Estimating and Interpreting The Instantaneous Frequency of a Signal—Part 1: Fundamentals

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The frequency of a sinusoidal signal is a well defined quantity. However, often in practice, signals are not truly sinusoidal, or even aggregates of sinusoidal components. Nonstationary signals in particular do not lend themselves well to decomposition into sinusoidal components. For such signals, the notion of frequency loses its effectiveness, and one needs to use a parameter which accounts for the time-varying nature of the process. This need has given rise to the idea of instantaneous frequency.

The instantaneous frequency (IF) of a signal is a parameter which is often of significant practical importance. In many situations such as seismic, radar, sonar, communications, and biomedical applications, the IF is a good descriptor of some physical phenomenon.

This paper discusses the concept of instantaneous frequency, its definitions, and the correspondence between the various mathematical models formulated for representation of IF. The paper also considers the extent to which the IF corresponds to our intuitive expectation of reality.

A historical review of the successive attempts to define the IF is presented. Then the relationships between the IF and the group-delay, analytic signal, and bandwidth-time (BT) product are explored, as well as the relationship with time-frequency distributions. Finally, the notions of monocomponent and multicomponent signals, and instantaneous bandwidth are discussed. It is shown that all these notions are well described in the context of the theory presented.

I. Introduction

This paper presents a tutorial review of the theory necessary to understand and utilize the notion of the instantaneous frequency (IF) of a signal. Like many other signal processing concepts, the IF was originally defined in the context of FM modulation theory in communications.

The importance of the IF concept stems from the fact that in many applications the signal analyst is confronted with the task of processing signals whose spectral characteristics (in particular the frequency of the spectral peaks) are varying with time. These signals are often referred to as "nonstationary," a simple example being the chirp signal. The latter may be conceptualized crudely as a sine wave whose frequency sweeps with time. In seismic processing these signals (referred to as "vibroseis" in geophysics) are used as an alternative to explosive type signals, and have the advantage that their spectral characteristics can be accurately controlled in almost every regard including duration, bandwidth and energy. These signals are also used extensively for estimation of Doppler frequency shift in radar returns, and for tracking the narrow-band components of passive sonar. They also appear in various natural situations, such as in the echo-location systems of bats.

For these signals, the IF is an important characteristic; it is a time-varying parameter which defines the location of the signal's spectral peak as it varies with time. Conceptually it may be interpreted as the frequency of a sine wave which locally fits the signal under analysis. Physically, it has meaning only for monocomponent signals, where there is only one frequency or a narrow range of frequencies varying as a function of time. For multicomponent signals, the notion of a single-valued IF becomes meaningless, and a break-down into its components is needed.

A theoretical treatment of instantaneous frequency is presented in this paper, while in the sequel, the discrete instantaneous frequency (DIF) is defined, the estimation procedures for the IF are discussed, and a review of applications is presented.

II. The Concept of Instantaneous Frequency

A. The Concept of Frequency

In mechanics, the frequency of vibratory motion is defined as the number of oscillations per unit time, where vibratory motion is any to-and-fro motion and an oscillation is a complete to-and-fro motion. In one oscillation, the vibrating body moves from the equilibrium position to one end of the path, then to the other end of path, and finally back to the equilibrium position. Using this process as a model, then, frequency may be defined for any arbitrary vibratory motion.

A special type of vibratory motion is a simple harmonic motion in which the acceleration is proportional to the displacement and always directed toward the equilibrium position. When a body moves with uniform speed in a circle, the projection of this motion on a diameter is simple

Manuscript received January 25, 1991; revised November 6, 1991.
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IEEE Log Number 9107702.

harmonic motion (Fig. 1). At instant, t, the projection, P, has a displacement, velocity, and acceleration given by (1), (2), and (3) respectively:

$$s(t) = a_0 \cos\theta = a_0 \cos\omega t \quad (1)$$

$$s'(t) = -a_0\omega \sin\omega t \quad (2)$$

$$s''(t) = -a_0\omega^2 \cos\omega t = -\omega^2 s(t) \quad (3)$$

where the primes indicate the order of derivative. We can relate the frequency, $f = \omega/2\pi$, to displacement by solving differential equation (3). The solution is $z(t)$ given by

$$z(t) = \alpha e^{j2\pi ft} \quad (4)$$

where $\omega = 2\pi f$ is a uniform angular speed and α is an arbitrary constant. Equations (1)–(4) relate the concept of frequency to a practical example.

In many applications there are traveling waves in materials (solid bodies, atmosphere, etc.) in which the motion of a particle at any fixed point can be described by a simple harmonic motion. The frequency f of the wave motion is defined as the number of waves which pass by any fixed point per unit time. The frequency, f, of electric current in a circuit may be defined similarly as the number of cycles per unit time.

Let us now consider a signal $s(t)$ which presents a weighted sum (a mixture) of harmonic vibrations. Given such a signal, the signal analyst wants to find out its exact spectral decomposition. This can be done by using the Fourier transform (FT) of the signal defined as

$$S(f) = \int_{-\infty}^{+\infty} s(t)e^{-j2\pi ft}dt. \quad (5a)$$

The values of $S(f)$ characterize entirely the signal $s(t)$ so that it can be reconstructed using the inverse Fourier transform (IFT) given by

$$s(t) = \int_{-\infty}^{+\infty} S(f)e^{j2\pi ft}df. \quad (5b)$$

The analysis equation (5a) and the synthesis equation (5b) are meaningful only for stationary signals, i.e., for signals with spectrum $S(f)$ constant in time. Any stationary signal can be represented as the weighted sum of sine and cosine waves with particular frequencies, amplitudes and phases (note that for a particular frequency f, amplitude and phase of sine and cosine are constant).

Clearly, the concept of frequency is unambiguous. We will see in the next section that things are not so definite in the nonstationary case. The sections that follow attempt to clarify the issue and to remove a number of apparent ambiguities.

B. Generalization of the Concept of Frequency to Nonstationary Signals

Since frequency usually defines the number of cycles or vibrations undergone during one unit of time by a body in periodic motion, there is an apparent paradox in associating

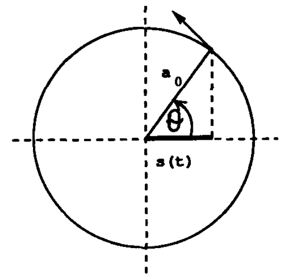


Fig. 1. Simple harmonic motion $s(t) = a_0 \cos\theta(t)$, where $\theta(t) = \omega_0 t$.

the words instantaneous and frequency. For this reason the definition of IF is controversial, application-related, and empirically assessed. This section presents a review of early contributions to the study of IF in an attempt to clarify the concept and its interpretation, and to address this paradox.

Carson and Fry in 1937 [11] considered a variable frequency in an electric circuit theory context and then applied the concept to frequency modulated (FM) signals. They defined an FM wave as

$$w(t) = \exp\left(j(\omega_0 t + \lambda \int_0^t m(t)dt)\right) \quad (6)$$

where $\omega_0 = 2\pi f_0$ is a constant carrier frequency, λ is a real parameter termed "the modulation index," and $m(t)$ represents a low-frequency signal to be transmitted ($|m(t)| \leq 1$). They defined the instantaneous angular frequency as

$$\Omega(t) = \omega_0 + \lambda m(t) \quad (7)$$

where $m(t)$ has the dimension of frequency, and the instantaneous cyclic frequency as

$$f_i(t) = f_0 + \frac{\lambda}{2\pi} m(t). \quad (8)$$

They argued that the notion of IF is a generalization of the definition of constant frequency, i.e., it is the rate of change of phase angle at time t. In 1946 Van der Pol [41] approached the problem of formulating a definition for the instantaneous frequency by analyzing an expression for simple harmonic motion:

$$s(t) = a\cos(2\pi ft + \theta) \quad (9)$$

where a is amplitude, f is frequency of the oscillation, θ is a phase constant, and the argument of the cosine function, namely $(2\pi ft + \theta)$, is the phase $\phi(t)$. He defined amplitude modulation by making the amplitude, a, vary as a function of t, as described in (10):

$$a(t) = a_0[1 + \mu g(t)] \text{ (amplitude modulation)} \quad (10)$$

where $g(t)$ is the modulating signal (Fig. 2). Similarly, he defined the phase modulation by

$$\theta(t) = \theta_0[1 + \mu g(t)] \text{ (phase modulation)} \quad (11)$$

so that the phase, which is the argument of the cosine function in (9), becomes: $\phi(t) = 2\pi ft + \theta(t)$ (Fig. 2).

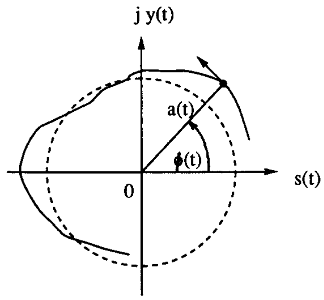


Fig. 2. Envelope and instantaneous phase of FM signal. FM signal is $s(t) = a(t) \cdot \cos\phi(t)$; $a(t)$—envelope; $\phi(t)$—phase.

In order to obtain an expression for frequency modulation, Van der Pol noted that it would be erroneous simply to substitute f in (9) by

$$f_i(t) = f_0[1 + \mu g(t)] \tag{12}$$

because it would lead to physical inconsistencies, since by substituting (12) into (9), the resultant phase does not yield (11). Instead, he reasoned that expression (9) for harmonic oscillations must be rewritten in the form:

$$s(t) = a\cos\left[\int_0^t 2\pi f_i(t)dt + \theta\right] \tag{13}$$

where the whole argument of the cosine function is the phase $\phi(t)$. This reasoning led him to the definition of the instantaneous frequency:

$$f_i(t) = \frac{1}{2\pi}\frac{d\phi(t)}{dt}. \tag{14}$$

Thus using a real representation of the signal, he arrived at a similar definition of IF to that of Carson and Fry [11] who used a complex representation of the signal.

The next important step in the study of IF was made by Gabor [20] who proposed a method for generating a unique complex signal from a real one. His method for doing so is first to find the FT of the real signal and then to "suppress the amplitudes belonging to negative frequencies and multiply the amplitudes of positive frequencies by two." He also showed that this procedure is equivalent to the following time-domain procedure:

$$z(t) = s(t) + jH[s(t)] \tag{15a}$$

$$= a(t)e^{j\phi(t)} \tag{15b}$$

where $z(t)$ is Gabor's complex signal, $s(t)$ is the real signal and H is the Hilbert transform (HT) (see Appendix A for its properties), defined as

$$H[s(t)] = \text{p.v.}\int_{-\infty}^{+\infty} \frac{s(t-\tau)}{\pi\tau}d\tau \tag{16}$$

where p.v. denotes the Cauchy principle value of the integral [22b]. Signals, $s(t)$ and $H[s(t)]$, are often said to be in quadrature, because in theory they are out of phase by $\pi/2$. However, in reality this is true only under certain conditions (discussed in Section III-A-1)). Gabor's motivation to define and work with a complex signal was that only by doing so was he able to define the central moments of frequency of the signal:

$$\langle f^n\rangle = \frac{\int_{-\infty}^{+\infty} f^n|Z(f)|^2 df}{\int_{-\infty}^{+\infty} |Z(f)|^2 df}. \tag{17}$$

Here $Z(f)$ is the spectrum of the complex signal. If the spectrum of the real signal was used in (17), all odd moments would be zero since $|S(f)|^2$ is even, and this would not fit well with physical reality. Gabor's complex signal is referred to as the "analytic signal" (see Appendix A).

Ville [42] unified the work done by Carson and Fry [11] and Gabor [20] and defined the IF of a signal expressed by $s(t) = a(t)\cos\phi(t)$ as

$$f_i(t) = \frac{1}{2\pi}\frac{d}{dt}[\arg z(t)] \tag{18}$$

where $z(t)$ is the analytic signal given by (15). Ville went further and noted that since the IF was time-varying, there should intuitively be some instantaneous spectrum associated with it—with the mean value of the frequencies in this instantaneous spectrum being the IF. Using Gabor's average measures [20], he showed that the average frequency in a signal's spectrum was equal to the time average of the IF [42]:

$$\langle f\rangle = \langle f_i\rangle \tag{19}$$

where

$$\langle f\rangle = \frac{\int_{-\infty}^{+\infty} f|Z(f)|^2 df}{\int_{-\infty}^{+\infty} |Z(f)|^2 df} \tag{20a}$$

$$\langle f_i\rangle = \frac{\int_{-\infty}^{+\infty} f_i(t)|z(t)|^2 dt}{\int_{-\infty}^{+\infty} |z(t)|^2 dt}. \tag{20b}$$

(Note that (20a) is an averaging over frequencies and that (20b) is an averaging over time). Using these results, Ville formulated a distribution of the signal in time and frequency now commonly referred to as the Wigner–Ville Distribution (WVD) [42]:

$$W(t,f) = \int_{-\infty}^{+\infty} z(t+\tau/2)z^*(t-\tau/2)e^{-j2\pi f\tau}d\tau. \tag{21}$$

(Note that $W(t,f)$ is the FT with respect to τ of the product $z(t+\tau/2)\cdot z^*(t-\tau/2)$ and is numerically evaluated using FFT algorithms [6]). Ville showed [42] that the first moment of the WVD with respect to frequency yields the IF:

$$f_i(t) = \frac{\int_{-\infty}^{+\infty} fW(t,f)df}{\int_{-\infty}^{+\infty} W(t,f)df} \tag{22}$$

1) Interpretation of Instantaneous Frequency: To provide insight into the meaning of the IF, let us consider the problem of positioning a signal, $s(t)$, in the frequency domain. We construct the analytic signal, $z(t) = a(t) \cdot e^{j\phi(t)}$, as defined in (15). Its spectrum $Z(f)$ is given by

$$Z(f) = \int_{-\infty}^{+\infty} z(t) e^{-j2\pi f t} dt \tag{23}$$

$$= \int_{-\infty}^{+\infty} a(t) e^{j[\phi(t) - 2\pi f t]} dt. \tag{24}$$

The application of the stationary phase principle (see Appendix B) indicates that this integral will have its largest value at the frequency, f_s, for which the phase is stationary, i.e., for f_s such that

$$\frac{d}{dt}[\phi(t) - 2\pi f_s t] = 0 \tag{25}$$

which leads to

$$f_s = \frac{1}{2\pi} \frac{d\phi(t)}{dt.} \tag{26}$$

This indicates that if f_s is a function of t, $f_s(t)$ provides a measure of the frequency domain signal energy concentration as a function of time. This measure equals the IF of the signal; this property explains the importance of the IF in signal recognition, tracking, estimation, and modeling.

However, the interpretation of the IF is often a subject of controversy. Shekel [35], for example, argued that the IF defined by

$$f_i(t) = \frac{1}{2\pi} \frac{d\phi(t)}{dt} \tag{27}$$

is not a unique function of time, since any amplitude modulated (AM) wave, written in complex form, may be expressed as either $m(t)e^{j2\pi f t}$ or $m_0 e^{j\phi(t)}$. The latter expression represents a wave with a constant amplitude and a complicated IF law, while the former expression represents a wave with complicated amplitude law and a constant IF. The implication is that there are many ways of constructing a complex signal starting from a real one. A unique complex representation of a signal is obtained by using the HT, as Gabor and Ville have noted [20], [42], but whether or not it corresponds to any physical reality is another question. This point is discussed further in Section III.

Mandel [28] also challenged any physical interpretation of the IF. He discussed its relationship with spectral frequency in terms of Fourier decomposition. He argued that there is no one-to-one relationship between these two kinds of frequencies and provided examples of signals to prove it. He also claimed that the only similarity between the IF and Fourier decomposition frequency is that the average frequency of the Fourier spectrum equals the time average of the IF. This coincidence, however, does not extend to higher order moments (as Ville [42] showed for the second moment). Let us consider the signal Mandel chose to illustrate his point:

$$z(t) = a_1 e^{(j\omega_0 - \Delta\omega/2)t} + a_2 e^{j(\omega_0 + \Delta\omega/2)t}. \tag{28}$$

If we express $z(t)$ in terms of envelope and phase $a(t) \cdot e^{j\phi(t)}$ we have

$$\phi(t) = \tan^{-1} \frac{(-a_1 + a_2) \sin(\Delta\omega/2)t}{(a_1 + a_2)\cos(\Delta\omega/2)t} \tag{29}$$

and now the instantaneous frequency given by Ville's definition (18) is

$$f_i(t) = f_0 + \frac{\Delta f}{2} \frac{-a_1{}^2 + a_2{}^2}{a_1{}^2 + a_2{}^2 + 2a_1 a_2 \cos \Delta\omega t}. \tag{30}$$

The Fourier spectrum of $z(t)$ is obvious from (28)—it consists of two components symmetrically placed with respect to f_0. The IF given by (30) varies with time, but excursions of $f_i(t)$ about f_0 are not symmetrical; "they are entirely upwards if $a_2 > a_1$ and entirely downwards if $a_1 > a_2$" [28]. However, in this example, we argue that the analytic signal in (28) corresponds to the bicomponent real signal $s(t) = s_1(t) + s_2(t)$. Hence, the IF is not meaningful for $s(t)$ but only for the single component signals $s_1(t)$ and $s_2(t)$ taken separately. Therefore, the IF expressed by (30) is outside the scope of the original definition (see Section II-C).

Mandel strongly promoted the idea that the IF and Fourier frequencies are different quantities, and that one source of their mutual confusion is the same name—frequency—attached to both of them. Finally, Mandel asks a question: Which of these two quantities is most closely related to measurements? He also provides the answer: It strongly depends "on the nature of the experiment."

Priestley indicated [32] that a nonstationary process in general cannot be represented in a meaningful way by the simple Fourier expansion as described by (5b). For example, consider the nonstationary signal with time-varying amplitude:

$$y(t) = A \cdot e^{-t^2/\alpha^2} \cos(2\pi f_0 t + \phi_0). \tag{31}$$

The FT of $y(t)$ consists of two Gaussian functions centered at f_o and $-f_o$ and thus it contains Fourier components at *all* frequencies. It is possible to use an alternative form for representing $y(t)$: it consists of just two "frequency" components (at f_o and $-f_o$), with each component having a time-varying amplitude $A \cdot e^{-t^2/\alpha^2}$. These two representations of $y(t)$ are equally valid. They correspond to different "families" of basic orthogonal functions used for representation. In the former case, the family consists of sines and cosines with constant amplitudes, and in latter case it consists of sines and cosines with time-varying amplitudes.

According to the conventional definition, the term "frequency" is associated specifically with the sine and cosine functions. In order to apply the notion of frequency in the analysis of nonstationary signals, it is necessary to introduce a new basic family of functions which must be nonstationary and still have an oscillatory form so that the notion of "frequency" is applicable. Thus Priestley suggested that an arbitrary nonstationary signal $s(t)$ can be represented as

$$s(t) = \int_{-\infty}^{+\infty} \gamma_t(f) S(f) df \tag{32}$$

where

$$\gamma_t(f) = A_t(f) \cdot e^{j2\pi f t}. \tag{33}$$

Priestley's approach leads to a model where signals are locally represented by a frequency and a spread about that frequency. In Ville's approach, frequency is always defined as the first derivative of the phase, regardless of stationarity. The two models are related, but Priestley's is readily applicable to multicomponent signals, while Ville's is valid only for monocomponent signals.

Another understanding of the meaning of the IF has been proposed by Cohen. He developed a generalized formulation for the distribution of energy in time and frequency [14]. The IF can then be considered to be the average of the frequencies that exist in the time-frequency plane at a given time [17]. This can be expressed as

$$\langle f_i \rangle_t = \frac{\phi'(t)}{2\pi}. \tag{34}$$

Several other authors have also contributed to the study of IF [1], [21], [22], [24], [34]. Rihaczek [34] investigated the points in the time-frequency plane of his complex energy distribution where the signal energy is concentrated. He showed that the energy was concentrated in frequency about the IF. Ackroyd [1] extended this result by showing that the first moment of the Rihaczek Distribution yields the IF, regardless of how fast the IF varies. The problems associated with defining the IF are analogous to the problems associated with defining the instantaneous amplitude or envelope [34a]. A review of frequency, amplitude and phase in the context of oscillation theory is provided in [40a]. Other authors have contributed to the development of the discrete IF, which will be dealt with in the sequel [8].

C. Model, Notations, Definitions

The review in the previous section shows that there are many ways of generalizing stationary models to the nonstationary case. In the following sections of this paper, we will restrict nonstationary signals to the class of FM signals, i.e., signals $s(t)$ which can be represented by the following model:

$$s(t) = \sum_{k=1}^{N} s_k(t) + n(t) \tag{35}$$

where $n(t)$ is a noise representing any undesirable components, and the $s_k(t)$ are N single component nonstationary signals described by envelopes $a_k(t)$ and instantaneous frequencies $f_{i_k}(t)$ such that the analytic signal $z_k(t)$ associated with $s_k(t)$ may be expressed as

$$z_k(t) = a_k(t) \cdot e^{j\phi_k(t)} \tag{36}$$

where

$$\phi_k(t) = 2\pi \int_{-\infty}^{t} f_{i_k}(\tau)\, d\tau. \tag{37}$$

In this model if $k = 1$, the signal may be referred to as monocomponent signal; if $k \geq 2$, the signal is referred to as multicomponent signal. The model defined by (35), (36), and (37) applies for multicomponent signals and allows the modeling of N time-varying frequency laws. Priestley's model is different in that it considers all frequencies to exist and assigns an amplitude to each one of them and, if they are nonexistent, they are assigned zero amplitude.

III. Instantaneous Frequency, Group Delay, the HT and the Analytic Signal

A. The HT and the Analytic Signal

Although Gabor provided a good intuitive justification for his introduction of the HT-based analytic signal, it is important to note that one may define other methods for constructing a complex signal from a real one. For example, one may use the alternative procedure of taking the quadrature component of $s(t)$ as the imaginary part of the complex signal as will be described in later sections. This leads in general to a signal different from that defined by Gabor and Ville. Further insight into this ambiguity may be gained by realizing that, given some real data represented by **one** function of time, we want to determine **two** functions of time that characterize the data. These two functions may represent either the envelope and phase or the real and imaginary parts. The nonuniqueness of the transformation can be seen by considering the following signal:

$$s(t) = \cos(2\pi f_1 t)\cos(2\pi f_2 t); \qquad f_2 > f_1. \tag{38}$$

One could either define the phase as $2\pi f_1 t$ with amplitude modulation, $a(t) = \cos 2\pi f_2 t$, or the phase as $2\pi f_2 t$ with amplitude modulation, $a(t) = \cos 2\pi f_1 t$.

It is not surprising, then, that this so-called analytic signal does not always correspond to a signal plus its quadrature component, a fact which is significant, given the widespread use of this method in areas such as communication theory. The reason is that constructing the analytic signal via the HT operation is equivalent to eliminating negative frequencies of the spectrum. If there is any significant leakage of the "positive" spectral components into the negative spectral region, then the HT will not yield the quadrature component of the input signal. As a consequence, any use of the IF may lead to doubtful results if one is not careful.

1) Conditions under which the HT Exactly Generates the Quadrature Component: Let us consider a real FM signal of the form, $a(t) \cdot \cos \phi(t)$, and let us see under what conditions the following equation can be verified:

$$a(t)\cos\phi(t) + jH[a(t)\cos\phi(t)] = a(t)e^{j\phi(t)}. \tag{39}$$

The problem was investigated for complex finite-energy signals in [2], [9], [30], and [33]. The solution is found by using Bedrosian's product theorem (BPT) summarized in Appendix C. It leads to the following result:

Equation (39) is valid if the spectrum $A(f) = F\{a(t)\}$ lies entirely in the region $|f| < f_0$ and the spectrum $F\{\cos\phi(t)\}$ exists only outside this region.

Proof: If the first condition of the BPT (see Appendix C) is fulfilled we can write:

$$\begin{aligned} a(t)\cos\phi(t) + jH\ \{a(t)\cos\phi(t)\} &= a(t)\cos\phi(t) + ja(t)H\ \{\cos\phi(t)\} \\ &= a(t)\cos\phi(t) + ja(t)\sin\phi(t) \\ &= a(t)\cdot e^{j\phi(t)} \end{aligned}$$

It can be seen from these equations that the HT-based analytic signal generator is a type of high frequency selector, as the predominantly high frequency portion of the signal becomes the complex phase term [24], [42]. This is illustrated further by considering again the signal in (38):

$$s(t) = \cos(2\pi f_1 t)\cos(2\pi f_2 t); f_2 > f_1 \quad (40)$$

The analytic signal produced via the HT is of the form:

$$z(t) = \cos(2\pi f_1 t)e^{j2\pi f_2 t}. \quad (41)$$

Thus the HT method inherently selects the highest frequency cosine for replacement by an exponential.

2) Interpretation of the HT-Generated Analytic Signal?: The HT and the analytic signal are not always interpretable in a way which is meaningful and representative of physical phenomena. As explained previously, if we have a modulated signal of the form, $a(t)e^{j\phi(t)}$, where physical meaning is attached to $a(t)$ and $\phi(t)$, and if the spectra of $a(t)$ and $\phi(t)$ are not separated in frequency, then the HT will be a result of overlapping and phase-distorted functions. Although the analytic signal will be of the form:

$$a_z(t)e^{j\phi_z(t)} \quad (42)$$

and will be unique, $a_z(t)$ and $\phi_z(t)$ will have questionable practical meaning. This leads to the assertion:

The amplitude $a(t)$ and phase $\phi(t)$ of a signal may only be considered independently if the spectra of $a(t)$ and $\cos\phi(t)$ are separated in frequency.

Thus the more closely a signal approaches a narrowband condition, the better the Hilbert Transformed signal approximates the quadrature signal, and the more likely the HT-based analytic signal is to provide an accurate model of a real system with a particular IF; also the better in general will be the estimate of instantaneous frequency. This will be illustrated in Section IV-D by examples.

B. Instantaneous Frequency and Group Delay

1) Definitions: This section relates the widely used concepts of instantaneous frequency and group-delay. Consider a signal $s(t)$, with $z(t)$ its corresponding analytic signal. The instantaneous frequency of $s(t)$ is defined by the derivative of the phase of $z(t)$ as expressed in (18), or in a form which will be useful for discrete-time implementation (see [8, pt. 2]):

$$\begin{aligned} f_i(t) = \lim_{\delta t \to 0} \frac{1}{4\pi\delta t}(\arg[z(t+\delta t)] \\ - \arg[z(t-\delta t)]) \bmod 2\pi \end{aligned} \quad (43)$$

where the notation $\pmod{2\pi}$ represents a modulo 2π operation to account for $\arg[z(t)]$ being defined on $[-\pi, +\pi)$. The complex signal $z(t) = a(t)e^{j\phi(t)}$ has a complex spectrum of the form:

$$Z(f) = A(f)e^{j\theta(f)} \quad (44)$$

where $a(t)$ and $A(f)$ are positive functions. Another quantity of interest often used to describe the signal is the group delay (GD) defined by

$$\tau_g(f) = -\frac{1}{2\pi}\frac{d}{df}\theta(f). \quad (45)$$

The group-delay may be interpreted as representing the variation of the propagation time as a function of the frequency of an impulse travelling through a linear filter with impulse response, $h(t) = s(t)$. Equation (45) can be rewritten similarly to (43) as

$$\begin{aligned} \tau_g(f) = \lim_{\delta f \to 0} -\frac{1}{4\pi\delta f}(\arg[Z(f+\delta f)] \\ - \arg[Z(f-\delta f)]) \bmod 2\pi. \end{aligned} \quad (46)$$

Equations (43) and (46) are useful because they can be translated directly to the discrete-time case. The relationship between the laws can be seen by noting that for a general complex signal, the following relationships hold:

$$\begin{aligned} F\{a(t)e^{j\phi(t))}\} &= F\{a(t)\} \underset{f}{*} F\{e^{j\phi(t)}\} \\ &= A(f)e^{j\theta(f)} \end{aligned} \quad (47)$$

$$\begin{aligned} F^{-1}\{A(f)e^{j\theta(f)}\} &= F^{-1}\{A(f)\} \underset{t}{*} F^{-1}\{e^{j\theta(f)}\} \\ &= a(t)e^{j\phi(t)} \end{aligned} \quad (48)$$

where F and F^{-1} denote the FT and the IFT, respectively, and where $\underset{t}{*}$ and $\underset{f}{*}$ denote the convolution in time and frequency, respectively.

It is clear that the phase spectrum, $\theta(f)$, and hence the GD, depend on both the phase and amplitude of the time signal; and the signal phase, $\phi(t)$, and hence the IF, also depend on both phase spectrum and amplitude spectrum.

2) Interpretation of Group-Delay: Given a signal $s(t)$, the value of $t = \tau_g$, when assumed to be unique, is considered to describe the localization of the signal in the time domain. If τ_g is a function of f, then it describes the localization of various spectral components of the signal in the time domain.

Proof: Consider $z(t)$ expressed in terms of its FT, $Z(f) = |Z(f)|e^{j\theta(f)}$

$$z(t) = \int_0^\infty Z(f)\ e^{j2\pi ft}\ df \quad (49)$$

$$= \int_0^\infty |Z(f)|\ e^{j[\theta(f)+2\pi ft]}df. \quad (50)$$

In application of the stationary phase principle (see Appendix B), if $|Z(f)|$ varies slowly with f while $\cos[\theta(f) + 2\pi f t]$ varies rapidly with f (i.e., it has many alternations), then the largest value of the integral (50) is obtained for the value of t which satisfies:

$$\frac{d}{df}[2\pi f t + \theta(f)] = 2\pi t + \frac{d\theta(f)}{df} = 0 \tag{51}$$

for which the solution is

$$\tau_g = -\frac{1}{2\pi}\frac{d\theta(f)}{df}. \tag{52}$$

Q.E.D.

IV. Influence of the Bandwidth-Time (*BT*) Product on the Notions of Instantaneous Frequency and Group Delay

This section relates the previous concepts under discussion to the BT product of the signal. This is done so that one may take into account the practical limitations of signals such as finite duration and finite bandwidth in the definitions and estimation procedures for instantaneous frequency and related quantities.

A. Bandwidth, Duration, and Asymptotic Signals

1) B and T: The duration, T, and the bandwidth, B, are broad characteristics of the signal and respectively provide some measure of the "epoch" of the signal and its spectrum. In practice, if the value of the signal in the time domain decreases below a certain threshold, then, we can say that the signal is negligible. This threshold is determined by ambient noise that is always present. Sometimes, we consider that the signal is negligible even before its level goes below that of the ambient noise; in other cases, even if the signal is immersed in noise, a signal can be detected. Note that similar reasoning applies to the frequency domain. A threshold determined by the noise levels influences decisions regarding the definition of the signal bandwidth.

Finally, the problem of defining the duration of a signal is a question of convention. It is the same as for the bandwidth. Everything depends on the particular application considered, and one should adapt definitions in the way which is most appropriate for the particular question being considered (see [7] for more details).

Several definitions are available for the measure of B and T [37a], [22a]. A definition commonly used was proposed by Gabor [20]: Let $s(t)$ be a zero-mean finite-energy signal; the effective duration, T_s, and the effective bandwidth, B_s, are respectively given by

$$T_S{}^2 = \frac{\int_{-\infty}^{\infty} t^2|s(t)|^2 dt}{\int_{-\infty}^{\infty} |s(t)|^2 dt} \tag{53}$$

$$B_S{}^2 = \frac{\int_{-\infty}^{\infty} f^2|S(f)|^2 df}{\int_{-\infty}^{\infty} |S(f)|^2 df}. \tag{54}$$

2) Relationship between B and T: Let us consider a signal $s(t)$ of duration T in the strict sense, and let us dilate (stretch) or compress $s(t)$ along the time axis, but without changing the amplitude of the signal.

If $s_k(t) = s(kt)$ is the dilated (stretched) signal, then the spectrum $S_k(f)$ is given by

$$S_k(f) = \frac{1}{k}S\left(\frac{f}{k}\right), \quad (k > 0).$$

If the time scale is multiplied by k, the frequency scale is divided by k, i.e., a dilation of the signal in time leads to a compression in frequency. This suggests the possibility of a relationship of the type BT = constant, the constant depending on the signal. With the definitions of equivalent bandwidth and equivalent duration given in (53) and (54) respectively, we know that this product is a constant, and that the minimum value of $1/4\pi$ is attained for a Gaussian signal [20], [36].

Several authors have tried to develop a theoretical basis for the imposition of finite duration and finite bandwidth. A review is given in [7]. Lacoume and Kofman [27] formally defined a signal class C_{BT} as the class of finite energy signals whose energy is approximately localized in the band $[-B/2, B/2]$ and in time $[-T/2, T/2]$. Almost all signals encountered in practical applications belong to this class. One may also define a subset of this class, to encompass what are known as "asymptotic" signals. These signals, in addition to having finite duration, bandwidth and energy, are characterized by a large BT product. We will consider signals with a BT product sufficiently high $(BT > 10)$ so that the approximation error involved in assuming band and time limited functions is very small. (Landau and Pollak [38] have found that more than 99% of signal energy is preserved within the limits of B and T if $BT > 5$). Note that FM signals used in communications and seismic belong to this class. All signals considered subsequently will be assumed to be asymptotic.

Another means of understanding the influence of the BT product on the IF and GD is to think of the BT product as providing an indication of the "richness" of information contained in the signal. This product could be compared with the number of realizations needed to estimate the probability density function (PDF) of a stochastic process. With only a few experiments available, the estimate does not really make sense, and it would make a poor basis for defining the process. Similarly, if there is a short signal, characterization by its IF has limited meaning, because there is not enough data to observe any variation. On the other hand, if the signal has a longer duration, then its IF becomes more meaningful.

B. Useful Definitions and Properties for Asymptotic Signals

We give, as follows, a number of definitions and results that have proved useful in practice when dealing with the IF's of real signals.

Definition 1: An asymptotic signal, $s(t)$, is referred to as a monocomponent (or invertible) signal if for that signal, the instantaneous frequency, $f_i(t)$, accurately represents the frequency modulation law of the signal and is single-valued and invertible, so that the function $f_i^{-1}(f)$ exists (see Fig. 3(a))

Definition 2: An asymptotic signal, $s(t)$, is referred to as a multicomponent signal if there exists a finite number, N, of monocomponent signals, $s_i(t), i = 1, N$ such that the relation $s(t) = \sum_{i=1}^{N} s_i(t)$ holds for all values of t for which $s(t)$ is defined, i.e., if $s(t)$ can be characterized as the sum of several monocomponent signals, and such that the decomposition is meaningful. Only one of $s_i(t)$ need to be asymptotic. This decomposition is not unique; it is application dependent (see Figs. 3(b) and (c)).

Often, in the process of breaking up a signal into meaningful components, it is the continuous pattern representing the variation as a function of time of the energy concentration in the $t - f$ domain which determines whether there is only one or several components. This energy concentration may be measured by the local bandwidth or spread about the instantaneous frequency of the signal (or its subcomponents in the case of a multicomponent signal). This is illustrated in Figs. 4(a) and (b), and the question is studied in detail in [7], [10], and [16]–[18]. See also Section V-A for a discussion in the context of time-frequency distributions.

Property 1: The energy distribution of an asymptotic signal, $s(t)$, is concentrated in a finite domain (time-bandwidth) of the time-frequency $(t - f)$ plane about the IF, and the degree of concentration is a function of the BT product [7], [38] (see Fig. 5).

Property 2: For a monocomponent asymptotic signal with BT large and with IF law monotonic, it was shown that $f_i(t)$ approaches $\tau_g^{-1}(f)$ [9]; i.e., these two functions are the inverse of each other. In this case, the $t-f$ laws have a physical meaning: the instantaneous frequency, $f_i(t)$, describes the frequency modulation law of the signal, $s(t)$, and $\tau_g(f)$ represents the time delay of the signal (see Fig. 6(b)).

Property 3: For a monocomponent signal, if BT is small, then $f_i g \neq \tau_g^{-1}$; i.e., these two functions are not the inverse of each other. In this case, no physical meaning can be associated with these $t - f$ laws, although they are mathematically well defined [26] (Fig. 6(a)). This question is also studied in [5].

Note that there are two requirements for the IF and GD laws to be the inverse of one another. The first is that the variations in time of the IF be monotonic and the second is that BT be large. The first condition ensures that every image of mapping f_i has only one preimage in time space T. Large BT, the second condition, ensures that every element in T is mapped to a single image in B. This requirement can be understood via the stationary phase principle which states that for large BT value signals, at each instant of time t_0, only one frequency (say f_0) is dominant in the spectrum of signal. For small BT value signals, the IF does not admit a one-to-one mapping because at particular time instant, there is no dominant frequency.

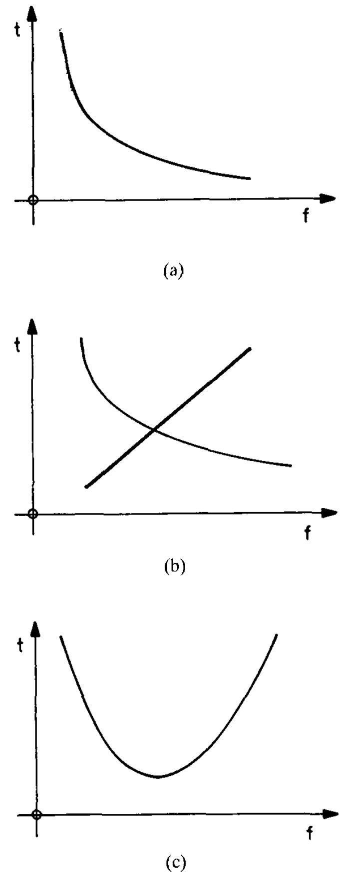
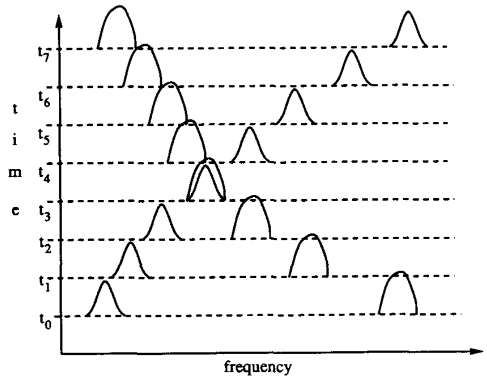


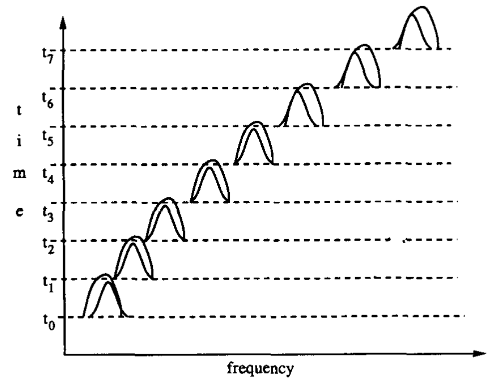
Fig. 3. Monocomponent and multicomponent signals in time-frequency plane (a) example of monocomponent signal; (b), (c) examples of multicomponent signals.

The fact that the IF and the GD do not map to a single law for low BT signals may seem surprising, since intuitively, the two quantities portray the same information—the time-frequency characterization of the signal. The discrepancy can be explained by the way the GD is defined. It is based on the phase of the FT, which for small BT signals, exhibits significant oscillation due to the Gibbs phenomenon [31]. It is tempting to try to define the GD via a different spectral representation which might remove the "spurious" effects. These oscillation are real, though, and can be related to some physical limitations of filters. The differences then, between the IF and the GD laws for low BT, are in some senses a necessary reflection of the practical limitations for short duration sequences.

When a signal has a defined inverse function of IF, that inverse function is identical to GD [7], [9]. However, there still remains the question: What is the relationship between GD and IF when conditions: 1) monotonic temporal variation of IF; 2) large BT, are not satisfied? This question is not answered yet. The results of time-frequency signal analysis (using time-frequency distributions), intuitively suggest that in the frequency domain the GD should be the



(a)



(b)

Fig. 4. Separability of signal components (a) multicomponent signal—overall the components are well separated; (b) monocomponent signal—overall the components are not well separated.

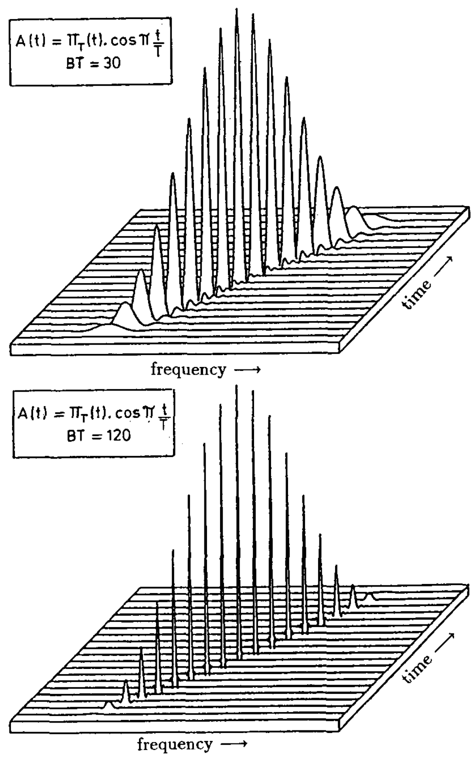


Fig. 5. Time-frequency representation for signals with different BT value. Concentration of time-frequency distribution increases with the BT.

corresponding concept to that of the IF in the time domain. The problem is that if the IF law is not monotonic, the GD is not a single-valued function and therefore can not be expressed in explicit form, so that definition of the GD (45) is not correct. In addition, one can argue that even the definition of the IF (18) is not correct for small BT signals and for multiple component signals.

Figure 7 presents the results of an experiment set up to gain a better understanding of the relationship between IF and GD. In this experiment, we construct three signals of the form given by (13) generated by N successive delayed sinusoids whose frequencies increase linearly as a function of time so that the IF can be expressed as

$$f_i(t) = f_1 + (i-1)\frac{f_2 - f_1}{N} \qquad (55)$$

where $((i-1)T)/N < t < iT/N; B = f_2 - f_1 > 0$ and $i = 1, 2, \ldots, N$. The amplitudes of the sinusoids are constants. These signals are shown in Figs. 7(d), (e), and (f) for N equal to 5, 10, and "infinity," respectively. Their spectra are plotted in Figs. 7(a), (b), and (c). The signal shown in Fig. 7(d) is constructed by adding five sinusoids which are delayed linearly. The signal in Fig. 7(e) is constructed by adding ten such sinusoids. The signal in Fig. 7(f) is constructed by adding an "infinite" number of such sinusoids, i.e., by making the frequency increasing linearly with time. The estimated IF and GD laws of these three signals given by (13) and (55) are shown on Figs. 7(g), (h), and (i). The solid line represents the IF (in coordinate system—frequency versus time) whereas the dotted line represents the GD (in coordinate system—time versus frequency) using the same axes and the scales. By construction, the instantaneous frequency of these signals varies linearly as a function of time. One can also observe on Figs. 7(g), (h), and (i) that as a consequence the group delay is also linear. This may seem to be a natural result valid for any value of the BT product, except for the fact that one cannot construct a pure frequency within a short duration. A large duration is needed to construct the proposed signal, i.e., one needs a large BT product.

C. Influence of the BT Product and the Signal Central Frequency on the Analytic Signal Formation

This section extends the results of Section III to the class of double limited signals, that is signals which are almost band-limited and time-limited. For this class, we can state the following theorems.

Theorem 1: For a signal of the form, $s(t) = a(t)\cos(2\pi f_0 t + \phi(t))$, where f_0 is the constant or tonal frequency,

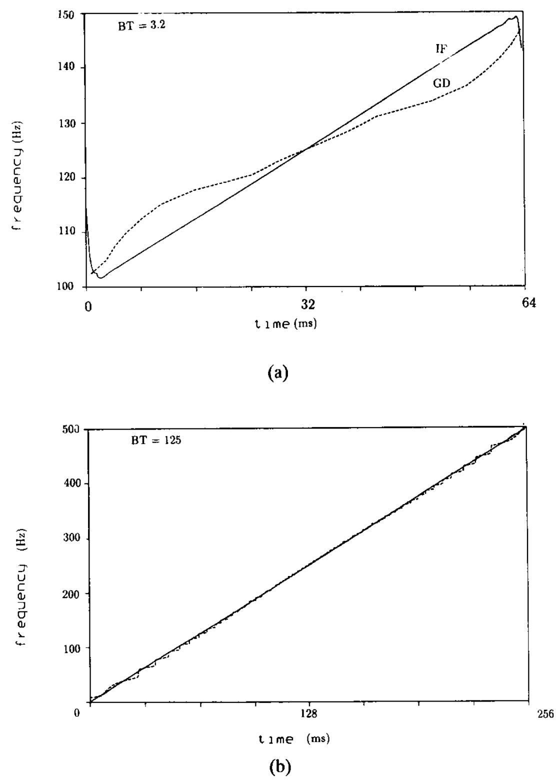


Fig. 6. Relationship between instantaneous frequency and group-delay. Estimated laws of instantaneous frequency (solid line) and group-delay (dashed line) for (a) small BT signal; (b) large BT signal. Note that IF and GD are plotted in the same coordinate system, such that *frequency* is the ordinate for IF and *time* is the ordinate for GD.

with an arbitrary BT value, the analytic signal derived using a HT approaches an exact representation of the signal plus its imaginary quadrature part, $a(t)e^{j(2\pi f_0 t+\phi(t))}$, as f_0 becomes large and approaches infinity.

Proof: The proof is based on work by Nuttall [30], where he shows that the difference in energy between the HT-generated analytic signal and the signal plus its imaginary quadrature part is the energy in the spectrum of $S_0(f) = F\{a(t)e^{j\phi(t)}\}$, below $f = -f_0$. Thus the representation becomes exact as f_0 approaches infinity. This result is clearly applicable to the class of signals considered here.

Theorem 2: If the signal, $s(t) = a(t)cos\phi(t)$, has a monotonic purely positive frequency law ($d\phi/dt \geq 0$, for all t), and a large BT product, then the analytic signal generated by applying the HT approaches $z(t) = a(t)e^{j\phi(t)}$ asymptotically as BT approaches infinity.

Proof: Jones [25] gave a full proof. The basic steps are as follows. First we find the FT:

$$S(f) = F\{a(t)\cos\phi(t)\} = F\left\{\frac{1}{2}a(t)[e^{j\phi(t)} + e^{-j\phi(t)}]\right\}. \quad (56)$$

This can be done using the stationary phase principle. A large BT value implies a very dispersive phase characteristic $\phi(t)$, so that condition (B5) from Appendix B is fulfilled (note that $BT \gg 1$ and condition (B5) are asymptotically equivalent). If instantaneous frequency $\phi\prime(t)/2\pi$ is a monotonic function of time (i.e., there is only one stationary phase point) [40] then the Fourier integrals in (56) can be approximated as

$$S(f) = S_1(f) + S_2(f) \quad (57)$$

$$S(f) = \frac{a(t_0)}{2}\left(\frac{2\pi}{|\phi''(t_0)|}\right)^{1/2} e^{j[\phi(t_0)-2\pi f t_0 \pm \pi/4]} + \frac{a(t_1)}{2}\left(\frac{2\pi}{|\phi''(t_1)|}\right)^{1/2} e^{j[-\phi(t_1)-2\pi f t_1 \pm \pi/4]} \quad (58)$$

where t_o and t_1 are stationary phase points for the first and the second term, respectively. Here the plus sign applies when $\phi''(t_0) > 0$, and the minus sign when $\phi''(t_0) < 0$. The stationary phase points satisfy:

$$\phi'(t_o) = 2\pi f \quad (59)$$

$$\phi'(t_1) = -2\pi f. \quad (60)$$

Due to the assumption that instantaneous frequency $\phi\prime(t)/2\pi$ is positive, (59) has solutions only for positive frequencies. Thus $S_1(f) = 0$ for $f \leq 0$. By similar reasoning one can show that $S_2(f) = 0$ for $f \geq 0$. Now since the FT of the HT-based analytic signal $z(t)$ is [31]:

$$Z(f) = \begin{cases} 0; & f < 0 \\ S(f); & f = 0 \\ 2S(f); & f > 0 \end{cases} \quad (61)$$

it follows that

$$Z(f) = 2S_1(f) \quad (62)$$

and recalling that $S_1(f) = 1/2\ F\{a(t)e^{j\phi(t)}\}$ we prove that $z(t) = a(t)e^{j\phi(t)}$.

D. Practical Consideration of Theorems for forming the Analytic Signal

In order to illustrate the effect of the BT value of the signal (Theorem 2), and the central frequency of the signal (Theorem 1) on the analytic signal generation and IF estimation accuracy, two experiments were performed by computer simulations. In both of them we consider signal of the form:

$$s(t) = \Pi_T(t - T/2) \cdot \cos(2\pi f_o t + 1/2\alpha t^2) \quad (63)$$

(with positive and monotonic IF law), where $\Pi_T(t)$ is a box car function indicating that $s(t)$ has a finite duration T, i.e., $\Pi_T(t) = 1$ for $-T/2 \leq t \leq T/2$ and $\Pi_T(t) = 0$ elsewhere.

There are two types of complex signals related to the real signal $s(t)$. One is the so called "real plus imaginary quadrature" (RQ) complex signal, obtained by substitution of cosine in (63) with exponential (note that in practice only a set of real data is given, and this type of substitution is not possible because the mathematical expression (63)

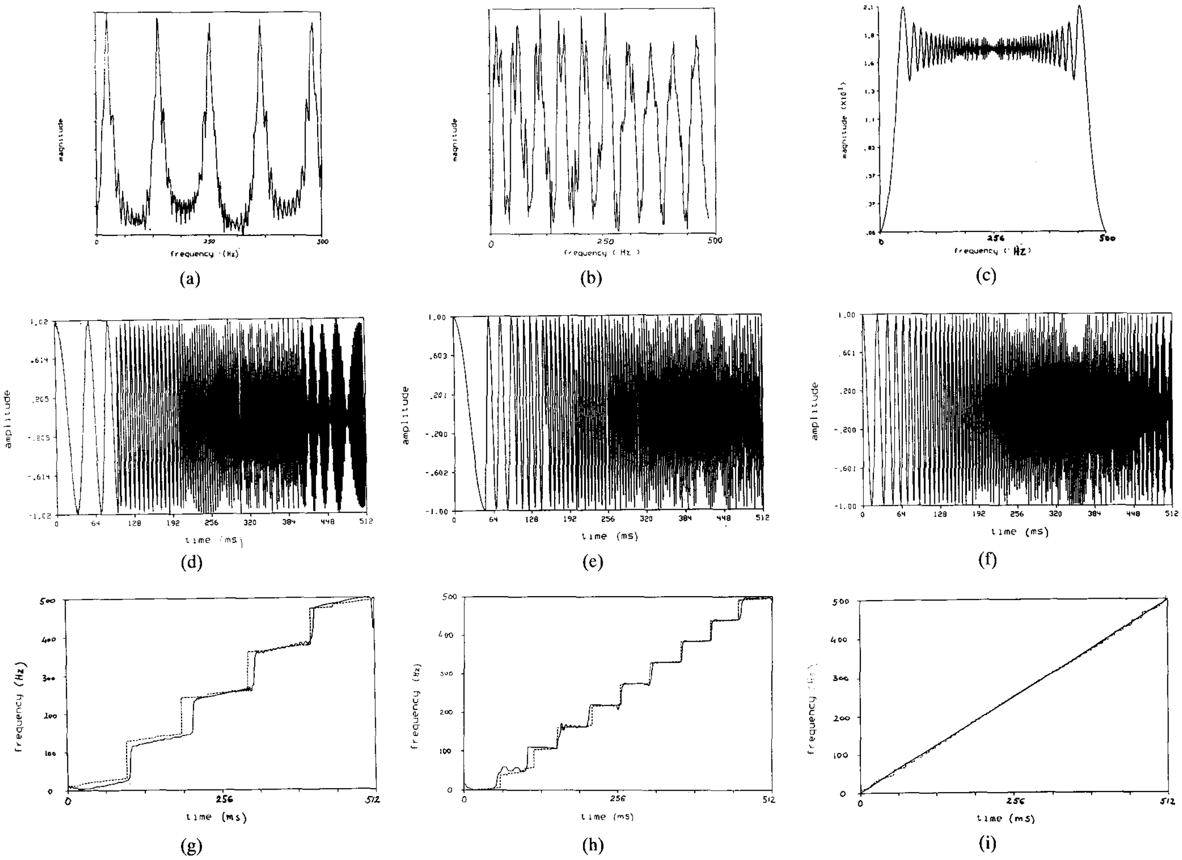


Fig. 7. Instantaneous frequency and group delay of sinusoidal sweeps. Estimated laws of instantaneous frequency (solid line) and group-delay (dashed line) for: (a),(d),(g) 5 steps; (b),(e),(h) 10 steps; (c),(f),(i) "infinite" number of steps. The two laws approach each other asymptotically.

is not available). Another complex signal related to $s(t)$ is the analytic signal generated by the HT. The RQ complex signal is considered to be the ideal signal that one would want to use. The analytic signal correctly approximates the RQ complex signal under the conditions stated by the Theorems 1 and 2.

Experiment 1: We have designed this experiment in order to illustrate the influence of the central frequency f_0 on the formation of the analytic signal (Theorem 1). The experiment shows that a good approximation of the RQ complex signal $\Pi_T(t) \cdot \exp(j2\pi f_0 t + j\alpha t^2)$ can be achieved even for a low BT signal provided its central frequency is large and positive. In Fig. 8 it is shown that by increasing the central frequency f_0, the spectral leakage of the RQ complex signal is substantially reduced, thus enabling the analytic signal to be its good approximation. The effect of this operation on IF estimation accuracy is illustrated in Fig. 9.

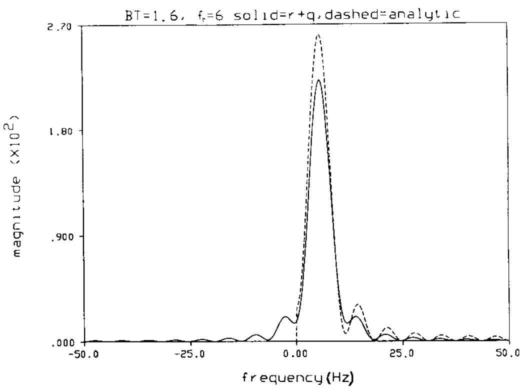
Experiment 2: This experiment is designed in order to illustrate the influence of BT on the formation of the analytic signal. The experiment shows that the analytic signal is a good approximation of the RQ complex signal if BT is large (Theorem 2). Figure 10 illustrates the amplitude spectra of the RQ complex and analytic signal for a small and large BT. For a small BT signal one can observe that there is a substantial spectral leakage of the RQ signal to negative frequencies, so that the analytic signal approximates it poorly. The effect of this poor approximation on IF estimation is destructive—Fig. 11 illustrates this point.

These two theorems relate the apparent difference in approach between the two methods of defining and measuring the IF: the IF describing the FM law in communication theory, and the signal analyst's IF [23]. Although the same quantity is being estimated, the different approaches adopted may lead to different results:

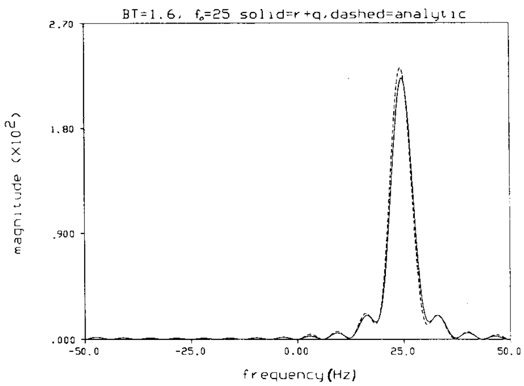
i) Consider the typical FM transmission signal in communications systems:

$$s(t) = a(t) \cdot \cos 2\pi (f_0 t + \int_{-\infty}^{t} m(\tau) d\tau) \qquad (64)$$

Here, the IF is a known physically meaningful quantity—it corresponds to the constant carrier frequency plus the varying frequency carrying the message. The signal may be demodulated using a device such as a phase-locked-loop or a zero-crossing detector, and

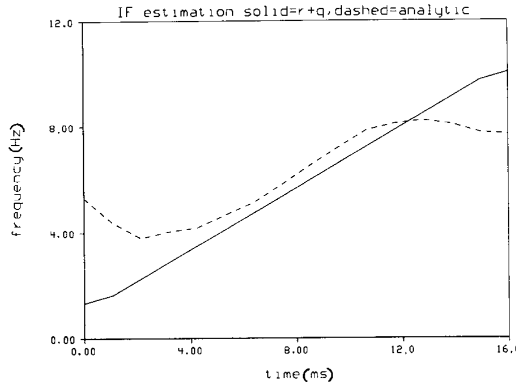


(a)

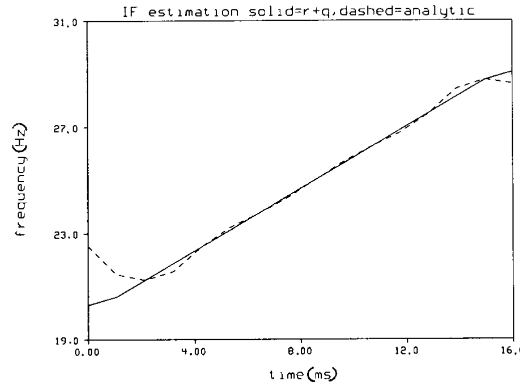


(b)

Fig. 8. Spectra of analytic signal and real+quadrature complex signal for large and small central frequency. Spectra of analytic signal generated by HT (dashed line) and RQ complex signal $a(t)\exp\phi(t)$ (solid line) for: (a) small central frequency f_0; (b) large central frequency f_0 (N. B. *Small* and *large* f_0 are given with respect to the sampling frequency which is in this case 100 Hz).



(a)



(b)

Fig. 9. Analytic signal versus real+quadrature complex signal—The effect of increased central frequency f_0 on if estimation. Instantaneous frequency laws estimated from the signals considered in Fig. 8. (a) small central frequency; (b) large central frequency. Estimation of IF from analytic signal is better for large f_0.

one finds

$$f_i(t) = f_0 + m(t) \quad (65)$$

The IF variations about f_0 are small (FM signal is narrowband with bandwidth B); f_0 is in general large; $a(t)$ has a finite duration T such that in general BT is large. Then the analytic signal is a good approximation of the RQ signal $a(t)\cdot\exp[j2\pi f_0 t + j\int m(\tau)d\tau]$.

ii) In the general signal-based definition of Ville the analytic signal is first generated and the IF becomes the derivative of the phase as given by eq.(43) which leads to practical IF estimators that can be used to demodulate the signal in a high SNR environment [7], [8].

The previous theorems indicate under what conditions the two representations of IF described in i) and ii) approach each other closely.

E. Calculation of the Instantaneous Frequency and Group Delay: An Example

Consider the following chirp signal with duration, T, and bandwidth, B, satisfying the condition $BT \gg 1$,

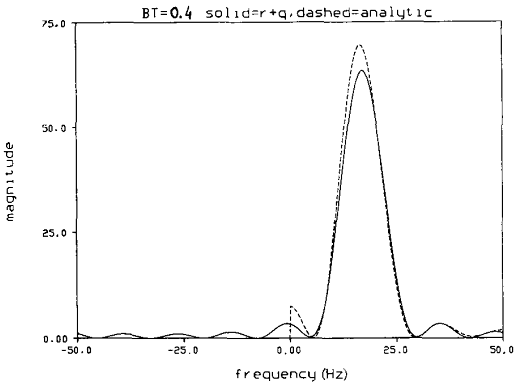
$$s(t) = \Pi_T(t)\cos\phi(t) \text{ where } \phi(t) = 2\pi(f_c t + 1/2\alpha t^2) \quad (66)$$

with $\alpha = B/T$ and $f_c = f_0 + B/2$. Theorem 2 states that the analytic signal associated with the large BT signal, $s(t)$, is approximately obtained by replacing the cosine function by an exponential function. Thus

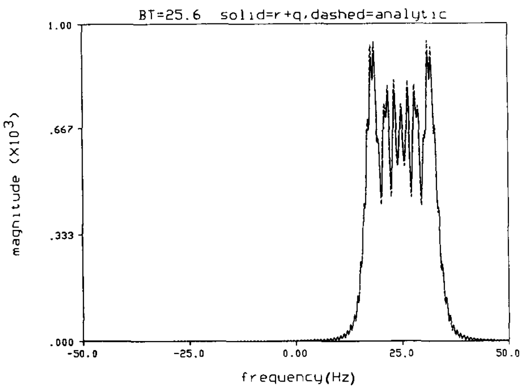
$$z(t) = \Pi_T(t)e^{j\phi(t)} \text{ and } f_i(t) = \frac{1}{2\pi}\frac{d\phi}{dt} = f_c + \alpha t. \quad (67)$$

Then, $f_i(t)$ can be considered as an operator mapping the space T_t of time instants, t, to the space B_f of frequencies, f. The inverse operator, $g(t) = f_i^{-1}(t)$, represents the inverse mapping and is easily defined by extracting t as a function of $f_i(t)$ in (67):

$$g(t) = \alpha^{-1}(f_i(t) - f_c). \quad (68)$$

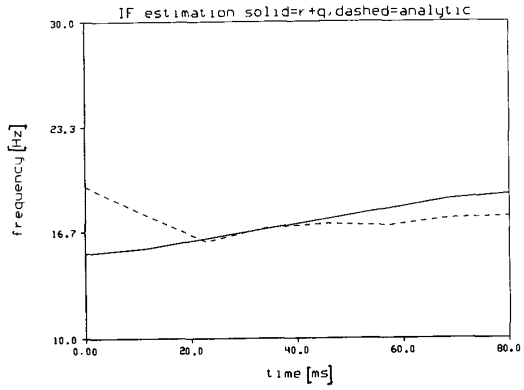


(a)

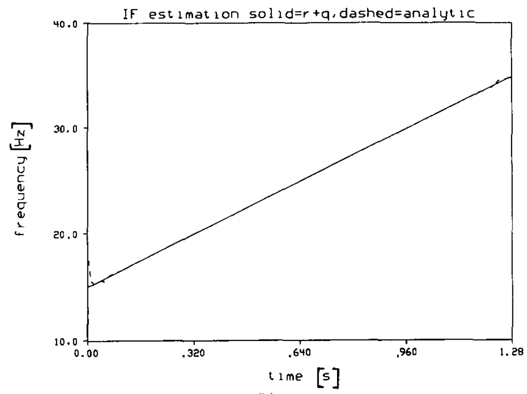


(b)

Fig. 10. Spectra of analytic signal and real+quadrature complex signal for large and small BT signal. Spectra of analytic signal generated by HT (dashed line) and RQ complex signal $a(t) \cdot \exp \phi(t)$ (solid line) for: (a) small BT; (b) large BT (note that for large BT we can not distinguish solid and dashed line as they are exactly superimposed).



(a)



(b)

Fig. 11. Analytic signal versus real+quadrature complex signal—The effect of BT value on IF estimation. Instantaneous frequency laws estimated from the signals considered in Fig. 10. Note that estimation of IF from the analytic signal is valid for large BT signals.

This result is now compared with the time delay, $\tau_g(f)$, of the signal. The FT of the signal, $z(t)$, is [7]:

$$Z(f) \simeq \frac{1}{\sqrt{\alpha}} \Pi_B(f - f_c) e^{j\pi(1/4 - 2\alpha^{-1}(f - f_c)^2)}. \quad (69)$$

The time delay, $\tau_g(f)$, is easily derived [7] as

$$\tau_g(f) = \frac{1}{2\alpha} 2(f - f_c) = \frac{1}{\alpha}(f - f_c) \quad (70)$$

and can be seen to represent the same operator as $g = f_i^{-1}$.

For the chirp signal, (67)–(70) show that $f_i(t)$ is a unique time-frequency characteristic of the signal. IF estimation is therefore an important problem in this case $(BT \gg 1)$ where it provides the unique $t - f$ law of the signal.

On the other hand, for small BT signals, two different time-frequency laws can be identified, specifically $f_i(t)$ and $\tau_g(f)$, and therefore, their interpretation becomes less meaningful (see Fig. 6). Although this does not necessarily preclude the use of the estimation methods, care must be taken in their interpretation [26].

Estimation of f_i: Although the IF provides information regarding the "internal organization of the signal," other relevant information, such as the spread of the signal around its $t - f$ law [17], is not available. An alternate method based on time-frequency distributions (TFD's) has the convenience that all the information regarding $t - f$ law, spread, etc., is presented in one single representation, and the IF law may be simply observed from its peaks. The following section relates the IF to TFD's.

V. Relationship Between IF and Time-Frequency Distributions

A. General Results

This and the following sections examine the relation between the IF of a signal and its TFD. TFD's were introduced as a means of representing signals whose frequency content is varying with time, and for which both time domain representations and frequency domain representations are inadequate to describe the signal appropriately [7]. Ideally, one would expect from a time-frequency representation of a signal that it peaks about the IF, with a spread related to the FT of the envelope of the signal. Accordingly, for monocomponent signals of the form of $a(t)e^{j\phi(t)}$ it would be intuitively satisfying to generate a TFD of the following form [9]

$$\rho(t,f) = A(t,f) \underset{f}{*} \delta(f - f_i(t)) \tag{71}$$

where $A(t,f)$ is the time-frequency representation of the amplitude function, $a(t)$. Thus the amplitude and phase would be separable, providing an easily interpretable distribution; the distribution is centered around the time-varying IF with the amplitude information distributed in time and frequency.

Few distributions considered at present provide a TFD of the form given in (71). The Wigner–Ville distribution (WVD) of a signal with a large BT product yields a distribution of the following general form [3]:

$$W(t,f) = kA_i\left(\left[\frac{32\pi^2}{f_i''(t)}\right]^{1/3}(f - f_i(t))\right) \tag{72}$$

where A_i is the Airy function, and k ensures that $\int_{-\infty}^{+\infty} W(t,f)df = a^2(t)$.

Thus it can be seen that the frequency location information is present but in a very complicated way. Only for the case where the 3rd order derivative of the phase function is zero, does the WVD produce a distribution of the form of (71). In other cases where the phase is more complicated, the distributions are distorted by the FT's of the higher order phase derivative terms.

Other TFD's may be calculated by a single generalized formula, derived by Cohen [14]:

$$\rho(t,f) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} e^{j2\pi v(u-t)} g(v,\tau) z(u+\tau/2) \cdot z^*(u-\tau/2) e^{-j2\pi f\tau} dv du d\tau \tag{73}$$

where $g(v,\tau)$ is a function which defines the particular TFD chosen, and where u and τ have the dimensions of time and v has the dimension of frequency. Any bilinear distribution can be generated from (73) by choosing different kernels $g(v,\tau)$. Now for any TFD $\rho(t,f)$ the conditional average value of frequency at particular time, say t, is

$$\langle f \rangle_t = \frac{\int_{-\infty}^{+\infty} f\rho(t,f)df}{\int_{-\infty}^{+\infty} \rho(t,f)df}. \tag{74}$$

One can show that the conditional frequency moment, $\langle f \rangle_t$, at a particular time equals the IF, $f_i(t)$, for TFD's whose kernels satisfy the following condition [4], [13]:

$$g(v,0) = 1 \tag{75a}$$

$$\left.\frac{\partial g(v,\tau)}{\partial \tau}\right|_{\tau=0} = 0. \tag{75b}$$

The result given by (74) and (75) is a generalization of the result obtained by Ville in 1948 for the WVD [42]. Note that if $g(v,\tau) = 1$, $\rho(t,f)$ reduces to the WVD. Similarly, the GD will be given as the first moment in time of any TFD characterized by a function $g(v,\tau)$ which satisfies (76). Again the WVD satisfies this condition:

$$g(0,\tau) = 1 \tag{76a}$$

$$\left.\frac{\partial g(v,\tau)}{\partial v}\right|_{v=0} = 0. \tag{76b}$$

Many distributions yield the IF by correct first moment calculation as discussed previously, but this is often computationally expensive and adversely affected by noise [8]. It would be an advantageous feature of a distribution to allow estimation of the IF simply by peak detection. With signals whose phase functions are quadratic (i.e., 3rd order derivative zero), the WVD of the signal will be of the form of (71) and thus IF estimation can be achieved by peak detection. In addition, one can also observe the spread about the IF at each instant of time. The same reasoning holds for the group-delay as well.

B. Instantaneous Bandwidth

Since the IF may be considered to be the average of the frequencies present at a given time instant, it seems reasonable to enquire about the standard deviation, or spread at that time. This spread may be interpreted as an instantaneous bandwidth (IB)—related to the IF in a way analogous to the frequency/bandwidth relationship.

A related concept introduced in [34] is that of "elementary cells" which specify the signal concentration, the dimension of the cells depending on the FM law of the signal and the BT product of the signal (Fig. 12). Rihaczek defined the dimensions of these cells by introducing the notion of "relaxation time," T_r, and "dynamic bandwidth," B_d, as [34]

$$T_r(t) = \left|\frac{df_i(t)}{dt}\right|^{-1/2} \tag{77}$$

$$B_d(f) = \left|\frac{d\tau_g(f)}{df}\right|^{-1/2}. \tag{78}$$

Since the classical bandwidth definition is the spread of frequencies about the average:

$$\sigma_{BW}^2 = \frac{\int_{-\infty}^{+\infty} (f - \langle f \rangle)^2 |S(f)|^2 df}{\int_{-\infty}^{+\infty} |S(f)|^2 df} \tag{79}$$

it is natural to expect that the IB should be of the form

$$\sigma_f^2(t) = \frac{\int_{-\infty}^{+\infty} (f - f_i(t))^2 \rho(t,f) df}{\int_{-\infty}^{+\infty} \rho(t,f) df} \tag{80}$$

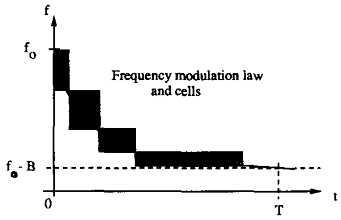


Fig. 12. Elementary cells in time-frequency plane.

where $\rho(t,f)$ is as defined in (73).

As is the case with the true bandwidth of (79) (which is purely a function of the magnitude spectrum), it is expected that the IB will simply be a function of the time amplitude, $a(t)$.

Since the WVD provides high signal energy concentration in time and frequency, it is tempting to try to use it to measure the spread of frequencies with time. Unfortunately, however, the spread of the IF for the WVD is only positive for certain types of signals (which includes those with positive definite amplitude functions, such as Gaussian signals). Even when the spread is positive some negative distribution values may appear in the calculation, and thus its usefulness is questionable [25].

Other TFD's may provide more useful measures of spread. Cohen and Lee have shown that by setting $g''(0) = 1/4$ in the general formulation for TFD's, a positive measure of spread is assured. They also showed that for all such TFD's the spread is given by [17]

$$\rho_f^2(t) = \frac{1}{4\pi^2}\left[\frac{a'(t)}{a(t)}\right]^2. \tag{81}$$

This positive measure of spread equals the second (frequency) moment of any distribution whose smoothing function satisfies $g''(0) = 1/4$. Cohen and Lee gave additional arguments in [16] and [18a] to justify this definition. It is helpful to compare this result with the relation between the global second moments of the frequency and the IF, derived by Mandel [28]:

$$\begin{aligned}\sigma_{BW}^2 &= \frac{\int_{-\infty}^{+\infty}(f-\langle f\rangle)^2|S(f)|^2df}{\int_{-\infty}^{+\infty}|S(f)|^2df}\\ &= \frac{\int_{-\infty}^{+\infty}\left[(f_i(t)-\langle f\rangle)^2+1/4\pi^2\left[\frac{a'(t)}{a(t)}\right]^2\right]|s(t)|^2dt}{\int_{-\infty}^{+\infty}|s(t)|^2dt}.\end{aligned} \tag{82), (83}$$

Mandel originally used this relationship to emphasize the distinction between the frequency and the IF as their second and higher global moments do not coincide. We consider the relationship for a different reason. The equation seems to suggest that the frequency spread or bandwidth may be broken into two components. The first component is due to the variation of the IF with time, and the second component is due to the amplitude envelope variations. This second component is of the same form as Cohen and Lee's measure of IB.

It should be noted that although Cohen and Lee's TFD-based spread in (81) is always positive, it is derived from distributions which may become negative. Hence, there is some question as to the validity of such a measure. In consideration of this latter complication, it is instructive to examine the nature of the spread of the IF in a purely positive TFD. This has been done for the Short-Time Fourier Transform (STFT) by Cohen [18]. He showed that if the terms due to windowing effects are removed, the spread of the IF for the STFT again reduces to that given in (81).

While it is difficult to establish the expression in (81) conclusively as the best measure for the IB, it does fit neatly into our intuitive understanding. If the envelope, $a(t)$, is constant (no amplitude modulation), there is no spread about the IF, i.e., the IF is the only existing frequency at a particular time. If amplitude modulation is present, the effect is to produce many neighboring frequencies, with the instantaneous frequency being the average of these frequencies (see Priestley [32]). The notion that there is a spread of frequencies evolving in time also partly answers the questions raised by Mandel. For an arbitrary distribution of a given variable, there is no guarantee that the mean, the median or the mode will coincide. Hence, to require that the mean spectral spread equals the mean instantaneous IF spread seems an unreasonable expectation.

Cohen and Lee's expression for the IB, however, raises a concern with the current use of TFD's because none of the TFD's used in practice today obey the condition that $g''(0) = 1/4$. Additionally, it has been shown that if a TFD is positive everywhere (e.g., the STFT), its first moment does not yield the IF, but a smeared version thereof [4], [7]. There appears room, then, for improvements in the commonly accepted representations of time-varying signals.

C. Multicomponent Signals

The concept of IF as previously defined loses its meaning when a multicomponent signal is considered. In this case, the signal can be modeled as a weighted sum of monocomponent signals, each one with its own IF (see Section II-C). For example, consider the signal:

$$x_m(t) = a_1(t)\cos\phi_1(t) + a_2(t)\cos\phi_2(t). \tag{84}$$

Assuming that each individual signal has a large BT product (with a purely positive IF), application of the linear HT will produce an analytic signal of the form:

$$z_m(t) = a_1(t)e^{j\phi_1(t)} + a_2(t)e^{j\phi_2(t)}. \tag{85}$$

The envelope of $z_m(t)$ is:

$$\begin{aligned}a_m(t) =& [a_1(t)^2 + a_2(t)^2 - 2a_1(t)a_2(t)\\ &\cdot\cos(\phi_1(t)-\phi_2(t))]^{1/2}.\end{aligned} \tag{86}$$

This envelope could be fast varying. In addition, the phase has resulted from a nonlinear combination of the two phase factors and may possess erratic behavior. Previous discussions and results therefore do not apply.

To answer the question of what makes a signal monocomponent or multicomponent it is helpful to consider first a general TFD of Cohen's Class [15]. The bilinear kernel, $z(u + \tau/2)z^*(u - \tau/2)$, of (43) introduces cross-terms when two or more analytic signals are present. This is even true for the manifestly positive STFT [9]. The cross-terms actually appear in the neighborhood of the auto-terms, with their extent and size being dependent on the interaction of the auto-term ambiguity functions and the window—the greater the separation of the ambiguity functions the more diminished will be the cross-terms [43]. Thus one could define a multicomponent signal as being one which has cross-terms present. Such a definition, however, is hard to test in practice.

Perhaps a more practical way of testing for whether a signal is monocomponent or multicomponent would be to test whether or not the energy distribution at a particular time is spread about only one frequency or whether there is a clear separation of the energy about two or more frequencies.

It should be noted that decomposition of a multicomponent signal is not unique, unless the components do not coincide at any point in the time-frequency plane. As an example, Fig. 13 shows two different separations of the monocomponent signals. The decomposition is application dependent.

VI. Conclusion

In this paper we have investigated the basic concepts necessary for the interpretation and use of the instantaneous frequency. This is an important problem in signal processing, and it raises fundamental questions in signal representation, many of which have been addressed here. The relationship between the IF and TFD's has been stressed. The points covered include the HT, analytic signal, group delay, monocomponent and multicomponent signals, instantaneous bandwidth and relationships of all the latter with the instantaneous frequency. While a number of key questions are resolved in this area, there is still much work remaining to be done before these notions become standard tools in the engineering community. This paper is a contribution towards this aim.

Apppendix A

A. Properties of the HT and the Analytic Signal

1) Formulations and Basic Properties: Gabor [20] first used the HT to generate the complex analytic signal according to:

$$s(t) = s(t) + jy(t) \tag{A1}$$

where $y(t) = H[s(t)]$ represents the HT of $s(t)$ defined as

$$H[s(t)] = \text{p.v.} \int_{-\infty}^{+\infty} \frac{s(t-\tau)}{\pi\tau} d\tau \tag{A2}$$

and satisfies the following properties [36]:

i) $$y(t) = H[s(t)] \tag{A3}$$

ii) $$s(t) = -H[y(t)] \tag{A4}$$

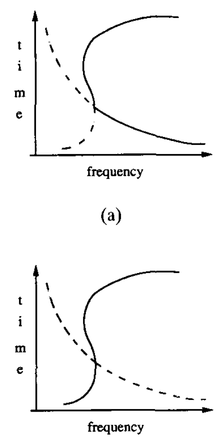


Fig. 13. Two different separations of the multicomponent signal. If two IF laws cross each other in the time-frequency plane, their separation is not unique in general case. The dashed lines in (a) and (b) lead to two different possibilities.

iii) $$s(t) = -H^2[s(t)] \tag{A5}$$

iv) $s(t)$ and $y(t)$ contain the same spectral components.

Note that if $s(t)$ admits an associated analytic signal $z(t)$, it has only one and vice-versa. Under some nonrestrictive conditions the function, $z(\tau)$, of the complex variable, $\tau = t + ju$, can be shown to be an analytic function in the upper half plane, $\text{Im}(\tau) \geq 0$. That is, (A3), (A4), (A5) satisfy the Cauchy–Riemann conditions [36]. Moreover, the real part of $z(\tau)$ equals $s(t)$ on the real axis:

$$z(\tau) = s(t, u) + jy(t, u) \tag{A6}$$

when $\tau \to t$, i.e., when $u \to 0$, we obtain:

$$z(t) = s(t) + jy(t). \tag{A7}$$

The imaginary part of $z(\tau)$ takes on the value $y(t)$, and is often called the quadrature signal. The analytic signal may also be defined in the frequency domain as follows.

An analytic signal is a function $z(\tau)$ of a complex variable $\tau = t + ju$ defined for $\text{Im}(\tau) \geq 0$, for which there exists a complex function, $Z(f)$, of the real variable, f, defined for $f \geq 0$, so that $z(\tau)$ is its FT:

$$z(\tau) = \int_0^{\infty} Z(f) e^{j2\pi f\tau} df. \tag{A8}$$

By definition, an analytic signal has therefore a spectrum limited to positive frequencies only. We consider that $Z(0) = S(0) = 0$. This property can be used to determine the analytic signal associated with a given real signal in the frequency domain. A useful property of analytic signals which follows immediately from the fact that there are no negative frequencies present, is that one only needs to sample at half the Nyquist rate [36].

B. Envelope and Instantaneous Phase

Given the signal, $s(t) = a(t)\cos\phi(t)$, subject to some conditions discussed later, we may express $z(t)$ in polar coordinates as

$$z(t) = a(t)e^{j\phi(t)} \tag{A9}$$

where

$$a(t) = \sqrt{s^2(t) + y^2(t)} \tag{A10}$$

and

$$\phi(t) = \arctan\left\{\frac{y(t)}{s(t)}\right\}. \tag{A11}$$

$a(t)$ is referred to as the instantaneous amplitude or envelope and $\phi(t)$ is referred to as the instantaneous phase (see Fig. 2). From (A10) we obtain that $a(t) \geq |s(t)|$, which indicates that the curve C_1 representing the absolute value of the signal, $|s(t)|$, does not intersect with the curve C_2 representing $a(t)$. Further we can show that $a(t) = s(t)$ when $y(t) = 0$ and these two functions have the same tangent [36]. The reason for the use of the term, *signal envelope* is that the curve C_1 envelopes the curve C_2.

Appendix B

A. Stationary Phase Principle

The stationary phase principle is a useful tool in developing the concept of instantaneous frequency. It was used by Carson and Fry [11] to estimate the limits of IF variations for FM signals, by Rihaczek [34] to show that the energy is concentrated in frequency about the IF. It was also used by Vakman [39] to explain a seemingly paradoxical situation of instantaneous frequency: although the IF is a local concept (indicated by term "instantaneous"), to calculate it is necessary to determine the HT of the signal, that is one has to know the entire history of the signal in the time domain. The stationary phase principle is also used to prove many results presented in [9] and in this paper.

B. Formulation [19]

We have to calculate:

$$I_\alpha = \int_{x_1}^{x_2} U(x)\cos\phi_\alpha(x)\,dx \tag{B1}$$

$$= \mathrm{Re}\left\{\int_{x_1}^{x_2} U(x)e^{j\phi_\alpha(x)}\,dx\right\} \tag{B2}$$

$\phi_\alpha(x) = \phi(\alpha, x)$ is parameterized by α. If $U(x)$ varies slowly, while $\phi_\alpha(x)$ varies over 2π, then the positive values of $\cos\phi_\alpha(x)$ tend to compensate its negative values, so as to make the result of the integral very small. However, if $\phi_\alpha(x)$ has stationary values for which $d\phi_\alpha(x)/dx = 0$, then for these values, the integral will be large (Fig. 14). The large values of the integral, I_α, will be obtained for the values of α which make $\phi(x, \alpha)$ stationary, i.e.,

$$\frac{d}{dx}(\phi(x,\alpha)) = 0 \tag{B3}$$

C. Application

The principle can be applied whenever it is necessary to determine a Fourier integral of the form

$$Z(f) = F\left\{a(t)e^{j\phi(t)}\right\} = \int_{-\infty}^{+\infty} a(t)e^{j[\phi(t)-2\pi ft]}dt \tag{B4}$$

providing that the exponential factor is oscillating rapidly compared to the amplitude factor of the integral (B4) in vicinity of the stationary phase point t_0, i.e.,

$$|\phi'(t_0)| >> |a'(t_0)| \tag{B5}$$

where the values of stationary phase points are found from the equation

$$\frac{d}{dt}[\phi(t) - 2\pi ft] = 0. \tag{B6}$$

If (B6) has only one solution (at $t = t_0$), then the approximation of integral (B4) is given by

$$Z(f) \simeq a(t_0)\left(\frac{2\pi}{|\phi''(t_0)|}\right)^{1/2} e^{j[\phi(t_0)-2\pi ft_0 \pm\pi/4]}. \tag{B7}$$

Here the plus sign applies when $\phi''(t_0) > 0$, and the minus sign when $\phi''(t_0) < 0$. Equation (B6) will indeed have a solitary solution if the instantaneous frequency law $\phi'(t)/2\pi$ changes monotonically. Note also that condition (B5) can be interpreted using Cohen's definition of IB [16] as

$$|f_i(t_0)| >> \sigma_f(t_0).$$

Appendix C

A. Bedrosian's Theorem for Complex Signals

Let $x(t)$ and $y(t)$ denote generally complex finite energy signals of the real variable t. Their FT's are $X(f) = F\{x(t)\}$ and $Y(f) = F\{y(t)\}$. If

$$i) \quad X(f) = 0 \text{ for } |f| > a \text{ and } Y(f) = 0 \text{ for } |f| < b, \text{ where } b \geq a \geq 0 \tag{C1}$$

or

$$ii) \quad X(f) = 0 \text{ for } f < -a \text{ and } Y(f) = 0 \text{ for } f < b, \text{ where } b \geq a \geq 0 \tag{C2}$$

then

$$H\{x(t)y(t)\} = x(t).H\,\{y(t)\} \tag{C3}$$

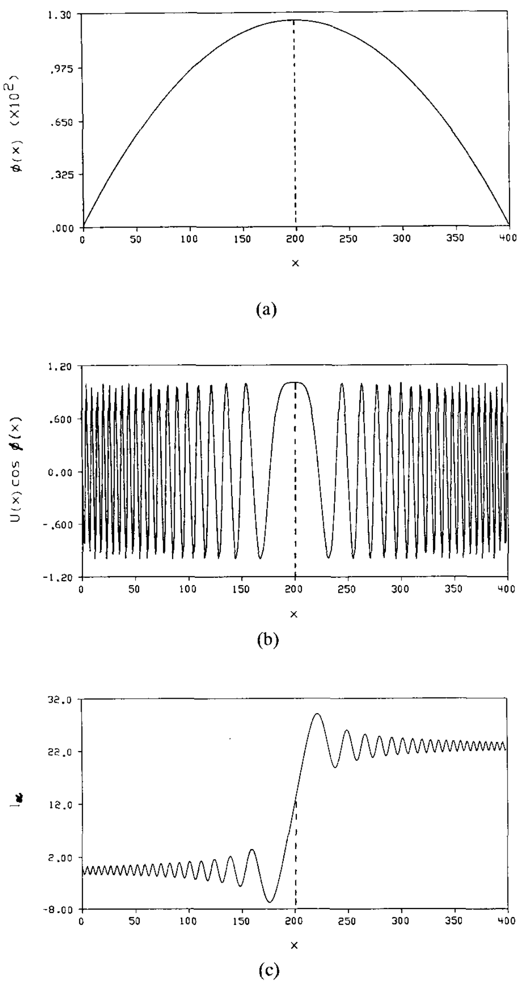


Fig. 14. The illustration of the stationary phase principle: (a) phase; (b) $U(x)\cos\phi(x)$; (c) integral I_α.

B. Discussion:

a) Conditions *i)* and *ii)* are necessary and sufficient [12].

b) Condition *i)* has very important practical meaning because it is applied when $x(t)$ and $y(t)$ are real functions. In this case, spectra $X(f)$ and $Y(f)$ must be disjoint (Fig. 15(a)).

c) Condition *ii)* is applied when $x(t)$ and $y(t)$ are complex. In this case spectra $X(f)$ and $Y(f)$ both must be right-sided and need not be disjoint (see Fig. 15(b)).

d) A special case of condition *ii)* is when $a = b = 0$. Then, both $x(t)$ and $y(t)$ are analytic signals and (C1) can be extended as follows:

$$H[x(t)y(t)] = x(t) \cdot H[y(t)] = y(t) \cdot H\,[x(t)]. \quad \text{(C4)}$$

ACKNOWLEDGMENT

The author acknowledges the support of the Australian Defence Science and Technology Organization (DSTO) and the Australian Research Council (ARC). In addition, the author wishes to thank Peter O'Shea, Graeme Jones, and Branko Ristic for their contribution and help in preparing this manuscript, and Leon Cohen, University of New York, for helpful discussions and suggestions. The author also acknowledges the review of the final document by former colleague Dave Mackerras. In addition, the author thanks the anonymous reviewers for their helpful comments and suggestions.

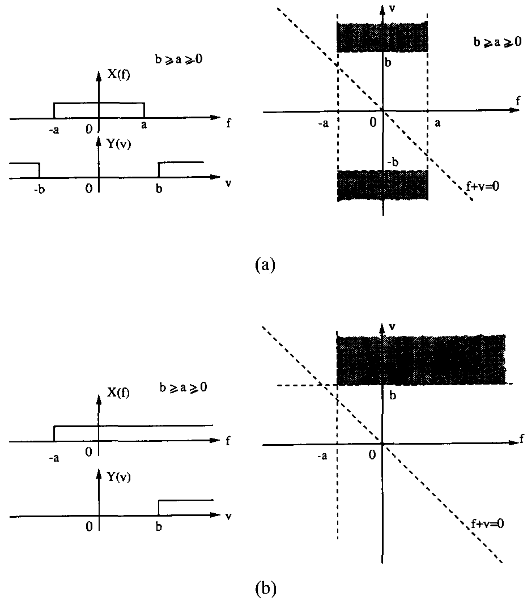


Fig. 15. Regions of integration for Bedrosian product theorem (a) case for real signals; (b) case for complex signals.

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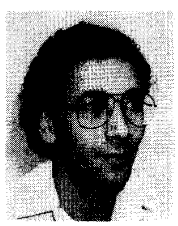

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