V \setminus V(G\{W\})$. By the Edge Principle, this extension is an eigenvector of G affording λ. □

Note that linearly independent eigenvectors of $G\{W\}$ extend to linearly independent eigenvectors of G. Thus, if λ is an eigenvalue of $G\{W\}$ of multiplicity m, it is an eigenvalue of G of multiplicity not less than m.

Suppose $i \in V(G)$. Let $N(i) = \{j \in V(G): \{i,j\} \in E(G)\}$ be the set of its *neighbors*. If $W \subset V(G)$,

$$N(W) = \bigcup_{i \in W} N(i)$$

consists of those vertices adjacent to at least one vertex of W.

Example 3.7. Suppose W is an independent set of vertices in G, each having (the same) degree d. Then the reduced graph $G\{W\}$ is bipartite with bipartition $V(G\{W\}) = W \cup N(W)$. Suppose x is an eigenvector of $G\{W\}$ supported by W, i.e., $x(k) = 0$, $k \in N(W)$. Because x cannot be identically 0, satisfaction of the eigenvector condition at all vertices of W requires that $\lambda = d$. The remaining conditions are

$$\sum_{\substack{i \in W \\ \{i,k\} \in E}} x(i) = 0, \quad k \in N(W), \tag{8}$$

a homogeneous system of $o(N(W))$ equations in $o(W)$ unknowns. If $q = o(W) - o(N(W)) > 0$, Eq. (8) has (at least) q linearly independent solutions and it follows from the Principle of Reduction and Extension that the multiplicity of $\lambda = d$ as an eigenvalue of G is not less than q. When $G\{W\}$ is a *complete* bipartite graph, it has $o(W) - 1$ linearly independent eigenvectors of the form $x(i) = -1$, $x(j) = 1$, and $x(k) = 0$, $k \in V(H) \setminus \{i, j\}$, where i is a fixed but arbitrary vertex of W and j runs over $W \setminus \{i\}$. In this case, the multiplicity of $\lambda = d$ as an eigenvalue of G is at least $o(W) - 1$, a result first proved by Isabel Faria [7].

Suppose G is a connected graph on $n > 2$ vertices. Let λ be an eigenvalue of G afforded by an eigenvector x that is nonzero for precisely two vertices, i and j. If $k \in N(j)$ and $k \neq i$ then $x(k) = 0$, and the eigenvector condition forces both $k \in N(i)$ and $x(i) = -x(j)$. Because G is connected, one (at least) of i and j is adjacent to a third vertex. Hence, $y = x/x(i)$ is an eigenvector of G that affords λ and whose only nonzero values are $y(i) = 1$ and $y(j) = -1$. Any such eigenvector is called a *Faria vector.*

Theorem 3.8 (Alternating Principle). *Let x be an eigenvector of G affording λ. Suppose the vertices in $Z = \{k \in V(G): \ x(k) \neq 0\}$ can be paired up in such a way that if i and j are paired vertices then $x(i) = -x(j)$. Suppose further that all paired vertices in Z are adjacent (not adjacent) to each other. Let G' be the graph obtained from G by removing (adding new) edges between all paired vertices of Z. Then x is an eigenvector of G' affording $\lambda - 2$* (affording $\lambda + 2$).

Proof. Let $i \in V$ be a fixed but arbitrary vertex. Let $d = d_G(i)$. If $i \notin Z$, the eigenvector condition $(d - (\lambda + 2))(0) = 0$ is satisfied because the neighbors of i in G' are the same as its neighbors in G. If $i \in Z$ then it is paired with some $j \in z$ and the eigenvector condition

$$((d \mp 1) - (\lambda \mp 2))x(i) = (d - \lambda)x(i) \pm x(i) = \mp x(j) + \Sigma_i(G),$$

is satisfied in G'. $\square$

The Alternating Principle is illustrated in Fig. 10.

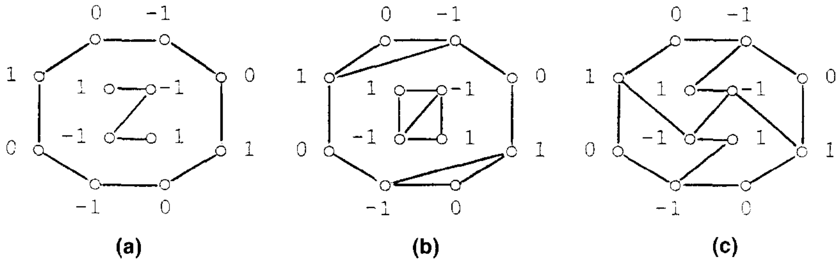

Fig. 10. (a) $\lambda = 2$. (b) $\lambda = 4$. (c) $\lambda = 4$.

Corollary 3.9. *Let x be a Faria vector of* $G = (V, E)$ *affording eigenvalue* λ, *where* $x(i) = 1 = -x(j)$. *Let* $e = \{i, j\}$. *Suppose* $e \in E$ $(e \notin E)$. *Let* $G' = G - e$ $(G' = G + e)$. *(By the Alternating Principle, x is an eigenvector of G′ corresponding to* $\lambda \mp 2$.*) Suppose y is an eigenvector of G affording* μ. *If y is orthogonal to x, then y is an eigenvector of G′ affording* μ.

Proof. Since y is orthogonal to x if and only if $y(k) = y(j)$, the result is a consequence of the Edge Principle. □

Because $L(G)$ is symmetric, G has a family $\mathscr{F}$ of n orthogonal eigenvectors. Suppose $x \in \mathscr{F}$ is a Faria vector affording λ. Let G' be the graph obtained from G by switching the adjacency of the supporting vertices of x. By Corollary 3.9, $\mathscr{F}$ is an orthogonal family of eigenvectors of G'. Moreover, apart from x, these eigenvectors afford the same eigenvalues for both G' and G. Thus, apart from replacing λ with $\lambda \mp 2$, G and G' have the same eigenvalues.

Suppose $G = K_{2,3}$, the graph illustrated in Fig. 1(a). A Faria vector for G is exhibited in Fig. 4(b). Adding an edge between the supporting vertices of this eigenvector produces the graph in Fig. 1(b). Thus, Corollary 3.9 "explains" the spectral relationship illustrated in Fig. 1. If x is the Faria vector of $K_{2,3}$ illustrated in Fig. 4(c) the adding an edge between its supporting vertices produces the graph in Fig. 2(c). While the vector y in Fig. 4 (d) is not orthogonal to x it can be replaced with the eigenvector $y' = 2y - x$. Evidently, in passing from Fig. 1(a) to Fig. 2(c), it is one eigenvalue that goes up by 2. (Not only does Corollary 3.9 answer this question of clarity, it clarifies the meaning of the question!)

Theorem 3.10 (Contraction Principle). *Let G be a graph on n vertices and x an eigenvector of G affording* λ. *Suppose* $x_i = 0 = x_j$, *where* $N(i) \cap N(j) = \emptyset$. *If* $\{i, j\}$ *is an edge of G, delete it. Let G′ be the graph on* $n - 1$ *vertices obtained by identifying vertices i and j, that is, by coalescing them into a single vertex (which is adjacent in G′ precisely to those vertices that are adjacent in G to i or to j). If x′ is the* $(n - 1)$*-dimensional vector obtained from x by deleting its jth coordinate, then x′ is an eigenvector of G′ affording* λ.

Proof. At all but the coalesced vertex of G', the eigenvector condition is the same as it is for G. Because $\Sigma_i(G) = 0 = \Sigma_j(G)$, the eigenvector condition is valid for the coalesced vertex of G' as well. □

Example 3.11. The eigenvector illustrated in Fig. 11(a) may be obtained by contracting two vertices in Fig. 9(a) or by contracting two vertices in Fig. 11(b).

4. Application to isospectral graphs

Graph G is *spectrally unique* if it is determined, up to isomorphism, by $S(G)$.

Theorem 4.1. Threshold graphs are spectrally unique.

Proof. It is proved in [14] that G is a threshold graph if and only if $\lambda_j(G)$ is the cardinality of $\{i\colon d_G(i) \geqslant j\}$, $1 \leqslant j \leqslant n$, that is, apart from isolated vertices, $S(G) = d^\star(G)$, the partition of $2m$ "conjugate" to $d(G)$. The result follows because threshold graphs are uniquely determined by their degree sequences. □

Graphs that are not spectrally unique belong to one or more isospectral pairs. One such pair was constructed in Example 2.7. It is proved in [13] (also see [3]) that for "almost all" trees T, there is a nonisomorphic tree T', such that T and T' are isospectral. Exponentially large families of nonisomorphic, isospectral, Laplacian integral graphs have been constructed [17]. When G is regular of degree r, λ is an eigenvalue of $L(G)$ if and only if $r - \lambda$ is an eigenvalue of $A(G)$. Thus, the smallest pair of nonisomorphic, adjacency cospectral, regular graphs exhibited in [6], p. 79 is also the smallest pair of non-isomorphic, isospectral, regular graphs. The complements of these graphs comprise another pair. A family of four 12-regular isospectral graphs on 28 vertices obtained from modifications of the line graph of K_8 is exhibited in [6], Example 1.1.2. However, in none of these examples is there a single isospectral pair that does not also share the same degree sequence. (In Example 2.7, G_i has six vertices of degree 8 and six of

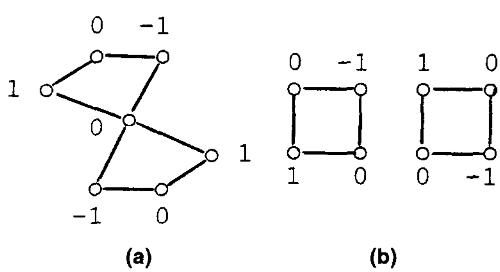

Fig. 11. (a) $\lambda = 2$. (b) $\lambda = 2$.

degree 9, $i = 1, 2$.) The main purpose of this section is to construct a pair of isospectral graphs that do not share the same degree sequence.

Let $H_1 = K_{2,3} = H_2$. Then the graph H exhibited in Fig. 12(a) can be obtained from $H_1 + H_2$ by adding three edges. Let x be the Faria vector for H_1 illustrated in Fig. 4(b). By the Edge Principle, x can be extended to a Faria vector of H that affords $\lambda = 3$. If y is the Faria vector for H_2 illustrated in Fig. 4(b), then it too extends to a Faria vector of H affording $\lambda = 3$. Because these extensions of x and y are linearly independent, the multiplicity of $\lambda = 3$ as an eigenvalue of H is at least two.

Applying the Edge Principle (three times) to two copies of Fig. 4(a) produces an eigenvector of H affording $\lambda = 5$. The same approach using Figs. 4(c) and 4(d) generates two linearly independent eigenvectors affording $\lambda = 2$. Indeed, Figs. 4(c) and 4(d) is also good for two linearly independent eigenvectors corresponding to $\lambda = 4$. These are obtained from a combination of the Alternating and Edge Principles.

The regular bipartite eigenvector, exhibited in Fig. 12 (b), affords $\lambda = 6$, and e_{10} produces $\lambda = 0$. The last eigenvalue, $\lambda = 1$, can be obtained from Eq. (2) by counting spanning trees in H, or by the ad hoc construction of an eigenvector such as the one exhibited in Fig. 12(c). Thus,

$$S(H) = (6, 5, 4, 4, 3, 3, 2, 2, 1, 0). \tag{9}$$

(The graph H is one of exactly 13 3-regular, Laplacian integral graphs [4,19].)

From Example 2.2, $S(K_{2,3}) = (5, 3, 2, 2, 0)$, so $S((K_{2,3} + K_1)^c) = (6, 4, 4, 3, 1, 0)$. If $G_1 = K_{2,3} + (K_{2,3} + K_1)^c$, then

$$S(G_1) = (6, 5, 4, 4, 3, 3, 2, 2, 1, 0, 0). \tag{10}$$

Comparing Eqs. (9) and (10), we see that G_1 and $G_2 = H + K_1$ are isospectral graphs. Because H is 3-regular, the degree sequence of G_2 is $d(G_2) = (3^{10}, 0)$, where the superscript denotes multiplicity. Since $d(K_{2,3}) = (3, 3, 2, 2, 2)$, $d((K_{2,3} + K_1)^c) = (5, 3, 3, 3, 2, 2)$. Thus, $d(G_1) = (5, 3^5, 2^5) \neq d(G_2)$. Going beyond degree sequences, note that G_1 is decomposable but not bipartite, while G_2 is bipartite but not decomposable. Finally, G_1^c and G_2^c are connected isospectral graphs with different degree sequences.

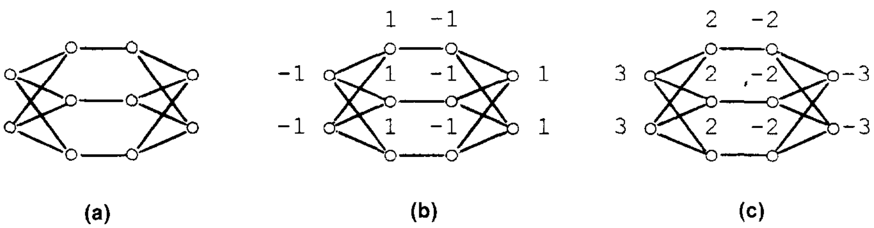

Fig. 12. (b) $\lambda = 6$. (c) $\lambda = 1$.

From any pair of isospectral graphs one may, of course, construct others by making substitutions in graph unions, joins, and/or products.

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