The Multidimensional Generalization of Quadrature Filters using Vector Field Theory

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1 Introduction

- What is this presentation about? **Phase**
- Why is phase important? **It carries structural information.**

Our interest is the **Local Phase**: spatial domain counterpart of the phase spectrum.
2 Local Phase in 1D

• The local phase is the local decomposition of the signal according to its symmetries.
• In 1D the local phase is properly defined through the analytic signal [Gabor 1946].
• The analytic signal is constructed from the Hilbert transform.

Hilbert Transform

\[ f_H(y) = f(x) * \frac{1}{\pi x} \Rightarrow F_H(u) = -i \cdot \text{sign}(u) \cdot F(u) = -i \cdot \frac{u}{|u|} \cdot F(u) \]

Analytic Signal

\[ f_A(x) = f(x) + i \cdot H[f(x)] = |f_A(x)|e^{i\Phi f_A(x)} \Rightarrow F_A(u) = 2 \cdot \Pi(u) \cdot F(u), \]
Interpretation

- Local Amplitude: $|f_A(x)|$
- Local Phase: $\phi_{f_A}(x)$

Example
3 Local Phase in nD

- In nD spaces we have a new degree of freedom: **orientation**.

- Some Hilbert transform extensions to nD:
  1. Total Hilbert transform [Stark 1971]
  2. Partial Hilbert transform [Knutsson 1995]

- Attempts lack of enough complexity: complex algebra is not enough.
4 Hilbert Transform and Vector Field Theory

- Analytic functions related to harmonic functions.
- An harmonic function $p$ satisfies Laplace’s equation.
- 1D Hilbert transform relates the components of a 2D harmonic field $g = \nabla p(x_1, x_2)$.

**Hilbert transform derivation [Felsberg 2001]**

- **Problem:** Laplace equation in the open domain $x_2 < 0$ with Neumann boundary conditions

  \[
  \begin{align*}
  \Delta p(x_1, x_2) &= 0, \\
  g_1(x_1, 0) &= f(x_1)
  \end{align*}
  \]

- **Statement:**

  \[
  g_2(x_1, 0) = \mathcal{H}(g_1(x_1, 0)) = \mathcal{H}(f(x_1))
  \]
Harmonic function

\[ p(x_1, x_2) \]

\[ f_H(x_1) \]

\[ f(x_1) \]

\[ g_2 \]

\[ g_1 \]

\[ x_2 = 0 \]

\[ x_1 \]

\[ x_2 \]

\[ f_H(x_1) \]

\[ f(x_1) \]
• **Proof:**

Rewriting Laplace equation in the transform domain for the coordinate $x_1$.

\[-4\pi^2 u_1^2 P(u_1, x_2) + \frac{\partial^2 P(u_1, x_2)}{\partial x_2^2} = 0.\]

The solution for $x_2 < 0$ is given by

\[P(u_1, x_2) = C(u_1)e^{2\pi|u_1|x_2}.\]

Then

\[G_1(u_1, x_2) = i2\pi u_1 P(u_1, x_2) \quad G_2(u_1, x_2) = 2\pi|u_1|P(u_1, x_2).\]

$G_1$ related to $G_2$

\[G_1(u_1, x_2) = i\frac{u_1}{|u_1|}G_2(u_1, x_2) = H_1(u_1)G_2(u_1, x_2)\]
5 Generalized Hilbert Transform: Riesz Transform

- (n+1)D extension of Laplace problem. \( p(x_{(n+1)}): \mathbb{R}^{n+1} \to \mathbb{R} \).

- Notation:

\[
\begin{align*}
  x_n &= [x_1, x_2, \cdots, x_n], \\
  x_{(n+1)} &= [x_1, x_2, \cdots, x_n, x_{n+1}] \\
  g(x_{(n+1)}) &= [g_1(x_{(n+1)}), g_2(x_{(n+1)}), \cdots, g_{(n+1)}(x_{(n+1)})] = \nabla p(x_{(n+1)}) = \nabla p(x_{(n+1)}).
\end{align*}
\]

- **Problem:** (n + 1)D Laplace problem in the domain \( x_{(n+1)} < 0 \) with Neumann boundary condition

\[
\begin{align*}
  \Delta p(x_{(n+1)}) &= 0, \\
  g_{(n+1)}(x_n, 0) &= f(x_n).
\end{align*}
\]

- **Statement:** nD Riesz transform relates the last component of \( g \) to the previous ones.
2D case → 3D Laplace equation

\[ p(x) \]

harmonic function

solution space

\[ g_1(x_1, x_2, 0) \]

\[ g_2(x_1, x_2, 0) \]

\[ g_3(x_1, x_2, 0) = f \]
• **Proof:**

Let us perform the calculation in the frequency domain for \( x_n \), keeping \( x_{n+1} \) coordinate in the spatial domain

\[
\frac{\partial^2 P(u_n, x_{n+1})}{\partial^2 x_{n+1}} = 4\pi^2 |u_n|^2 P(u_n, x_{n+1})
\]

The solution for \( x_{(n+1)} \)

\[
P(u_n, x_{n+1}) = C(u_n) e^{2\pi |u_n| x_{n+1}}
\]

\( C(u_n) \) is a function independent of \( x_{n+1} \)

\[
\begin{align*}
G_k(u_n, x_{n+1}) &= i 2\pi u_k P(u_n, x_{n+1}) & 1 \leq k \leq n \\
G_{n+1}(u_n, x_{n+1}) &= \frac{\partial P}{\partial x_{n+1}} = 2\pi |u_n| C(u_n) P(u_n, x_{n+1}).
\end{align*}
\]

The relation between \( G_k \) and \( G_{n+1} \)

\[
G_k(u_n, x_{n+1}) = i \frac{u_k}{|u_n|} O(u_n) G_{n+1}(u_n, x_{n+1}) & 1 \leq k \leq n,
\]

From this result is possible to construct a vector function \( F_R(u) : \mathbb{R}^n \to \mathbb{R}^n \) such as

\[
F_R(u) = [G_1(u, 0), G_2(u, 0), \cdots, G_n(u, 0)]^T.
\]
Riesz transform of $f$ in the Fourier domain:

$$F_{\mathcal{R}}(u) = i \frac{u}{|u|} O(u) F(u)$$

Riesz transform of $f$ in the spatial domain:

$$f_{\mathcal{R}}(x) = - \frac{x}{2\pi |x|^n} \ast o(x) \ast f(x)$$

2D case $\rightarrow$ 3D Riesz Transform
6 Monogenic Signal

- nD extension of analytic signal.
- Embedding of nD signal $f(x) : \mathbb{R}^n \rightarrow \mathbb{R}$ in a (n+1)D dimensional space

$$f_M : \mathbb{R}^n \rightarrow \mathbb{R}^{(n+1)} \quad f_M(x) = [-f_R(x), f(x)]^T$$

Interpretation

Local Amplitude

$$\|f_M(x)\| = \sqrt{f^2(x) + \|f_R(x)\|^2}.$$  

Local Phase Vector

$$\varphi(x) = \frac{f_R(x)}{\|f_R(x)\|} \arctan \left( \frac{\|f_R(x)\|}{f(x)} \right)$$
Example
7 Generalized Quadrature Filters

• How to estimate analytic signal? ➞ quadrature filters [Granlund and Knutsson 1995].
• How to estimate monogenic signal? ➞ Generalized quadrature filters [Knutsson 2003]
• Spherical separable: \( Q(u) = R(|u|)D(u) \)

\[
D(u) = (\hat{u}^T \hat{n})^{2a} \left( -i \frac{u}{|u|} \right)
\]

• \( a = 0 \): spherical QF [Felsberg 2001]

• \( a = 1 \): loglets QF [Knutsson 2003]
8  Results

MRI
Ultrasound
9 Conclusion

- nD generalization of Hilbert Transform.
- Local phase analysis of nD signals
- Applications:
  - Phase based registration
  - Phase based segmentation
  - Local Structure Tensor Estimation