CHAPTER V

EXTERIOR DIFFERENTIAL FORMS

5.1. SCOPE OF THE CHAPTER

Studies of differential forms has started with the works of Grassmann and efforts to extend the integral theorems in classical vector analysis has played a significant part in the development of the theory. Several elemental concepts, for instance the exterior product, has been introduced by French mathematician Jules Henri Poincaré (1854-1912). However, it was French mathematician Élie Joseph Cartan (1869-1951) who enormously contributed in the period from 1899 to 1926 to the establishment of the theoretical framework of exterior forms on differentiable manifolds by identifying exterior differential forms as exterior products of differentials of coordinates (exterior derivatives) and thus equipping them with an algebraic structure.

In Sec. 5.2, the exterior differential forms on differentiable manifolds and exterior algebra formed by them are defined and it is shown that they constitute a module. Sec. 5.3 deals with some useful algebraic properties concerning 1-forms. In Sec. 5.4 the interior product of a vector with an exterior form is defined, various properties of this operation that reduces the degree of the form by one are revealed and criteria for the existence of a divisor of a form are established by making use of the interior product. To replace the natural basis of the exterior algebra, we consider in Sec. 5.5 a top-down generation of a new basis from the volume form, which has the highest degree on a given manifold, by its appropriate interior products with natural basis vectors of the tangent bundle. We examine relations between these bases in detail. In some cases, the use of these bases turns out to be quite advantageous. Sec. 5.6 is concerned with certain subalgebras of the exterior algebra called ideals and characteristic vectors of an exterior form and also of an ideal are introduced. It is shown in Sec. 5.7 that a smooth mapping between two differentiable manifolds gives rise to an additive pull-back operator that transports exterior forms on the range of the mapping to forms on its domain by preserving their degrees. Moreover certain

Exterior Analysis, DOI: 10.1016/B978-0-12-415902-0.50005-0

properties of this operator are emphasised. The exterior derivative which is one of the fundamental operators acting on exterior forms is defined in Sec. 5.8 and its properties are discussed there. Closed and exact forms are introduced as well in this section. Sec. 5.9 deals with Riemannian manifolds endowed with a metric tensor that makes it possible to measure distances between points of a manifold. Metric tensor also helps us to relate covariant components of a tensor with its contravariant components and vice versa. Utilising this opportunity, we define the Hodge dual of a form and the Hodge star operation generating this form. Then, we discuss its properties and scrutinise the co-differential, Laplace-de Rham and Laplace-Beltrami operators. Sec. 5.10 is concerned with closed ideals, the forms belonging to which have exterior derivatives remaining in the ideal and conditions leading to a closed ideal are examined. The Lie derivative of an exterior form that measures the variation in this form along the flow generated by a vector field on a manifold is considered in Sec. 5.11 and the Cartan formula that makes it possible to calculate Lie derivatives of forms relatively easily is derived. We define in Sec. 5.12 isovector fields of an ideal and show that the ideal remains invariant under the flow produced by an isovector field and prove that isovectors constitute a Lie subalgebra of the tangent bundle. Finally, we investigate in Sec. 5.13 the mappings, or submanifolds, annihilating an ideal. The notion of complete integrability is introduced, conditions providing its existence are discussed and the theorems of Cartan and Frobenius, that play a pivotal part in comprehending this concept, are proven. Sec. 5.14 is devoted to an overview of some properties of exterior forms defined on a Lie group which is also a smooth manifold. Left- and right-invariant 1-forms are defined by using certain pull-back mappings on the exterior algebra built on the Lie group. These mappings are generated by left and right translations in the group. It is shown further that left-invariant 1-forms called Maurer-Cartan forms constitute the dual of the Lie algebra of the Lie group and they satisfy a system of exterior differential equations depending on structure constants of the Lie algebra.

5.2. EXTERIOR DIFFERENTIAL FORMS

We have seen in Sec. 4.3 that a k-exterior differential form field on an m-dimensional smooth manifold M is defined as a completely antisymmetric k-covariant tensor field or as an alternating k-linear functional and it can be represented in natural coordinates $\mathbf{x} = \varphi(p)$ in a chosen chart as follows

$$\omega(p) = \frac{1}{k!}\,\omega_{i_1 i_2 \cdots i_k}(\mathbf{x})\,dx^{i_1} \wedge dx^{i_2} \wedge \cdots \wedge dx^{i_k} \tag{5.2.1}$$

where smooth functions $\omega_{i_1 i_2 \cdots i_k} \in \Lambda^0(M)$ are completely antisymmetric in their indices. We call k as the degree of the form. If we identify the sum $\omega = \omega_1 + \omega_2$ of two forms ω_1 and ω_2 of the same degree k by employing the following completely antisymmetric components

$$\omega_{i_1 i_2 \cdots i_k}(\mathbf{x}) = \omega^1_{i_1 i_2 \cdots i_k}(\mathbf{x}) + \omega^2_{i_1 i_2 \cdots i_k}(\mathbf{x}) \in \Lambda^0(M),$$

then we deduce that ω is a k-form as well. Similarly the scalar multiplication $f\omega$ where $f \in \Lambda^0(M)$ is a k-form specified by smooth functions

$$f(\mathbf{x})\omega_{i_1 i_2 \cdots i_k}(\mathbf{x}) \in \Lambda^0(M).$$

Therefore, k-exterior differential forms constitute a module over the commutative ring $\Lambda^0(M)$. Henceforth, we denote this module by $\Lambda^k(M)$. Naturally, $\Lambda^k(M)$ reduces to a vector space over the field of real numbers. When $k > m$*, it is evident that exterior forms vanish identically*. The basis of this module are the following linearly independent k-forms:

$$\{dx^{i_1} \wedge dx^{i_2} \wedge \cdots \wedge dx^{i_k} : i_1, \ldots, i_k = 1, \ldots, m\}$$

whose number is $\binom{m}{k} = \dfrac{m!}{k!\,(m-k)!}$. This basis is expressed more concretely in terms of *essential components* through ordered indices in the form $\{dx^{i_1} \wedge dx^{i_2} \wedge \cdots \wedge dx^{i_k} : 1 \le i_1 < i_2 < \cdots < i_k \le m\}$. In this case (5.2.1) can also be written as

$$\omega(p) = \sum_{1 \le i_1 < i_2 < \cdots < i_k \le m} \omega_{i_1 i_2 \cdots i_k}(\mathbf{x})\, dx^{i_1} \wedge dx^{i_2} \wedge \cdots \wedge dx^{i_k}.$$

Instead of m natural basis dx^j of $T^*(M)$ associated with local coordinates x^j in local charts at every points of the manifold we can of course choose m linearly independent 1-forms prescribed by

$$\theta^i = \theta^i_j(\mathbf{x})\, dx^j \in \Lambda^1(M),\ i, j = 1, \ldots, m;\ \ \det\left[\theta^i_j(\mathbf{x})\right] \neq 0$$

as a basis and represent a k-form in terms of this basis in the following manner

$$\omega(p) = \frac{1}{k!}\, \Omega_{i_1 i_2 \cdots i_k}(\mathbf{x})\, \theta^{i_1} \wedge \theta^{i_2} \wedge \cdots \wedge \theta^{i_k}$$

where

$$\Omega_{i_1 i_2 \cdots i_k}(\mathbf{x}) = \omega_{j_1 j_2 \cdots j_k}(\mathbf{x}) \Theta^{[j_1}_{i_1} \Theta^{j_2}_{i_1} \cdots \Theta^{j_k]}_{i_1}.$$

Here $[\Theta^i_j(\mathbf{x})]$ is the inverse of the matrix $\left[\theta^i_j(\mathbf{x})\right]$.

Just like in Sec. 1.5 we can define the operation of the exterior product of exterior differential forms $\alpha \in \Lambda^k(M), \beta \in \Lambda^l(M)$ by

$$\begin{aligned}\gamma = \alpha \wedge \beta &= \frac{1}{k!\,l!}\,\alpha_{i_1\cdots i_k}\beta_{j_1\cdots j_l}dx^{i_1}\wedge\cdots\wedge dx^{i_k}\wedge dx^{j_1}\wedge\cdots\wedge dx^{j_l}\\ &= \frac{1}{(k+l)!}\,\gamma_{i_1\cdots i_k j_1\cdots j_l}dx^{i_1}\wedge\cdots\wedge dx^{i_k}\wedge dx^{j_1}\wedge\cdots\wedge dx^{j_l}\end{aligned}$$

where $\wedge\, : \Lambda^k(M)\times\Lambda^l(M)\to\Lambda^{k+l}(M)$ assigns now a $(k+l)$-form to k- and l-forms. Here the functions $\gamma_{i_1\cdots i_k j_1\cdots j_l}(\mathbf{x})\in\Lambda^0(M)$ are given by

$$\gamma_{i_1\cdots i_k j_1\cdots j_l} = \frac{(k+l)!}{k!\,l!}\alpha_{[i_1\cdots i_k}\beta_{j_1\cdots j_l]}$$

[*see* (1.5.1)]. If we regard a function $f\in\Lambda^0(M)$ as a 0-form, we can write

$$f\wedge\omega = f\omega\in\Lambda^k(M)$$

for a k-form ω. It is straightforward to observe that the exterior product possesses the following properties:

$$\begin{aligned}&\alpha\wedge(\beta+\gamma) = \alpha\wedge\beta+\alpha\wedge\gamma,\\ &(\alpha+\beta)\wedge\gamma = \alpha\wedge\gamma+\beta\wedge\gamma,\\ &\alpha\wedge(\beta\wedge\gamma) = (\alpha\wedge\beta)\wedge\gamma = \alpha\wedge\beta\wedge\gamma,\\ &\qquad\beta\wedge\alpha = (-1)^{kl}\alpha\wedge\beta,\ \ \alpha\in\Lambda^k(M),\ \beta\in\Lambda^l(M).\end{aligned}\tag{5.2.2}$$

It is thus seen that the exterior product is associative and distributive, but it is generally not commutative. Whenever kl is an even number one has $\beta\wedge\alpha=\alpha\wedge\beta$, whereas $\beta\wedge\alpha=-\alpha\wedge\beta$ when it is an odd number. *If* $\omega\in\Lambda^k(M)$ *and k is an odd number*, then we find that

$$\omega\wedge\omega = (-1)^{k^2}\omega\wedge\omega = -\,\omega\wedge\omega$$

since k^2 is also an odd number. Thus the *square* of such a form vanishes

$$\omega^2 = \omega\wedge\omega = 0.$$

The set of exterior differential forms of all degrees on a manifold M constitute the ***exterior algebra*** $\Lambda(M)$ with the binary operation of exterior product. The exterior algebra is expressible as the direct sum

$$\begin{aligned}\Lambda(M) &= \Lambda^0(M)\oplus\Lambda^1(M)\oplus\cdots\oplus\Lambda^k(M)\oplus\cdots\oplus\Lambda^m(M)\\ &= \bigoplus_{k=0}^{m}\Lambda^k(M)\end{aligned}$$

of modules $\Lambda^k(M), k = 0, 1, \ldots, m$. Hence $\Lambda(M)$ is a ***graded algebra***. Of course, only the sum of forms of the same degree is really meaningful. Smooth coefficient functions belong to the ring $\Lambda^0(M)$ and the natural basis of the exterior algebra $\Lambda(M)$ is given by

$$\{1\} \cup \{dx^i\} \cup \{dx^i \wedge dx^j, i < j\} \cup \cdots \cup \{dx^{i_1} \wedge \cdots \wedge dx^{i_k}, i_1 < \cdots < i_k\} \\ \cup \cdots \cup \{dx^1 \wedge dx^2 \wedge \cdots \wedge dx^m\}.$$

Thus the dimension of the exterior algebra is

$$\sum_{k=0}^{m} \binom{m}{k} = 2^m.$$

The value of a form $\omega \in \Lambda^k(M)$ on vectors $U_1, U_2, \ldots, U_k \in T(M)$ is computed as we have mentioned in p. 26 [*see* (1.4.4)] by the relation

$$\omega(U_1, U_2, \ldots, U_k) = \omega_{i_1 i_2 \cdots i_k} u_1^{i_1} u_2^{i_2} \cdots u_k^{i_k} \tag{5.2.3}$$

where we wrote $U_\alpha = u_\alpha^i(\mathbf{x}) \dfrac{\partial}{\partial x^i}, i = 1, 2, \ldots, m; \alpha = 1, 2, \ldots, k$. It then immediately follows that coefficient functions are determined by

$$\omega_{i_1 i_2 \cdots i_k} = \omega\Big(\frac{\partial}{\partial x^{i_1}}, \frac{\partial}{\partial x^{i_2}}, \ldots, \frac{\partial}{\partial x^{i_k}}\Big). \tag{5.2.4}$$

On an m-dimensional manifold M, the module $\Lambda^m(M)$ is 1-dimensional. Hence, every m-form is represented as

$$\omega = f(\mathbf{x})\, dx^1 \wedge dx^2 \wedge \cdots \wedge dx^m, \quad f \in \Lambda^0(M),$$

The form

$$\mu = dx^1 \wedge dx^2 \wedge \cdots \wedge dx^m \in \Lambda^m(M) \tag{5.2.5}$$

is called the ***volume form***. Indeed if we consider m linearly independent vector fields $V_1 = \Delta v^1 \dfrac{\partial}{\partial x^1}, \ldots, V_m = \Delta v^m \dfrac{\partial}{\partial x^m}$, we obtain

$$\mu(V_1, V_2, \ldots, V_m) = \begin{vmatrix} \Delta v^1 & 0 & \cdots & 0 \\ 0 & \Delta v^2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \Delta v^m \end{vmatrix} = \Delta v^1 \Delta v^2 \cdots \Delta v^m$$

and this is the volume of a rectangular parallelepiped in $\mathbb{R}^m$.

We are not compelled to employ *the natural basis* $\{dx^i\} \subset T^*(M)$

and *its reciprocal basis* $\{\partial/\partial x^i\} \subset T(M)$. Let us introduce *a reciprocal basis* $\{\theta^i\} \subset T^*(M)$ and *a basis* $\{V_i\} \subset T(M)$. Therefore the relations $\theta^i(V_j) = \delta^i_j, i, j = 1, \dots, m$ are to be satisfied. A form $\omega \in \Lambda^k(M)$ can now be represented by

$$\omega(p) = \frac{1}{k!}\,\omega_{i_1 i_2 \cdots i_k}(\mathbf{x})\,\theta^{i_1} \wedge \theta^{i_2} \wedge \cdots \wedge \theta^{i_k}$$

where coefficients $\omega_{i_1 i_2 \cdots i_k}$ must of course be completely antisymmetric. Then we obtain

$$\omega(V_{i_1}, V_{i_2}, \dots, V_{i_k}) = \frac{1}{k!}\,\omega_{j_1 j_2 \cdots j_k} \begin{vmatrix} \theta^{j_1}(V_{i_1}) & \theta^{j_1}(V_{i_2}) & \cdots & \theta^{j_1}(V_{i_k}) \\ \theta^{j_2}(V_{i_1}) & \theta^{j_2}(V_{i_2}) & \cdots & \theta^{j_2}(V_{i_k}) \\ \vdots & \vdots & & \vdots \\ \theta^{j_k}(V_{i_1}) & \theta^{j_k}(V_{i_2}) & \cdots & \theta^{j_k}(V_{i_k}) \end{vmatrix}$$

$$= \frac{1}{k!}\,\omega_{j_1 j_2 \cdots j_k} \begin{vmatrix} \delta^{j_1}_{i_1} & \delta^{j_1}_{i_2} & \cdots & \delta^{j_1}_{i_k} \\ \delta^{j_2}_{i_1} & \delta^{j_2}_{i_2} & \cdots & \delta^{j_2}_{i_k} \\ \vdots & \vdots & & \vdots \\ \delta^{j_k}_{i_1} & \delta^{j_k}_{i_2} & \cdots & \delta^{j_k}_{i_k} \end{vmatrix} = \frac{1}{k!}\,\omega_{j_1 j_2 \cdots j_k}\delta^{j_1 j_2 \cdots j_k}_{i_1 i_2 \cdots i_k}$$

Therefore, we again conclude that

$$\omega(V_{i_1}, V_{i_2}, \dots, V_{i_k}) = \omega_{i_1 i_2 \cdots i_k}. \tag{5.2.6}$$

5.3. SOME ALGEBRAIC PROPERTIES

We say that a k-form $\Omega \in \Lambda^k(M)$ is a ***simple form*** if it is expressible as an exterior product of k linearly independent 1-forms [*see p.* 36]. Hence, if we can write

$$\Omega = \omega^1 \wedge \omega^2 \wedge \cdots \wedge \omega^k \in \Lambda^k(M)$$

where $\omega^r \in \Lambda^1(M), r = 1, \dots, k \leq m$ are linearly independent, then Ω is a simple k-form.

Theorem 5.3.1. $\omega^1, \omega^2, \dots, \omega^k \in \Lambda^1(M)$ *are linearly independent* 1-*forms if and only if* $\Omega = \omega^1 \wedge \omega^2 \wedge \cdots \wedge \omega^k \neq 0$.

Let us suppose first that $\Omega \neq 0$. We consider the linear combination $c_r\omega^r = c_1\omega^1 + c_2\omega^2 + \cdots + c_k\omega^k = 0$ where $c_1, c_2, \dots, c_k \in \Lambda^0(M)$ are arbitrary coefficient functions. The exterior product of this form by the $(k-1)$-form $\omega^2 \wedge \cdots \wedge \omega^k$ yields $c_1\Omega = 0$ because square of a 1-form vanishes. We thus find $c_1 = 0$. In a similar fashion, we deduce that all coefficients must be zero. Hence, 1-forms $\omega^1, \omega^2, \dots, \omega^k$ are linearly independent. Conversely, let us choose k linearly independent 1-forms $\omega^1, \omega^2, \dots, \omega^k$ that are represented by

$$\omega^r = a_i^r \, dx^i, \; r = 1, \dots, k \leq m; \; i = 1, \dots, m.$$

Hence, the rank of the $k \times m$ matrix $[a_i^r]$ should be k so that this matrix must have at least one $k \times k$ submatrix with non-vanishing determinant. On the other hand, the k-form that is the exterior products of these 1-forms can be written as follows:

$$\begin{aligned}\Omega &= \omega^1 \wedge \omega^2 \wedge \cdots \wedge \omega^k \\ &= a_{i_1}^1 a_{i_2}^2 \cdots a_{i_k}^k dx^{i_1} \wedge dx^{i_2} \wedge \cdots \wedge dx^{i_k} \\ &= a_{[i_1}^1 a_{i_2}^2 \cdots a_{i_k]}^k \, dx^{i_1} \wedge dx^{i_2} \wedge \cdots \wedge dx^{i_k}.\end{aligned}$$

One immediately sees that for a particular choice of indices $i_1, \dots, i_k$, the coefficient of the form $dx^{i_1} \wedge \cdots \wedge dx^{i_k}$ will be the determinant of a $k \times k$ submatrix of the matrix $[a_i^r]$. Therefore, the form Ω is the sum of such k-forms. However, in this sum at least one term is different from zero. Hence, we obtain $\Omega \neq 0$. □

Theorem 5.3.2. *If the forms* $\alpha^r, \beta^r \in \Lambda^1(M), r = 1, \dots, k$ *are connected by the expression*

$$\beta^r = c_s^r \alpha^s, c_s^r \in \Lambda^0(M), \; r, s = 1, \dots, k,$$

then there exists the relation

$$\beta^1 \wedge \beta^2 \wedge \cdots \wedge \beta^k = (\det [c_s^r]) \alpha^1 \wedge \alpha^2 \wedge \cdots \wedge \alpha^k$$

among them.

In fact, it is readily found that the relation

$$\begin{aligned}\beta^1 \wedge \beta^2 \wedge \cdots \wedge \beta^k &= c_{s_1}^1 c_{s_2}^2 \cdots c_{s_k}^k \alpha^{s_1} \wedge \alpha^{s_2} \wedge \cdots \wedge \alpha^{s_k} \\ &= c_{s_1}^1 c_{s_2}^2 \cdots c_{s_k}^k \delta_{12\cdots k}^{s_1 s_2 \cdots s_k} \alpha^1 \wedge \alpha^2 \wedge \cdots \wedge \alpha^k \\ &= (\det [c_s^r]) \alpha^1 \wedge \alpha^2 \wedge \cdots \wedge \alpha^k\end{aligned}$$

is obtained. □

Theorem 5.3.3. *If* 1*-forms* $\omega^r \in \Lambda^1(M), r = 1, \dots, k$ *are linearly independent and if* 1*-forms* $\gamma_r \in \Lambda^1(M), \; r = 1, \dots, k$ *satisfy the relation*

$$\gamma_r \wedge \omega^r = \gamma_1 \wedge \omega^1 + \gamma_2 \wedge \omega^2 + \cdots + \gamma_k \wedge \omega^k = 0,$$

then every form γ_r *belongs to the submodule generated by the forms* $\omega^1, \omega^2, \dots, \omega^k$. *Hence one is able to write*

$$\gamma_r = a_{rs}\,\omega^s,\;\; a_{rs} \in \Lambda^0(M),\;\; r,s = 1,\dots,k$$

where the matrix $\mathbf{A} = [a_{rs}]$ *is symmetric, namely,* $a_{rs} = a_{sr}$.

Exterior product of the relation $\gamma_r \wedge \omega^r = 0$ with the $(k-1)$-form $\omega^2 \wedge \cdots \wedge \omega^k$ yields $\gamma_1 \wedge \Omega = 0$. $\Omega = \omega^1 \wedge \omega^2 \wedge \cdots \wedge \omega^k \neq 0$ because 1-forms ω^r are linearly independent. It then follows that the form γ_1 is linearly dependent on the forms $\omega^1, \omega^2, \dots, \omega^k$. In a similar fashion, we find $\gamma_r \wedge \Omega = 0$ for each r. Therefore, the forms γ_r are linear combinations of the forms ω^r. Thus, one can write

$$\gamma_r = a_{rs}\,\omega^s.$$

On the other hand, the relation

$$0 = \gamma_r \wedge \omega^r = a_{rs}\,\omega^s \wedge \omega^r = a_{[rs]}\,\omega^s \wedge \omega^r$$

leads to $a_{[rs]} = 0$, and consequently to the symmetry relation $a_{rs} = a_{sr}$. This theorem is also known as the ***Cartan lemma***. □

5.4. INTERIOR PRODUCT

We have seen that new elements of the exterior algebra $\Lambda(M)$ over an m-dimensional manifold M are generated by exterior products of its elements. But the exterior product is an operation that raises the degrees of forms. Nevertheless, we can obtain at most forms of degree m with an operation raising degrees because we know that forms of degrees higher than m vanish identically. Since it is evident that it is not possible to obtain a form with a lesser degree than a given form by resorting to the exterior product, we need to introduce a new operation to achieve this task. We further wish that this operation possesses appropriate properties. We devise this operation by means of a vector field. We call it the ***interior product*** of a vector field $V \in T(M)$ with an exterior form field $\omega \in \Lambda(M)$. To this end, we introduce the interior product operator $\mathbf{i}$ in the following form

$$\mathbf{i} : T(M) \times \Lambda^k(M) \to \Lambda^{k-1}(M),$$

or

$$\mathbf{i}_V : \Lambda^k(M) \to \Lambda^{k-1}(M)$$

where the vector V is now specified. We further impose the conditions that the operator $\mathbf{i}_V$ has to satisfy the following rules:

$$
\begin{aligned}
&(i).\, \mathbf{i}_V(f) = 0, V \in T(M), f \in \Lambda^0(M). && (5.4.1)\\
&(ii).\, \mathbf{i}_V(\omega) = \omega(V) = \langle \omega, V \rangle = \omega_i v^i \in \Lambda^0(M), V \in T(M), \omega \in \Lambda^1(M).\\
&(iii).\, \mathbf{i}_V(\alpha + \beta) = \mathbf{i}_V(\alpha) + \mathbf{i}_V(\beta),\ \ V \in T(M),\ \alpha, \beta \in \Lambda^k(M).\\
&(iv).\, \mathbf{i}_V(\alpha \wedge \beta) = \mathbf{i}_V(\alpha) \wedge \beta + (-1)^{deg\,(\alpha)} \alpha \wedge \mathbf{i}_V(\beta),\\
&\qquad\qquad V \in T(M),\ \ \alpha, \beta \in \Lambda(M).
\end{aligned}
$$

Here k can only take the values $1, \dots, m$. Since we can interpret the function $f \in \Lambda^0(M)$ as a 0-degree form so that we can write $f \wedge \omega = f\omega$, the rules (i) and (iv) result in $\mathbf{i}_V(f\omega) = f\mathbf{i}_V(\omega)$. It is readily verified that the above rules would suffice to determine the operator $\mathbf{i}_V$ uniquely. Let us assume that there exists a second operator $\mathbf{i}'_V$ accommodating to these rules. Then, it would be necessary to write $\mathbf{i}_V(f) = \mathbf{i}'_V(f) = 0, \mathbf{i}_V(\omega) = \mathbf{i}'_V(\omega) = \omega(V)$ for each $f \in \Lambda^0(M)$ and $\omega \in \Lambda^1(M)$. We thus find that $\mathbf{i}'_V|_{\Lambda^0(M)} = \mathbf{i}_V|_{\Lambda^0(M)}$, $\mathbf{i}'_V|_{\Lambda^1(M)} = \mathbf{i}_V|_{\Lambda^1(M)}$. But, the rules (iii) and (iv) assure us that actions of these two operators will also be the same on 2-, 3-, $\dots, m$-forms. Consequently, we write $\mathbf{i}'_V|_{\Lambda(M)} = \mathbf{i}_V|_{\Lambda(M)}$ over the entire exterior algebra so that we get $\mathbf{i}_V = \mathbf{i}'_V$. The rule (iv) indicates clearly that the interior product is an ***antiderivation***. The interior product operator $\mathbf{i}_V$ is sometimes symbolised by the *hook operator* $\rfloor$. In that case, the form $\mathbf{i}_V(\omega)$ will be denoted by $V\rfloor\omega$.

Let $f \in \Lambda^0(M)$. We take $\omega = df \in \Lambda^1(M)$ so $\big($5.4.1 $(ii)\big)$ results in

$$\mathbf{i}_V(df) = df(V) = f_{,i}v^i = V(f).$$

We shall now try to evaluate explicitly the action of the interior product $\mathbf{i}_V : \Lambda(M) \to \Lambda(M)$, which maps the exterior algebra into itself, by the aid of the above rules. Suppose that a form field $\omega \in \Lambda^k(M)$ and a vector field $V \in T(M)$ are given by

$$
\begin{aligned}
\omega &= \frac{1}{k!}\, \omega_{i_1 i_2 \cdots i_k}(\mathbf{x})\, dx^{i_1} \wedge dx^{i_2} \wedge \cdots \wedge dx^{i_k},\\
V &= v^i(\mathbf{x}) \frac{\partial}{\partial x^i}.
\end{aligned}
$$

Because of the relation $\mathbf{i}_V\big(\omega_{i_1 i_2 \cdots i_k}(\mathbf{x})\big) = 0$ we can write

$$\mathbf{i}_V(\omega) = \frac{1}{k!}\, \omega_{i_1 i_2 \cdots i_k} \mathbf{i}_V(dx^{i_1} \wedge dx^{i_2} \wedge \cdots \wedge dx^{i_k}).$$

On the other hand, the rule (ii) dictates that $\mathbf{i}_V(dx^{i_r}) = V(dx^{i_r}) = v^{i_r}$. Hence, according to (iv) we get

$$
\begin{aligned}
\mathbf{i}_V(dx^{i_1}\wedge dx^{i_2}\wedge\cdots\wedge dx^{i_k}) &= \mathbf{i}_V(dx^{i_1})\wedge dx^{i_2}\wedge\cdots\wedge dx^{i_k}\\
-\,dx^{i_1}\wedge\mathbf{i}_V(dx^{i_2}\wedge\cdots\wedge dx^{i_k}) &= \mathbf{i}_V(dx^{i_1})\wedge dx^{i_2}\wedge\cdots\wedge dx^{i_k}\\
&\quad -dx^{i_1}\wedge\mathbf{i}_V(dx^{i_2})\wedge dx^{i_3}\wedge\cdots\wedge dx^{i_k}\\
&\quad +dx^{i_1}\wedge dx^{i_2}\wedge\mathbf{i}_V(dx^{i_3}\wedge\cdots\wedge dx^{i_k})\\
&=\cdots= v^{i_1}dx^{i_2}\wedge\cdots\wedge dx^{i_k}-v^{i_2}dx^{i_1}\wedge dx^{i_3}\wedge\cdots\wedge dx^{i_k}+\cdots\\
&\quad +(-1)^{k-1}v^{i_k}dx^{i_1}\wedge dx^{i_2}\wedge\cdots\wedge dx^{i_{k-1}}\\
&=\sum_{l=1}^{k}(-1)^{l-1}v^{i_l}dx^{i_1}\wedge dx^{i_2}\wedge\cdots\wedge dx^{i_{l-1}}\wedge dx^{i_{l+1}}\wedge\cdots\wedge dx^{i_k}.
\end{aligned}
$$

In the last line above, we adopted the convention $dx^{i_0}=1$. So we find that

$$
\begin{aligned}
&\mathbf{i}_V(\omega)=\\
&\quad\frac{1}{k!}\sum_{l=1}^{k}(-1)^{l-1}\,\omega_{i_1\cdots i_{l-1}i_li_{l+1}\cdots i_k}v^{i_l}dx^{i_1}\wedge\cdots\wedge dx^{i_{l-1}}\wedge dx^{i_{l+1}}\wedge\cdots\wedge dx^{i_k}\\
&=\frac{1}{k!}\sum_{l=1}^{k}(-1)^{2(l-1)}\,v^{i_l}\omega_{i_li_1\cdots i_{l-1}i_{l+1}\cdots i_k}dx^{i_1}\wedge\cdots\wedge dx^{i_{l-1}}\wedge dx^{i_{l+1}}\wedge\cdots\wedge dx^{i_k}\\
&=\frac{1}{k!}\sum_{l=1}^{k}v^{i}\omega_{ii_1i_2\cdots i_{k-1}}dx^{i_1}\wedge dx^{i_2}\wedge\cdots\wedge dx^{i_{k-1}}\\
&=\frac{k}{k!}v^{i}\omega_{ii_1i_2\cdots i_{k-1}}dx^{i_1}\wedge dx^{i_2}\wedge\cdots\wedge dx^{i_{k-1}}.
\end{aligned}
$$

In the third line, on making use of the complete antisymmetry of coefficients, we have written $\omega_{i_1\cdots i_{l-1}i_li_{l+1}\cdots i_k}=(-1)^{l-1}\omega_{i_li_1\cdots i_{l-1}i_{l+1}\cdots i_k}$. We have gone into the fourth line by appropriately renaming the dummy indices. We finally deduce that, by means of the operator $\mathbf{i}_V$, a k-form $\omega\in\Lambda^k(M)$ is transformed into a $(k-1)$- form $\mathbf{i}_V(\omega)\in\Lambda^{k-1}(M)$ defined by

$$
\mathbf{i}_V(\omega)=\frac{1}{(k-1)!}v^{i}\omega_{ii_1i_2\cdots i_{k-1}}dx^{i_1}\wedge dx^{i_2}\wedge\cdots\wedge dx^{i_{k-1}}. \qquad (5.4.2)
$$

This expression can also be rewritten in term of essential components as

$$
\mathbf{i}_V(\omega)=\sum_{1\le i_1<\cdots<i_{k-1}\le m}v^{i}\omega_{ii_1i_2\cdots i_{k-1}}dx^{i_1}\wedge dx^{i_2}\wedge\cdots\wedge dx^{i_{k-1}}.
$$

When we recall, together with the rule (5.4.1.(iii)), that $\mathbf{i}_V(f\omega)=f\mathbf{i}_V(\omega)$ for a function $f\in\Lambda^0(M)$, we immediately see that *the operator* $\mathbf{i}_V$ *is linear over the module* $\mathfrak{V}(M)$. Next, let us consider k arbitrary vector fields V,

$V_1, V_2, \ldots, V_{k-1} \in T(M)$. We know that the value of a form $\omega \in \Lambda^k(M)$ on these vectors is given by

$$\omega(V, V_1, V_2, \ldots, V_{k-1}) = \omega_{ii_1\cdots i_{k-1}} v^i v_1^{i_1} \cdots v_{k-1}^{i_{k-1}}.$$

On the other hand, according to (5.4.2) the value of the form $\mathbf{i}_V(\omega) \in \Lambda^{k-1}(M)$ on vectors $V_1, V_2, \ldots, V_{k-1}$ is found as

$$\mathbf{i}_V(\omega)(V_1, V_2, \ldots, V_{k-1}) = \omega_{ii_1\cdots i_{k-1}} v^i v_1^{i_1} \cdots v_{k-1}^{i_{k-1}}.$$

Therefore, for every vector fields $V, V_1, V_2, \ldots, V_{k-1}$ the equality

$$\mathbf{i}_V(\omega)(V_1, V_2, \ldots, V_{k-1}) = \omega(V, V_1, V_2, \ldots, V_{k-1}) \tag{5.4.3}$$

holds. *Actually, it can be shown that this relation may be employed to define the interior product operator*.

Example 5.4.1. Let the form $\omega \in \Lambda^2(M)$ be given by

$$\omega = \frac{1}{2}\,\omega_{ij}\,dx^i \wedge dx^j,\ \omega_{ji} = -\,\omega_{ij}.$$

Interior product of this form with a vector field V becomes

$$\mathbf{i}_V(\omega) = v^i \omega_{ij}\,dx^j \in \Lambda^1(M)$$ ■

Let us now calculate the interior product of the form $\omega \in \Lambda^k(M)$ with two vector fields V_1 and V_2 successively. It follows from (5.4.2) by renaming dummy indices that

$$\mathbf{i}_{V_2}\big(\mathbf{i}_{V_1}(\omega)\big) = (\mathbf{i}_{V_2} \circ \mathbf{i}_{V_1})(\omega) = \frac{1}{(k-2)!} v_2^j v_1^i \omega_{iji_1\cdots i_{k-2}}\,dx^{i_1} \wedge \cdots \wedge dx^{i_{k-2}}.$$

It is clear that $(\mathbf{i}_{V_2} \circ \mathbf{i}_{V_1})(\omega) \in \Lambda^{k-2}(M)$. Let us now change the order of the vectors in the interior product. On recalling that the coefficients $\omega_{iji_3\cdots i_k}$ are antisymmetric with respect to indices i and j, we get

$$\begin{aligned}(\mathbf{i}_{V_2} \circ \mathbf{i}_{V_1})(\omega) &= -\frac{1}{(k-2)!} v_1^i v_2^j \omega_{jii_1\cdots i_{k-2}}\,dx^{i_1} \wedge \cdots \wedge dx^{i_{k-2}} \\ &= -\,(\mathbf{i}_{V_1} \circ \mathbf{i}_{V_2})(\omega).\end{aligned}$$

Since this relation must be valid for every form $\omega \in \Lambda(M)$, we arrive at the anticommutativity property of the interior product:

$$\mathbf{i}_{V_1} \circ \mathbf{i}_{V_2} = -\,\mathbf{i}_{V_2} \circ \mathbf{i}_{V_1}. \tag{5.4.4}$$

Thus for every vector V, we get the result

$$\mathbf{i}_V \circ \mathbf{i}_V = \mathbf{i}_V^2 = 0. \tag{5.4.5}$$

The successive interior products of a k-form with l vector fields where $l \leq k$ is the $(k-l)$-form given below:

$$(\mathbf{i}_{V_l} \circ \cdots \circ \mathbf{i}_{V_1})(\omega) = \frac{1}{(k-l)!} v_1^{i_1} \cdots v_l^{i_l} \omega_{i_1 \cdots i_l i_{l+1} \cdots i_k} dx^{i_{l+1}} \wedge \cdots \wedge dx^{i_k}.$$

Evidently the operator $\mathbf{i}_{V_l} \circ \cdots \circ \mathbf{i}_{V_1}$ is completely antisymmetric:

$$\mathbf{i}_{V_l} \circ \cdots \circ \mathbf{i}_{V_p} \circ \cdots \circ \mathbf{i}_{V_q} \circ \cdots \circ \mathbf{i}_{V_1} = -\,\mathbf{i}_{V_l} \circ \cdots \circ \mathbf{i}_{V_q} \circ \cdots \circ \mathbf{i}_{V_p} \circ \cdots \circ \mathbf{i}_{V_1}.$$

It is readily observed that for k vector fields $V_1, \dots, V_l, V_{l+1}, \dots, V_k$, we obtain

$$(\mathbf{i}_{V_l} \circ \cdots \circ \mathbf{i}_{V_1})(\omega)(V_{l+1}, \dots, V_k) = \omega(V_1, \dots, V_l, V_{l+1}, \dots, V_k). \tag{5.4.6}$$

If we take $l = k$, we conclude that

$$(\mathbf{i}_{V_k} \circ \cdots \circ \mathbf{i}_{V_1})(\omega) = v_1^{i_1} v_2^{i_2} \cdots v_k^{i_k} \omega_{i_1 i_2 \cdots i_k} = \omega(V_1, V_2, \dots, V_k).$$

Thus the successive interior products of a k-form with k ordered vector fields yields the value of this form on these vectors. If $l > k$, then the successive interior products of a k-form with l vectors vanishes identically.

It follows from the definition (5.4.2) that

$$\begin{aligned}
\mathbf{i}_{V_1+V_2}(\omega) &= \frac{1}{(k-1)!}(v_1^i + v_2^i)\, \omega_{i i_1 i_2 \cdots i_{k-1}} dx^{i_1} \wedge dx^{i_2} \wedge \cdots \wedge dx^{i_{k-1}} \\
&= \mathbf{i}_{V_1}(\omega) + \mathbf{i}_{V_2}(\omega), \\
\mathbf{i}_{fV}(\omega) &= \frac{1}{(k-1)!} f v^i \omega_{i i_1 i_2 \cdots i_{k-1}}\, dx^{i_1} \wedge dx^{i_2} \wedge \cdots \wedge dx^{i_{k-1}} = f \mathbf{i}_V(\omega).
\end{aligned}$$

Since these relations are valid for every form $\omega \in \Lambda(M)$, then we reach to the following properties:

$$\mathbf{i}_{V_1+V_2} = \mathbf{i}_{V_1} + \mathbf{i}_{V_2}, \quad \mathbf{i}_{fV} = f \mathbf{i}_V. \tag{5.4.7}$$

Next, let us assume that the forms ω and Ω satisfy the degree condition $deg\,(\Omega) \leq deg\,(\omega)$. If we can find a form ω_1 so that one is able to write $\omega = \omega_1 \wedge \Omega$, the form Ω is called a ***divisor*** of the form ω. It is obvious that $deg\,(\omega_1) = deg\,(\omega) - deg\,(\Omega)$.

Theorem 5.4.1. *A* 1*-form* $\Omega \neq 0$ *is a divisor of a form* $\omega \in \Lambda(M)$ *with non-vanishing degree if and only if* $\omega \wedge \Omega = 0$.

Evidently, this is the necessary condition. If we can write $\omega = \omega_1 \wedge \Omega$, then we obtain $\omega \wedge \Omega = \omega_1 \wedge \Omega \wedge \Omega = 0$ since $\Omega \in \Lambda^1(M)$. We now prove that it is also the sufficient condition. Let us write $\Omega = \Omega_i\, dx^i, i = 1, \dots, m$. Since $\Omega \neq 0$, at least one of the coefficients should be different from zero. By changing the ordering, if necessary, we take $\Omega_1 \neq 0$. Let us choose a new basis in $T^*(M)$ as follows

$$\theta^1 = \Omega,\ \theta^2 = dx^2,\ \dots,\ \theta^m = dx^m.$$

The transformation of bases is designated by

$$\begin{bmatrix} \theta^1 \\ \theta^2 \\ \vdots \\ \theta^m \end{bmatrix} = \begin{bmatrix} \Omega_1 & \Omega_2 & \cdots & \Omega_m \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix} \begin{bmatrix} dx^1 \\ dx^2 \\ \vdots \\ dx^m \end{bmatrix}.$$

Since $\Omega_1 \neq 0$, the determinant of the matrix of transformation does not vanish. Hence, the inverse transformation becomes

$$dx^1 = \frac{1}{\Omega_1}\,\Omega - \frac{\Omega_2}{\Omega_1}\,\theta^2 - \cdots - \frac{\Omega_m}{\Omega_1}\,\theta^m,\ dx^i = \theta^i, i = 2, \dots, m$$

On inserting these 1-forms into ω and noting that the square of a 1-form is zero, we arrive at the expression

$$\omega = \omega_1 \wedge \Omega + \omega_2$$

where we must have $deg\,(\omega_1) = deg\,(\omega) - 1$ and $deg\,(\omega_2) = deg\,(\omega)$. The form Ω is not included in forms ω_1 and ω_2. We thus get

$$0 = \omega \wedge \Omega = \omega_1 \wedge \Omega \wedge \Omega + \omega_2 \wedge \Omega = \omega_2 \wedge \Omega$$

whence we deduce that $\omega_2 = 0$. Hence, one writes $\omega = \omega_1 \wedge \Omega$. □

An immediate corollary of this theorem can be expressed in the following manner: *If linearly independent forms $\Omega^1, \Omega^2, \dots, \Omega^r \in \Lambda^1(M)$ are divisors of a form $\omega \in \Lambda^k(M)$, then the form $\Omega^1 \wedge \Omega^2 \wedge \cdots \wedge \Omega^r$ is also a divisor of ω.*

Indeed if Ω^1 is a divisor, then we write $\omega \wedge \Omega^1 = 0$ and $\omega = \omega_1 \wedge \Omega^1$. Since Ω^2 is also a divisor, the relation $0 = \omega \wedge \Omega^2 = \omega_1 \wedge \Omega^1 \wedge \Omega^2$ should be satisfied. But Ω^1 and Ω^2 are linearly independent so that $\Omega^1 \wedge \Omega^2 \neq 0$. Consequently, we find $\omega_1 \wedge \Omega^2 = 0$. Thus Ω^2 must be a divisor of ω_1. Hence, we have to write $\omega_1 = \omega_2 \wedge \Omega^2$. Continuing this way, we reach to the result

$$\omega = \omega_{\underline{r}} \wedge \Omega^1 \wedge \Omega^2 \wedge \cdots \wedge \Omega^{\underline{r}}. \qquad \square$$

If $\omega \in \Lambda^k(M)$, then the condition $\omega \wedge \Omega = 0$ which secures that 1-form Ω is a divisor of ω is cast into the relation

$$\frac{1}{k!}\omega_{i_1 i_2\cdots i_k}\Omega_i\, dx^{i_1}\wedge dx^{i_2}\wedge\cdots\wedge dx^{i_k}\wedge dx^i = 0$$

whence we deduce that the following $\begin{pmatrix} m \\ k+1 \end{pmatrix}$ expressions

$$\Omega_{[i}\omega_{i_1 i_2\cdots i_k]} = 0. \tag{5.4.8}$$

should be satisfied.

We can easily identify through the interior product whether a given k-form is simple.

Theorem 5.4.2. *Let $\omega \in \Lambda^k(M)$ be a non-zero form. We construct a form $\Omega \in \Lambda^1(M)$ as follows*

$$\Omega = (\mathbf{i}_{V_{k-1}}\circ\cdots\circ\mathbf{i}_{V_2}\circ\mathbf{i}_{V_1})(\omega)$$

where $V_1, V_2, \dots, V_{k-1} \in T(M)$. The form ω is simple if and only if $\omega\wedge\Omega = 0$ for all vector fields $V_1, V_2, \dots, V_{k-1} \in T(M)$.

To show that this is the necessary condition, let us suppose that ω is a simple form, in other words, it is expressible as $\omega = \omega^1\wedge\omega^2\wedge\cdots\wedge\omega^k$ where $\omega^r \in \Lambda^1(M)$, $r = 1,\dots,k$. Next, we shall try to determine a basis $\{U_1,\dots, U_k, U_{k+1}, \dots, U_m\}$ of the tangent bundle $T(M)$ in such a way that they possess the following properties:

$$\mathbf{i}_{U_\alpha}(\omega^r) = \delta^r_\alpha,\ r = 1,\dots,k;\alpha = 1,\dots,k,k+1,\dots,m.$$

To this end, let us write $U_\alpha = u^i_\alpha\partial_i$ and $\omega^r = \omega^r_i\, dx^i$ in terms of local coordinates. Since $\omega \neq 0$, the forms ω^r are linearly independent. Therefore, the rank of the $k\times m$ matrix $[\omega^r_i]$ is k. We then split the relation $\mathbf{i}_{U_\alpha}(\omega^r) = \omega^r_i u^i_\alpha = \delta^r_\alpha, i = 1,\dots,m$ into following expressions

$$\begin{aligned} &\omega^r_A u^A_s + \omega^r_\Gamma u^\Gamma_s = \delta^r_s,\ r,s,A = 1,\dots,k;\ \Gamma = k+1,\dots,m, \\ &\omega^r_A u^A_\Delta + \omega^r_\Gamma u^\Gamma_\Delta = 0,\ r,A = 1,\dots,k;\ \Gamma,\Delta = k+1,\dots,m. \end{aligned} \tag{5.4.9}$$

We may assume without loss of generality that det $[\omega^r_A] \neq 0$. We thus obtain from (5.4.9) that

$$\begin{aligned} u^A_s &= (\omega^{-1})^A_s - (\omega^{-1})^A_r\,\omega^r_\Gamma u^\Gamma_s, \\ u^A_\Delta &= -\,(\omega^{-1})^A_r\,\omega^r_\Gamma u^\Gamma_\Delta. \end{aligned}$$

On defining

$$\Omega^A_\Gamma = -\,(\omega^{-1})^A_r\,\omega^r_\Gamma,$$

we find that

$$u^A_r = (\omega^{-1})^A_r + \Omega^A_\Gamma u^\Gamma_r, \quad u^A_\Delta = \Omega^A_\Gamma u^\Gamma_\Delta.$$

Hence, the basis vectors U_α meeting the desired conditions can now be expressed as

$$\begin{aligned} U_r &= u^i_r \frac{\partial}{\partial x^i} = u^A_r \frac{\partial}{\partial x^A} + u^\Gamma_r \frac{\partial}{\partial x^\Gamma} \\ &= \big[(\omega^{-1})^A_r + \Omega^A_\Gamma u^\Gamma_r\big] \frac{\partial}{\partial x^A} + u^\Gamma_r \frac{\partial}{\partial x^\Gamma}, \\ U_\Gamma &= u^i_\Gamma \frac{\partial}{\partial x^i} = u^A_\Gamma \frac{\partial}{\partial x^A} + u^\Delta_\Gamma \frac{\partial}{\partial x^\Delta} \\ &= u^\Delta_\Gamma \big[\frac{\partial}{\partial x^\Delta} + \Omega^A_\Delta \frac{\partial}{\partial x^A}\big]. \end{aligned}$$

If we introduce vectors W_A and W_Γ by

$$W_A = \frac{\partial}{\partial x^A}, \quad W_\Gamma = \frac{\partial}{\partial x^\Gamma} + \Omega^A_\Gamma \frac{\partial}{\partial x^A}$$

we obtain

$$U_r = (\omega^{-1})^A_r W_A + u^\Gamma_r W_\Gamma, \quad U_\Gamma = u^\Delta_\Gamma W_\Delta$$

where $[u^\Gamma_r]$ and $[u^\Delta_\Gamma]$ are arbitrary matrices. We observe at once that m vectors $\{W_A, W_\Gamma\}$ are linearly independent. If we restrict the arbitrariness of the square matrix $[u^\Delta_\Gamma]$ such that it has a non-zero determinant, then the vectors $\{U_\alpha\}$ turn out to be linearly independent. Consequently, any vector field V_A with $A = 1, \dots, k$ can now be expressed as a linear combination

$$V_A = c^\alpha_A U_\alpha = c^1_A U_1 + \cdots + c^m_A U_m$$

where $c^\alpha_A, \alpha = 1, \dots, m; A = 1, \dots, k$ are arbitrary coefficient functions from which we get

$$\begin{aligned} \mathbf{i}_{V_A}(\omega) &= \sum_{r=1}^k (-1)^{r-1} \mathbf{i}_{V_A}(\omega^r)\, \omega^1 \wedge \cdots \wedge \omega^{r-1} \wedge \omega^{r+1} \wedge \cdots \wedge \omega^k \\ &= \sum_{r=1}^k (-1)^{r-1} c^\alpha_A \delta^r_\alpha\, \omega^1 \wedge \cdots \wedge \omega^{r-1} \wedge \omega^{r+1} \wedge \cdots \wedge \omega^k \\ &= \sum_{r=1}^k (-1)^{r-1} c^r_A\, \omega^1 \wedge \cdots \wedge \omega^{r-1} \wedge \omega^{r+1} \wedge \cdots \wedge \omega^k. \end{aligned}$$

Therefore, the $(k-1)$-form $\mathbf{i}_{V_1}(\omega)$ is now a linear combination of k *simple* $(k-1)$-forms. When we apply the operator $\mathbf{i}_{V_2}$ to this form, we see that the $(k-2)$-form $(\mathbf{i}_{V_2} \circ \mathbf{i}_{V_1})(\omega)$ is the linear combination of k *simple* $(k-2)$-forms. On continuing this way by applying the operators $\mathbf{i}_{V_1}, \dots, \mathbf{i}_{V_{k-1}}$ successively to the form ω, we reduce the form Ω to the linear combination of k number of 1-forms ω^r:

$$\Omega = (\mathbf{i}_{V_{k-1}} \circ \cdots \circ \mathbf{i}_{V_1})(\omega^1 \wedge \cdots \wedge \omega^k) = \lambda_r \omega^r = \lambda_1 \omega^1 + \cdots + \lambda_k \omega^k.$$

We thus conclude that

$$\omega \wedge \Omega = \omega^1 \wedge \cdots \wedge \omega^k \wedge (\lambda_1 \omega^1 + \cdots + \lambda_k \omega^k) = 0.$$

In order to show sufficiency, we consider the k-form

$$\omega = \frac{1}{k!}\, \omega_{i_1 \cdots i_k}(\mathbf{x})\, dx^{i_1} \wedge \cdots \wedge dx^{i_k} \in \Lambda^k(M)$$

and the 1-form

$$\Omega = (\mathbf{i}_{V_{k-1}} \circ \cdots \circ \mathbf{i}_{V_1})(\omega) = \omega_{i_1 \cdots i_{k-1} i_k} v_1^{i_1} \cdots v_{k-1}^{i_{k-1}} dx^{i_k} \in \Lambda^1(M)$$

which is made up by interior products with arbitrary vector fields $V_1, \dots, V_{k-1}$. Let us then write

$$\omega \wedge \Omega = \frac{1}{k!}\, \omega_{i_1 \cdots i_k} \omega_{j_1 \cdots j_{k-1} j_k} v_1^{j_1} \cdots v_{k-1}^{j_{k-1}}\, dx^{i_1} \wedge \cdots \wedge dx^{i_k} \wedge dx^{j_k} = 0.$$

Since this equality must be satisfied for all vector fields $V_1, \dots, V_{k-1}$, we arrive at the conditions

$$\omega_{i_1 \cdots i_k} \omega_{j_1 \cdots j_{k-1} j_k} dx^{i_1} \wedge \cdots \wedge dx^{i_k} \wedge dx^{j_k} = 0$$

leading to

$$\omega_{j_1 \cdots j_{k-1} [j_k} \omega_{i_1 \cdots i_k]} = 0. \tag{5.4.10}$$

These conditions require that the completely antisymmetric coefficients $\omega_{i_1 \cdots i_k}$ have to satisfy certain quadratic equations whose number is clearly $\binom{m}{k-1}\binom{m}{k+1}$. We shall now attempt to recognise the result brought about by these equation in a somewhat indirect way. Since we have presumed that $\omega \neq 0$, we can select $\omega_{12 \cdots k} \neq 0$ by renaming, if necessary, reciprocal basis vectors. We then define the following 1-forms

$$\Omega^1 = \omega_{i23\cdots k}\, dx^i,\ \Omega^2 = \omega_{1i3\cdots k}\, dx^i, \ldots, \Omega^k = \omega_{123\cdots k-1i}\, dx^i.$$

Therefore, we can write with $r = 1, \ldots, k$ and $\Gamma = k+1, \ldots, m$

$$\begin{aligned}\Omega^r &= \omega_{123\cdots k}\, dx^r + \omega_{123\cdots \underset{r}{\Gamma} \cdots k}\, dx^\Gamma \\ &= \omega_{123\cdots k}\, dx^r + \omega_{123\cdots \underset{r}{k+1} \cdots k}\, dx^{k+1} + \cdots + \omega_{123\cdots \underset{r}{m} \cdots k}\, dx^m\end{aligned} \tag{5.4.11}$$

These forms are linearly independent. In fact, if we write $c_r\Omega^r = 0$ where $c_r, r = 1, \ldots, k$ are arbitrary coefficient functions, the relation

$$c_r\Omega^r = \omega_{123\cdots k} c_r\, dx^r + \sum_{r=1}^{k} \omega_{123\cdots \underset{r}{\Gamma} \cdots k} c_r\, dx^\Gamma = 0$$

requires that $c_r = 0, r = 1, \ldots, k$. On the other hand, a proper choice of indices $j_1, j_2, \ldots, j_{k-1}$ in (5.4.10) leads to the relations

$$\omega_{23\cdots k[i}\omega_{i_1\cdots i_k]} = 0, \quad \omega_{13\cdots k[i}\omega_{i_1\cdots i_k]} = 0, \quad \ldots \quad , \\ \omega_{123\cdots (k-1)[i}\omega_{i_1\cdots i_k]} = 0.$$

In view of (5.4.8), we infer that the 1-forms $\Omega^1, \Omega^2, \ldots, \Omega^k$ are divisors of the form ω. Since these forms are linearly independent, we conclude that $\omega = \lambda\Omega^1 \wedge \cdots \wedge \Omega^k$. The factor λ can be found by equating coefficients of the form $dx^1 \wedge \cdots \wedge dx^k$ in both sides of this expression. Utilising (5.4.11), we end up with

$$\lambda = \frac{1}{(\omega_{123\cdots k})^{k-1}}.$$

Hence, on defining $\omega^1 = \lambda\Omega^1, \omega^2 = \Omega^2, \ldots, \omega^k = \Omega^k$, we get

$$\omega = \omega^1 \wedge \omega^2 \wedge \cdots \wedge \omega^k$$ □

Example 5.4.2. We consider the form $\omega = \frac{1}{2}\,\omega_{ij}\, dx^i \wedge dx^j \in \Lambda^2(M)$. The requirement that this form is to be a simple form can be written from (5.4.10) as follows

$$\omega_{i[j}\,\omega_{kl]} = 0 \ \text{ or } \ \omega_{ij}\,\omega_{kl} + \omega_{ik}\,\omega_{lj} + \omega_{il}\,\omega_{jk} = 0.$$

When this condition is met, we obtain

$$\Omega^1 = \omega_{i2}\, dx^i, \quad \Omega^2 = \omega_{1i}\, dx^i$$

if we take $\omega_{12} \neq 0$. Then we find that

$$\Omega^1 \wedge \Omega^2 = \omega_{i2}\omega_{1j}\, dx^i \wedge dx^j = \omega_{[i2}\omega_{1j]}\, dx^i \wedge dx^j$$
$$= \frac{1}{2}(\omega_{i2}\omega_{1j} - \omega_{j2}\omega_{1i})\, dx^i \wedge dx^j.$$

On the other hand, the coefficients ω_{ij} are satisfying the relations

$$\omega_{i2}\omega_{1j} + \omega_{i1}\omega_{j2} + \omega_{ij}\omega_{21} = \omega_{i2}\omega_{1j} - \omega_{j2}\omega_{1i} - \omega_{ij}\omega_{12} = 0$$

so that we obtain $\omega_{i2}\omega_{1j} - \omega_{j2}\omega_{1i} = \omega_{12}\omega_{ij}$. This yields

$$\omega = \Omega^1 \wedge \Omega^2/\omega_{12}.$$

Hence, if we choose

$$\omega^1 = \Omega^1/\omega_{12} \quad \text{and} \quad \omega^2 = \Omega^2,$$

we find that

$$\omega = \omega^1 \wedge \omega^2.$$ ■

5.5. BASES INDUCED BY THE VOLUME FORM

The non-zero m-volume form μ on an m-dimensional manifold M was introduced by (5.2.5). On using Levi-Civita symbols defined in p. 31, this form can also be expressed as

$$\mu = dx^1 \wedge dx^2 \wedge \cdots \wedge dx^m$$
$$= \frac{1}{m!}\, e_{i_1 i_2 \cdots i_m}\, dx^{i_1} \wedge dx^{i_2} \wedge \cdots \wedge dx^{i_m}.$$

Our aim is to derive a new set of basis forms for the exterior algebra that may prove to be more advantageous in certain cases than the natural basis. However, to fulfil this task, we have to reveal some novel properties of the generalised Kronecker deltas introduced previously by the expression (1.4.6):

$$\delta^{i_1 i_2 \cdots i_k}_{j_1 j_2 \cdots j_k} = \begin{vmatrix} \delta^{i_1}_{j_1} & \delta^{i_1}_{j_2} & \cdots & \delta^{i_1}_{j_k} \\ \delta^{i_2}_{j_1} & \delta^{i_2}_{j_2} & \cdots & \delta^{i_2}_{j_k} \\ \vdots & \vdots & & \vdots \\ \delta^{i_k}_{j_1} & \delta^{i_k}_{j_2} & \cdots & \delta^{i_k}_{j_k} \end{vmatrix}. \tag{5.5.1}$$

If we expand the $k \times k$ symbolic determinant (5.5.1) with respect to its first row, we obtain the following expression by adopting the convention that $\delta^{i_r}_{j_0}$

does not exist

$$\begin{aligned}
\delta^{i_1 i_2 \cdots i_k}_{j_1 j_2 \cdots j_k} &= \sum_{l=1}^{k} (-1)^{l-1} \delta^{i_1}_{j_l} \begin{bmatrix} \delta^{i_2}_{j_1} & \cdots & \delta^{i_2}_{j_{l-1}} & \delta^{i_2}_{j_{l+1}} & \cdots & \delta^{i_2}_{j_k} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \delta^{i_k}_{j_1} & \cdots & \delta^{i_k}_{j_{l-1}} & \delta^{i_k}_{j_{l+1}} & \cdots & \delta^{i_k}_{j_k} \end{bmatrix} \\
&= \sum_{l=1}^{k} (-1)^{l-1} \delta^{i_1}_{j_l} \delta^{i_2 \cdots i_{l-1} i_l i_{l+1} \cdots i_k}_{j_1 \cdots j_{l-1} j_{l+1} \cdots j_k} \\
&= \delta^{i_1}_{j_1} \delta^{i_2 \cdots i_k}_{j_2 \cdots j_k} + \sum_{l=2}^{k} (-1)^{l-1} \delta^{i_1}_{j_l} \delta^{i_2 \cdots i_{l-1} i_l i_{l+1} \cdots i_k}_{j_1 \cdots j_{l-1} j_{l+1} \cdots j_k}.
\end{aligned}$$

On the other hand, for $l \geq 2$ we can write

$$\begin{aligned}
(-1)^{l-1} \delta^{i_2 \cdots i_{l-1} i_l i_{l+1} \cdots i_k}_{j_1 j_2 \cdots j_{l-1} j_{l+1} \cdots j_k} &= (-1)^{l-1+l-2} \delta^{i_2 \cdots i_{l-1} i_l i_{l+1} \cdots i_k}_{j_2 \cdots j_{l-1} j_1 j_{l+1} \cdots j_k} \\
&= (-1)^{2l-3} \delta^{i_2 \cdots i_{l-1} i_l i_{l+1} \cdots i_k}_{j_2 \cdots j_{l-1} j_1 j_{l+1} \cdots j_k} \\
&= - \delta^{i_2 \cdots i_{l-1} i_l i_{l+1} \cdots i_k}_{j_2 \cdots j_{l-1} j_1 j_{l+1} \cdots j_k}
\end{aligned}$$

and find

$$\begin{aligned}
\delta^{i_1 i_2 \cdots i_k}_{j_1 j_2 \cdots j_k} &= \delta^{i_1}_{j_1} \delta^{i_2 \cdots i_k}_{j_2 \cdots j_k} - \sum_{l=2}^{k} \delta^{i_1}_{j_l} \delta^{i_2 \cdots i_{l-1} i_l i_{l+1} \cdots i_k}_{j_2 \cdots j_{l-1} j_1 j_{l+1} \cdots j_k} \qquad (5.5.2) \\
&= \delta^{i_1}_{j_1} \delta^{i_2 \cdots i_k}_{j_2 \cdots j_k} - \delta^{i_1}_{j_2} \delta^{i_2 i_3 \cdots i_k}_{j_1 j_3 \cdots j_k} - \delta^{i_1}_{j_3} \delta^{i_2 i_3 i_4 \cdots i_k}_{j_2 j_1 j_4 \cdots j_k} - \cdots - \delta^{i_1}_{j_k} \delta^{i_2 i_3 \cdots i_{k-1} i_k}_{j_2 j_3 \cdots j_{k-1} j_1}.
\end{aligned}$$

On making a contraction on the indices i_1 and j_1 in (5.5.2) by taking $i_1 = j_1$, we arrive at

$$\begin{aligned}
\delta^{i_1 i_2 \cdots i_k}_{i_1 j_2 \cdots j_k} &= m \delta^{i_2 \cdots i_k}_{j_2 \cdots j_k} - (k-1) \delta^{i_2 \cdots i_k}_{j_2 \cdots j_k} \qquad (5.5.3) \\
&= (m - k + 1) \delta^{i_2 \cdots i_k}_{j_2 \cdots j_k}.
\end{aligned}$$

When we repeat this operation r times, we conclude that

$$\delta^{i_1 \cdots i_r i_{r+1} \cdots i_k}_{i_1 \cdots i_r j_{r+1} \cdots j_k} = \qquad (5.5.4)$$
$$(m-k+1)(m-k+2) \cdots (m-k+r) \delta^{i_{r+1} \cdots i_k}_{j_{r+1} \cdots j_k}.$$

Let us next take $k = m$ in the expression above. We thus conclude that (5.5.4) then yields

$$\delta^{i_1 \cdots i_r i_{r+1} \cdots i_m}_{i_1 \cdots i_r j_{r+1} \cdots j_m} = r! \, \delta^{i_{r+1} \cdots i_m}_{j_{r+1} \cdots j_m} \qquad (5.5.5)$$

so one deduces that

$$\delta^{i_1 i_2 \cdots i_m}_{i_1 i_2 \cdots i_m} = m!. \tag{5.5.6}$$

We know from (1.4.16) that we can write

$$\delta^{i_1 i_2 \cdots i_m}_{j_1 j_2 \cdots j_m} = e^{i_1 i_2 \cdots i_m} e_{j_1 j_2 \cdots j_m}. \tag{5.5.7}$$

Hence, making use of (5.5.5) we can reach to the relation

$$\delta^{i_1 \cdots i_r}_{j_1 \cdots j_r} = \frac{1}{(m-r)!} e^{i_1 \cdots i_r i_{r+1} \cdots i_m} e_{j_1 \cdots j_r i_{r+1} \cdots i_m}.$$

We now define m number of $(m-1)$-forms as follows

$$\mu_i = \mathbf{i}_{\partial_i}(\mu) = \frac{1}{(m-1)!}\, e_{i i_2 \cdots i_m} dx^{i_2} \wedge \cdots \wedge dx^{i_m} \in \Lambda^{m-1}(M). \tag{5.5.8}$$

Let us next evaluate the exterior product of a form μ_i with dx^j to obtain

$$\begin{aligned} dx^j \wedge \mu_i &= \frac{1}{(m-1)!}\, e_{i i_2 \cdots i_m}\, dx^j \wedge dx^{i_2} \wedge \cdots \wedge dx^{i_m} \qquad (5.5.9) \\ &= \frac{1}{(m-1)!}\, e_{i i_2 \cdots i_m} e^{j i_2 \cdots i_m}\, dx^1 \wedge dx^2 \wedge \cdots \wedge dx^m \\ &= \frac{1}{(m-1)!} \delta^{j i_2 \cdots i_m}_{i i_2 \cdots i_m} \mu = \frac{(m-1)!}{(m-1)!} \delta^j_i \mu. \\ &= \delta^j_i \mu \in \Lambda^m(M) \end{aligned}$$

We now write $c^i \mu_i = 0$ where c^i are arbitrary functions. The exterior product of this zero form with dx^j is

$$0 = c^i dx^j \wedge \mu_i = c^i \delta^j_i \mu = c^j \mu.$$

Since, μ does not vanish we deduce that $c^j = 0, j = 1, \ldots, m$. Thus m forms $\mu_i \in \Lambda^{m-1}(M)$ are linearly independent and they constitute a basis for the module $\Lambda^{m-1}(M)$.

We shall now try to determine top-down generated bases for the modules $\Lambda^{m-k}(M)$ for $k = 0, 1, \ldots, m$ in an exactly similar fashion. To this end, we introduce the forms

$$\begin{aligned} \mu_{i_k i_{k-1} \cdots i_1} &= (\mathbf{i}_{\partial_{i_k}} \circ \mathbf{i}_{\partial_{i_{k-1}}} \circ \cdots \circ \mathbf{i}_{\partial_{i_1}})(\mu) \qquad (5.5.10) \\ &= \frac{1}{(m-k)!}\, e_{i_1 \cdots i_k i_{k+1} \cdots i_m}\, dx^{i_{k+1}} \wedge \cdots \wedge dx^{i_m} \in \Lambda^{m-k}(M). \end{aligned}$$

Because of the properties of the interior product, these forms have to be completely antisymmetric:

$$\mu_{i_k i_{k-1}\cdots i_1} = \mu_{[i_k i_{k-1}\cdots i_1]}.$$

Therefore, the number of their independent components is $\binom{m}{k} = \binom{m}{m-k}$ which is equal to the dimension of the module $\Lambda^{m-k}(M)$. By adopting the convention $\mu_{i_0} = \mu$, the definition (5.5.10) leads to

$$\mu_{i_k i_{k-1}\cdots i_1} = \mathbf{i}_{\partial_{i_k}}(\mu_{i_{k-1}\cdots i_1}), \quad 1 \le k \le m. \tag{5.5.11}$$

On using Levi-Civita symbols, we obtain from (5.5.10) that

$$\begin{aligned} e^{i_1\cdots i_k j_{k+1}\cdots j_m}\mu_{i_k\cdots i_1} &= \frac{1}{(m-k)!}\,\delta^{i_1\cdots i_k j_{k+1}\cdots j_m}_{i_1\cdots i_k i_{k+1}\cdots i_m} dx^{i_{k+1}} \wedge \cdots \wedge dx^{i_m} \\ &= \frac{k!}{(m-k)!}\,\delta^{j_{k+1}\cdots j_m}_{i_{k+1}\cdots i_m} dx^{i_{k+1}} \wedge \cdots \wedge dx^{i_m} \\ &= k!\; dx^{[j_{k+1}} \wedge \cdots \wedge dx^{j_m]} \\ &= k!\; dx^{j_{k+1}} \wedge \cdots \wedge dx^{j_m} \end{aligned}$$

where we have employed (1.4.8). We thus find the inverse relation

$$dx^{i_{k+1}} \wedge \cdots \wedge dx^{i_m} = \frac{1}{k!}\, e^{i_1\cdots i_k i_{k+1}\cdots i_m}\mu_{i_k\cdots i_1}. \tag{5.5.12}$$

Let us now choose $m-(k-l) \le m$, namely, $l \le k$. In this case the form

$$dx^{j_1} \wedge \cdots \wedge dx^{j_l} \wedge \mu_{i_k\cdots i_1}$$

becomes obviously a $(m-k+l)$-form. The explicit evaluation of that form by making use of (5.5.10) and (5.5.12) gives

$$\begin{aligned} &dx^{j_1} \wedge \cdots \wedge dx^{j_l} \wedge \mu_{i_k\cdots i_1} \\ &\quad = \frac{1}{(m-k)!}\, e_{i_1\cdots i_k i_{k+1}\cdots i_m} dx^{j_1} \wedge \cdots \wedge dx^{j_l} \wedge dx^{i_{k+1}} \wedge \cdots \wedge dx^{i_m} \\ &\quad = \frac{1}{(m-k)!}\frac{1}{(k-l)!}\, e_{i_1\cdots i_k i_{k+1}\cdots i_m} e^{s_1\cdots s_{k-l} j_1\cdots j_l i_{k+1}\cdots i_m}\mu_{s_{k-l}\cdots s_1} \\ &\quad = \frac{1}{(k-l)!}\,\delta^{s_1\cdots s_{k-l} j_1\cdots j_l}_{i_1 i_2\cdots i_{k-2} i_{k-1} i_k}\mu_{s_{k-l}\cdots s_1}. \end{aligned} \tag{5.5.13}$$

If we take $l = k$, then (5.5.13) leads to

$$dx^{j_1} \wedge dx^{j_2} \wedge \cdots \wedge dx^{j_k} \wedge \mu_{i_k\cdots i_2 i_1} = \delta^{j_1 j_2\cdots j_k}_{i_1 i_2\cdots i_k}\,\mu \tag{5.5.14}$$

since we have assumed that $\mu_{s_0} = \mu$. After having this relation on hand, we can easily demonstrate that the forms $\mu_{i_k \cdots i_1}$ constitute a basis for the module $\Lambda^{m-k}(M)$. Let us write

$$c^{i_1 \cdots i_k} \mu_{i_k \cdots i_1} = 0$$

where $c^{i_1 \cdots i_k}$ are arbitrary smooth functions. It is obvious that we can select the coefficient functions $c^{i_1 \cdots i_k}$ as being completely antisymmetric, that is, satisfying relations

$$c^{i_1 \cdots i_k} = c^{[i_1 \cdots i_k]}$$

without loss of generality. The exterior product of the above linear combination with the form $dx^{j_1} \wedge \cdots \wedge dx^{j_k}$ yields due to (5.5.14)

$$\begin{aligned} \delta^{j_1 j_2 \cdots j_k}_{i_1 i_2 \cdots i_k} c^{i_1 \cdots i_k} \mu &= k!\, c^{[i_1 \cdots i_k]} \mu \\ &= k!\, c^{i_1 \cdots i_k} \mu = 0. \end{aligned}$$

Since $\mu \neq 0$, we then deduce that all coefficients vanish, i.e., $c^{i_1 \cdots i_k} = 0$. Therefore, the forms $\mu_{i_k \cdots i_1}$ are linearly independent so they constitute a basis of the module $\Lambda^{m-k}(M)$. Consequently, we obtain the following sequence of ***top down generated bases*** for modules $\Lambda^m(M)$, $\Lambda^{m-1}(M), \ldots,$ $\Lambda^2(M)$, $\Lambda^1(M)$, $\Lambda^0(M)$ from the volume form μ:

$$\begin{aligned} \Lambda^m(M) &: \mu = \frac{1}{m!}\, e_{i_1 i_2 \cdots i_m} dx^{i_1} \wedge dx^{i_2} \wedge \cdots \wedge dx^{i_m}, \\ \Lambda^{m-1}(M) &: \mu_i = \mathbf{i}_{\partial_i}(\mu) = \frac{1}{(m-1)!}\, e_{i i_2 \cdots i_m} dx^{i_2} \wedge \cdots \wedge dx^{i_m}, \\ \Lambda^{m-2}(M) &: \mu_{ji} = \mathbf{i}_{\partial_j}(\mu_i) = \frac{1}{(m-2)!}\, e_{i j i_3 \cdots i_m} dx^{i_3} \wedge \cdots \wedge dx^{i_m}, \\ &\vdots \\ \Lambda^{m-k}(M) &: \mu_{i_k i_{k-1} \cdots i_1} = \mathbf{i}_{\partial_{i_k}}(\mu_{i_{k-1} \cdots i_1}) \\ &\qquad = \frac{1}{(m-k)!}\, e_{i_1 \cdots i_k i_{k+1} \cdots i_m} dx^{i_{k+1}} \wedge \cdots \wedge dx^{i_m}, \\ &\vdots \\ \Lambda^1(M) &: \mu_{i_{m-1} \cdots i_1} = \mathbf{i}_{\partial_{i_{m-1}}}(\mu_{i_{m-2} \cdots i_1}) = e_{i_1 \cdots i_{m-1} i_m}\, dx^{i_m}, \\ \Lambda^0(M) &: \mu_{i_m \cdots i_1} = e_{i_1 \cdots i_m} = \pm 1. \end{aligned}$$

If we take $l = 1$ in (5.5.13) and utilise (5.5.2) the following result comes out

$$
\begin{aligned}
dx^i \wedge \mu_{i_k\cdots i_1} &= \frac{1}{(k-1)!}\delta^{j_1\cdots j_{k-1}i}_{i_1\cdots i_{k-1}i_k}\mu_{j_{k-1}\cdots j_1} \\
&= \frac{1}{(k-1)!}\delta^{ij_1\cdots j_{k-1}}_{i_k i_1\cdots i_{k-1}}\mu^{j_1 j_2\cdots j_{k-1}}_{j_{k-1}\cdots j_1} \\
&= \frac{1}{(k-1)!}\Big[\delta^i_{i_k}\delta^{j_1 j_2\cdots j_{k-1}}_{i_1 i_2\cdots i_{k-1}} - \delta^i_{i_1}\delta_{i_k i_2\cdots i_{k-1}} \\
&\quad - \delta^i_{i_2}\delta^{j_1 j_2\cdots j_{k-1}}_{i_1 i_k\cdots i_{k-1}} - \cdots - \delta^i_{i_{k-1}}\delta^{j_1 j_2\cdots j_{k-1}}_{i_1 i_2\cdots i_k}\Big]\mu_{j_{k-1}\cdots j_1} = \\
&\delta^i_{i_k}\mu_{[i_{k-1}\cdots i_2 i_1]} - \delta^i_{i_1}\mu_{[i_{k-1}\cdots i_2 i_k]} - \delta^i_{i_2}\mu_{[i_{k-1}\cdots i_k i_1]} - \cdots - \delta^i_{i_{k-1}}\mu_{[i_k\cdots i_2 i_1]} \\
&= \delta^i_{i_k}\mu_{i_{k-1}\cdots i_2 i_1} - \delta^i_{i_1}\mu_{i_{k-1}\cdots i_2 i_k} \\
&\qquad - \delta^i_{i_2}\mu_{i_{k-1}\cdots i_k i_1} - \cdots - \delta^i_{i_{k-1}}\mu_{i_k\cdots i_2 i_1}.
\end{aligned}
$$

Finally, we observe that we can write

$$
dx^i \wedge \mu_{i_k\cdots i_1} = k\,\delta^i_{[i_k}\mu_{i_{k-1}\cdots i_2 i_1]} \tag{5.5.15}
$$

because of the complete antisymmetry of forms $\mu_{i_{k-1}\cdots i_2 i_1}$ with respect to its $k-1$ indices. Indeed, we find that

$$
\begin{aligned}
k\,\delta^i_{[i_k}\mu_{i_{k-1}\cdots i_2 i_1]} &= \frac{k}{k!}\,\delta^{j_1\cdots j_{k-1}j_k}_{i_1\cdots i_{k-1}i_k}\,\delta^i_{j_k}\mu_{j_{k-1}\cdots j_1} \\
&= \frac{1}{(k-1)!}\,\delta^{j_1\cdots j_{k-1}i}_{i_1\cdots i_{k-1}i_k}\mu_{j_{k-1}\cdots j_1}.
\end{aligned}
$$

For instance, we have the relations

$$
\begin{aligned}
dx^i \wedge \mu_{jk} &= 2\delta^i_{[j}\mu_{k]} = \delta^i_j\mu_k - \delta^i_k\mu_j, \\
dx^l \wedge \mu_{kji} &= 3\delta^l_{[k}\mu_{ji]} = \delta^l_k\mu_{ji} + \delta^l_j\mu_{ik} + \delta^l_i\mu_{kj}.
\end{aligned} \tag{5.5.16}
$$

Thus, a form $\omega \in \Lambda^{m-k}(M)$ is also expressible as

$$
\omega = \frac{1}{k!}\,\omega^{i_1 i_2\cdots i_k}(\mathbf{x})\,\mu_{i_k\cdots i_2 i_1} \tag{5.5.17}
$$

where the functions $\omega^{i_1 i_2\cdots i_k} \in \Lambda^0(M)$ are completely antisymmetric, that is, they satisfy the relation $\omega^{i_1 i_2\cdots i_k} = \omega^{[i_1 i_2\cdots i_k]}$.

On utilising this representation, we can readily prove that *every form in* $\Lambda^{m-1}(M)$ *is simple*. A non-zero form $\omega \in \Lambda^{m-1}(M)$ can now be expressed as $\omega = \omega^i\mu_i$. If 1-form $\Omega = \Omega_j\,dx^j$ is a divisor of the form ω, then the relation $\Omega \wedge \omega = 0$ or $\Omega_j\omega^i dx^j \wedge \mu_i = \Omega_j\omega^i\delta^j_i\mu = \Omega_i\omega^i\mu = 0$ must hold. This means that $\Omega_i\omega^i = \Omega_1\omega^1 + \Omega_2\omega^2 + \cdots + \Omega_m\omega^m = 0$. Since we have supposed that $\omega \neq 0$, then at least one coefficient does not vanish. Without loss of generality, we may choose that the coefficient ω^m is different from zero. We thus obtain

$$\Omega_m = -\frac{\omega^1}{\omega^m}\Omega_1 - \frac{\omega^2}{\omega^m}\Omega_2 - \cdots - \frac{\omega^{m-1}}{\omega^m}\Omega_{m-1}$$

and inserting this expression into the form Ω, we get

$$\Omega = \Omega_1(dx^1 - \frac{\omega^1}{\omega^m}\,dx^m) + \cdots + \Omega_{m-1}(dx^{m-1} - \frac{\omega^{m-1}}{\omega^m}\,dx^m).$$

Next, we define $m-1$ linearly independent 1-forms by

$$\Omega^1 = \omega^m\,dx^1 - \omega^1\,dx^m, \Omega^2 = dx^2 - \frac{\omega^2}{\omega^m}\,dx^m, \ldots, \Omega^{m-1} = dx^{m-1} - \frac{\omega^{m-1}}{\omega^m}\,dx^m$$

Each one of these forms divides the form ω. Hence, we can write

$$\omega = \Omega^1 \wedge \Omega^2 \wedge \cdots \wedge \Omega^{m-1}. \qquad \square$$

The interior product of a vector $V = v^i\partial_i$ with a form $\omega \in \Lambda^{m-k}(M)$ can now be expressed as follows

$$\begin{aligned}\mathbf{i}_V(\omega) &= \frac{1}{k!}\,v^i\omega^{i_1i_2\cdots i_k}\mathbf{i}_{\partial_i}(\mu_{i_k\cdots i_2i_1}) = \frac{1}{k!}\,v^i\omega^{i_1i_2\cdots i_k}\mu_{ii_k\cdots i_2i_1}\\ &= \frac{k+1}{(k+1)!}\,v^{[i}\omega^{i_1i_2\cdots i_k]}\mu_{ii_k\cdots i_2i_1} \in \Lambda^{m-(k+1)}(M).\end{aligned}$$

It is clear that a form $\omega \in \Lambda^{m-k}(M)$ can hereby be represented by resorting to two different bases as given below:

$$\omega = \frac{1}{k!}\,\omega^{i_1\cdots i_k}\mu_{i_k\cdots i_1} = \frac{1}{(m-k)!}\,\omega_{i_{k+1}\cdots i_m}dx^{i_{k+1}} \wedge \cdots \wedge dx^{i_m}.$$

When we employ (5.5.10) it follows from this expression that

$$\frac{1}{k!}\,\frac{1}{(m-k)!}\,\omega^{i_1\cdots i_k}\,e_{i_1\cdots i_ki_{k+1}\cdots i_m}dx^{i_{k+1}} \wedge \cdots \wedge dx^{i_m} = \frac{1}{(m-k)!}\,\omega_{i_{k+1}\cdots i_m}dx^{i_{k+1}} \wedge \cdots \wedge dx^{i_m}$$

so that coefficient functions are interrelated by

$$\omega_{i_{k+1}\cdots i_m} = \frac{1}{k!}\,e_{i_1\cdots i_ki_{k+1}\cdots i_m}\omega^{i_1\cdots i_k} \tag{5.5.18}$$

After having performed some operations involving Levi-Civita symbols, we readily get

$$\begin{aligned}\omega_{i_{k+1}\cdots i_m}e^{j_1\cdots j_k i_{k+1}\cdots i_m} &= \frac{1}{k!}\,\delta^{j_1\cdots j_k i_{k+1}\cdots i_m}_{i_1\cdots i_k i_{k+1}\cdots i_m}\omega^{i_1\cdots i_k}\\ &= \frac{1}{k!}(m-k)!\,\delta^{j_1\cdots j_k}_{i_1\cdots i_k}\omega^{i_1\cdots i_k}\\ &= (m-k)!\,\omega^{[j_1\cdots j_k]}\\ &= (m-k)!\,\omega^{j_1\cdots j_k}\end{aligned}$$

and we finally reach to the inverse relation

$$\omega^{j_1\cdots j_k} = \frac{1}{(m-k)!}\,e^{j_1\cdots j_k i_{k+1}\cdots i_m}\omega_{i_{k+1}\cdots i_m}. \tag{5.5.19}$$

Let us consider a form $\omega \in \Lambda^k(M)$ given by

$$\omega = \frac{1}{k!}\,\omega_{i_1\cdots i_k}(\mathbf{x})\,dx^{i_1}\wedge\cdots\wedge dx^{i_k}$$

in the natural basis. On using the same functions $\omega_{i_1\cdots i_k}$, but transferring lower indices to upper indices to comply with the Einstein summation convention in its usual fashion, we may define a form $*\omega \in \Lambda^{m-k}(M)$ associated with the form $\omega \in \Lambda^k(M)$ by the relation

$$*\omega = \frac{1}{k!}\,\omega^{i_1 i_2\cdots i_k}\mu_{i_k\cdots i_2 i_1}. \tag{5.5.20}$$

The form $*\omega$ so obtained will called the ***Hodge dual*** of the form ω. This concept was first introduced by English mathematician William Vallance Douglas Hodge (1903-1975). We investigate properties of the Hodge dual a little bit later within the context of the Riemannian manifolds in detail and put the operation of raising the indices of component functions on a more solid foundation. Let us just point out that, according to (5.5.14) one is able to write

$$\begin{aligned}\omega\wedge *\omega &= \Big(\frac{1}{k!}\Big)^2\omega_{j_1\cdots j_k}\,\omega^{i_1\cdots i_k}dx^{j_1}\wedge\cdots\wedge dx^{j_k}\wedge\mu_{i_k\cdots i_1}\\ &= \Big(\frac{1}{k!}\Big)^2\delta^{j_1\cdots j_k}_{i_1\cdots i_k}\omega_{j_1\cdots j_k}\,\omega^{i_1\cdots i_k}\mu\\ &= \frac{1}{k!}\omega_{[i_1\cdots i_k]}\,\omega^{i_1\cdots i_k}\mu\\ &= \frac{1}{k!}\omega_{i_1\cdots i_k}\omega^{i_1\cdots i_k}\mu.\end{aligned}$$

As an example, consider a 1-form $\omega = \omega_i dx^i$. We then obtain

$$*\omega = \omega^i \mu_i = \frac{1}{(m-1)!}\, e_{ii_2\cdots i_m} \omega^i dx^{i_2} \wedge \cdots \wedge dx^{i_m} \in \Lambda^{m-1}(M),$$

and consequently

$$\omega \wedge *\omega = \omega^i \omega_i \mu.$$

5.6. IDEALS OF THE EXTERIOR ALGEBRA $\Lambda(M)$

Since $\Lambda(M)$ is an algebra, it is quite natural that we look for its ideals. A subset, or more precisely a subalgebra, of the exterior algebra $\Lambda(M)$ is called an ***ideal*** $\mathcal{I}$ (*homogeneous ideal*) of $\Lambda(M)$ if it satisfies the conditions below:

(i). *For every forms* $\alpha, \beta \in \mathcal{I}$ *of the same degree, one has* $\alpha + \beta \in \mathcal{I}$.

(ii). *If* $\alpha \in \mathcal{I}$, *then one has* $\gamma \wedge \alpha = (-1)^{(deg\,\gamma)(deg\,\alpha)} \alpha \wedge \gamma \in \mathcal{I}$ *for all* $\gamma \in \Lambda(M)$.

We see that only the sum of forms of the same degree in $\mathcal{I}$ is allowed. That is the reason why we call the ideal $\mathcal{I}$ as a *homogeneous ideal*. It is quite obvious that it is not possible for elements of the ideal to escape outside this subalgebra by means of exterior product.

Let us now consider some r members $\alpha_1, \alpha_2, \ldots, \alpha_r$ of the exterior algebra $\Lambda(M)$ that can be of diverse degrees and construct all forms in the following shape

$$\beta = \gamma^1 \wedge \alpha_1 + \cdots + \gamma^r \wedge \alpha_r = \gamma^a \wedge \alpha_a,\ \gamma^a \in \Lambda(M),\ a = 1, \ldots, r.$$

If the degree of the form β is p, then it is evident that the degree conditions given below must hold

$$deg\,\gamma^a + deg\,\alpha_a = p, \quad deg\,\alpha_a \le p, \quad a = 1, \ldots, r.$$

We denote the collection of all members of $\Lambda(M)$ constructed this way by $\mathcal{I}(\alpha_1, \alpha_2, \ldots, \alpha_r)$. Let two forms β_1 and β_2 of the same degree belong to $\mathcal{I}$. Hence, we can write

$$\beta_1 = \gamma^a_{(1)} \wedge \alpha_a, \quad \beta_2 = \gamma^a_{(2)} \wedge \alpha_a,\ \gamma^a_{(1)}, \gamma^a_{(2)} \in \Lambda(M)$$

so that we obtain

$$\beta_1 + \beta_2 = (\gamma^a_{(1)} + \gamma^a_{(2)}) \wedge \alpha_a.$$

Since $\gamma^a_{(1)} + \gamma^a_{(2)} \in \Lambda(M)$, we see that $\beta_1 + \beta_2 \in \mathcal{I}$. Similarly, if $\beta \in \mathcal{I}$ and $\sigma \in \Lambda(M)$ we have to write

$$\sigma \wedge \beta = \sigma \wedge (\gamma^a \wedge \alpha_a) = (\sigma \wedge \gamma^a) \wedge \alpha_a$$

where $\gamma^a \in \Lambda(M)$. Since $\sigma \wedge \gamma^a \in \Lambda(M)$, we find that $\sigma \wedge \beta \in \mathcal{I}$. These clearly results indicate that the set $\mathcal{I}(\alpha_1, \alpha_2, \dots, \alpha_r)$ so constructed by given forms that may be of various degrees is an ideal of the exterior algebra $\Lambda(M)$. The forms $\alpha_1,\ \alpha_2, \dots, \alpha_r$ are then naturally called the ***generators*** of the ideal $\mathcal{I}$.

We say that an ideal $\mathcal{I}$ is generated by the forms $\alpha_1, \alpha_2,\ \dots, \alpha_r$ if each member of which is expressible as the sum of terms admitting at least one member of the set $\{\alpha_1, \alpha_2, \dots, \alpha_r\}$ as an exterior factor.

Example 5.6.1. Let us consider the exterior algebra $\Lambda(\mathbb{R}^4)$ and the coordinate cover $\{x^i\} = \{x, y, z, t\}$ for the manifold $\mathbb{R}^4$. We want to determine the members of the ideal generated by the forms

$$\begin{aligned}
\alpha_1 &= 2\, dx - 3y\, dz, \\
\alpha_2 &= x\, dy - z\, dt, \\
\alpha_3 &= x^2 t\, dx \wedge dt - t\, dy \wedge dz.
\end{aligned}$$

Since the lowest degree of the generating forms is 1, then this ideal cannot contain 0-forms, namely, smooth functions. Forms with degrees higher than 4 are identically zero. We can classify the forms in the ideal according to their degrees as follows:

1-*forms*: $\beta = f(2\, dx - 3y\, dz) + g(x\, dy - z\, dt), \quad f, g \in \Lambda^0(\mathbb{R}^4)$

2-*forms*:

$$\beta = \gamma^1 \wedge \alpha_1 + \gamma^2 \wedge \alpha_2 + f\alpha_3$$

where

$$\gamma^a = f^a dx + g^a dy + h^a dz + k^a dt,\ \ f, f^a, g^a, h^a, k^a \in \Lambda^0(\mathbb{R}^4), a = 1, 2$$

so that we get

$$\begin{aligned}
\beta = &-(2g^1 - xf^2)\, dx \wedge dy - (3yf^1 + 2h^1)\, dx \wedge dz \\
&- (zf^2 + 2k^1 - x^2 tf)\, dx \wedge dt - (3yg^1 + xh^2 + tf)\, dy \wedge dz \\
&- (zg^2 + xk^2)\, dy \wedge dt + (3k^1 y - zh^2)\, dz \wedge dt
\end{aligned}$$

3-*forms*:

$$\beta = \gamma^1 \wedge \alpha_1 + \gamma^2 \wedge \alpha_2 + \gamma \wedge \alpha_3$$

where

$$\begin{aligned}\gamma^a &= f^a dx\wedge dy + g^a dx\wedge dz + h^a dx\wedge dt + k^a dy\wedge dz\\ &\quad + l^a dy\wedge dt + m^a dz\wedge dt, f^a, g^a, h^a, k^a, l^a, m^a \in \Lambda^0(\mathbb{R}^4), a = 1,2,\\ \gamma &= f\,dx + g\,dy + h\,dz + k\,dt, \quad f,g,h,k\in\Lambda^0(\mathbb{R}^4),\end{aligned}$$

so that

$$\begin{aligned}\beta = &-(3yf^1 - 2k^1 + xg^2 + tf)\,dx\wedge dy\wedge dz,\\ &+(2l^1 - zf^2 - xh^2 - x^2tg)\,dx\wedge dy\wedge dt,\\ &+(3yh^1 + 2m^1 - zg^2 - x^2th)\,dx\wedge dz\wedge dt,\\ &+(3yl^1 - zk^2 + xm^2 - tk)\,dy\wedge dz\wedge dt.\end{aligned}$$

4-*forms*:

$$\beta = \gamma^1\wedge\alpha_1 + \gamma^2\wedge\alpha_2 + \gamma\wedge\alpha_3$$

where

$$\begin{aligned}\gamma^a &= f^a dx\wedge dy\wedge dz + g^a dx\wedge dy\wedge dt + h^a dx\wedge dz\wedge dt\\ &\quad + k^a dy\wedge dz\wedge dt,\ \ f^a, g^a, h^a, k^a\in\Lambda^0(\mathbb{R}^4), a=1,2,\\ \gamma &= f\,dx\wedge dy + g\,dx\wedge dz + h\,dx\wedge dt + k\,dy\wedge dz + l\,dy\wedge dt\\ &\quad + m\,dz\wedge dt + l\,dy\wedge dt + m\,dz\wedge dt,\ f,g,h,k,l,m\in\Lambda^0(\mathbb{R}^4),\end{aligned}$$

so that

$$\beta = (3yg^1 - 2k^1 - zf^2 + xh^2 - th + x^2tk)\,dx\wedge dy\wedge dz\wedge dt. \qquad \blacksquare$$

Let $\mathcal{I}$ be an ideal. If two forms $\alpha,\beta\in\Lambda(M)$ of the same degree are related by $\alpha-\beta\in\mathcal{I}$, we write $\alpha = \beta \bmod \mathcal{I}$ or, amounting to the same thing, $\alpha-\beta = 0 \bmod \mathcal{I}$. When we consider such kind of forms α and β, it becomes clear that we may use the representation $\gamma\wedge(\alpha-\beta) = 0 \bmod \mathcal{I}$ for all forms $\gamma\in\Lambda(M)$.

The ***characteristic vector fields*** of a form $\omega\in\Lambda(M)$ are defined as vector fields satisfying the condition

$$\mathbf{i}_V(\omega) = 0. \tag{5.6.1}$$

These vectors belong to a subbundle of the tangent bundle $T(M)$. Indeed, in view of (5.4.7), if $\mathbf{i}_V(\omega)=0$ we then obtain $\mathbf{i}_{fV}(\omega) = f\mathbf{i}_V(\omega) = 0$ for all $f\in\Lambda^0(M)$. Likewise, if $\mathbf{i}_{V_1}(\omega) = \mathbf{i}_{V_2}(\omega) = 0$ we get $\mathbf{i}_{V_1+V_2}(\omega) = \mathbf{i}_{V_1}(\omega) + \mathbf{i}_{V_2}(\omega) = 0$. Therefore vectors fV and V_1+V_2 are also characteristic vectors of the form ω. We can easily demonstrate that if the rank of the form defined in Sec. 1.6 is r, the number of linearly independent characteristic vector fields turns out to be $m - r$. Let us take the form $\omega \in \Lambda^k(M)$ into account. Then the relation

$$\mathbf{i}_V(\omega) = \frac{1}{(k-1)!} v^i \omega_{ii_1 i_2 \cdots i_{k-1}} dx^{i_1} \wedge dx^{i_2} \wedge \cdots \wedge dx^{i_{k-1}} = 0$$

results in $v^i \omega_{ii_1 i_2 \cdots i_{k-1}} = 0, 1 \leq i_1, i_2, \ldots, i_{k-1} \leq m$. If we note that these relations are identical with equations (1.6.3), we arrive at the fact that if the form possesses $m - r$ linearly independent characteristic vector fields, then its rank must be r. This amounts to say that there are exactly r linearly independent forms $\theta^\alpha \in \Lambda^1(M), \alpha = 1, \ldots, r$ so that ω is represented just as in (1.6.6) by the expression

$$\omega = \frac{1}{k!}\, \omega_{\alpha_1 \alpha_2 \cdots \alpha_k} \theta^{\alpha_1} \wedge \theta^{\alpha_2} \wedge \cdots \wedge \theta^{\alpha_k}. \tag{5.6.2}$$

When the rank r is equal to m, then the characteristic vector can only be the zero vector.

Let $\mathcal{I}$ be an ideal of the exterior algebra $\Lambda(M)$. If a vector field $V \in T(M)$ satisfies the condition $\mathbf{i}_V(\omega) \in \mathcal{I}$ for all forms $\omega \in \mathcal{I}$, then it is called a ***characteristic vector field of the ideal***[1]. If we recall the definition of an ideal and properties of the interior product, we immediately recognise that characteristic vector fields of an ideal form a submodule $\mathcal{S}(\mathcal{I}) \subseteq \mathfrak{V}(M)$ that is called the ***characteristic subspace of the ideal***. We thus symbolically write $\mathbf{i}_V(\mathcal{I}) \subseteq \mathcal{I}$ whenever $V \in \mathcal{S}(\mathcal{I})$.

Theorem 5.6.1. *Let $\mathcal{I}(\omega^1, \omega^2, \ldots, \omega^r)$ be an ideal of the exterior algebra $\Lambda(M)$ generated by the forms $\omega^1, \omega^2, \ldots, \omega^r \in \Lambda^k(M)$ of the same degree. A vector field $V \in T(M)$ is a characteristic vector field of the ideal $\mathcal{I}$ if and only if $\mathbf{i}_V(\omega^a) = 0, a = 1, 2, \ldots, r$.*

We suppose that $\mathbf{i}_V(\omega^a) = 0, a = 1, \ldots, r$. If $\alpha \in \mathcal{I}$, then we need to write $\alpha = \gamma_a \wedge \omega^a$ where all forms $\gamma_a \in \Lambda(M)$ ought to have the same degree. We thus obtain

$$\mathbf{i}_V(\alpha) = \mathbf{i}_V(\gamma_a) \wedge \omega^a + (-1)^{deg\, \gamma_a} \gamma_a \wedge \mathbf{i}_V(\omega^a) = \mathbf{i}_V(\gamma_a) \wedge \omega^a \in \mathcal{I}.$$

Conversely, let us assume that $\mathbf{i}_V(\alpha) \in \mathcal{I}$ for all $\alpha \in \mathcal{I}$. Consequently, this property is also valid for the forms $\alpha = f_a\, \omega^a \in \Lambda^k(M)$ where the functions $f_a \in \Lambda^0(M)$ are arbitrary. However, it is not possible for $(k-1)$-forms to belong to the ideal. Therefore, we can only write $\mathbf{i}_V(\alpha) = 0$. Hence, we conclude that

[1]Sometimes it is called a ***Cauchy characteristic vector field*** after Cauchy who had introduced the concept of characteristics to partial differential equations.

$$\mathbf{i}_V(\alpha) = f_a \,\mathbf{i}_V(\omega^a) = 0$$

and $\mathbf{i}_V(\omega^a) = 0, a = 1, \dots, r$ because the functions f_a are arbitrary. □

Naturally, Theorem 5.6.1 would also prevail for an ideal generated by the forms $\omega^1, \omega^2, \dots, \omega^r \in \Lambda^1(M)$. The characteristic vectors of such a special ideal will be called *the characteristic vectors of the exterior system* $\{\omega^a \in \Lambda^1(M), a = 1, \dots, r\}$.

Theorem 5.6.2. *The characteristic vectors of an exterior system* $\{\omega^a \in \Lambda^1(M), a = 1, \dots, r\}$ *engender a submodule* $\mathcal{S}$ *of* $\mathfrak{V}(M$. *If the forms* ω^a *are linearly independent, namely, if* $\Omega = \omega^1 \wedge \omega^2 \wedge \dots \wedge \omega^r \neq 0$, *then the dimension of* $\mathcal{S}$ *is* $m - r$.

We know that characteristic vectors of any ideal constitute a *characteristic subspace* $\mathcal{S}$. If $\Omega = \omega^1 \wedge \omega^2 \wedge \dots \wedge \omega^r \neq 0$, then the 1-forms $\omega^1, \dots,$ ω^r are linearly independent. If we add $m - r$ linearly independent 1-forms $\omega^{r+1}, \dots,$ $\omega^m \in \Lambda^1(M)$ to those forms, then the forms $\omega^1, \dots, \omega^m$ can now be chosen as a basis for $\Lambda^1(M) = T^*(M)$. As is well known, we can select a basis $\{V_i\}$ in $T(M)$ so that $\{\omega^i\}$ becomes reciprocal basis satisfying the relations

$$\mathbf{i}_{V_j}(\omega^i) = \omega^i(V_j) = \delta^i_j, \; i, j = 1, \dots, m.$$

We thus get

$$\mathbf{i}_{V_j}(\omega^i) = 0, \; i = 1, \dots, r; \; j = r + 1, \dots, m.$$

Therefore, $m - r$ linearly independent vectors $V_{r+1}, \dots, V_m$ are actually characteristic vectors of the exterior system. On the other hand, because of the relations

$$\mathbf{i}_{V_1}(\omega^1) = \mathbf{i}_{V_2}(\omega^2) = \cdots = \mathbf{i}_{V_r}(\omega^r) = 1$$

the vectors $V_1, \dots, V_r$ cannot be characteristic vectors of the exterior system. Hence, the dimension of the characteristic subspace $\mathcal{S}$ becomes $m - r$. □

It is seen right away from above that the relations

$$\mathbf{i}_{V_j}(\omega^i) = 0, \; i = r + 1, \dots, m; \; j = 1, \dots, r$$

together with

$$\mathbf{i}_{V_{r+1}}(\omega^{r+1}) = \mathbf{i}_{V_{r+2}}(\omega^{r+2}) = \cdots = \mathbf{i}_{V_m}(\omega^m) = 1$$

are satisfied as well. *This amounts to say that the vector fields* $V_1, \dots, V_r$ *are in turn characteristic vectors of the exterior system* $\{\omega^{r+1}, \dots, \omega^m\}$ *while vectors* $V_{r+1}, \dots, V_m$ *cannot be characteristic vectors of that system*. This means that the dimension of the characteristic subspace of the exterior system $\{\omega^{r+1}, \ldots, \omega^m\}$ is r. We can summarise the foregoing results by the symbolic relations

$$\Lambda^1(M) = \Lambda^1_{(r)}(M) \oplus \Lambda^1_{(m-r)}(M), \;\; T(M) = T_{(m-r)}(M) \oplus T_{(r)}(M).$$

Moreover, if we denote the interior product by the hook operator $\rfloor$, we can also write

$$T_{(m-r)}(M) \rfloor \Lambda^1_{(r)}(M) = 0, \;\; T_{(r)}(M) \rfloor \Lambda^1_{(m-r)}(M) = 0$$

whence we readily reach to the following conclusion:

Let $\mathcal{I}(\omega^a)$ be an ideal generated by 1-forms and let V be a characteristic vector field of this ideal. If one has $\mathbf{i}_V(\omega) \neq 0$ for a form $\omega \in \Lambda^1(M)$, then this form cannot belong to the ideal $\mathcal{I}(\omega^a)$ or, conversely, it is not possible to get $\mathbf{i}_V(\omega) = 0$ if $\omega \notin \mathcal{I}(\omega^a)$.

Let the ideal $\mathcal{I}$ be generated by forms $\omega^1, \ldots, \omega^r \in \Lambda(M)$ of diverse degrees. Then we can provide the theorem below for a systematic determination of its characteristic vectors.

Theorem 5.6.3. *The necessary and sufficient conditions for a vector $V \in T(M)$ to be a characteristic vector of the ideal $\mathcal{I}(\omega^1, \omega^2, \ldots, \omega^r)$ is the existence of forms $\lambda^a_b \in \Lambda(M)$ of suitable degrees such that the relations*

$$\mathbf{i}_V(\omega^a) = \lambda^a_b \wedge \omega^b, \;\; a, b = 1, 2, \ldots, r$$

are satisfied.

Let us suppose the vector field V holds the foregoing conditions. If ω is a member of the ideal, we can write $\omega = \gamma_a \wedge \omega^a$, $\gamma_a \in \Lambda(M)$. Clearly, one must have $deg\,(\gamma_a) + deg\,(\omega^a) = deg\,(\omega)$. We thus deduce that

$$\begin{aligned}\mathbf{i}_V(\omega) &= \mathbf{i}_V(\gamma_a) \wedge \omega^a + (-1)^{deg\,(\gamma_{\underline{a}})} \gamma_a \wedge \mathbf{i}_V(\omega^a) \\ &= \big(\mathbf{i}_V(\gamma_b) + (-1)^{deg\,(\gamma_{\underline{a}})} \gamma_a \wedge \lambda^a_b\big) \wedge \omega^b \in \mathcal{I}\end{aligned}$$

which means that V is a characteristic vector. Conversely, if V is a characteristic vector, then its interior product with any form in the ideal should lie within the ideal. This rule will of course be valid for the generators ω^a so that one must find forms λ^a_b so much so that the relations $\mathbf{i}_V(\omega^a) = \lambda^a_b \wedge \omega^b$ will hold. □

If $\mathcal{S}(\mathcal{I}) \subseteq T(M)$ is an r-dimensional characteristic subspace of an ideal $\mathcal{I}$, then for all *linearly independent* vectors $V_1, \ldots, V_k \in \mathcal{S}, 1 \leq k \leq r$ and a form $\omega \in \mathcal{I}$ we clearly get

$$(\mathbf{i}_{V_k} \circ \cdots \circ \mathbf{i}_{V_1})(\omega) \in \mathcal{I}, \;\; 1 \leq k \leq r.$$

We consider an ideal $\mathcal{I}(\omega^1,\omega^2,\dots,\omega^s)$ of $\Lambda(M)$ generated by forms of diverse degrees and assume that $\mathcal{S}(\mathcal{I})$ is its characteristic subspace with dimension $m-r$. $\mathcal{S}(\mathcal{I})$ is brought forth by linearly independent vector fields $V_{r+1},\dots,V_m$. We can supply this set with arbitrary linearly independent vector fields $V_1,\dots,V_r$ to obtain a basis in the tangent bundle $T(M)$. We now pursue the path used in proving Theorem 5.6.2 to determine the reciprocal basis $\theta^1,\dots,\theta^m \in \Lambda^1(M)$ in the cotangent bundle $T^*(M)$ in such a way that we have

$$\mathbf{i}_{V_j}(\theta^i) = \theta^i(V_j) = \delta^i_j,\ \ i,j = 1,2,\dots,m.$$

We thus obtain

$$\mathbf{i}_{V_a}(\theta^\alpha) = \delta^\alpha_a\ = 0,\ \ a = r+1,\dots,m,\ \alpha = 1,\dots,r. \tag{5.6.3}$$

This means that the same vectors $V_a, a = r+1,\dots,m$ span the $(m-r)$-dimensional characteristic subspace of the ideal $\mathcal{J}(\theta^\alpha)$ generated by 1-forms $\theta^\alpha, \alpha = 1,\dots,r$. In other words, we conclude that $\mathcal{S}(\mathcal{I}) = \mathcal{S}(\mathcal{J})$. *The number r is called the rank of the ideal $\mathcal{I}$*. Within this context, we can prove the following theorem.

Theorem 5.6.4. *Let $\mathcal{S}(\mathcal{I})$ be the $(m-r)$-dimensional characteristic subspace of an ideal $\mathcal{I}(\omega^A)$ generated by forms ω^A, $A = 1,\dots,s$ of various degrees. There exist linearly independent 1-forms $\theta^\alpha, \alpha = 1,\dots,r$ and if the ideal generated by these 1-forms is $\mathcal{J}(\theta^\alpha)$, then one finds $\mathcal{I}(\omega^A) \subseteq \mathcal{J}(\theta^\alpha)$.*

If $V_{r+1},\dots,V_m \in T(M)$ is a basis of the characteristic subspace $\mathcal{S}(\mathcal{I})$, we first complete to a full basis of $T(M)$ as we have mentioned above, then we can construct the reciprocal basis $\theta^1,\dots,\theta^m \in \Lambda^1(M)$ of $T^*(M$. We define $m-r$ degree preserving mappings $h_a : \Lambda(M) \to \Lambda(M)$ where $a = r+1,\dots,m$ by the rule

$$\sigma_a = h_a(\omega) = \omega - \theta^{\underline{a}} \wedge \mathbf{i}_{V_{\underline{a}}}(\omega) \tag{5.6.4}$$

Let us remember that the summation convention will be disabled on underscored indices. It is clear that $\sigma_a = h_a(\omega) \in \mathcal{I}$ whenever $\omega \in \mathcal{I}$. Next, we consider a generator ω^A of the ideal $\mathcal{I}$. Let us now introduce the forms $\sigma^A_a = h_a(\omega^A) = \omega^A - \theta^{\underline{a}} \wedge \mathbf{i}_{V_{\underline{a}}}(\omega^A) \in \mathcal{I}$ to find

$$\mathbf{i}_{V_{\underline{a}}}(\sigma^A_{\underline{a}}) = \mathbf{i}_{V_a}(\omega^A) - \mathbf{i}_{V_{\underline{a}}}(\theta^{\underline{a}})\,\mathbf{i}_{V_{\underline{a}}}(\omega^A) + \theta^{\underline{a}} \wedge \mathbf{i}^2_{V_{\underline{a}}}(\omega^A) = 0$$

where we have employed the relations $\mathbf{i}_{V_{\underline{a}}}(\theta^{\underline{a}}) = 1$ and $\mathbf{i}^2_{V_{\underline{a}}} = 0$. We see that the definition $\sigma^A_{ba} = h_b \circ h_a(\omega^A) = h_b(\sigma^A_a) = \sigma^A_a - \theta^{\underline{b}} \wedge \mathbf{i}_{V_{\underline{b}}}(\sigma^A_a) \in \mathcal{I}$ leads similarly to $\mathbf{i}_{V_{\underline{b}}}(\sigma^A_{\underline{ba}}) = 0$. Furthermore, since $\mathbf{i}_{V_{\underline{a}}}(\sigma^A_{\underline{a}}) = 0$ we obtain

$$\begin{aligned}\mathbf{i}_{V_{\underline{a}}}(\sigma^A_{b\underline{a}}) &= \mathbf{i}_{V_{\underline{a}}}(\sigma^A_{\underline{a}}) - \delta^b_a\,\mathbf{i}_{V_{\underline{b}}}(\sigma^A_a) + \theta^{\underline{b}} \wedge \mathbf{i}_{V_{\underline{a}}} \circ \mathbf{i}_{V_{\underline{b}}}(\sigma^A_a)\\ &= -\,\theta^{\underline{b}} \wedge \mathbf{i}_{V_{\underline{b}}} \circ \mathbf{i}_{V_{\underline{a}}}(\sigma^A_a) = 0\end{aligned}$$

for $b \neq a$. These results clearly indicate that the forms

$$\sigma^A = \sigma^A_{m\cdots r+1} = (h_m \circ \cdots \circ h_{r+1})(\omega^A) \in \mathcal{I} \tag{5.6.5}$$

will satisfy the relations

$$\mathbf{i}_{V_a}(\sigma^A) = 0, \;\; a = r+1, \ldots, m, \; A = 1, \ldots, s.$$

Thus, for all vectors $V \in \mathcal{S}(\mathcal{I})$ we find that

$$\mathbf{i}_V(\sigma^A) = 0, \;\; A = 1, \ldots, s. \tag{5.6.6}$$

The rule of formation of the forms σ^A, which are of the same degree as the forms ω^A implies that $\mathcal{I}(\omega^A) = \mathcal{I}(\sigma^A)$. We now assume that $\sigma^A \in \Lambda^k(M)$. When we choose the 1-forms $\{\theta^i : i = 1, \ldots, m\}$ as a basis of $T^*(M)$, we can of course write

$$\sigma^A = \frac{1}{k!}\,\sigma^A_{i_1\cdots i_k}\theta^{i_1} \wedge \cdots \wedge \theta^{i_k}.$$

If we express a vector $V \in T(M)$ as $V = v^i V_i$ and pay attention that the vectors $\{V_i\}$ and the forms $\{\theta^i\}$ are reciprocal bases in $T(M)$ and $T^*(M)$, respectively, then we can describe the interior product of the form σ^A with the vector V as follows

$$\mathbf{i}_V(\sigma^A) = \frac{1}{(k-1)!}v^i\sigma^A_{ii_1i_2\cdots i_{k-1}}\theta^{i_1} \wedge \theta^{i_2} \wedge \cdots \wedge \theta^{i_{k-1}}$$

just as expressed in (5.4.2). On the other hand, when $V \in \mathcal{S}(\mathcal{I})$ we have to write $V = v^a V_a$. We thus get

$$\mathbf{i}_V(\sigma^A) = \frac{1}{(k-1)!}v^a\sigma^A_{ai_1i_2\cdots i_{k-1}}\theta^{i_1} \wedge \theta^{i_2} \wedge \cdots \wedge \theta^{i_{k-1}} = 0$$

since $v^i = 0$ for $i = 1, \ldots, r$. That yields $v^a\sigma^A_{ai_1i_2\cdots i_{k-1}} = 0$. Because this equality must be valid for every choice of functions $v^a \in \Lambda^0(M)$, we find at last that $\sigma^A_{ai_1i_2\cdots i_{k-1}} = 0$. Due to the complete antisymmetry of these functions with respect to its k indices, these relations would be met for all positions of indices. This is tantamount to say that

$$\sigma^A_{i_1i_2\cdots i_k} = 0, \; r+1 \le i_1, i_2, \ldots, i_k \le m.$$

Therefore the forms σ^A have to possess the following structure

$$\sigma^A = \frac{1}{k!}\,\sigma^A_{\alpha_1\cdots\alpha_k}\theta^{\alpha_1}\wedge\cdots\wedge\theta^{\alpha_k},\ 1\le\alpha_1,\alpha_2\ldots,\alpha_k\le r$$

which implies that $\sigma^A\in\mathcal{J}(\theta^\alpha)$. This result means of course $\mathcal{I}(\sigma^A)\subseteq\mathcal{J}(\theta^\alpha)$ and consequently $\mathcal{I}(\omega^A)\subseteq\mathcal{J}(\theta^\alpha)$. This proves the theorem. □

Example 5.6.1. Let us take the form $\omega=\omega_i(\mathbf{x})\,dx^i\in\Lambda^1(M)$ into account. A vector $V=v^i(\mathbf{x})\,\partial_i\in T(M)$ is a characteristic vector of the form ω if it meets the condition $\mathbf{i}_V(\omega)=v^i\omega_i=0$. If we take $\omega_1\neq 0$, we see that there are $m-1$ linearly independent vectors

$$V_k=\omega_1\frac{\partial}{\partial x^k}-\omega_k\frac{\partial}{\partial x^1},\ \ k=2,3,\ldots,m$$

satisfying this condition. ■

Example 5.6.2. An exterior system is given by the forms

$$\omega^1=dx-y\,dz\in\Lambda^1(\mathbb{R}^4),\ \ \omega^2=dx-x\,dy+t\,dz\in\Lambda^1(\mathbb{R}^4).$$

If $V=v^x\partial_x+v^y\partial_y+v^z\partial_z+v^t\partial_t$ is a characteristic vector of this system, then the following equations should be satisfied:

$$v^x-yv^z=0,\ v^x-xv^y+tv^z=0.$$

We thus obtain

$$v^x=yv^z,\ v^y=\frac{y+t}{x}v^z.$$

Hence, two linearly independent characteristic vectors are found to be

$$V_1=y\,\frac{\partial}{\partial x}+\frac{y+t}{x}\frac{\partial}{\partial y}+\frac{\partial}{\partial z},\ \ V_2=\frac{\partial}{\partial t}.$$ ■

Example 5.6.3. We consider the ideal generated by the forms

$$\omega^1=dx-y\,dz\in\Lambda^1(\mathbb{R}^4),\ \ \omega^2=t\,dx\wedge dz-x\,dy\wedge dt\ \in\Lambda^1(\mathbb{R}^4)$$

Its characteristic vector field V must satisfy the relations $\mathbf{i}_V(\omega^1)=0$ and $\mathbf{i}_V(\omega^2)=\lambda(\mathbf{x})\omega^1$ where $\lambda\in\Lambda^0(\mathbb{R}^4)$ that can be written explicitly as

$$v^x-yv^z=0,\ tv^x dz-tv^z dx-xv^y dt+xv^t dy=\lambda\,dx-\lambda y\,dz$$

whence we find that

$$\lambda=\ -tv^z,\ v^x=yv^z,\ \ v^y=v^t=0.$$

Thus 1-dimensional characteristic subspace of the ideal is spanned by the vector field

$$V = y\,\frac{\partial}{\partial x} + \frac{\partial}{\partial z}.$$

On the other hand, the characteristic vectors of the forms ω^1 and ω^2 are determined through the relations $\mathbf{i}_V(\omega^1) = 0$ and $\mathbf{i}_U(\omega^2) = 0$ leading to

$$v^x - yv^z = 0,\ u^x = u^z = u^y = u^t = 0.$$

Hence, characteristic vectors are

$$V_1 = y\,\frac{\partial}{\partial x} + \frac{\partial}{\partial z},\ V_2 = \frac{\partial}{\partial y},\ V_3 = \frac{\partial}{\partial t};\ U = 0.$$ ■

5.7. EXTERIOR FORMS UNDER MAPPINGS

Let M^m and N^n be two differentiable manifolds and $\phi : M \to N$ be a smooth mapping. We know that the mapping $\phi^* : \Lambda^0(N) \to \Lambda^0(M)$ derived from ϕ via the rule $\phi^* g = g \circ \phi$ assigns a smooth function $f = \phi^* g \in \Lambda^0(M)$ to a smooth function $g \in \Lambda^0(N)$ [*see p.* 98]. We shall now show that ϕ gives rise in general to a mapping $\phi^* : \Lambda(N) \to \Lambda(M)$. Let us take a form $\omega \in \Lambda^k(N)$ into consideration. If we denote local coordinates associated with a chart at the point $q \in N$ by $\mathbf{y} = \{y^\alpha\} = \{y^1, y^2, \dots, y^n\}$, we may write

$$\omega(q) = \frac{1}{k!}\,\omega_{\alpha_1\alpha_2\cdots\alpha_k}(\mathbf{y})\,dy^{\alpha_1} \wedge dy^{\alpha_2} \wedge \cdots \wedge dy^{\alpha_k} \in \Lambda^k(N).$$

Here the indices $\alpha_1, \dots, \alpha_k$ take values $1, \dots, n$. On the other hand, if local coordinates in a chart at a point $p \in M$ are $\mathbf{x} = \{x^i\} = \{x^1, x^2, \dots, x^m\}$, we know that the mapping $q = \phi(p)$ elicits a mapping $\Phi : \mathbb{R}^m \to \mathbb{R}^n$ in the functional form $\mathbf{y} = \Phi(\mathbf{x})$ or $y^\alpha = \Phi^\alpha(x^1, \dots, x^m), \alpha = 1, \dots, n$. The differential $d\phi : T_p(M) \to T_{\phi(p)}(N)$ of ϕ at the point p carries a vector at that point p over a vector at the point $q = \phi(p)$. We now define a form $\omega^* = \phi^*\omega$ at the point p corresponding to a form ω at the point $\phi(p)$ in such a way that the numerical equality

$$(\phi^*\omega)(V_1, \dots, V_k) = \omega\big(d\phi(V_1), \dots, d\phi(V_k)\big) \tag{5.7.1}$$

will be satisfied for all vectors $V_1, V_2, \dots, V_k \in T_p(M)$. This relation will actually determine a mapping in the form $\phi^* : \Lambda^k(N) \to \Lambda^k(M)$. In fact, the vector $V^* = d\phi(V)$ is represented in view of (2.7.4) by

$$V^* = v^i \frac{\partial \Phi^\alpha}{\partial x^i}\frac{\partial}{\partial y^\alpha} = v^{*\alpha}\frac{\partial}{\partial y^\alpha}$$

where $V = v^i \dfrac{\partial}{\partial x^i}$. Therefore, we obtain

$$\omega^*(V_1,\dots,V_k) = \omega^*_{i_1\cdots i_k} v_1^{i_1}\cdots v_k^{i_k} =$$
$$\omega(V_1^*,\dots,V_k^*) = \omega_{\alpha_1\cdots\alpha_k} v_1^{*\alpha_1}\cdots v_k^{*\alpha_k} = \omega_{\alpha_1\cdots\alpha_k}\frac{\partial \Phi^{\alpha_1}}{\partial x^{i_1}}\cdots\frac{\partial \Phi^{\alpha_k}}{\partial x^{i_k}} v_1^{i_1}\cdots v_k^{i_k}.$$

Since, this expression would be valid for all vectors $V_1,\dots,V_k$, we reach to the conclusion

$$\begin{aligned}\omega^*_{i_1\cdots i_k}(\mathbf{x}) &= \omega_{\alpha_1\cdots\alpha_k}\big(\Phi(\mathbf{x})\big)\frac{\partial \Phi^{\alpha_1}}{\partial x^{i_1}}\cdots\frac{\partial \Phi^{\alpha_k}}{\partial x^{i_k}} \\ &= \omega_{\alpha_1\cdots\alpha_k}\big(\Phi(\mathbf{x})\big)\frac{\partial \Phi^{[\alpha_1}}{\partial x^{i_1}}\cdots\frac{\partial \Phi^{\alpha_k]}}{\partial x^{i_k}}.\end{aligned} \tag{5.7.2}$$

We have to note that the complete antisymmetry on indices α causes the complete antisymmetry on indices i. Accordingly, the ***pull-back***, or ***reciprocal image*** $\omega^*(p)$ of a form $\omega(q) \in \Lambda^k(N)$, where $q = \phi(p) \in N$ and $p \in M$, is the k-form given by

$$\begin{aligned}\omega^*(p) = \phi^*\omega(q) &= \frac{1}{k!}\,\omega_{\alpha_1\cdots\alpha_k}\big(\Phi(\mathbf{x})\big)\frac{\partial \Phi^{\alpha_1}}{\partial x^{i_1}}\cdots\frac{\partial \Phi^{\alpha_k}}{\partial x^{i_k}}\, dx^{i_1}\wedge\cdots\wedge dx^{i_k} \\ &= \frac{1}{k!}\,\omega^*_{i_1\cdots i_k}(\mathbf{x})\, dx^{i_1}\wedge\cdots\wedge dx^{i_k} \in \Lambda^k(M).\end{aligned}$$

ϕ^* is called the ***pull-back operator*** and it can also be expressed in the usual form $\phi^*\omega = \omega\circ\phi$. However, this operation must be interpreted this time in a broader sense. We simply realise that the form $\phi^*\omega$ is obtainable from the form ω by inserting into ω the differential transformation

$$dy^\alpha = \frac{\partial y^\alpha}{\partial x^i}dx^i = \frac{\partial \Phi^\alpha}{\partial x^i}dx^i$$

in addition to the mapping $\omega_{\alpha_1\cdots\alpha_k}\circ\phi$. It is clear that ϕ^* is a degree preserving mapping. If $n \geq k > m$, then it is evident that $\phi^*\omega = 0$ identically.

Let us consider the forms $\alpha,\beta \in \Lambda^k(N)$. If we notice the relation (5.7.2) we find that

$$\phi^*(\alpha+\beta) = \phi^*\alpha + \phi^*\beta. \tag{5.7.3}$$

Hence the operator ϕ^* is additive. Furthermore, if $\omega \in \Lambda^k(N)$, $\sigma \in \Lambda^l(N)$, then the form $\gamma = \omega \wedge \sigma \in \Lambda^{k+l}(N)$ becomes

$$\gamma = \frac{1}{k!\,l!}\,\omega_{\alpha_1\cdots\alpha_k}\sigma_{\beta_1\cdots\beta_l}\,dy^{\alpha_1}\wedge\cdots\wedge dy^{\alpha_k}\wedge dy^{\beta_1}\wedge\cdots\wedge dy^{\beta_l}$$

and the form $\phi^*\gamma \in \Lambda^{k+l}(M)$ is cast into

$$\phi^*\gamma = \frac{1}{k!\,l!}\,\omega^*_{i_1\cdots i_k}\sigma^*_{j_1\cdots j_l}\,dx^{i_1}\wedge\cdots\wedge dx^{i_k}\wedge dx^{j_1}\wedge\cdots\wedge dx^{j_l}.$$

We thus reach to the conclusion

$$\phi^*(\omega\wedge\sigma) = \phi^*\omega\wedge\phi^*\sigma. \tag{5.7.4}$$

When $g \in \Lambda^0(N)$, we get from (5.7.4)

$$\phi^*(g\,\omega) = (\phi^*g)\phi^*\omega.$$

If $g \in \mathbb{R}$, one finds $\phi^*(g\,\omega) = g\,\phi^*\omega$. Therefore, ϕ^* *reduces to a linear operator only on the field of real numbers*. On recalling (5.7.4), we recognise that the mapping ϕ^* is a homomorphism on the exterior algebra $\Lambda(N)$. If ϕ is a diffeomorphism, then it becomes clear that the operator ϕ^* will be an algebra isomorphism.

Let M_1, M_2 and M_3 be smooth manifolds, and $\phi : M_1 \to M_2$ and $\psi : M_2 \to M_3$ be smooth mappings. These mappings give rise to pull-back operators $\psi^* : \Lambda(M_3) \to \Lambda(M_2)$ and $\phi^* : \Lambda(M_2) \to \Lambda(M_1)$ so that one has $\psi^*\omega \in \Lambda^k(M_2)$ and $\phi^*(\psi^*\omega) \in \Lambda^k(M_1)$ for a form $\omega \in \Lambda^k(M_3)$. On the other hand, it is straightforward to see that we can write $\psi\circ\phi : M_1 \to M_3$ and $(\psi\circ\phi)^* : \Lambda(M_3) \to \Lambda(M_1)$. In appropriate local coordinates, we have

$$\begin{aligned}
\omega &= \frac{1}{k!}\,\omega_{a_1\cdots a_k}(\mathbf{z})\,dz^{a_1}\wedge\cdots\wedge dz^{a_k},\\
\psi^*\omega &= \frac{1}{k!}\,\omega_{a_1\cdots a_k}\big(\mathbf{z}(\mathbf{y})\big)\frac{\partial z^{a_1}}{\partial y^{\alpha_1}}\cdots\frac{\partial z^{a_k}}{\partial y^{\alpha_k}}\,dy^{\alpha_1}\wedge\cdots\wedge dy^{\alpha_k},\\
\phi^*(\psi^*\omega) &= \frac{1}{k!}\,\omega_{a_1\cdots a_k}\big[\mathbf{z}\big(\mathbf{y}(\mathbf{x})\big)\big]\frac{\partial z^{a_1}}{\partial y^{\alpha_1}}\cdots\frac{\partial z^{a_k}}{\partial y^{\alpha_k}}\frac{\partial y^{\alpha_1}}{\partial x^{i_1}}\cdots\frac{\partial y^{\alpha_k}}{\partial x^{i_k}}\,dx^{i_1}\wedge\cdots\wedge dx^{i_k}.
\end{aligned}$$

But, the chain rule of differentiation

$$\frac{\partial z^{a_r}}{\partial y^{\alpha_r}}\frac{\partial y^{\alpha_r}}{\partial x^{i_r}} = \frac{\partial z^{a_r}}{\partial x^{i_r}}$$

implies that

$$\phi^*(\psi^*\omega) = \frac{1}{k!}\,\omega_{a_1\cdots a_k}\big(\mathbf{z}(\mathbf{x})\big)\frac{\partial z^{a_1}}{\partial x^{i_1}}\cdots\frac{\partial z^{a_k}}{\partial x^{i_k}}\,dx^{i_1}\wedge\cdots\wedge dx^{i_k} = (\psi\circ\phi)^*\omega.$$

Since this relation must be valid for all forms $\omega \in \Lambda(M_3)$, we arrive at the composition rule

$$(\psi \circ \phi)^* = \phi^* \circ \psi^*. \tag{5.7.5}$$

If the mapping $\phi : M \to N$ is a *diffeomorphism*, then the mapping $\phi^{-1} : N \to M$ is also a diffeomorphism. Thus the relations $\phi^{-1} \circ \phi = i_M$, $\phi \circ \phi^{-1} = i_N$ and $i_M^* = i_{\Lambda(M)}$, $i_N^* = i_{\Lambda(N)}$ leads, according to (5.7.5), to

$$(\phi^{-1} \circ \phi)^* = i_{\Lambda(M)} = \phi^* \circ (\phi^{-1})^*,\ (\phi \circ \phi^{-1})^* = i_{\Lambda(N)} = (\phi^{-1})^* \circ \phi^*$$

which implies in this case that $(\phi^*)^{-1} = (\phi^{-1})^* : \Lambda(M) \to \Lambda(N)$.

We have so far seen that the mapping $\phi : M \to N$ generates both the differential mapping $d\phi = \phi_* : T(M) \to T(N)$ and the pull-back operator $\phi^* : \Lambda(N) \to \Lambda(M)$. Let us now consider a form $\omega \in \Lambda^k(N)$ at a point $\phi(p) \in N$ corresponding to a point $p \in M$ and a vector $V = v^i \partial_i \in T_p(M)$. We know that the vector $V^* = \phi_*(V) = d\phi(V) \in T_{\phi(p)}(N)$ is given by

$$d\phi(V) = v^\alpha \frac{\partial}{\partial y^\alpha},\quad v^\alpha = v^i \frac{\partial \Phi^\alpha}{\partial x^i}.$$

The interior product of the form ω with this vector is of course

$$\mathbf{i}_{d\phi(V)}(\omega) = \frac{1}{(k-1)!}\, \omega_{\alpha_1\alpha_2\cdots\alpha_k}(\mathbf{y}) v^{\alpha_1}\, dy^{\alpha_2} \wedge \cdots \wedge dy^{\alpha_k}.$$

The pull-back of that form then becomes

$$\begin{aligned}
\phi^*\big(\mathbf{i}_{d\phi(V)}(\omega)\big) &= \frac{1}{(k-1)!}\, \omega_{\alpha_1\alpha_2\cdots\alpha_k} v^{i_1} \frac{\partial \Phi^{\alpha_1}}{\partial x^{i_1}} \frac{\partial \Phi^{\alpha_2}}{\partial x^{i_2}} \cdots \frac{\partial \Phi^{\alpha_k}}{\partial x^{i_k}}\, dx^{i_2} \wedge \cdots \wedge dx^{i_k} \\
&= \frac{1}{(k-1)!}\, \omega_{i_1 i_2 \cdots i_k}(\mathbf{x}) v^{i_1}\, dx^{i_2} \wedge \cdots \wedge dx^{i_k} \\
&= \mathbf{i}_V(\phi^* \omega).
\end{aligned}$$

Since this relation would be true for all forms $\omega \in \Lambda(N)$, we conclude that for all vectors $V \in T(M)$ we get the rule

$$\phi^* \circ \mathbf{i}_{\phi_*(V)} = \phi^* \circ \mathbf{i}_{V^*} = \mathbf{i}_V \circ \phi^* : \Lambda^k(N) \to \Lambda^{k-1}(M). \tag{5.7.6}$$

If the operator ϕ_*^{-1} exists, then (5.7.6) means that the relation

$$\phi^* \circ \mathbf{i}_U = \mathbf{i}_{\phi_*^{-1} U} \circ \phi^* \tag{5.7.7}$$

will also be valid for all vectors $U \in T(N)$.

If $\phi : M \to M$, then ϕ maps the manifold M into itself. When ϕ is a diffeomorphism, it produces a coordinate transformation on M that can be represented locally by functions $\mathbf{y} = \Phi(\mathbf{x})$ or $y^\alpha = \Phi^\alpha(x^1, \dots, x^m)$, $\alpha = 1, \dots, m$ where $\Phi = \varphi \circ \phi \circ \varphi^{-1}$ with φ being the local homeomorphism of the associated chart. If we denote the Jacobian determinant by $J = \det \phi = \det [\partial \Phi^\alpha / \partial x^i]$, then we must have, in this case, $J \neq 0$. We would like now to investigate the transformation of bases induced by the volume form under such a mapping ϕ. The transformation of a generic basis form

$$\mu_{\alpha_k \cdots \alpha_1} = \frac{1}{(m-k)!}\, e_{\alpha_1 \cdots \alpha_k \alpha_{k+1} \cdots \alpha_m} dy^{\alpha_{k+1}} \wedge \cdots \wedge dy^{\alpha_m} \in \Lambda^{m-k}(M)$$

yields

$$\phi^* \mu_{\alpha_k \cdots \alpha_1} = \frac{1}{(m-k)!}\, e_{\alpha_1 \cdots \alpha_k \alpha_{k+1} \cdots \alpha_m} \frac{\partial y^{\alpha_{k+1}}}{\partial x^{i_{k+1}}} \cdots \frac{\partial y^{\alpha_m}}{\partial x^{i_m}} dx^{i_{k+1}} \wedge \cdots \wedge dx^{i_m}$$

from which we write

$$\frac{\partial y^{\alpha_1}}{\partial x^{i_1}} \cdots \frac{\partial y^{\alpha_k}}{\partial x^{i_k}} \phi^* \mu_{\alpha_k \cdots \alpha_1} = \frac{1}{(m-k)!}\, e_{\alpha_1 \cdots \alpha_k \alpha_{k+1} \cdots \alpha_m} \frac{\partial y^{\alpha_1}}{\partial x^{i_1}} \cdots \frac{\partial y^{\alpha_k}}{\partial x^{i_k}} \frac{\partial y^{\alpha_{k+1}}}{\partial x^{i_{k+1}}} \cdots \frac{\partial y^{\alpha_m}}{\partial x^{i_m}} dx^{i_{k+1}} \wedge \cdots \wedge dx^{i_m}.$$

According to (1.4.18), we have

$$e_{\alpha_1 \cdots \alpha_m} \frac{\partial y^{\alpha_1}}{\partial x^{i_1}} \cdots \frac{\partial y^{\alpha_m}}{\partial x^{i_m}} = e_{i_1 \cdots i_m} \det \Big[\frac{\partial y^\alpha}{\partial x^i} \Big].$$

Therefore, we find that

$$\frac{\partial y^{\alpha_1}}{\partial x^{i_1}} \cdots \frac{\partial y^{\alpha_k}}{\partial x^{i_k}} \phi^* \mu_{\alpha_k \cdots \alpha_1} = \det \Big[\frac{\partial y^\alpha}{\partial x^i} \Big] \frac{1}{(m-k)!}\, e_{i_1 \cdots i_k i_{k+1} \cdots i_m} dx^{i_{k+1}} \wedge \cdots \wedge dx^{i_m}$$

and finally, owing to (5.5.10)

$$\mu_{i_k \cdots i_1} = \Big(\det \Big[\frac{\partial y^\alpha}{\partial x^i} \Big] \Big)^{-1} \frac{\partial y^{\alpha_1}}{\partial x^{i_1}} \cdots \frac{\partial y^{\alpha_k}}{\partial x^{i_k}}\, \phi^* \mu_{\alpha_k \cdots \alpha_1}. \tag{5.7.8}$$

Let us now consider a form $\omega \in \Lambda^{m-k}(M)$ described by

$$\omega = \frac{1}{k!}\, \omega^{\alpha_1 \cdots \alpha_k}(\mathbf{y}) \mu_{\alpha_k \cdots \alpha_1}. \tag{5.7.9}$$

The pull-back of ω thus becomes

$$\phi^*\omega = \frac{1}{k!}\,\omega^{i_1\cdots i_k}(\mathbf{x})\,\mu_{i_k\cdots i_1} = \frac{1}{k!}\,\phi^*\omega^{\alpha_1\cdots\alpha_k}(\mathbf{y})\,\phi^*\mu_{\alpha_k\cdots\alpha_1}$$

and (5.7.8) gives

$$(\phi^*\omega^{\alpha_1\cdots\alpha_k})(\mathbf{x}) = \Big(\det\Big[\frac{\partial y^\alpha}{\partial x^i}\Big]\Big)^{-1}\frac{\partial y^{\alpha_1}}{\partial x^{i_1}}\cdots\frac{\partial y^{\alpha_k}}{\partial x^{i_k}}\,\omega^{i_1\cdots i_k}(\mathbf{x}). \quad (5.7.10)$$

In the module $\Lambda^m(M)$ bases are the volume forms

$$\mu_{\mathbf{x}} = dx^1\wedge\cdots\wedge dx^m,\ \ \mu_{\mathbf{y}} = dy^1\wedge\cdots\wedge dy^m$$

and (5.7.8) leads to

$$\phi^*\mu_{\mathbf{y}} = (\det\phi)\mu_{\mathbf{x}} = \det\Big[\frac{\partial y^\alpha}{\partial x^i}\Big]\mu_{\mathbf{x}} = J\,dx^1\wedge\cdots\wedge dx^m. \quad (5.7.11)$$

Conversely, if the relation (5.7.11) is valid, then we must find $\det\phi\neq 0$. In consequence, the celebrated implicit function theorem states that the mapping ϕ is locally a diffeomorphism. Any form $\omega(\mathbf{y})\in\Lambda^m(M)$ is now expressible as $\omega(\mathbf{y}) = g(\mathbf{y})\mu_{\mathbf{y}}$. Thus, under coordinates transformation we obtain the form $\phi^*\omega = (g\circ\phi)\det\big[\partial y^\alpha/\partial x^i\big]\,\mu_{\mathbf{x}}$.

Next, we consider a submanifold S of dimension $r<m$ of the manifold M. We suppose that we describe this submanifold by a smooth mapping $\phi: S\to M$. In local coordinates, this mapping will be prescribed as a coordinate transformation

$$x^i = \Phi^i(u^\alpha),\ \ i = 1,\dots,m;\ \alpha = 1,\dots,r. \quad (5.7.12)$$

The pull-back $\phi^*\omega\in\Lambda^k(S)$ of a form $\omega\in\Lambda^k(M)$ on S is given by

$$\phi^*\omega = \frac{1}{k!}\,\omega_{\alpha_1\cdots\alpha_k}(\mathbf{u})\,du^{\alpha_1}\wedge\cdots\wedge du^{\alpha_k} \quad (5.7.13)$$

where the coefficients $\omega_{\alpha_1\cdots\alpha_k}(\mathbf{u})$ are determined through the relations

$$\omega_{\alpha_1\cdots\alpha_k}(\mathbf{u}) = \omega_{i_1\cdots i_k}\big(\Phi(\mathbf{u})\big)\frac{\partial\Phi^{i_1}}{\partial u^{\alpha_1}}\cdots\frac{\partial\Phi^{i_k}}{\partial u^{\alpha_k}}. \quad (5.7.14)$$

If the form ω does not vanish identically on M, then the submanifold S, consequently the mapping $\phi: S\to M$, satisfying the condition $\phi^*\omega = 0$ is called *a solution of the exterior equation* $\omega = 0$. When $k>r$, then $\phi^*\omega\equiv 0$ identically, that is, any submanifold whose dimension is less than k is automatically a solution of this equation. If $k\le r$, then the mapping ϕ that gives rise to an r-dimensional solution submanifold is determined, in view of

(5.7.13) and (5.7.14), through the equations $\omega_{\alpha_1\cdots\alpha_k}(\mathbf{u}) = 0$. We then call ϕ as the ***resolvent mapping*** for the exterior equation.

We can introduce another interpretation to a solution of an exterior equation. The differential $d\phi : T(S) \to T(M)$ of the mapping $\phi : S \to M$ push a vector field in the tangent bundle of S up to a vector field in the tangent bundle of M. Let $V \in T(S)$, then we can write

$$V = v^\alpha \frac{\partial}{\partial u^\alpha}, \;\; d\phi(V) = v^i \frac{\partial}{\partial x^i} \in T(M),$$

where

$$v^i = \frac{\partial \Phi^i}{\partial u^\alpha} v^\alpha.$$

According to (5.7.1), every k linearly independent vector fields selected from $T(S)$ of dimension $r \geq k$ must satisfy the relation

$$\omega\big(d\phi(V_1), \dots, d\phi(V_k)\big) = (\phi^*\omega)(V_1, \dots, V_k) = 0$$

since $\phi^*\omega = 0$. Hence, in order to determine *locally* an r-dimensional solution submanifold through a point $p \in M$, all we need to do is to find a subspace $T_p(S)$ of the tangent space $T_p(M)$ annihilating the form ω. We know from the Frobenius theorem that the distribution made up by those local subspaces should be involutive so that the local tangent spaces can be patched together to generate a smooth submanifold.

Example 5.7.1. We take $M = \mathbb{R}^2$ and $\omega = x\,dy - 3y\,dx \in \Lambda^1(\mathbb{R}^2)$. Our aim is to determine a mapping $\phi : \mathbb{R} \to \mathbb{R}^2$ so as $\phi^*\omega = 0$. Let us write

$$x = \alpha(u), \;\; y = \beta(u).$$

Then we get $\phi^*\omega = (\alpha\beta' - 3\beta\alpha')du = 0$ and the condition $\alpha\beta' = 3\beta\alpha'$. This differential equation can be cast into the form

$$\frac{\beta'}{\beta} = 3\frac{\alpha'}{\alpha} \quad \text{or} \quad (\log\beta)' = 3(\log\alpha)'$$

so that we obtain $\beta(u) = C\alpha(u)^3$. Therefore, the curves prescribed by parametric equations $x = \alpha(u)$, $y = C\alpha(u)^3$ where $\alpha(u)$ is an arbitrary function solve the exterior equation $\omega = 0$.

Let us now consider a vector $V = v^u(u)\dfrac{\partial}{\partial u} \in T(\mathbb{R})$. We then have

$$d\phi(V) = \alpha' v^u \frac{\partial}{\partial x} + \beta' v^u \frac{\partial}{\partial y} \in T(\mathbb{R}^2).$$

Hence the equation $\omega\big(d\phi(V)\big) = xv^y - 3yv^x = (\alpha\beta' - 3\beta\alpha')v^u = 0$ leads similarly to the above expression and to

$$v^x = \alpha' f(u),\;\; v^y = 3C\alpha^2\alpha' f(u) = C\big(\alpha(u)^3\big)' f(u).$$

where we defined $v^u(u) = f(u)$. ■

Example 5.7.2. We consider $M = \mathbb{R}^3$ and the form

$$\omega = P\,dx^1 \wedge dx^2 + Q\,dx^1 \wedge dx^3 + R\,dx^2 \wedge dx^3 \in \Lambda^2(\mathbb{R}^3)$$

where $P, Q, R \in \Lambda^0(\mathbb{R}^3)$. We define a 2-dimensional solution submanifold by the parametric equations $x^i = \phi^i(u^1, u^2), i = 1, 2, 3$. We denote the functional determinant by

$$\frac{\partial(\phi^i, \phi^j)}{\partial(u^\alpha, u^\beta)} = \frac{\partial\phi^i}{\partial u^\alpha}\frac{\partial\phi^j}{\partial u^\beta} - \frac{\partial\phi^i}{\partial u^\beta}\frac{\partial\phi^j}{\partial u^\alpha}$$

we then attain at the result

$$\phi^*\omega = \left[P\frac{\partial(\phi^1, \phi^2)}{\partial(u^1, u^2)} + Q\frac{\partial(\phi^1, \phi^3)}{\partial(u^1, u^2)} + R\frac{\partial(\phi^2, \phi^3)}{\partial(u^1, u^2)}\right] du^1 \wedge du^2.$$

Therefore, in order to satisfy $\phi^*\omega = 0$ we have to find the solution of the following non-linear partial differential equation

$$P\frac{\partial(\phi^1, \phi^2)}{\partial(u^1, u^2)} + Q\frac{\partial(\phi^1, \phi^3)}{\partial(u^1, u^2)} + R\frac{\partial(\phi^2, \phi^3)}{\partial(u^1, u^2)} = 0$$

where $P = P(\phi^1, \phi^2, \phi^3), Q = Q(\phi^1, \phi^2, \phi^3), R = R(\phi^1, \phi^2, \phi^3)$. ■

5.8. EXTERIOR DERIVATIVE

We define an operator $d : \Lambda(M) \to \Lambda(M)$ on a smooth manifold M mapping the exterior algebra $\Lambda(M)$ into itself in such a way that it holds the following rules:

(i). $d(\omega + \sigma) = d\omega + d\sigma,\;\; d(\lambda\omega) = \lambda\, d\omega;\;\; \omega, \sigma \in \Lambda(M), \lambda \in \mathbb{R}$.
(ii). $d(\omega \wedge \sigma) = d\omega \wedge \sigma + (-1)^{deg\,(\omega)}\omega \wedge d\sigma$.
(iii). $d^2 = d \circ d = 0$, *i.e.*, $d(d\omega) = d^2\omega = 0$ *for all* $\omega \in \Lambda(M)$.
(iv). *If* $f \in \Lambda^0(M)$, *then* $df = f_{,i}\,dx^i \in \Lambda^1(M)$.

The rule (i) means that d is a linear operator on $\mathbb{R}$ whereas the rule (iv) implies that the 1-form df is the classical differential of the smooth function $f \in \Lambda^0(M)$. Here, we have introduced the notation

$$\frac{\partial(\,\cdot\,)}{\partial x^i} \doteq (\,\cdot\,)_{,i} \tag{5.8.1}$$

which we shall employ frequently henceforth. The rule (iii) shows that d is a ***nilpotent operator***. d so defined is called the ***exterior derivative operator*** and the form $d\omega$ is the ***exterior derivative*** of the form ω.

Theorem 5.8.1. *The foregoing rules* (i) - (iv) *determine the exterior derivative operator* d *uniquely.*

We know that an exterior form $\omega \in \Lambda^k(M)$ on a manifold M is expressible in local coordinates on an open set $U \subseteq M$ as follows

$$\omega = \frac{1}{k!}\,\omega_{i_1 i_2\cdots i_k} dx^{i_1} \wedge dx^{i_2} \wedge \cdots \wedge dx^{i_k},\;\; \omega_{i_1 i_2\cdots i_k} \in \Lambda^0(M).$$

Since $\omega_{i_1 i_2\cdots i_k}$ is a 0-form, we obtain

$$d\omega = \frac{1}{k!}\big[d\omega_{i_1\cdots i_k} \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_k} + \omega_{i_1 i_2\cdots i_k} d(dx^{i_1} \wedge \cdots \wedge dx^{i_k})\big]$$

in view of (ii). We shall now demonstrate by mathematical induction that

$$d(dx^{i_1} \wedge \cdots \wedge dx^{i_k}) = 0.$$

If $k = 1$, because of $(iii$-$iv)$ we find $d(dx^{i_1}) = d^2 x^{i_1} = 0$. Let us assume that the above relation is valid for $k - 1$. Hence, we deduce from the rules of exterior differentiation

$$\begin{aligned} d(dx^{i_1} \wedge \cdots \wedge dx^{i_k}) &= \\ d^2 x^{i_1} \wedge (dx^{i_2} \wedge \cdots \wedge dx^{i_k}) &- dx^{i_1} \wedge d(dx^{i_2} \wedge \cdots \wedge dx^{i_k}) \\ &= -\,dx^{i_1} \wedge d(dx^{i_2} \wedge \cdots \wedge dx^{i_k}) = 0. \end{aligned}$$

so that this relation is also valid for k. Therefore, the exterior derivative of the form $\omega \in \Lambda^k(M)$ is designated *uniquely* in local coordinates as follows

$$\begin{aligned} d\omega &= \frac{1}{k!}\,d\omega_{i_1\cdots i_k} \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_k} \\ &= \frac{1}{k!}\frac{\partial \omega_{i_1\cdots i_k}}{\partial x^i}\,dx^i \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_k} \\ &= \frac{1}{k!}\omega_{[i_1\cdots i_k,i]}\,dx^i \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_k} \in \Lambda^{k+1}(M). \end{aligned} \tag{5.8.2}$$

Thus the operator d is of the form $d : \Lambda^k(M) \to \Lambda^{k+1}(M)$ and increases the degree of the form by one. The form $d\omega \in \Lambda^{k+1}(M)$ can be written in the standard form in the following manner

$$d\omega = \frac{1}{(k+1)!}\,\omega_{ii_1\cdots i_k} dx^i \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_k}$$

where we obviously have

$$\omega_{ii_1\cdots i_k} = (k+1)\,\omega_{[i_1\cdots i_k,i]} \in \Lambda^0(M). \tag{5.8.3}$$

In order that this definition of the exterior derivative to be meaningful it should not depend on the chosen local coordinates, namely, the chosen chart of the atlas. To observe this property, let us consider the coordinate transformation $x^i = x^i(y^j)$ in overlapping charts. We thus write

$$\begin{aligned}\omega &= \frac{1}{k!}\,\omega_{i_1\cdots i_k}(\mathbf{x})\,dx^{i_1} \wedge \cdots \wedge dx^{i_k}\\ &= \frac{1}{k!}\,\omega_{i_1\cdots i_k}(\mathbf{x}(\mathbf{y}))\,dx^{i_1}(\mathbf{y}) \wedge \cdots \wedge dx^{i_k}(\mathbf{y})\end{aligned}$$

so that the exterior derivatives with respect to **y**- and **x**-coordinates are found to be related by

$$\begin{aligned}d_{\mathbf{y}}\omega &= \frac{1}{k!}\frac{\partial\omega_{i_1\cdots i_k}\big(\mathbf{x}(\mathbf{y})\big)}{\partial y^j}\,dy^j \wedge dx^{i_1}(\mathbf{y}) \wedge \cdots \wedge dx^{i_k}(\mathbf{y})\\ &= \frac{1}{k!}\frac{\partial\omega_{i_1\cdots i_k}(\mathbf{x})}{\partial x^i}\frac{\partial x^i}{\partial y^j}\,dy^j \wedge dx^{i_1}(\mathbf{y}) \wedge \cdots \wedge dx^{i_k}(\mathbf{y})\\ &= \frac{1}{k!}\frac{\partial\omega_{i_1\cdots i_k}(\mathbf{x})}{\partial x^i}\,dx^i \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_k} = d_{\mathbf{x}}\omega.\end{aligned}$$

This relation is valid for all $\omega \in \Lambda(M)$. Hence, we obtain $d_{\mathbf{y}} = d_{\mathbf{x}}$ showing that the operator d is intrinsically defined. □

After having defined the exterior derivative by the expression (5.8.2), it is straightforward to see that the rules (i)-(iv) are automatically satisfied. That (i) becomes valid is obvious. To show (ii), let us consider the forms $\omega \in \Lambda^k(M)$ and $\sigma \in \Lambda^l(M)$ given by

$$\begin{aligned}\omega &= \frac{1}{k!}\,\omega_{i_1\cdots i_k} dx^{i_1} \wedge \cdots \wedge dx^{i_k},\\ \sigma &= \frac{1}{l!}\,\sigma_{i_1\cdots i_l}\,dx^{i_1} \wedge \cdots \wedge dx^{i_l}\end{aligned}$$

and evaluate the exterior derivative of $\omega \wedge \sigma$. We obtain

$$d(\omega \wedge \sigma) = d\Big[\frac{1}{k!}\frac{1}{l!}\,\omega_{i_1\cdots i_k}\sigma_{j_1\cdots j_l}\,dx^{i_1} \wedge \cdots \wedge dx^{i_k} \wedge dx^{j_1} \wedge \cdots \wedge dx^{j_l}\Big]$$

$$
\begin{aligned}
&= \frac{1}{k!}\, d\omega_{i_1\cdots i_k} \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_k} \wedge \frac{1}{l!}\,\sigma_{j_1\cdots j_l}\, dx^{j_1} \wedge \cdots \wedge dx^{j_l} \\
&\quad + \frac{1}{k!}\,\omega_{i_1\cdots i_k} \frac{1}{l!}\, d\sigma_{j_1\cdots j_l} \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_k} \wedge dx^{j_1} \wedge \cdots \wedge dx^{j_l} \\
&= \frac{1}{k!}\, d\omega_{i_1\cdots i_k} \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_k} \wedge \frac{1}{l!}\,\sigma_{j_1\cdots j_l}\, dx^{j_1} \wedge \cdots \wedge dx^{j_l} \\
&\qquad + (-1)^k \frac{1}{k!}\,\omega_{i_1\cdots i_k}\, dx^{i_1} \wedge \cdots \wedge dx^{i_k} \wedge \frac{1}{l!}\, d\sigma_{j_1\cdots j_l} \wedge dx^{j_1} \wedge \cdots \wedge dx^{j_l}
\end{aligned}
$$

and we thus get

$$d(\omega \wedge \sigma) = d\omega \wedge \sigma + (-1)^k \omega \wedge d\sigma.$$

Similarly, we find

$$d^2\omega = \frac{1}{k!}\,\omega_{i_1\cdots i_k,ij}\, dx^i \wedge dx^j \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_k} \in \Lambda^{k+2}(M).$$

But the exterior product is antisymmetric with respect to indices i and j, while the second partial derivatives are symmetric. Therefore, summations over these indices from 1 to m become zero and we get $d^2\omega = 0$. The rule (iv) is retrieved immediately form the definition (5.8.2).

We can provide a more explicit expression for coefficient functions $\omega_{ii_1\cdots i_k} \in \Lambda^0(M)$ specifying the form $d\omega \in \Lambda^{k+1}(M)$. If we take notice of the relation (5.5.2) we readily arrive at

$$
\begin{aligned}
\omega_{ii_1\cdots i_k} &= \frac{k+1}{(k+1)!}\,\delta^{j_1 j_2\cdots j_k j}_{i_1 i_2\cdots i_k i}\,\omega_{j_1\cdots j_k,j} = \frac{1}{k!}\Big[\delta^j_i \delta^{j_1 j_2\cdots j_k}_{i_1 i_2\cdots i_k} - \delta^j_{i_1}\delta^{j_1 j_2\cdots j_k}_{i i_2\cdots i_k} \\
&\qquad - \delta^j_{i_2}\delta^{j_1 j_2\cdots j_k}_{i_1 i\cdots i_k} - \cdots - \delta^j_{i_k}\delta^{j_1 j_2\cdots j_k}_{i_1 i_2\cdots i_{k-1} i}\Big]\omega_{j_1\cdots j_k,j} \\
&= \frac{1}{k!}\Big[\delta^{j_1 j_2\cdots j_k}_{i_1 i_2\cdots i_k}\omega_{j_1\cdots j_k,i} - \delta^{j_1 j_2\cdots j_k}_{i i_2\cdots i_k}\omega_{j_1\cdots j_k,i_1} - \delta^{j_1 j_2\cdots j_k}_{i_1 i\cdots i_k}\omega_{j_1\cdots j_k,i_2} \\
&\qquad - \cdots - \delta^{j_1 j_2\cdots j_k}_{i_1 i_2\cdots i_{k-1} i}\omega_{j_1\cdots j_k,i_k}\Big].
\end{aligned}
$$

Since $\omega_{j_1\cdots j_k}$ is completely antisymmetric, this expression may be transformed into the following form:

$$
\begin{aligned}
\omega_{ii_1\cdots i_k} &= \omega_{i_1\cdots i_k,i} - \omega_{ii_2\cdots i_k,i_1} - \omega_{i_1 i\cdots i_k,i_2} - \cdots - \omega_{i_1 i_2\cdots i,i_k} \qquad (5.8.4) \\
&= \omega_{i_1\cdots i_k,i} - \sum_{r=1}^{k}\omega_{i_1\cdots i_{r-1} i i_{r+1}\cdots i_k,i_r}.
\end{aligned}
$$

Example 5.8.1. The exterior derivative of the form $\omega = \omega_j\, dx^j \in \Lambda^1(M)$ will be

$$d\omega = \omega_{j,i}\, dx^i \wedge dx^j = \frac{1}{2}\, \omega_{ij}\, dx^i \wedge dx^j,$$
$$\omega_{ij} = 2\, \omega_{[j,i]} = \omega_{j,i} - \omega_{i,j}.$$

Let us take $M = \mathbb{R}^3$. In this case, the number of the independent components of the coefficients ω_{ij} is three and this matrix can be represented by an axial vector. One can then write $\omega = \mathbf{V} \cdot d\mathbf{r} = X_1\, dx^1 + X_2\, dx^2 + X_3\, dx^3$ where we employed the notation of the classical vector algebra to denote $\mathbf{V} = X_1\mathbf{e}_1 + X_2\mathbf{e}_2 + X_3\mathbf{e_3}$ and $d\mathbf{r} = dx_1\mathbf{e}_1 + dx_2\mathbf{e}_2 + dx_3\mathbf{e}_3$. $(\,\cdot\,)$ is the usual scalar product and $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ are orthonormal basis vectors of $\mathbb{R}^3$. The exterior derivative of the form ω becomes

$$d\omega = \Big(\frac{\partial X_3}{\partial x^2} - \frac{\partial X_2}{\partial x^3}\Big)\, dx^2 \wedge dx^3 + \Big(\frac{\partial X_1}{\partial x^3} - \frac{\partial X_3}{\partial x^1}\Big)\, dx^3 \wedge dx^1$$
$$+ \Big(\frac{\partial X_2}{\partial x^1} - \frac{\partial X_1}{\partial x^2}\Big)\, dx^1 \wedge dx^2.$$

Evidently the coefficients of the form $d\omega$ is nothing but the components of ***the curl*** of the vector $\mathbf{V}$, i.e., $\mathbf{W} = \mathrm{curl}\,\mathbf{V} = \boldsymbol{\nabla} \times \mathbf{V}$. This vector is also expressible as

$$\mathbf{W} = W_i\mathbf{e}_i = e_{ijk}\frac{\partial X_k}{\partial x^j}\mathbf{e}_i = e_{ijk}X_{k,j}\mathbf{e}_i, \;\; i, j, k = 1, 2, 3.$$

On the other hand, if we consider the forms

$$\omega_1 = \mathbf{V}_1 \cdot d\mathbf{r} \;\text{ and }\; \omega_2 = \mathbf{V}_2 \cdot d\mathbf{r}$$

we see that their exterior product is

$$\omega_1 \wedge \omega_2 = (X_2Y_3 - X_3Y_2)\, dx^2 \wedge dx^3 + (X_3Y_1 - X_1Y_3)\, dx^3 \wedge dx^1$$
$$+ (X_1Y_2 - X_2Y_1)\, dx^1 \wedge dx^2$$

the coefficients of which are components of the usual vectorial product $\mathbf{W} = \mathbf{V}_1 \times \mathbf{V}_2$. This vector can also be written as follows

$$\mathbf{W} = W_i\mathbf{e}_i = e_{ijk}X_jY_k\mathbf{e}_i, \;\; i, j, k = 1, 2, 3.$$

Let us next calculate the exterior derivative of the form $\omega = f\mathbf{V} \cdot d\mathbf{r}$ where $f \in \Lambda^0(\mathbb{R}^3)$, we easily reach to the relation

$$\mathrm{curl}\, f\mathbf{V} = \mathrm{grad}\, f \times \mathbf{V} + f\mathrm{curl}\,\mathbf{V}.$$ ■

Example 5.8.2. We consider the form $\omega = \frac{1}{2!}\, \omega_{jk}\, dx^j \wedge dx^k \in \Lambda^2(M)$ whose exterior derivative becomes

$$d\omega = \frac{1}{2!}\,\omega_{jk,i}\,dx^i \wedge dx^j \wedge dx = \frac{1}{3!}\,\omega_{ijk}\,dx^i \wedge dx^j \wedge dx^k \in \Lambda^3(M).$$

According to (5.8.4), the coefficients of this form are given by

$$\omega_{ijk} = \omega_{jk,i} - \omega_{ik,j} - \omega_{ji,k} = \omega_{jk,i} + \omega_{ki,j} + \omega_{ij,k}.$$

Let us choose again $M = \mathbb{R}^3$ and write

$$\omega = X_1\,dx^2 \wedge dx^3 + X_2\,dx^3 \wedge dx^1 + X_3\,dx^1 \wedge dx^2 \in \Lambda^2(\mathbb{R}^3)$$

in terms of essential components. We observe at once that

$$d\omega = \Big(\frac{\partial X_1}{\partial x^1} + \frac{\partial X_2}{\partial x^2} + \frac{\partial X_3}{\partial x^3}\Big)dx^1 \wedge dx^2 \wedge dx^3 \in \Lambda^3(\mathbb{R}^3),$$

namely, the coefficient of this form is just the ***divergence*** $\boldsymbol{\nabla} \cdot \mathbf{V} = \operatorname{div} \mathbf{V}$ of the vector field $\mathbf{V} = X_1\mathbf{e}_1 + X_2\mathbf{e}_2 + X_3\mathbf{e_3}$ which can also be written as follows

$$\boldsymbol{\nabla} \cdot \mathbf{V} = \frac{\partial X_i}{\partial x^i} = X_{i,i}.$$

If we take into account the forms ω_1 and ω_2 defined in Example 5.8.1, then the relation $d(\omega_1 \wedge \omega_2) = d\omega_1 \wedge \omega_2 - \omega_1 \wedge d\omega_2$ yields the equality

$$\operatorname{div}(\mathbf{V}_1 \times \mathbf{V}_2) = \mathbf{V}_2 \cdot \operatorname{curl} \mathbf{V}_1 - \mathbf{V}_1 \cdot \operatorname{curl} \mathbf{V_2}. \qquad \blacksquare$$

We know that a form $\omega \in \Lambda^{m-k}(M)$ is expressible as in (5.5.17) by using a basis induced by the volume form. Since $d\mu_{i_k\cdots i_2 i_1} = 0$, the exterior derivative of this form is given by

$$d\omega = \frac{1}{k!}\,\omega^{i_1 i_2\cdots i_k}{}_{,i}\,dx^i \wedge \mu_{i_k\cdots i_2 i_1} = \frac{k}{k!}\,\omega^{i_1 i_2\cdots i_k}{}_{,i}\,\delta^i_{[i_k}\mu_{i_{k-1}\cdots i_2 i_1]}$$

where we employed the relation (5.5.15). Because of the antisymmetry of $\omega^{i_1 i_2\cdots i_k}$, we conclude that

$$\begin{aligned} d\omega &= \frac{1}{(k-1)!}\,\omega^{i_1 i_2\cdots i_k}{}_{,i}\,\delta^i_{i_k}\mu_{i_{k-1}\cdots i_2 i_1} \\ &= \frac{1}{(k-1)!}\,\omega^{i_1 i_2\cdots i_{k-1} i}{}_{,i}\,\mu_{i_{k-1}\cdots i_2 i_1} \in \Lambda^{m-(k-1)}(M). \end{aligned} \tag{5.8.5}$$

It is clear that one has

$$\omega^{i_1 i_2\cdots i_{k-1} i}{}_{,i} = \frac{\partial \omega^{i_1 i_2\cdots i_{k-1} 1}}{\partial x^1} + \frac{\partial \omega^{i_1 i_2\cdots i_{k-1} 2}}{\partial x^2} + \cdots + \frac{\partial \omega^{i_1 i_2\cdots i_{k-1} m}}{\partial x^m}.$$

We thus see that the coefficients of the form $d\omega$ is evaluated as a kind of *divergence*.

Let $\phi : M \to N$ be a differentiable mapping between the smooth manifolds M and N. We know that this mapping conduces toward the pullback mapping $\phi^* : \Lambda(N) \to \Lambda(M)$ which assigns a form $\phi^*\omega \in \Lambda^k(M)$ to a form $\omega \in \Lambda^k(N)$.

Theorem 5.8.2. *If $\phi : M \to N$ is a smooth mapping, then we have the relation $d(\phi^*\omega) = \phi^* d\omega$ for all forms $\omega \in \Lambda(N)$. Consequently, one has the following rule of composition*

$$d \circ \phi^* = \phi^* \circ d : \Lambda^k(N) \to \Lambda^{k+1}(M)$$

which means that the operators d and ϕ^* commute.

We prove this theorem by explicitly calculating both sides. Let us consider a form

$$\omega = \frac{1}{k!}\,\omega_{\alpha_1\alpha_2\cdots\alpha_k} dy^{\alpha_1} \wedge dy^{\alpha_2} \wedge \cdots \wedge dy^{\alpha_k} \in \Lambda^k(N).$$

Its exterior derivative is

$$d\omega = \frac{1}{k!}\,\omega_{\alpha_1\alpha_2\cdots\alpha_k,\alpha}\, dy^{\alpha} \wedge dy^{\alpha_1} \wedge dy^{\alpha_2} \wedge \cdots \wedge dy^{\alpha_k}.$$

We thus obtain

$$\phi^* d\omega = \\ \frac{1}{k!}\Big(\frac{\partial \omega_{\alpha_1\alpha_2\cdots\alpha_k}}{\partial y^\alpha} \circ \phi\Big)\, \frac{\partial \Phi^\alpha}{\partial x^i}\frac{\partial \Phi^{\alpha_1}}{\partial x^{i_1}}\cdots\frac{\partial \Phi^{\alpha_k}}{\partial x^{i_k}}\, dx^i \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_k}.$$

where the functions $y^\alpha = \Phi^\alpha(x^i)$ is generated by the mapping ϕ through local charts at the points $p \in M$ and $q = \phi(p) \in N$. On the other hand, due to the symmetry of second derivatives and antisymmetry of exterior products, we get

$$\begin{aligned} d(\phi^*\omega) &= \frac{1}{k!}\, d\Big(\omega_{\alpha_1\cdots\alpha_k}\big(\Phi(\mathbf{x})\big)\frac{\partial \Phi^{\alpha_1}}{\partial x^{i_1}}\cdots\frac{\partial \Phi^{\alpha_k}}{\partial x^{i_k}}\Big)\, dx^{i_1} \wedge \cdots \wedge dx^{i_k} \\ &= \frac{1}{k!}\Big[\frac{\partial \omega_{\alpha_1\cdots\alpha_k}}{\partial y^\alpha}\frac{\partial \Phi^\alpha}{\partial x^i}\frac{\partial \Phi^{\alpha_1}}{\partial x^{i_1}}\cdots\frac{\partial \Phi^{\alpha_k}}{\partial x^{i_k}} + \omega_{\alpha_1\cdots\alpha_k}\frac{\partial^2 \Phi^{\alpha_1}}{\partial x^i \partial x^{i_1}}\cdots\frac{\partial \Phi^{\alpha_k}}{\partial x^{i_k}} \\ &\quad + \cdots + \omega_{\alpha_1\cdots\alpha_k}\frac{\partial \Phi^{\alpha_1}}{\partial x^{i_1}}\cdots\frac{\partial^2 \Phi^{\alpha_k}}{\partial x^i \partial x^{i_k}}\Big]\, dx^i \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_k} \\ &= \frac{1}{k!}\frac{\partial \omega_{\alpha_1\cdots\alpha_k}\big(\Phi(\mathbf{x})\big)}{\partial y^\alpha}\,\frac{\partial \Phi^\alpha}{\partial x^i}\frac{\partial \Phi^{\alpha_1}}{\partial x^{i_1}}\cdots\frac{\partial \Phi^{\alpha_k}}{\partial x^{i_k}}\, dx^i \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_k} \end{aligned}$$

Therefore, we find that $d(\phi^*\omega) = \phi^* d\omega$ for any $\omega \in \Lambda(N)$. □

If the exterior derivative of a form $\omega \in \Lambda(M)$ vanishes, that is, if $d\omega = 0$, then ω is called a ***closed form***. Thus, closed forms constitute the *null space* or *kernel* of the operator d:

$$\mathcal{N}(d) = \text{Ker}\,(d) = \{\omega \in \Lambda(M) : d\omega = 0\}.$$

If for a form $\omega \in \Lambda^k(M)$, there exists a form $\sigma \in \Lambda^{k-1}(M)$ such that $\omega = d\sigma$, then ω is called an ***exact form***. Obviously, this means that exact forms occupy the range of the operator d:

$$\mathcal{R}(d) = \text{Im}\,(d) = \{\omega \in \Lambda(M) : \omega = d\gamma, \gamma \in \Lambda(M)\}$$

If ω is an exact form, we have $d\omega = d^2\sigma = 0$. Hence, an exact form is naturally a closed form. However, the converse statement is not always true. This subject will be investigated in detail in Chapter VI through the homotopy operator.

If $\omega \in \Lambda^m(M)$ we get $d\omega = 0$ because every $(m+1)$-form is identically zero. Therefore, *every m-form will be closed* on an m-dimensional manifold.

Theorem 5.8.3. *The closed and exact forms in the module* $\Lambda^k(M)$ *constitute linear vector spaces* $\mathcal{C}^k(M)$ *and* $\mathcal{E}^k(M)$, *respectively, over real numbers.*

Let us consider the closed forms $\omega, \sigma \in \Lambda^k(M)$ satisfying $d\omega = d\sigma = 0$. Let $f, g \in \Lambda^0(M$ be arbitrary functions. Then we find that

$$d(f\omega + g\sigma) = df \wedge \omega + dg \wedge \sigma.$$

Hence, this expression vanishes if and only if $df = dg = 0$. Thus if only if f and g are constants, then the form $f\omega + g\sigma$ is closed. In other words, closed forms constitute a linear vector space $\mathcal{C}^k(M)$ only on $\mathbb{R}$.

This time, let us take the exact forms $\omega, \sigma \in \Lambda^k(M)$ into consideration. Hence, there are forms $\alpha, \beta \in \Lambda^{k-1}(M)$ such that $\omega = d\alpha, \sigma = d\beta$. Since we can write

$$\omega + \sigma = d\alpha + d\beta = d(\alpha + \beta)$$

we see that the form $\omega + \sigma$ is exact. Next, let us consider the form

$$f\omega = f d\alpha = d(f\alpha) - df \wedge \alpha$$

for an arbitrary function $f \in \Lambda^0(M)$. This means that the form $f\omega$ can be exact if only $df = 0$, or f is a constant. Thus exact forms constitute a linear vector space $\mathcal{E}^k(M)$ only on $\mathbb{R}$. Since every exact form is closed, it is evident that $\mathcal{E}^k(M) \subseteq \mathcal{C}^k(M)$. □

Next, we define the sets $\mathcal{C}(M) = \overset{m}{\underset{k=0}{\oplus}} \mathcal{C}^k(M)$ and $\mathcal{E}(M) = \overset{m}{\underset{k=1}{\oplus}} \mathcal{E}^k(M)$. We can easily verify that *they form graded subalgebras of the exterior algebra* $\Lambda(M)$ *on* $\mathbb{R}$. In fact, if $\omega, \sigma \in \mathcal{C}(M)$, we have $d\omega = d\sigma = 0$ and consequently, $d(\omega \wedge \sigma) = d\omega \wedge \sigma + (-1)^{deg(\omega)}\omega \wedge d\sigma = 0$ so we find that $\omega \wedge \sigma \in \mathcal{C}(M)$. On the other hand, if $\omega, \sigma \in \mathcal{E}(M)$, then we have to write $\omega = d\alpha,\ \sigma = d\beta$ so that we obtain $\omega \wedge \sigma = d\alpha \wedge d\beta = d(\alpha \wedge d\beta)$ leading to $\omega \wedge \sigma \in \mathcal{E}(M)$.

Example 5.8.3. We consider a form $\omega \in \Lambda^1(M)$. If $\omega \in \mathcal{E}^1(M)$, then there must exist a function $\Omega \in \Lambda^0(M)$ so that we can write $\omega = d\Omega$ or

$$\omega_i\, dx^i = \Omega_{,i}\, dx^i.$$

Hence, the relations $\omega_i = \Omega_{,i}$ must hold. Thus, the coefficients ω_i have to verify the integrability conditions $\omega_{i,j} - \omega_{j,i} = 0$ in order to be able to determine Ω. On the other hand, if the form ω is closed, then we get

$$d\omega = \omega_{i,j}\, dx^j \wedge dx^i = \omega_{[i,j]}\, dx^j \wedge dx^i = 0$$

from which we deduce that $\omega_{[i,j]} = 0$ or $\omega_{i,j} - \omega_{j,i} = 0$. Thus, if the form is exact, then the conditions to be closed is satisfied automatically. However, in order that a closed 1-form is to be exact we have to find the solution of $m(m-1)/2$ first order partial differential equations satisfied by m unknowns ω_i in the form $\omega_i = \Omega_{,i}$. The existence of the solution is, however, strongly dependent on the topology of the manifold. ■

Example 5.8.4. We consider a form $\omega \in \Lambda^2(M)$. This form will be exact if there exists a form $\alpha \in \Lambda^1(M)$ such that $\omega = d\alpha$. Let us then take $\omega = \frac{1}{2}\omega_{ij}\, dx^i \wedge dx^j$ and $\alpha = \alpha_j\, dx^j$. The relation

$$\frac{1}{2}\,\omega_{ij}\, dx^i \wedge dx^j = \alpha_{j,i}\, dx^i \wedge dx^j = \alpha_{[j,i]}\, dx^i \wedge dx^j$$

leads to $\omega_{ij} = 2\alpha_{[j,i]} = \alpha_{j,i} - \alpha_{i,j}$. In order that the functions α_i satisfying these conditions could be determined the 2-form ω must be closed. This becomes possible if the condition

$$d\omega = \frac{1}{2}\,\omega_{ij,k}\, dx^k \wedge\, dx^i \wedge dx^j = \frac{1}{2}\,\omega_{[ij,k]}\, dx^k \wedge\, dx^i \wedge dx^j = 0$$

is met. Therefore, the coefficients ω_{ij} must satisfy the following differential equations

$$\omega_{[ij,k]} = 0 \quad \text{or} \quad \frac{\partial \omega_{ij}}{\partial x^k} + \frac{\partial \omega_{jk}}{\partial x^i} + \frac{\partial \omega_{ki}}{\partial x^j} = 0.$$ ■

Example 5.8.5. Let us consider the form $\omega \in \Lambda^{m-k}(M)$. This form is exact if $\omega = d\alpha$ so that (5.8.5) yields

$$\begin{aligned}\omega &= \frac{1}{k!}\,\omega^{i_1\cdots i_k}\,\mu_{i_k\cdots i_1} = d\Big(\frac{1}{(k+1)!}\alpha^{i_1\cdots i_k i_{k+1}}\,\mu_{i_{k+1}i_k\cdots i_1}\Big)\\ &= \frac{1}{k!}\alpha^{i_1\cdots i_k i}{}_{,i}\,\mu_{i_k\cdots i_1}.\end{aligned}$$

Hence, the coefficients must satisfy a relation like

$$\omega^{i_1\cdots i_k} = \alpha^{i_1\cdots i_k i}{}_{,i}$$

whence we conclude that

$$\omega^{i_1\cdots i_{k-1}j}{}_{,j} = \alpha^{i_1\cdots i_{k-1}ji}{}_{,ij} = 0.$$

If $\omega \in \Lambda^{m-1}(M)$, the above conditions obviously reduce to

$$\omega^i = \alpha^{ij}{}_{,j} \quad \text{and} \quad \omega^i{}_{,i} = \alpha^{ij}{}_{,ij} = 0. \qquad \blacksquare$$

Let us finally consider the sequence of modules

$$\Lambda^0(M) \xrightarrow{d} \cdots \xrightarrow{d} \Lambda^k(M) \xrightarrow{d} \Lambda^{k+1}(M) \xrightarrow{d} \cdots \xrightarrow{d} \Lambda^m(M) \xrightarrow{d} 0 \qquad (5.8.6)$$

where homomorphisms between successive linear vector spaces are provided by the exterior derivative d on real numbers. *Since $d \circ d = d^2 = 0$, this sequence is evidently a cochain complex*. As we shall see later in Chapter VIII, this cochain complex will play quite a significant part in revealing some fundamental properties of closed and exact forms that connect some topological and analytical features.

5.9. RIEMANNIAN MANIFOLDS. HODGE DUAL

A 2-covariant tensor field $\boldsymbol{\mathcal{G}} \in \mathfrak{T}(M)^0_2$ on a smooth manifold M will be called a ***metric tensor*** if it obeys the following requirements:

(i). *$\boldsymbol{\mathcal{G}}$ is a symmetric tensor.*
(ii). *The bilinear form $\boldsymbol{\mathcal{G}}_p$ is not degenerate at every point $p \in M$, that is, $\boldsymbol{\mathcal{G}}_p(U, V) = 0$ for all $U \in T_p(M)$ if and only if $V = 0$ at the point p.*

A manifold equipped with such a metric tensor will be called a ***Riemannian manifold***. In local coordinates, the metric tensor is expressible as

$$\boldsymbol{\mathcal{G}} = g_{ij}(\mathbf{x})\,dx^i \otimes dx^j, \quad g_{ij} = g_{ji}. \qquad (5.9.1)$$

Consequently, the condition $\boldsymbol{\mathcal{G}}(U,V)=g_{ij}u^iv^j=0$ for all vectors $U=u^i\partial_i$ where $V=v^i\partial_i$ results in $g_{ij}v^j=0$. Whenever this homogeneous system of linear equations is satisfied if and only if $V=0$, then the matrix $\mathbf{G}=[g_{ij}]$ must be regular at every point, namely, its inverse must exist. Let us denote the inverse matrix by $\mathbf{G}^{-1}=[(g^{-1})^{ij}]=[g^{ij}]$. Hence, the relations $g^{ik}g_{kj}=g_{jk}g^{ki}=\delta^i_j$ will hold. By means of the metric tensor $\boldsymbol{\mathcal{G}}$, we can assign a field of 1-form in $T^*(M)$ to every vector field $V\in T(M)$ prescribed by $V=v^i\partial/\partial x^i$ where $v^i(\mathbf{x})$ denote the contravariant components of V through the relation

$$\omega_V=\boldsymbol{\mathcal{G}}(V)=g_{ij}v^j\,dx^i=v_i\,dx^i\in T^*(M)=\Lambda^1(M).$$

Thus the metric tensor gives rise to a linear mapping $\boldsymbol{\mathcal{G}}:T(M)\to T^*(M)$. The coefficients of the form ω_V given by

$$v_i=g_{ij}v^j\in\Lambda^0(M) \tag{5.9.2}$$

is called the *covariant components* of the vector V. If we make use of the inverse matrix $\mathbf{G}^{-1}$, (5.9.2) can be transformed into

$$v^i=g^{ij}v_j. \tag{5.9.3}$$

Thus a vector V can also be expressed as

$$V=v^j\frac{\partial}{\partial x^j}=g^{ji}v_i\frac{\partial}{\partial x^j}=v_ie^i.$$

Since the matrix $\mathbf{G}$ is regular, the vectors

$$e^i=g^{ij}\frac{\partial}{\partial x^j},\ \ i=1,\dots,m \tag{5.9.4}$$

constitute a basis for the tangent space as well. It then easily follows from (5.9.1) and (5.9.4) that

$$\boldsymbol{\mathcal{G}}(\partial_i,\partial_j)=g_{ij},\ \ \boldsymbol{\mathcal{G}}(e^i,e^j)=g_{kl}g^{ik}g^{jl}=g^{ij}. \tag{5.9.5}$$

Let us now consider a form $\omega=\omega_i\,dx^i\in\Lambda^1(M)$ and introduce a vector through the relation

$$V_\omega=g^{ij}\omega_j\frac{\partial}{\partial x^i}=\omega^i\frac{\partial}{\partial x^i}\in T(M),\ \ \omega^i=g^{ij}\omega_j.$$

We can readily verify that $\boldsymbol{\mathcal{G}}(V_\omega)=\omega$. Moreover, we can write

$$\boldsymbol{\mathcal{G}}(V_\omega,V_\sigma)=g_{ij}\omega^i\sigma^j=g_{ij}g^{ik}g^{jl}\omega_k\sigma_l=g^{kl}\omega_k\sigma_l. \tag{5.9.6}$$

These results reveal the fact that the metric tensor furnishes an isomorphism between bundles $T(M)$ and $T^*(M)$. The inverse operator is procured by the inverse matrix g^{ij}. Let us define a new set of basis vectors in $T^*(M)$ by

$$f_i = g_{ij}\,dx^j. \tag{5.9.7}$$

We then obtain

$$f_i(e^j) = g_{ik}\,dx^k(g^{jl}\partial_l) = g_{ik}g^{jl}\delta^k_l = g_{ik}g^{kj} = \delta^j_i$$

which means that $\{e^i\}$ and $\{f_i\}$ are reciprocal bases. On making use of (5.9.7) we can also write $dx^i = g^{ij}f_j$. Utilising (5.9.7), we easily get another representation of the metric tensor

$$\begin{aligned} g^{ij}f_i \otimes f_j &= g^{ij}g_{ik}g_{jl}\,dx^k \otimes dx^l = \delta^j_k g_{jl}\,dx^k \otimes dx^l \\ &= g_{kl}\,dx^k \otimes dx^l = \boldsymbol{\mathcal{G}}. \end{aligned}$$

When we consider a coordinate transformation such as $y^i = y^i(x^j)$ in a neighbourhood of a point $p \in M$ we arrive at the following rule of transformation

$$\begin{aligned} f'_i(\mathbf{y}) &= g'_{ij}\,dy^j = \frac{\partial x^k}{\partial y^i}\frac{\partial x^l}{\partial y^j}g_{kl}\frac{\partial y^j}{\partial x^m}dx^m = \frac{\partial x^k}{\partial y^i}g_{kl}dx^l \\ &= \frac{\partial x^k}{\partial y^i}f_k(\mathbf{x}). \end{aligned}$$

The inverse relation then obviously becomes

$$f_k(\mathbf{x}) = \frac{\partial y^i}{\partial x^k}f'_i(\mathbf{y})$$

Hence, the relation

$$\boldsymbol{\mathcal{G}} = g'^{ij}f'_i \otimes f'_j = g^{kl}f_k \otimes f_l = \frac{\partial y^i}{\partial x^k}\frac{\partial y^j}{\partial x^l}g^{kl}f'_i \otimes f'_j$$

leads to the transformation

$$g'^{ij} = \frac{\partial y^i}{\partial x^k}\frac{\partial y^j}{\partial x^l}g^{kl}$$

meaning that the coefficients g^{ij} are actually contravariant components of the tensor $\boldsymbol{\mathcal{G}}$.

If the tensor $\boldsymbol{\mathcal{G}}$ is positive definite, namely, if for every non-zero vector field $V \in T(M)$ one has

$$\boldsymbol{\mathcal{G}}(V,V) = g_{ij}v^i v^j > 0 \tag{5.9.8}$$

we say that the *Riemannian manifold* is ***complete*** and the *metric* is ***definite.*** If this condition does not hold, then M is a ***pseudo-Riemannian manifold*** or an ***incomplete Riemannian manifold*** and the metric is ***indefinite.*** When the metric on a Riemannian manifold verifies the constraint (5.9.8), then it becomes possible to define an ***inner product*** or, if we put it another way, a ***scalar product of two vectors*** on the tangent bundle $T(M)$ of the manifold through the relation

$$(U,V) = \boldsymbol{\mathcal{G}}(U,V) = g_{ij}u^i v^j, \quad U,V \in T(M). \tag{5.9.9}$$

It is a simple exercise to show that the above definition entirely complies with the rules concerning an inner product on a vector space. Hence, the finite-dimensional vector space $T_p(M)$ then becomes a real Hilbert space. $T(M)$ will then be the union of Hilbert spaces. The relations (5.9.9) and (5.9.8) makes it possible to associate with a vector a positive number that vanishes if and only if the vector is zero. We call this number as the ***length*** or the ***norm*** of the vector V:

$$\|V\| = \sqrt{(V,V)} = \sqrt{g_{ij}v^i v^j} > 0. \tag{5.9.10}$$

In like fashion, we can define an inner product on the dual space $T^*(M)$ by the relation

$$(\omega,\sigma) = g^{ij}\omega_i\sigma_j, \ \ \omega,\sigma \in \Lambda^1(M).$$

If $(U,V) = g_{ij}u^i v^j = 0$ for distinct vectors U and V, namely, if their inner product vanishes, we say that these vectors constitute an ***orthogonal*** set. When, in addition, their norms is equal to 1, then they form an ***orthonormal*** set. When we are provided with a set of orthogonal vectors, this set can obviously be cast into a set of orthonormal vectors by dividing each vector by its norm. In a finite-dimensional complete Riemannian manifold, we can always construct an orthonormal basis for $T(M)$ inductively. Let $U_i, i = 1,\dots,m$ be a linearly independent set of vectors. Let us start by taking $W_1 = U_1$ and construct the following sequence of vectors

$$W_i = U_i - \sum_{j=1}^{i-1}\frac{W_j(U_i,W_j)}{\|W_j\|^2}, \ V_i = \frac{W_i}{\|W_i\|}, \ i = 1,\dots,m.$$

It is straightforward to verify that the vectors $V_1, V_2, \dots, V_m$ form an orthonormal basis, that is, they possess the property

$$(V_i, V_j) = g_{kl} v_i^k v_j^l = \delta_{ij}.$$

This method that generates generally a set of orthonormal vectors from a given countable set of linearly independent vectors spanning the same subspace is known as the ***Gram-Schmidt orthonormalisation process*** after Danish mathematician Jørgen Pedersen Gram (1850-1916) and German mathematician Erhard Schmidt (1876-1959). They had developed it independently. However, it must be fair to mention that French mathematician Pierre-Simon Laplace (1749-1827) had presented this process much earlier than either Gram or Schmidt albeit in a somewhat limited context. Thus, we can always choose an orthonormal basis in the finite-dimensional $T(M)$ such that the components of the metric tensor become simply

$$\boldsymbol{\mathcal{G}}(V_i, V_j) = (V_i, V_j) = \delta_{ij}.$$

Indeed, if we choose a reciprocal basis $\{\theta^i\}$ in $T^*(M)$ in such way that the relations $\theta^i(V_j) = \delta^i_j$ are satisfied, then the metric tensor will be represented in the following form

$$\boldsymbol{\mathcal{G}} = \delta_{ij}\theta^i \otimes \theta^j = \theta^1 \otimes \theta^1 + \theta^2 \otimes \theta^2 + \cdots + \theta^m \otimes \theta^m.$$

We thus conclude that in a complete Riemannian manifold, there is always a *local basis* in $T(M)$ such that the metric tensor is locally given by an identity matrix. Such a manifold is also called ***locally Euclidean*** as far as the inner product properties are concerned.

If the metric is indefinite, we can still define a kind of inner product by (5.9.9), but, this time, the so-called *norm* of a vector V defined by

$$\|V\| = \sqrt{(V,V)} = \sqrt{g_{ij}v^i v^j} = \sqrt{g^{ij}v_i v_j}$$

may be a real or an imaginary number because the term $(V,V) = g_{ij}v^i v^j$ may be positive, negative or zero. If $g_{ij}v^i v^j = 0$, then $V \neq 0$ is called a ***null vector***. However, metric tensor is still symmetric and non-degenerate. Hence, its real eigenvalues cannot be zero and it has m linearly independent orthogonal eigenvectors $V_1, V_2, \dots, V_m$ so normalised that $(V_i, V_j) = 0$ if $i \neq j$ and $|(V_{\underline{i}}, V_{\underline{i}})| = 1$, or $(V_{\underline{i}}, V_{\underline{i}}) = \pm 1$. This means that we can write the relation

$$(V_i, V_j) = g_{kl} v_i^k v_j^l = \pm\, \delta_{ij}.$$

According to this definition a null vector will be orthogonal to itself. Hence, the components of the metric tensor with respect to such a basis are prescribed by

$$\mathcal{G}(V_i, V_j) = g_{kl} v_i^k v_j^l = (V_i, V_j) = \pm\,\delta_{ij}.$$

This amount to say that there is always a basis $\{V_i\}$ of $T(M)$ with respect to which the metric tensor is designated by a diagonal matrix whose entries are either $+1$ or -1. We then choose the reciprocal basis $\{\theta^i\}$ in $T^*(M)$ to express the metric tensor in the form

$$\mathcal{G} = \theta^1 \otimes \theta^1 + \cdots + \theta^r \otimes \theta^r - \theta^{r+1} \otimes \theta^{r+1} - \cdots - \theta^m \otimes \theta^m$$

by changing the ordering of basis vectors if necessary. The number $s = m - r$ is called the ***index*** of the metric tensor. We say that the sequence $+ \cdots + \; - \cdots -$ that consists of r number of $+$ and s number of $-$ is the ***signature of this tensor***. *The signature is even if* s *is an even number and is odd if* s *is an odd number*. A manifold endowed with such a metric is named as a ***locally Minkowskian manifold*** after German mathematician Hermann Minkowski (1864-1909) who had explored such manifolds within the context of the theory of general relativity. If the metric tensor is positive definite, we evidently have $s = 0$ and $r = m$.

The metric tensor provide a means to calculate the arc length of a curve on a manifold. We know that a curve on a manifold M is a differentiable mapping $\gamma : [a, b] \to M$ and the point $p(t) \in M$ on the curve are described by $p(t) = \gamma(t)$, $a \le t \le b$. If the tangent vector of the curve at a point p is $V\big(p(t)\big)$, then the elementary arc length may be defined as

$$ds^2 = \|V(t)\|^2 dt^2 = g_{ij} v^i(t) v^j(t)\, dt^2 = g_{ij}\, dx^i dx^j$$

and the arc length of the curve between the points $p(a)$ and $p(b)$ is consequently given by

$$l = \int_a^b \|V(t)\| dt = \int_a^b \sqrt{g_{ij} v^i(t) v^j(t)}\, dt.$$

If the Riemannian manifold is complete, then l is always a positive number.

The metric tensor also helps convert covariant components of a tensor to its contravariant components and vice versa. Let us consider the covariant tensor

$$\mathcal{T} = t_{j_1 j_2 \cdots j_k} dx^{j_1} \otimes dx^{j_2} \otimes \cdots \otimes dx^{j_k}$$

that can also be written in the form

$$\mathcal{T} = g^{i_1 j_1} \cdots g^{i_k j_k} t_{j_1 \cdots j_k} f_{i_1} \otimes \cdots \otimes f_{i_k} = t^{i_1 \cdots i_k} f_{i_1} \otimes \cdots \otimes f_{i_k}$$

if we use the inverse relation (5.9.7) as $dx^i = g^{ij} f_j$. Here we define

$$t^{i_1\cdots i_k} = g^{i_1 j_1}\cdots g^{i_k j_k} t_{j_1\cdots j_k}. \tag{5.9.11}$$

The coefficients $t^{i_1\cdots i_k}$ are obtained by performing k contractions on a tensor $\mathfrak{T}(M)^{2k}_k$ formed as the product of a $\mathfrak{T}(M)^0_k$ tensor and k times of a $\mathfrak{T}(M)^2_0$ tensor which is the inverse metric tensor. Hence, the quotient rule [*see p.* 212] states that they are nothing but the *contravariant components* of the same tensor $\mathcal{T}$. Thus the components of the inverse metric tensor prove to be useful in *raising* the indices in the tensorial components. Similarly, we can show that the components of the metric tensor can be instrumental in *lowering* indices in the tensorial components. Indeed, if a tensor $\mathcal{T}$ is given in the form

$$\mathcal{T} = t^{j_1\cdots j_k}\frac{\partial}{\partial x^{j_1}}\otimes\cdots\otimes\frac{\partial}{\partial x^{j_k}}$$

then inserting $\partial_i = g_{ij}\, e^j$ that follows from (5.9.4) into the above expression we find that

$$\mathcal{T} = g_{i_1 j_1}\cdots g_{i_k j_k} t^{j_1\cdots j_k}\, e^{i_1}\otimes\cdots\otimes e^{i_k} = t_{i_1\cdots i_k} e^{i_1}\otimes\cdots\otimes e^{i_k}$$

where the covariantly transforming coefficients

$$t_{i_1\cdots i_k} = g_{i_1 j_1}\cdots g_{i_k j_k} t^{j_1\cdots j_k} \tag{5.9.12}$$

are called the *covariant components* of the tensor $\mathcal{T}$. It is seen that the existence of the metric tensor effectively abolishes the distinction between covariant and contravariant tensors and provides a natural transition between components of such kind of tensors. It is clear that this procedure is applicable to any index of mixed components of a tensor.

Suppose that a tensor is defined as a contraction of a product of two tensors. In terms of components we can write for example

$$\begin{aligned} t_{i i_1\cdots i_k}\tau^{i j_1\cdots j_l} &= g_{ij}g^{ik}t^{j}{}_{i_1\cdots i_k}\tau_k{}^{j_1\cdots j_l} = \delta^k_j\, t^{j}{}_{i_1\cdots i_k}\tau_k{}^{j_1\cdots j_l} \\ &= t^{j}{}_{i_1\cdots i_k}\tau_j{}^{j_1\cdots j_l}. \end{aligned}$$

We thus reach to the conclusion that such a tensor does not change if we arbitrarily lower one and raise the other of contracted indices.

If we can find a form $\Omega\in\Lambda^m(M)$ on an m-dimensional manifold M such that $\Omega\neq 0$ at every point $p\in M$, then we say that M is an ***orientable manifold*** and Ω is a ***volume form***. In that case, it is clear that one is able to write $\Omega = f(\mathbf{x})\,dx^1\wedge\cdots\wedge dx^m$ where we must have $f\neq 0$ everywhere on M. When M is a *complete Riemannian manifold*, we get $g = \det[g_{ij}] > 0$. Under a coordinate transformation $y^i = y^i(x^j)$, we readily obtain in general

$$\det[g'_{ij}(\mathbf{y})] = \det\Big[\frac{\partial x^k}{\partial y^i}\frac{\partial x^l}{\partial y^j}g_{kl}(\mathbf{x})\Big] = \frac{\det[g_{kl}(\mathbf{x})]}{J^2}$$

where $J = \det[\partial y^i/\partial x^j] \neq 0$. Let us now define $g = \big|\det[g_{ij}]\big| > 0$ so that we can write $g'(\mathbf{y}) = g(\mathbf{x})/J^2$. We now introduce a volume form as follows

$$\mu(\mathbf{x}) = \sqrt{g}\, dx^1 \wedge \cdots \wedge dx^m. \tag{5.9.13}$$

If the Riemannian manifold is not complete, then $\det[g_{ij}]$ may be positive or negative although it cannot be zero because we have assumed that the metric tensor is non-degenerate. In that case, we always have $g = \big|\det[g_{ij}]\big| > 0$ in (5.9.13). Such a g has obviously the same transformation rule as that of given above. The form $\mu \in \Lambda^m(M)$ will be called the ***Riemannian volume form***. Under a coordinate transformation $y^i = y^i(x^j)$, this form is transformed in the following manner

$$\begin{aligned}
\mu(\mathbf{y}) &= \sqrt{g'}\, dy^1 \wedge \cdots \wedge dy^m \\
&= \frac{\sqrt{g}}{|J|}\frac{\partial y^1}{\partial x^{i_1}}\cdots\frac{\partial y^m}{\partial x^{i_m}}\, dx^{i_1} \wedge \cdots \wedge dx^{i_m} \\
&= \frac{\sqrt{g}}{|J|}\, e^{i_1\cdots i_m}\frac{\partial y^1}{\partial x^{i_1}}\cdots\frac{\partial y^m}{\partial x^{i_m}}\, dx^1 \wedge \cdots \wedge dx^m \\
&= \sqrt{g}\,\frac{J}{|J|}\, dx^1 \wedge \cdots \wedge dx^m \\
&= (\operatorname{sgn} J)\sqrt{g}\, dx^1 \wedge \cdots \wedge dx^m \\
&= (\operatorname{sgn} J)\, \mu(\mathbf{x})
\end{aligned}$$

where $\operatorname{sgn} J = J/|J|$ is $+1$ if $J > 0$ and -1 if $J < 0$. Clearly, this volume form remains invariant under coordinate transformations if $J > 0$. The form (5.9.13) can also be written as

$$\begin{aligned}
\mu &= \frac{1}{m!}\sqrt{g}\, e_{i_1\cdots i_m}\, dx^{i_1} \wedge \cdots \wedge dx^{i_m} \\
&= \frac{1}{m!}\, \epsilon_{i_1\cdots i_m}\, dx^{i_1} \wedge \cdots \wedge dx^{i_m}
\end{aligned} \tag{5.9.14}$$

where we defined the ***covariant Levi-Civita permutation tensor*** by the relation

$$\epsilon_{i_1\cdots i_m} = \sqrt{g}\, e_{i_1\cdots i_m}. \tag{5.9.15}$$

On the other hand, the expression

$$\begin{aligned} e^{j_1\cdots j_m}\mu &= \frac{1}{m!}\sqrt{g}\,\delta^{j_1\cdots j_m}_{i_1\cdots i_m}\,dx^{i_1}\wedge\cdots\wedge dx^{i_m} \\ &= \sqrt{g}\;dx^{[j_1}\wedge\cdots\wedge dx^{j_m]} \\ &= \sqrt{g}\;dx^{j_1}\wedge\cdots\wedge dx^{j_m} \end{aligned}$$

yields

$$dx^{i_1}\wedge\cdots\wedge dx^{i_m} = \frac{e^{i_1\cdots i_m}}{\sqrt{g}}\mu = \epsilon^{i_1\cdots i_m}\mu$$

where the ***contravariant Levi-Civita permutation tensor*** is defined by

$$\epsilon^{i_1\cdots i_m} = \frac{e^{i_1\cdots i_m}}{\sqrt{g}}. \tag{5.9.16}$$

In order to identify the tensorial character of these quantities let us start with the relations

$$\begin{aligned} e_{i_1i_2\cdots i_n}(\boldsymbol{y})J^{-1} &= e_{j_1j_2\cdots j_n}(\boldsymbol{x})\frac{\partial x^{j_1}}{\partial y^{i_1}}\frac{\partial x^{j_2}}{\partial y^{i_2}}\cdots\frac{\partial x^{j_n}}{\partial y^{i_n}} \\ e^{i_1i_2\cdots i_n}(\boldsymbol{y})J &= e^{j_1j_2\cdots j_n}(\boldsymbol{x})\frac{\partial y^{i_1}}{\partial x^{j_1}}\frac{\partial y^{i_2}}{\partial x^{j_2}}\cdots\frac{\partial y^{i_n}}{\partial x^{j_n}} \end{aligned}$$

from which we deduce the transformation rules of Levi-Civita symbols as

$$\begin{aligned} e_{i_1i_2\cdots i_n}(\boldsymbol{y}) &= J\frac{\partial x^{j_1}}{\partial y^{i_1}}\frac{\partial x^{j_2}}{\partial y^{i_2}}\cdots\frac{\partial x^{j_n}}{\partial y^{i_n}}e_{j_1j_2\cdots j_n}(\boldsymbol{x}), \\ e^{i_1i_2\cdots i_n}(\boldsymbol{y}) &= J^{-1}\frac{\partial y^{i_1}}{\partial x^{j_1}}\frac{\partial y^{i_2}}{\partial x^{j_2}}\cdots\frac{\partial y^{i_n}}{\partial x^{j_n}}e^{j_1j_2\cdots j_n}(\boldsymbol{x}). \end{aligned}$$

This means that $e_{i_1i_2\cdots i_n}$ and $e^{i_1i_2\cdots i_n}$ are actually ***tensor densities*** because the transformation rule depends on the Jacobian of the coordinate transformation. Since we can write $J = \operatorname{sgn} J|J|$, Levi-Civita tensors will satisfy

$$\begin{aligned} \epsilon_{i_1i_2\cdots i_n}(\boldsymbol{y}) &= \operatorname{sgn} J\frac{\partial x^{j_1}}{\partial y^{i_1}}\frac{\partial x^{j_2}}{\partial y^{i_2}}\cdots\frac{\partial x^{j_n}}{\partial y^{i_n}}\epsilon_{j_1j_2\cdots j_n}(\boldsymbol{x}), \\ \epsilon^{i_1i_2\cdots i_n}(\boldsymbol{y}) &= \operatorname{sgn} J\frac{\partial y^{i_1}}{\partial x^{j_1}}\frac{\partial y^{i_2}}{\partial x^{j_2}}\cdots\frac{\partial y^{i_n}}{\partial x^{j_n}}\epsilon^{j_1j_2\cdots j_n}(\boldsymbol{x}). \end{aligned}$$

So *Levi-Civita tensors* $\epsilon_{i_1i_2\cdots i_n}$ *and* $\epsilon^{i_1i_2\cdots i_n}$ *are* ***pseudotensors*** because the transformation rule changes sign depending on the Jacobian of the coordinate transformation. They behave like absolute tensors if $J > 0$. In order to understand how they are related, let us consider the relation

$$\begin{aligned} g^{i_1j_1}\cdots g^{i_nj_n}\epsilon_{j_1\cdots j_n} &= \sqrt{g}\,e_{j_1\cdots j_n}g^{i_1j_1}\cdots g^{i_nj_n} = \sqrt{g}\det[g^{ij}]e^{i_1\cdots i_n} \\ &= \frac{g}{\det[g_{ij}]}\epsilon^{i_1i_2\cdots i_n} = (\operatorname{sgn}\det[g_{ij}])\,\epsilon^{i_1i_2\cdots i_n} \end{aligned}$$

Similarly, we find that

$$g_{i_1j_1}\cdots g_{i_nj_n}\epsilon^{j_1\cdots j_n} = (\operatorname{sgn}\det[g_{ij}])\,\epsilon_{i_1\cdots i_n}.$$

Hence, they represent covariant and contravariant components of the same tensor if $\det[g_{ij}] > 0$. We also easily observe that we get the absolute tensor

$$\delta^{i_1\cdots i_m}_{j_1\cdots j_m} = e^{i_1\cdots i_m}e_{j_1\cdots j_m} = \epsilon^{i_1\cdots i_m}\epsilon_{j_1\cdots j_m}.$$

We can now fulfil the task of the top down generation of ordered bases for the exterior algebra $\Lambda(M)$ just like we have done in Sec. 5.5 by using the volume form defined by (5.9.14). Let us introduce similarly the ordered forms

$$\begin{aligned} \mu_{i_ki_{k-1}\cdots i_1} &= (\mathbf{i}_{\partial_{i_k}}\circ\mathbf{i}_{\partial_{i_{k-1}}}\circ\cdots\circ\mathbf{i}_{\partial_{i_1}})(\mu) \\ &= \mathbf{i}_{\partial_{i_k}}(\mu_{i_{k-1}\cdots i_1}) \\ &= \frac{1}{(m-k)!}\,\epsilon_{i_1\cdots i_ki_{k+1}\cdots i_m}dx^{i_{k+1}}\wedge\cdots\wedge dx^{i_m}\in\Lambda^{m-k}(M) \end{aligned} \tag{5.9.17}$$

where $1\le k\le m$. Following the path we have pursued in obtaining the relation (5.5.12), we easily deduce from (5.9.17) that

$$dx^{i_{k+1}}\wedge\cdots\wedge dx^{i_m} = \frac{1}{k!}\,\epsilon^{i_1\cdots i_ki_{k+1}\cdots i_m}\mu_{i_k\cdots i_1}. \tag{5.9.18}$$

It is straightforward to see that all expressions appearing between (5.5.13) and (5.5.18) remain without change if we replace μ by (5.9.14) and Levi-Civita symbols by Levi-Civita tensors. In like fashion, we can verify at once that the forms $\mu_{i_k\cdots i_1}$ defined in (5.9.17) constitute a basis of the module $\Lambda^{m-k}(M)$. Thus a form $\omega\in\Lambda^{m-k}(M)$ may be written again as

$$\omega = \frac{1}{k!}\,\omega^{i_1\cdots i_k}\mu_{i_k\cdots i_1}.$$

But, the exterior derivative of this form is now rather different from what is given in (5.8.5). This derivative is of course

$$d\omega = \frac{1}{k!}(\,\omega^{i_1\cdots i_k}{}_{,i}\,dx^i\wedge\mu_{i_k\cdots i_1} + \omega^{i_1\cdots i_k}d\mu_{i_k\cdots i_1}).$$

On the other hand, an explicit calculation leads to

$$
\begin{aligned}
d\mu_{i_k\cdots i_1}
&= \frac{1}{(m-k)!}\, e_{i_1\cdots i_k i_{k+1}\cdots i_m}(\sqrt{g})_{,i}\, dx^i \wedge dx^{i_{k+1}} \wedge \cdots \wedge dx^{i_m} \\
&= \frac{1}{(m-k)!}\frac{(\sqrt{g})_{,i}}{\sqrt{g}}\, \epsilon_{i_1\cdots i_k i_{k+1}\cdots i_m} dx^i \wedge dx^{i_{k+1}} \wedge \cdots \wedge dx^{i_m} \\
&= \frac{1}{(m-k)!}\frac{(\sqrt{g})_{,i}}{\sqrt{g}}\, \epsilon_{i_1\cdots i_k i_{k+1}\cdots i_m}\, \frac{1}{(k-1)!}\epsilon^{j_1\cdots j_{k-1} i i_{k+1}\cdots i_m}\mu_{j_{k-1}\cdots j_1} \\
&= \frac{1}{(k-1)!}\frac{1}{(m-k)!}\frac{(\sqrt{g})_{,i}}{\sqrt{g}}\delta^{j_1\cdots j_{k-1} i i_{k+1}\cdots i_m}_{i_1\cdots\, i_{k-1} i_k i_{k+1}\cdots i_m}\mu_{j_{k-1}\cdots j_1} \\
&= \frac{1}{(k-1)!}\frac{(\sqrt{g})_{,i}}{\sqrt{g}}\delta^{j_1\cdots j_{k-1} i}_{i_1\cdots i_{k-1} i_k}\mu_{j_{k-1}\cdots j_1} = k\frac{(\sqrt{g})_{,i}}{\sqrt{g}}\delta^{i}_{[i_k}\mu_{i_{k-1}\cdots i_2 i_1]}.
\end{aligned}
$$

Hence, according to (5.5.15) and due to the complete antisymmetry of functions $\omega^{i_1\cdots i_k}$ we obtain

$$
\begin{aligned}
d\omega &= \frac{1}{(k-1)!}\Big(\omega^{i_1\cdots i_k}{}_{,i} + \frac{(\sqrt{g})_{,i}}{\sqrt{g}}\,\omega^{i_1\cdots i_k}\Big)\delta^{i}_{[i_k}\mu_{i_{k-1}\cdots i_1]} \\
&= \frac{1}{(k-1)!}\frac{1}{\sqrt{g}}(\sqrt{g}\,\omega^{i_1\cdots i_k})_{,i}\delta^{i}_{[i_k}\mu_{i_{k-1}\cdots i_1]} \\
&= \frac{1}{(k-1)!}\frac{1}{\sqrt{g}}(\sqrt{g}\,\omega^{i_1\cdots i_{k-1} i})_{,i}\mu_{i_{k-1}\cdots i_1} = \frac{1}{(k-1)!}\,\omega^{i_1\cdots i_{k-1} i}{}_{;i}\,\mu_{i_{k-1}\cdots i_1}
\end{aligned}
$$

where we introduced the definition

$$
\omega^{i_1\cdots i_{k-1} i}{}_{;i} = \frac{1}{\sqrt{g}}(\sqrt{g}\,\omega^{i_1\cdots i_{k-1} i})_{,i} \tag{5.9.19}
$$

A semicolon in front of an index denotes the *covariant derivative* with respect to a variable depicted by this index. We discuss the concept of covariant derivative in Chapter VII in detail. Here we just confine ourselves to indicate that although the quantities $\omega^{i_1\cdots i_{k-1} i}{}_{,i}$ are not generally components of a tensor, the coefficients $\omega^{i_1\cdots i_{k-1} i}{}_{;i}$ of the form $d\omega$ are components of a $(k-1)$-contravariant tensor. We now suppose that a form

$$
\omega = \frac{1}{k!}\,\omega_{i_1\cdots i_k}(\mathbf{x})\, dx^{i_1} \wedge \cdots \wedge dx^{i_k} \in \Lambda^k(M)
$$

is given on an orientable Riemannian manifold. The ***Hodge dual*** or just simply the ***dual*** of this form is defined by

$$*\omega = \frac{1}{k!}\,\omega^{i_1\cdots i_k}(\mathbf{x})\,\mu_{i_k\cdots i_1} \in \Lambda^{m-k}(M) \tag{5.9.20}$$

where *contravariant components* are of course now prescribed by

$$\omega^{i_1\cdots i_k} = g^{i_1j_1}\cdots g^{i_kj_k}\omega_{j_1\cdots j_k}. \tag{5.9.21}$$

The operator $* : \Lambda^k(M) \to \Lambda^{m-k}(M)$ is known as the ***Hodge star operator***. The form (5.9.20) is expressible in the natural basis as

$$\begin{aligned} *\omega &= \frac{1}{k!}\frac{1}{(m-k)!}\,\epsilon_{i_1\cdots i_k i_{k+1}\cdots i_m}\omega^{i_1\cdots i_k}\,dx^{i_{k+1}}\wedge\cdots\wedge dx^{i_m} \\ &= \frac{1}{(m-k)!}\,*\omega_{i_{k+1}\cdots i_m}dx^{i_{k+1}}\wedge\cdots\wedge dx^{i_m} \end{aligned} \tag{5.9.22}$$

where we have defined

$$*\omega_{i_{k+1}\cdots i_m} = \frac{1}{k!}\,\epsilon_{i_1\cdots i_k i_{k+1}\cdots i_m}\omega^{i_1\cdots i_k}. \tag{5.9.23}$$

Hodge star operator is evidently a linear operator on the graded exterior algebra. On applying $*$ operator successively, it follows from (5.9.22) that

$$\begin{aligned} **\omega &= \frac{1}{(m-k)!}\,*\omega^{i_{k+1}\cdots i_m}\mu_{i_m\cdots i_{k+1}} \\ &= \frac{1}{(m-k)!}\frac{1}{k!}\epsilon^{i_1\cdots i_k i_{k+1}\cdots i_m}\omega_{i_1\cdots i_k}\mu_{i_m\cdots i_{k+1}} \\ &= \frac{1}{(m-k)!}\frac{1}{k!}(-1)^{k(m-k)}\epsilon^{i_{k+1}\cdots i_m i_1\cdots i_k}\omega_{i_1\cdots i_k}\mu_{i_m\cdots i_{k+1}} \\ &= (-1)^{k(m-k)}\frac{1}{k!}\,\omega_{i_1\cdots i_k}dx^{i_1}\wedge\cdots\wedge dx^{i_k} = (-1)^{k(m-k)}\omega. \end{aligned}$$

In order to reach to this result, we have raised and lowered the indices appropriately utilising the metric tensor. Consequently, if applied on k-forms, the inverse of the operator $*$ becomes

$$*^{-1} = (-1)^{k(m-k)}* = (-1)^{k(m-1)}* \tag{5.9.24}$$

because $k^2 - k$ is always an even number. It easily verified that the dual of the volume form (5.9.14) is

$$*\mu = \frac{1}{m!}\,\epsilon^{i_1\cdots i_m}\mu_{i_m\cdots i_1} = \frac{1}{m!}\,\epsilon^{i_1\cdots i_m}\epsilon_{i_1\cdots i_m} = 1. \tag{5.9.25}$$

If we take $k = m$, then (5.9.24) yields $*^{-1} = *$ and we obtain

$$*1 = **\mu = \mu. \tag{5.9.26}$$

Let us now consider the forms $\omega, \sigma \in \Lambda^k(M)$ given by

$$\omega = \frac{1}{k!}\,\omega_{i_1\cdots i_k}dx^{i_1}\wedge\cdots\wedge dx^{i_k},$$
$$\sigma = \frac{1}{k!}\,\sigma_{i_1\cdots i_k}dx^{i_1}\wedge\cdots\wedge dx^{i_k}.$$

In this situation, we have $\omega \wedge *\sigma \in \Lambda^m(M)$. If we evaluate this form explicitly, we obtain

$$\begin{aligned}\omega \wedge *\sigma &= \Big(\frac{1}{k!}\Big)^2 \omega_{i_1\cdots i_k}\sigma^{j_1\cdots j_k}dx^{i_1}\wedge\cdots\wedge dx^{i_k}\wedge \mu_{j_k\cdots j_1}\\ &= \Big(\frac{1}{k!}\Big)^2 \omega_{i_1\cdots i_k}\sigma^{j_1\cdots j_k}\delta^{i_1\cdots i_k}_{j_1\cdots j_k}\mu = \frac{1}{k!}\,\omega_{i_1\cdots i_k}\sigma^{[i_1\cdots i_k]}\mu\\ &= \frac{1}{k!}\,\omega_{i_1\cdots i_k}\sigma^{i_1\cdots i_k}\mu.\end{aligned}$$

On the other hand, since the same expression may be directly transformed into the form $\omega \wedge *\sigma = \dfrac{1}{k!}\,\sigma_{i_1\cdots i_k}\omega^{i_1\cdots i_k}\mu$, we arrive at the identity

$$\omega \wedge *\sigma = \sigma \wedge *\omega. \tag{5.9.27}$$

For a form $\omega \in \Lambda^k(M)$, we similarly find

$$\omega \wedge *\omega = \frac{1}{k!}\,\omega_{i_1\cdots i_k}\omega^{i_1\cdots i_k}\mu.$$

Next, we take a form $\omega \in \Lambda^k(M)$ into account and calculate the exterior derivative of its dual. Recalling the definition (5.9.19), we obtain

$$\begin{aligned}d(*\omega) &= \frac{1}{k!}\,d\big(\omega^{i_1\cdots i_k}\mu_{i_k\cdots i_1}\big) = \frac{1}{(k-1)!}\,\omega^{i_1\cdots i_{k-1}i}{}_{;i}\,\mu_{i_{k-1}\cdots i_1} \\ &= \frac{1}{(k-1)!}\frac{1}{(m-k+1)!}\omega^{i_1\cdots i_{k-1}i}{}_{;i}\,\epsilon_{i_1\cdots i_{k-1}i_k\cdots i_m}dx^{i_k}\wedge\cdots\wedge dx^{i_m}.\end{aligned} \tag{5.9.28}$$

It is clear that $d(*\omega) \in \Lambda^{m-k+1}(M)$. An operator $\delta : \Lambda^k(M) \to \Lambda^{k-1}(M)$ will now be defined as follows

$$\delta\omega = (-1)^{m(k+1)+1}{*d(*\omega)} = (-1)^k *^{-1}d(*\omega) \tag{5.9.29}$$

where we adopted the convention $\delta f = 0$ for $f \in \Lambda^0(M)$. Since δ is the composition of linear operators, it is a linear operator on $\mathbb{R}$. According to (5.9.29) we can write $\delta = \ \pm * d *$. If m is even or if m and k are odd, the sign is $-$, if m is odd and k is even, the sign is $+$. (5.9.29) is then expressed as

$$\delta\omega = \frac{(-1)^{m(k+1)+1}}{(m-k+1)!}\Big[\frac{1}{(k-1)!}\epsilon^{i_1\cdots i_{k-1}i_k\cdots i_m}\omega_{i_1\cdots i_{k-1}i;}{}^{i}\Big]\mu_{i_m\cdots i_k}$$

where we naturally define

$$\omega_{i_1\cdots i_{k-1}i;}{}^{i} = g_{i_1j_1}\cdots g_{i_{k-1}j_{k-1}}\omega^{j_1\cdots j_{k-1}i}{}_{;i}\,.$$

Since we can write

$$\begin{aligned}\frac{1}{(m-k+1)!}\epsilon^{i_1\cdots i_{k-1}i_k\cdots i_m}\mu_{i_m\cdots i_k} &= \frac{(-1)^{(k-1)(m-k+1)}}{(m-k+1)!}\epsilon^{i_k\cdots i_m i_1\cdots i_{k-1}}\mu_{i_m\cdots i_k}\\ &= (-1)^{(k-1)(m-k+1)}dx^{i_1}\wedge\cdots\wedge dx^{i_{k-1}}\end{aligned}$$

on using (5.9.18), we finally reach to the result

$$\delta\omega = \frac{(-1)^k}{(k-1)!}\,\omega_{i_1\cdots i_{k-1}i;}{}^{i}\,dx^{i_1}\wedge\cdots\wedge dx^{i_{k-1}} \tag{5.9.30}$$

after having omitted even numbers in the exponent $(k-1)(m-k+1)+m(k+1)+1$ of -1 in the above expression. Thus we can regard δ as a sort of *divergence operator*. Hence, the form $(-1)^k\delta\omega\in\Lambda^{k-1}(M)$ will be called the *divergence* of the form $\omega\in\Lambda^k(M)$. We shall call δ as the ***co-differential operator***. Various properties of this operator can easily be identified:

(i). *We have* $\delta\circ\delta\omega=\delta^2\omega=\pm *^{-1}d**^{-1}d*\omega=\pm *^{-1}d^2*\omega=0$ *for all* $\omega\in\Lambda(M)$ *so that we obtain* $\delta^2=0$.

(ii). *If* $\omega\in\Lambda^k(M)$, *we have* $*(\delta\omega)=(-1)^k\,d(*\omega)$.

Indeed (5.9.30) and (5.9.17) yield

$$\begin{aligned}*(\delta\omega) &= \frac{(-1)^k}{(k-1)!}\,\omega^{i_1\cdots i_{k-1}i}{}_{;i}\mu_{i_{k-1}\cdots i_1}\\ &= \frac{(-1)^k}{(k-1)!(m-k+1)!}\omega^{i_1\cdots i_{k-1}i}{}_{;i}\,\epsilon_{i_1\cdots i_{k-1}i_k\cdots i_m}dx^{i_k}\wedge\cdots\wedge dx^{i_m}.\end{aligned}$$

We then obtain the desired result in view of (5.9.28). We can also arrive at this result directly from the definition of the operator δ. Let us consider a form $\omega\in\Lambda^{k+1}(M)$. We find that

$$*\delta\omega = (-1)^{m(k+2)+1}**d*\omega = (-1)^{mk+1+k(m-1)}d*\omega = (-1)^{k+1}d*\omega.$$

Since the number $1\le k\le m$ is arbitrary, when we apply this operator to the form $\omega \in \Lambda^k(M)$, we get $*\delta = (-1)^k\, d*$.

(iii). *If* $\omega \in \Lambda^k(M)$, *we have* $\delta(*\omega) = (-1)^{k+1}{*d}(\omega)$.

In fact, discarding even numbers in the exponent of -1 we find

$$\begin{aligned}\delta(*\omega) &= (-1)^{m(m-k+1)+1}{*d}{*}{*}\omega = (-1)^{-mk+1+k(m-1)}{*d}(\omega)\\ &= (-1)^{-k+1}{*d}(\omega) = (-1)^{k+1}{*d}(\omega).\end{aligned}$$

Hence, we get $\delta* = (-1)^{k+1}{*d}$ when applied to the form $\omega \in \Lambda^k(M)$.

(iv). *The relations* $*\delta d = d\delta*$ *and* $*d\delta = \delta d*$ *are valid*:

Let us take $\omega \in \Lambda^k(M)$. By direct calculations, we find

$$\begin{aligned}*\delta d(\omega) &= (-1)^{m(k+2)+1}{*}{*}d{*}d\omega = (-1)^{k+1}d{*}d\omega,\\ d\delta(*\omega) &= (-1)^{m(m-k+1)+1}d{*}d{*}{*}\omega = (-1)^{k+1}d{*}d\omega.\end{aligned}$$

We thus conclude that $*\delta d(\omega) = d\delta*(\omega)$ for all $\omega \in \Lambda(M)$. Similarly, we obtain

$$\begin{aligned}*d\delta(\omega) &= (-1)^{m(k+1)+1}{*d}{*d}{*}\omega,\\ \delta d*(\omega) &= (-1)^{m(m-k+2)+1}{*d}{*d}{*}\omega = (-1)^{m(k+1)+1}{*d}{*d}{*}\omega\end{aligned}$$

where $\omega \in \Lambda^k(M)$. This implies that $*d\delta(\omega) = \delta d*(\omega)$ for all $\omega \in \Lambda(M)$ since it is valid for all degrees.

(v). *The relations* $\delta{*}d = d{*}\delta = 0$ *are valid.*

If $\omega \in \Lambda^k(M)$, we get

$$\begin{aligned}\delta{*}d(\omega) &= (-1)^{m(k+2)+1}{*d}{*}{*}d\omega = (-1)^{k+1}{*d}^2(\omega) = 0,\\ d{*}\delta(\omega) &= (-1)^{m(k+1)+1}d{*}{*}d{*}\omega = (-1)^{m-k+1}d^2(*\omega) = 0\end{aligned}$$

so that $\delta{*}d(\omega) = d{*}\delta(\omega) = 0$ for all $\omega \in \Lambda(M)$.

For a form $\omega \in \Lambda^1(M)$ we obtain $*(\delta\omega) = -\,\omega^i{}_{;i}\mu$ and $\delta\omega \in \Lambda^0(M)$ is given by $\delta\omega = -\,\omega^i{}_{;i}$. Let us define the form $\omega = \omega_i dx^i \in \Lambda^1(M)$ associated with a vector field $V = v^i\,\partial_i \in T(M)$ by taking $\omega_i = g_{ij}v^j$. Then, we naturally find $\omega^i = g^{ij}\omega_j = v^i$ so that we are able to write

$$v^i{}_{;i} = \operatorname{div} V = -\,\delta\omega$$

We now define an operator $\varDelta : \Lambda^k(M) \to \Lambda^k(M)$ that is linear on $\mathbb{R}$ by the following relation

$$\varDelta = \delta d + d\delta. \tag{5.9.31}$$

$\varDelta$ is called the ***Laplace-de Rham operator*** after Laplace and Swiss mathematician Georges de Rham (1903-1990). If we take a function $f \in \Lambda^0(M)$ into account, application of this operator yields

$$\varDelta f = \delta d f + d\delta f = \delta d f = \nabla^2 f \tag{5.9.32}$$

where $\nabla^2 = \delta d : \Lambda^0(M) \to \Lambda^0(M)$ is called the ***Laplace-Beltrami operator*** [Italian mathematician Eugenio Beltrami (1835-1900)]. Since we write $df = f_{,i}\, dx^i$, according to (5.9.30) and (5.9.19) we get

$$\nabla^2 f = -(f_{,i})_;{}^i = -\frac{1}{\sqrt{g}}(\sqrt{g}\, g^{ij} f_{,j})_{,i}. \tag{5.9.33}$$

In Cartesian coordinates, this expression takes the form

$$\nabla^2 f = -\sum_{i=1}^{m} \frac{\partial^2 f}{\partial {x^i}^2}.$$

We have to note that this operator is differing only in sign from the familiar one encountered in partial differential equations. The Laplace-Beltrami operator $\varDelta$ possesses the following properties that can easily be verified:

(i). *One has* $\varDelta = (d+\delta)^2$.

$$\varDelta = (d+\delta)\circ(d+\delta) = d^2 + d\delta + \delta d + \delta^2 = d\delta + \delta d.$$

(ii). *One has* $d\varDelta = \varDelta d = d\delta d$.

$$d\varDelta = d\delta d\ + d^2\delta = d\delta d, \quad \varDelta d = \delta d^2 + d\delta d = d\delta d.$$

(iii). *One has* $\delta\varDelta = \varDelta\delta = \delta d\delta$.

$$\delta\varDelta = \delta^2 d + \delta d\delta = \delta d\delta, \quad \varDelta\delta = \delta d\delta + d\delta^2 = \delta d\delta.$$

(iv). *One has* $*\varDelta = \varDelta *$.

$$\begin{aligned} *\varDelta &= *(\delta d + d\delta) = *\delta d + *d\delta = d\delta * + \delta d * = (d\delta + \delta d)* \\ &= \varDelta *. \end{aligned}$$

A form $\omega \in \Lambda^k(M)$ satisfying the equation $\varDelta\omega = 0$ will be called a ***harmonic form***. The set

$$\mathrm{H}^k(M) = \{\omega \in \Lambda^k(M) : \varDelta\omega = 0\} = \mathcal{N}(\varDelta)$$

is a subspace of $\Lambda^k(M)$ on $\mathbb{R}$.

Example 5.9.1. Let us take $M = \mathbb{R}^3$ and we introduce the spherical coordinates (r,θ,ϕ) connected to Cartesian coordinates by the relations

$$x = r\sin\theta\cos\phi,\ y = r\sin\theta\sin\phi,\ z = r\cos\theta,\ 0 \le \theta \le \pi, 0 \le \phi \le 2\pi.$$

Since the arc element is determined by

$$ds^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta \, d\phi^2,$$

the components of the metric tensor and its inverse are given by

$$g_{rr} = 1, \ g_{\theta\theta} = r^2, \quad g_{\phi\phi} = r^2 \sin^2\theta;$$
$$g^{rr} = 1, \ g^{\theta\theta} = 1/r^2, \ g^{\phi\phi} = 1/r^2 \sin^2\theta.$$

Thus the volume form becomes

$$\mu = r^2 \sin\theta \, dr \wedge d\theta \wedge d\phi$$

whence we produce the basis for $\Lambda^2(\mathbb{R}^3)$

$$\mu_r = \mathbf{i}_{\partial_r}(\mu) = r^2 \sin\theta \, d\theta \wedge d\phi, \ \ \mu_\theta = \mathbf{i}_{\partial_\theta}(\mu) = -r^2 \sin\theta \, dr \wedge d\phi,$$
$$\mu_\phi = \mathbf{i}_{\partial_\phi}(\mu) = r^2 \sin\theta \, dr \wedge d\theta$$

We can now represent a form $\omega \in \Lambda^1(\mathbb{R}^3)$ by

$$\omega = \omega_r \, dr + \omega_\theta \, d\theta + \omega_\phi \, d\phi$$

where coefficients are functions of variables r, θ, ϕ. The Hodge dual of the form ω will be given by

$$*\omega = \omega^r \mu_r + \omega^\theta \mu_\theta + \omega^\phi \mu_\phi$$

where the coefficients are calculated as follows

$$\omega^r = \omega_r, \ \ \omega^\theta = \frac{1}{r^2} \omega_\theta, \ \ \omega^\phi = \frac{1}{r^2 \sin^2\theta} \omega_\phi.$$

Therefore, we get

$$*\omega = \frac{\omega_\phi}{\sin\theta} dr \wedge d\theta - \omega_\theta \sin\theta \, dr \wedge d\phi + \omega_r r^2 \sin\theta \, d\theta \wedge d\phi.$$

We readily see that we obtain

$$\omega \wedge *\omega = \Big(\omega_r^2 + \frac{1}{r^2} \omega_\theta^2 + \frac{1}{r^2 \sin^2\theta} \omega_\phi^2\Big)\mu.$$

Let us now evaluate the exterior derivatives of the forms ω and $*\omega$. We find

$$d\omega = (\omega_{\theta,r} - \omega_{r,\theta}) dr \wedge d\theta + (\omega_{\phi,r} - \omega_{r,\phi}) dr \wedge d\phi + (\omega_{\phi,\theta} - \omega_{\theta,\phi}) d\theta \wedge d\phi$$
$$d{*\omega} = \big[(\omega_r r^2 \sin\theta)_{,r} + (\omega_\theta \sin\theta)_{,\theta} + \big(\frac{\omega_\phi}{\sin\theta}\big)_{,\phi}\big] dr \wedge d\theta \wedge d\phi$$
$$= \Big(\omega_{r,r} + \frac{2}{r} \omega_r + \frac{1}{r^2} \omega_{\theta,\theta} + \frac{\cos\theta}{r^2 \sin\theta} \omega_\theta + \frac{1}{r^2 \sin^2\theta} \omega_{\phi,\phi}\Big)\mu.$$

Since $*\mu = 1$, $m = 3$, $k = 1$, the co-differential of ω becomes

$$\delta\omega = -*d*\omega = -\Big(\frac{\partial\omega_r}{\partial r} + \frac{2}{r}\,\omega_r + \frac{1}{r^2}\frac{\partial\omega_\theta}{\partial\theta} + \frac{\cos\theta}{r^2\sin\theta}\,\omega_\theta + \frac{1}{r^2\sin^2\theta}\frac{\partial\omega_\phi}{\partial\phi}\Big).$$

Let us now consider the function $f \in \Lambda^0(\mathbb{R}^3)$. Its differential is the 1-form

$$df = f_{,r}\,dr + f_{,\theta}\,d\theta + f_{,\phi}\,d\phi.$$

Hence, if we write $\omega_r = f_{,r}$, $\omega_\theta = f_{,\theta}$, $\omega_\phi = f_{,\phi}$ the above relation leads to

$$\nabla^2 f = \delta df = -\Big(\frac{\partial^2 f}{\partial r^2} + \frac{2}{r}\frac{\partial f}{\partial r} + \frac{1}{r^2}\frac{\partial^2 f}{\partial\theta^2} + \frac{\cos\theta}{r^2\sin\theta}\frac{\partial f}{\partial\theta} + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 f}{\partial\phi^2}\Big)$$

that is the known result apart from a sign difference. ■

5.10. CLOSED IDEALS

Let $\mathcal{I}$ be an ideal of the exterior algebra $\Lambda(M)$. $\mathcal{I}$ is called a ***closed ideal*** if $d\omega \in \mathcal{I}$ for all forms $\omega \in \mathcal{I}$. This situation is symbolically expressed as $d\mathcal{I} \subset \mathcal{I}$. Sometimes, a closed ideal is also named as a ***differential ideal***. Let us consider the ideal $\mathcal{I}(\omega_1, \dots, \omega_r)$ generated by forms $\omega_1, \dots, \omega_r \in \Lambda(M)$. If the ideal $\mathcal{I}$ is not a closed one, then we can construct an *extended ideal* $\overline{\mathcal{I}}(\omega_1, \dots, \omega_r, d\omega_1, \dots, d\omega_r)$, which is called the ***closure*** of $\mathcal{I}$, that will be closed. Indeed, if $\omega \in \overline{\mathcal{I}}$, then there are appropriate forms γ^α and $\Gamma^\alpha, \alpha = 1, \dots, r$ such that we can write $\omega = \gamma^\alpha \wedge \omega_\alpha + \Gamma^\alpha \wedge d\omega_\alpha$. We then obtain

$$d\omega = d\gamma^\alpha \wedge \omega_\alpha + \big(d\Gamma^\alpha + (-1)^{deg\,\gamma^\alpha}\gamma^\alpha\big) \wedge d\omega_\alpha \in \overline{\mathcal{I}}.$$

Naturally, if the exterior derivatives of some generating forms are already inside the ideal, we have to discard these exterior derivatives as generators in determining the closure. More generally, let us denote the set of forms $d\omega$ corresponding to all forms $\omega \in \mathcal{I}$ by $d\mathcal{I}$. We immediately observe that the set $\overline{\mathcal{I}} = \mathcal{I} \cup d\mathcal{I}$ is a closed ideal in $\Lambda(M)$.

Next, we discuss the necessary and sufficient conditions for an ideal generated by finitely many forms to be closed.

Theorem 5.10.1. *Let $\mathcal{I}(\omega_1, \dots, \omega_r)$ be an ideal of the exterior algebra $\Lambda(M)$. The ideal $\mathcal{I}$ is closed if and only if appropriate forms $\Gamma_\alpha^\beta \in \Lambda(M)$, $\alpha, \beta = 1, \dots, r$ can be so found that the relations $d\omega_\alpha = \Gamma_\alpha^\beta \wedge \omega_\beta$ are satisfied.*

It is clear that the conditions $deg\,\omega_\alpha + 1 = deg\,\Gamma_\alpha^\beta + deg\,\omega_\beta$ or

$$deg\,\Gamma^{\beta}_{\alpha} = deg\,\omega_{\alpha} - deg\,\omega_{\beta} + 1 \geq 0$$

should be satisfied if the forms Γ^{β}_{α} exist. Hence, only the generating forms whose degrees are less than or equal to $deg\,\omega_{\alpha} + 1$ can take place in the sum. Let us first assume that the ideal $\mathcal{I}$ is closed. Then we must get $d\omega_{\alpha} \in \mathcal{I}$ when $\omega_{\alpha} \in \mathcal{I}$. This means that there exists forms Γ^{β}_{α} so that the relations $d\omega_{\alpha} = \Gamma^{\beta}_{\alpha} \wedge \omega_{\beta}$ are satisfied. For sufficiency, let us assume the existence of the relations $d\omega_{\alpha} = \Gamma^{\beta}_{\alpha} \wedge \omega_{\beta}$. If $\omega \in \mathcal{I}$, then we can find forms γ^{α} so that one is able to write $\omega = \gamma^{\alpha} \wedge \omega_{\alpha}$. In this case, the exterior derivative of ω is evaluated as

$$\begin{aligned} d\omega &= d\gamma^{\alpha} \wedge \omega_{\alpha} + (-1)^{deg\,\gamma^{\alpha}} \gamma^{\alpha} \wedge d\omega_{\alpha} \\ &= \big(d\gamma^{\beta} + (-1)^{deg\,\gamma^{\alpha}} \gamma^{\alpha} \wedge \Gamma^{\beta}_{\alpha}\big) \wedge \omega_{\beta} \end{aligned}$$

implying that $d\omega \in \mathcal{I}$. However, the forms Γ^{β}_{α} should be restricted because they have to satisfy the following compatibility conditions:

$$\begin{aligned} d^2\omega_{\alpha} &= d\Gamma^{\beta}_{\alpha} \wedge \omega_{\beta} + (-1)^{deg\,\Gamma^{\beta}_{\underline{\alpha}}} \Gamma^{\beta}_{\underline{\alpha}} \wedge d\omega_{\beta} \\ &= \big(d\Gamma^{\beta}_{\alpha} + (-1)^{deg\,\Gamma^{\gamma}_{\underline{\alpha}}} \Gamma^{\gamma}_{\underline{\alpha}} \wedge \Gamma^{\beta}_{\gamma}\big) \wedge \omega_{\beta} = 0. \end{aligned}$$

Evidently, in the above sums only forms complying the degree conformity can take place. □

Example 5.10.1. Let us consider the ideal $\mathcal{I}(\omega_1, \omega_2)$ of $\Lambda(\mathbb{R}^4)$ generated by the forms $\omega_1 = dx - y\,dz,\ \omega_2 = t\,dx \wedge dz - x\,dy \wedge dt$. We write

$$d\omega_1 = -\,dy \wedge dz = \Gamma^1_1 \wedge (dx - y\,dz) + \Gamma^2_1(t\,dx \wedge dz - x\,dy \wedge dt)$$

where $\Gamma^1_1 \in \Lambda^1(\mathbb{R}^4), \Gamma^2_1 \in \Lambda^0(\mathbb{R}^4)$. If we choose

$$\Gamma^1_1 = \gamma_1\,dx + \gamma_2\,dy + \gamma_3\,dz + \gamma_4\,dt$$

then we find

$$\begin{aligned} dy \wedge dz = (y\gamma_1 + \gamma_3 - t\Gamma^2_1)\,dx \wedge dz + \gamma_2\,dx \wedge dy + \gamma_4\,dx \wedge dt \\ + y\gamma_2\,dy \wedge dz + x\Gamma^2_1\,dy \wedge dt - y\,\gamma_4\,dz \wedge dt. \end{aligned}$$

Comparing both sides, we see that the following equations must hold

$$y\gamma_1 + \gamma_3 - t\Gamma^2_1 = \gamma_2 = \gamma_4 = x\Gamma^2_1 = y\,\gamma_4 = 0,\ \ y\gamma_2 = 1$$

from which we obtain $\Gamma^2_1 = \gamma_4 = 0,\ \gamma_3 = -\,y\gamma_1$. But, to satisfy the relations $\gamma_2 = 0$ and $y\gamma_2 = 1$ simultaneously is not possible. Hence, the form Γ^1_1 does not exist implying that $d\omega_1$ does not belong to $\mathcal{I}$. On the other hand, we have

$$d\omega_2 = dt \wedge dx \wedge dz - dx \wedge dy \wedge dt = \Big(\frac{1}{x}dx + \frac{1}{t}dt\Big) \wedge \omega_2.$$

Thus $d\omega_2$ is inside the ideal. In this case the closure of the ideal $\mathcal{I}$ should be designated by $\overline{\mathcal{I}}(\omega_1, \omega_2, dy \wedge dz)$. ■

When we are dealing with ideals whose generators are 1-forms, the condition of being closed is reduced to a much simpler form.

Theorem 5.10.2. *Let an ideal of the exterior algebra $\Lambda(M)$ generated by linearly independent 1-forms $\omega^1, \dots, \omega^r$ be $\mathcal{I}(\omega^1, \dots, \omega^r)$. If $r < m-1$, then the ideal $\mathcal{I}$ is closed if and only if the conditions $d\omega^\alpha \wedge \Omega = 0, \alpha = 1, \dots, r$ are satisfied where we defined $\Omega = \omega^1 \wedge \cdots \wedge \omega^r \neq 0$.*

If $\mathcal{I}$ is closed, that is, if there exist forms $\Gamma^\alpha_\beta \in \Lambda^1(M^m)$ so that we can write $d\omega^\alpha = \Gamma^\alpha_\beta \wedge \omega^\beta$, then it is evident that the relations $d\omega^\alpha \wedge \Omega = 0$ are automatically satisfied. Conversely, let us suppose that we get $d\omega^\alpha \wedge \Omega = 0$ for $1 \le \alpha \le r$. Next, we add $m - r$ linearly independent 1-forms $\omega^{r+1}, \dots,$ ω^m to the forms $\omega^1, \dots, \omega^r$ to make a basis $\{\omega^i\} = \{\omega^\alpha, \omega^a\}, a = r+1, \dots, m, i = 1, \dots, m$ of $\Lambda^1(M)$. In this situation, a basis for the module $\Lambda^2(M)$ becomes $\omega^i \wedge \omega^j, i < j$ and so long as $\lambda^\alpha_{ij} = -\lambda^\alpha_{ji}$, we can write

$$\begin{aligned} d\omega^\alpha &= \lambda^\alpha_{ij}\,\omega^i \wedge \omega^j = \lambda^\alpha_{\beta\gamma}\,\omega^\beta \wedge \omega^\gamma + \lambda^\alpha_{a\beta}\,\omega^a \wedge \omega^\beta + \lambda^\alpha_{\beta a}\,\omega^\beta \wedge \omega^a + \lambda^\alpha_{ab}\,\omega^a \wedge \omega^b \\ &= \lambda^\alpha_{\beta\gamma}\,\omega^\beta \wedge \omega^\gamma + 2\lambda^\alpha_{a\beta}\,\omega^a \wedge \omega^\beta + \lambda^\alpha_{ab}\,\omega^a \wedge \omega^b. \end{aligned}$$

whence we deduce that

$$d\omega^\alpha \wedge \Omega = \lambda^\alpha_{ab}\,\omega^a \wedge \omega^b \wedge \Omega, \;\; \lambda^\alpha_{ab} = -\lambda^\alpha_{ba}.$$

When $r \le m-2$, the foregoing expression is a $(r+2)$-form which is the sum of simple $(r+2)$-forms. Since the forms ω^α and ω^a are linearly independent none of the forms $\omega^a \wedge \omega^b \wedge \Omega$ vanishes if $a \neq b$. Thus the condition $d\omega^\alpha \wedge \Omega = 0$ can only be realised when $\lambda^\alpha_{ab} = 0$. In this case, we obtain

$$d\omega^\alpha = (\lambda^\alpha_{\gamma\beta}\,\omega^\gamma + 2\lambda^\alpha_{a\beta}\,\omega^a) \wedge \omega^\beta = \Gamma^\alpha_\beta \wedge \omega^\beta \in \mathcal{I}$$

Hence, the ideal $\mathcal{I}$ is closed. □

When $r \ge m-1$ the forms $d\omega^\alpha \wedge \Omega$ are identically zero because their degrees is higher than m. Therefore, they cannot provide a criterion to identify a closed ideal. However, the theorem below fills this gap.

Theorem 5.10.3. *An ideal of the exterior algebra $\Lambda(M^m)$ is closed if it is generated either by $r = m$ or $r = m-1$ linearly independent 1-forms.*

When $r = m$, the linearly independent 1-forms $\omega^1, \dots, \omega^m$ generating an ideal $\mathcal{I}$ constitute a basis of $\Lambda^1(M)$. Consequently, we can write

$$d\omega^\alpha = \lambda^\alpha_{\gamma\beta}\,\omega^\gamma \wedge \omega^\beta = \Gamma^\alpha_\beta \wedge \omega^\beta \in \mathcal{I}$$

where $\Gamma^\alpha_\beta = \lambda^\alpha_{\gamma\beta}\,\omega^\gamma$. Hence, the ideal $\mathcal{I}$ is closed. When $r = m-1$, we can choose a 1-form σ that is independent of those $m-1$ forms. Thus $\omega^1, \ldots, \omega^{m-1}, \sigma$ become a basis of $\Lambda^1(M)$. If we consider an ideal $\mathcal{I}$ generated by these forms, we get

$$d\omega^\alpha = \lambda^\alpha_{\gamma\beta}\,\omega^\gamma \wedge \omega^\beta + \lambda^\alpha_\beta\,\sigma \wedge \omega^\beta = (\lambda^\alpha_{\gamma\beta}\,\omega^\gamma + \lambda^\alpha_\beta\,\sigma) \wedge \omega^\beta = \Gamma^\alpha_\beta \wedge \omega^\beta \in \mathcal{I}.$$

Hence, the ideal again becomes closed. □

The following theorem is concerned with the closure $\overline{\mathcal{I}}(\omega^1, \ldots, \omega^r, d\omega^1, \ldots, d\omega^r)$ of an ideal $\mathcal{I}(\omega^1, \ldots, \omega^r)$.

Theorem 5.10.4. *The exterior derivative $d\omega$ of a form $\omega \in \Lambda^k(M)$ remains inside the closure $\overline{\mathcal{I}}$ of the ideal $\mathcal{I}$ if and only if we can find forms $\alpha \in \Lambda^k(M)$ and $\beta \in \mathcal{C}^{k+1}(M)$ in the ideal $\mathcal{I}$ such that $d(\omega + \alpha) = \beta$.*

If $\alpha, \beta \in \mathcal{I}$, then we can write $\alpha = \gamma_\alpha \wedge \omega^\alpha, \beta = \lambda_\alpha \wedge \omega^\alpha$ for appropriate forms γ_α and λ_α where $\alpha = 1, 2, \ldots, r$. Thus, we readily obtain for a k-form ω satisfying the relation $d(\omega + \alpha) = \beta$, the following expression

$$d\omega = -d\alpha + \beta = (-d\gamma_\alpha + \lambda_\alpha) \wedge \omega^\alpha + (-1)^{deg\,(\gamma_{\underline{\alpha}})}\gamma_\alpha \wedge d\omega^\alpha \in \overline{\mathcal{I}}.$$

The above equality requires that $d\beta = 0$, thus we must have $\beta \in \mathcal{C}^{k+1}(M)$. Conversely, let us assume that $d\omega \in \overline{\mathcal{I}}$. Consequently, we can write

$$d\omega = \lambda_\alpha \wedge \omega^\alpha + \mu_\alpha \wedge d\omega^\alpha$$

where $\lambda_\alpha \in \Lambda^{k+1-deg\,(\omega^\alpha)}(M)$, $\mu_\alpha \in \Lambda^{k-deg\,(\omega^\alpha)}(M)$. Because of the relation $d^2\omega = 0$, the forms λ_α and μ_α ought to meet the condition

$$d\lambda_\alpha \wedge \omega^\alpha + (d\mu_\alpha + (-1)^{deg\,(\lambda_{\underline{\alpha}})}\lambda_\alpha) \wedge d\omega^\alpha = 0.$$

We now define the forms ϕ_α as follows

$$(-1)^{deg\,(\lambda_{\underline{\alpha}})}\phi_\alpha = d\mu_\alpha + (-1)^{deg\,(\lambda_{\underline{\alpha}})}\lambda_\alpha,\ \ deg\,(\phi_\alpha) = deg\,(\lambda_\alpha).$$

If we insert the form $\lambda_\alpha = \phi_\alpha + (-1)^{deg\,(\lambda_{\underline{\alpha}})-1}d\mu_\alpha$ into above expressions and note that $deg\,(\mu_\alpha) = deg\,(\lambda_\alpha) - 1$ by definition, we obtain

$$\begin{aligned} d\omega &= \phi_\alpha \wedge \omega^\alpha + (-1)^{deg\,(\lambda_{\underline{\alpha}})-1}d\mu_\alpha \wedge \omega^\alpha + \mu_\alpha \wedge d\omega^\alpha \\ &= \phi_\alpha \wedge \omega^\alpha + (-1)^{deg\,(\lambda_{\underline{\alpha}})-1}d(\mu_\alpha \wedge \omega^\alpha), \\ 0 &= d\phi_\alpha \wedge \omega^\alpha + (-1)^{deg\,(\phi_{\underline{\alpha}})}\phi_\alpha \wedge d\omega^\alpha = d(\phi_\alpha \wedge \omega^\alpha). \end{aligned}$$

It will suffice now to introduce the forms $\alpha = (-1)^{deg\,(\lambda_{\underline{\alpha}})}\mu_\alpha \wedge \omega^\alpha \in \mathcal{I}$ and $\beta = \phi_\alpha \wedge \omega^\alpha \in \mathcal{I}$ to reach to the result $d(\omega + \alpha) = \beta$ and $d\beta = 0$. □

5.11. LIE DERIVATIVES OF EXTERIOR FORMS

Let us consider a congruence on a manifold M brought out by a vector field V and the *flow* $\phi_t : M \to M$ induced by this congruence. As is well known, this mapping carries a point $p \in M$ to a point $\overline{p}(t) = \phi_t(p) \in M$. On recalling the relation (2.9.11), we represent this mapping by $\overline{p}(t) = \phi_t(p) = e^{tV}(p)$. We can also write

$$\overline{x}^i(t) = e^{tV}(x^i) = x^i + tV(x^i) + \frac{t^2}{2!}V^2(x^i) + \cdots + \frac{t^n}{n!}V^n(x^i) + \cdots$$

in terms of local coordinates. We employed here only the symbol V for the vector field believing that it will no longer cause any ambiguity.

We suppose that a form field $\omega \in \Lambda^k(M)$ is given. Let us transport the form $\omega\big(\overline{p}(t)\big)$ at a point $\overline{p}(t)$ to a point p by pulling it back by the mapping ϕ_t^*. We thus obtain

$$\omega^*(p;t) = \omega \circ \phi_t(p) = (\phi_t^*\omega)(p) = (e^{tV})^*\omega.$$

As we have done before, we will now define the ***Lie derivative*** of a form field ω at a point p by the following limiting process:

$$£_V\omega = \lim_{t\to 0}\frac{(e^{tV})^*\omega - \omega}{t} = \lim_{t\to 0}\frac{(e^{tV})^* - i_\Lambda}{t}\,\omega \in \Lambda^k(M) \quad (5.11.1)$$

where $i_\Lambda : \Lambda(M) \to \Lambda(M)$ is the identity operator on the exterior algebra. This definition reveals immediately certain important properties of the Lie derivative.

(i). *We can write*

$$(e^{tV})^*\omega = \omega + t£_V\omega + o(t).$$

(ii). *When* $f \in \Lambda^0(M)$, *we have* [*see* (2.10.18)]

$$£_V f = v^i f_{,i} = V(f) = \mathbf{i}_V(df).$$

In fact, for small values of the parameter t we obtain

$$£_V f = \lim_{t\to 0}\frac{f\big(\overline{p}(t)\big) - f(p)}{t} = \lim_{t\to 0}\frac{f(\mathbf{x} + t\mathbf{v} + \boldsymbol{o}(t)) - f(\mathbf{x})}{t} = v^i f_{,i}.$$

(iii). *We have*

$$£_V(\omega + \sigma) = £_V\omega + £_V\sigma.$$

This is observed at once by noting he relation

$$(e^{tV})^*(\omega + \sigma) = (e^{tV})^*\omega + (e^{tV})^*\sigma.$$

(iv). *The Leibniz rule*

$$£_V(\omega \wedge \sigma) = (£_V\omega) \wedge \sigma + \omega \wedge (£_V\sigma)$$

is in effect.

Recalling the relation $(e^{tV})^*(\omega \wedge \sigma) = (e^{tV})^*\omega \wedge (e^{tV})^*\sigma$, we arrive at the desired result

$$\begin{aligned} £_V(\omega \wedge \sigma) &= \lim_{t \to 0} \frac{(e^{tV})^*\omega \wedge (e^{tV})^*\sigma - \omega \wedge \sigma}{t} \\ &= \lim_{t \to 0} \frac{(\omega + t£_V\omega + o(t)) \wedge (\sigma + t£_V\sigma + o(t)) - \omega \wedge \sigma}{t} \\ &= \lim_{t \to 0} \frac{t(£_V\omega \wedge \sigma + \omega \wedge £_V\sigma) + o(t)}{t} = £_V\omega \wedge \sigma + \omega \wedge £_V\sigma. \end{aligned}$$

This expression can easily be generalised to an arbitrary number of forms so that one is able to write

$$\begin{aligned} £_V(\omega_1 \wedge \omega_2 \wedge \ldots \wedge \omega_r) &= £_V\omega_1 \wedge \omega_2 \wedge \ldots \wedge \omega_r \\ &\quad + \omega_1 \wedge £_V\omega_2 \wedge \ldots \wedge \omega_r + \omega_1 \wedge \omega_2 \wedge \ldots \wedge £_V\omega_r. \end{aligned}$$

This relation offers essentially an approach to calculate the Lie derivative of any form once we determine the Lie derivatives of only 0- and 1-forms. We have already found the Lie derivative of 0-forms. We now try to evaluate the Lie derivative of a 1-form. Let us take

$$\omega = \omega_i\, dx^i \in \Lambda^1(M), \;\; V = v^i \frac{\partial}{\partial x^i} \in T(M).$$

Since we can write. $\overline{x}^i = x^i + tv^i + o(t)$, then the Taylor series about the point $\mathbf{x}$ yields

$$\begin{aligned} (e^{tV})^*\omega &= \omega_i\big(x^j + tv^j + o(t)\big)\big(dx^i + tv^i_{,k}\, dx^k + o(t)\big) \\ &= \Big(\omega_i(\mathbf{x}) + t\frac{\partial \omega_i}{\partial x^j}v^j + o(t)\Big)\Big(dx^i + t\frac{\partial v^i}{\partial x^k}\, dx^k + o(t)\Big) \\ &= \omega_i(\mathbf{x})\, dx^i + t\Big(\frac{\partial \omega_i}{\partial x^j}v^j\, dx^i + \omega_i\frac{\partial v^i}{\partial x^k}\, dx^k\Big) + o(t). \end{aligned}$$

On changing properly the names of dummy indices, we finally get

$$£_V\omega = \Big(\frac{\partial \omega_i}{\partial x^j}v^j + \omega_j\frac{\partial v^j}{\partial x^i}\Big)\, dx^i = (\omega_{i,j}v^j + \omega_j v^j_{,i})\, dx^i. \quad (5.11.2)$$

The coefficients $(£_V\omega)_i = \omega_{i,j}v^j + \omega_j v^j_{,i} \in \Lambda^0(M)$ totally specifies the 1-form $£_V\omega$. As a special example, let us consider the form $df = f_{,i}\,dx^i$ where $f \in \Lambda^0(M)$. Then, with $\omega_i = f_{,i}$ (5.11.2) leads to

$$£_V\,df = (f_{,ij}v^j + f_{,j}v^j_{,i})\,dx^i = (f_{,j}v^j)_{,i}\,dx^i = \big(V(f)\big)_{,i}\,dx^i = dV(f).$$

If we now select $f = x^k$, we reach to quite a significant conclusion

$$£_V\,dx^k = dV(x^k) = dv^k = v^k_{,i}\,dx^i. \tag{5.11.3}$$

Next, we take a form $\omega \in \Lambda^k(M)$ into account denoted by

$$\omega = \frac{1}{k!}\,\omega_{i_1\cdots i_k}(\mathbf{x})\,dx^{i_1}\wedge\cdots\wedge dx^{i_k}.$$

On utilising the above properties, we can now calculate the Lie derivative of this form as follows:

$$\begin{aligned}
£_V\omega &= \frac{1}{k!}\big[(£_V\omega_{i_1\cdots i_k})\,dx^{i_1}\wedge\cdots\wedge dx^{i_k} + \omega_{i_1i_2\cdots i_k}(£_V\,dx^{i_1})\wedge\cdots\wedge dx^{i_k}\\
&\quad + \cdots + \omega_{i_1\cdots i_k}dx^{i_1}\wedge\cdots\wedge(£_V\,dx^{i_k})\big]\\
&= \frac{1}{k!}\big[\omega_{i_1\cdots i_k,i}v^i\,dx^{i_1}\wedge\cdots\wedge dx^{i_k} + \omega_{i_1i_2\cdots i_k}v^{i_1}_{,i}\,dx^i\wedge dx^{i_2}\wedge\cdots\wedge dx^{i_k}\\
&\quad + \cdots + \omega_{i_1\cdots i_{k-1}i_k}v^{i_k}_{,i}\,dx^{i_1}\wedge\cdots\wedge dx^{i_{k-1}}\wedge dx^i\big]\\
&= \frac{1}{k!}\big[\omega_{i_1\cdots i_k,i}v^i + \omega_{ii_2\cdots i_k}v^i_{,i_1} + \cdots + \omega_{i_1\cdots i_{k-1}i}v^i_{,i_k}\big]dx^{i_1}\wedge\cdots\wedge dx^{i_k}.
\end{aligned}$$

Hence, the Lie derivative of a form $\omega \in \Lambda^k(M)$ is expressible as

$$£_V\omega = \frac{1}{k!}(£_V\omega)_{i_1i_2\cdots i_k}\,dx^{i_1}\wedge dx^{i_2}\wedge\cdots\wedge dx^{i_k} \in \Lambda^k(M)$$

where the *completely antisymmetric* coefficients $(£_V\omega)_{i_1i_2\cdots i_k} \in \Lambda^0(M)$ are determined by

$$\begin{aligned}
(£_V\omega)_{i_1i_2\cdots i_k} &= \omega_{i_1i_2\cdots i_k,i}v^i + \omega_{ii_2\cdots i_k}v^i_{,i_1} + \omega_{i_1ii_3\cdots i_k}v^i_{,i_2} + \cdots + \omega_{i_1i_2\cdots i_{k-1}i}v^i_{,i_k}\\
&= v^i\frac{\partial\omega_{i_1i_2\cdots i_k}}{\partial x^i} + \sum_{r=1}^{k}\omega_{i_1\cdots i_{r-1}ii_{r+1}\cdots i_k}\frac{\partial v^i}{\partial x^{i_r}}
\end{aligned} \tag{5.11.4}$$

It is clear that the complete antisymmetry in the coefficients $\omega_{i_1\cdots i_k}$ renders the coefficients in (5.11.4) completely antisymmetric. It is now clear that the Lie derivative $£_V : \Lambda(M) \to \Lambda(M)$ is a *degree preserving* mapping.

The expression (5.11.2) for Lie derivatives of 1-forms can be transformed into the following identical shape

$$\pounds_V\omega = \big[(\omega_{i,j} - \omega_{j,i})v^j + (v^j\omega_j)_{,i}\big]\,dx^i,\ \ \omega \in \Lambda^1(M).$$

On the other hand, since one has $d\omega = \omega_{i,j}\,dx^j \wedge dx^i$ we obtain

$$\begin{aligned}\mathbf{i}_V(d\omega) &= \omega_{i,j}v^j\,dx^i - \omega_{i,j}v^i\,dx^j = (\omega_{i,j} - \omega_{j,i})v^j\,dx^i,\\ d\mathbf{i}_V(\omega) &= (v^j\omega_j)_{,i}\,dx^i.\end{aligned}$$

We thus arrive at the expression

$$\pounds_V\omega = \mathbf{i}_V(d\omega) + d\mathbf{i}_V(\omega),\ \ \omega \in \Lambda^1(M)$$

known as the ***Cartan magic formula***. We shall now prove that this formula is valid for any form in the exterior algebra.

Theorem 5.11.1. *For any form $\omega \in \Lambda(M)$ and vector field $V \in T(M)$, the Lie derivative of this form is calculated by $\pounds_V\omega = \mathbf{i}_V(d\omega) + d\mathbf{i}_V(\omega)$.*

Let us consider a form $\omega = \dfrac{1}{k!}\,\omega_{i_1\cdots i_k}(\mathbf{x})\,dx^{i_1} \wedge \cdots \wedge dx^{i_k} \in \Lambda^k(M)$ and a vector field $V = v^i(\mathbf{x})\dfrac{\partial}{\partial x^i}$. The exterior derivative of ω is given by

$$\begin{aligned}d\omega &= \frac{1}{k!}\,\omega_{i_1\cdots i_k,i}\,dx^i \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_k}\\ &= \frac{1}{(k+1)!}(k+1)\,\omega_{[i_1\cdots i_k,i]}\,dx^i \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_k}.\end{aligned}$$

Therefore, we obtain

$$\mathbf{i}_V(d\omega) = \frac{1}{k!}(k+1)\,\omega_{[i_1\cdots i_k,i]}v^i\,dx^{i_1} \wedge \cdots \wedge dx^{i_k}.$$

On the other hand, we can write

$$\begin{aligned}\mathbf{i}_V(\omega) &= \frac{1}{(k-1)!}\,\omega_{ii_2\cdots i_k}v^i\,dx^{i_2} \wedge \cdots \wedge dx^{i_k},\\ d\mathbf{i}_V(\omega) &= \frac{1}{k!}k(\,\omega_{i[i_2\cdots i_k}v^i)_{,i_1]}\,dx^{i_1} \wedge dx^{i_2} \wedge \cdots \wedge dx^{i_k}.\end{aligned}$$

Hence, we find that

$$\mathbf{i}_V(d\omega) + d\mathbf{i}_V(\omega) = \frac{1}{k!}\Omega_{i_1 i_2\cdots i_k}dx^{i_1} \wedge dx^{i_2} \wedge \cdots \wedge dx^{i_k}$$

where the smooth functions $\Omega_{i_1 i_2\cdots i_k} \in \Lambda^0(M)$ are defined by

$$\Omega_{i_1 i_2\cdots i_k} = (k+1)\,\omega_{[i_1 i_2\cdots i_k,i]}v^i + k\,\omega_{i[i_2\cdots i_k,i_1]}v^i + k\,\omega_{i[i_2\cdots i_k}v^i_{,i_1]}.$$

In order to evaluate explicitly the coefficients $\Omega_{i_1 i_2 \cdots i_k}$, we resort to the relations (5.8.3) and (5.8.4) to get

$$\begin{aligned}\Omega_{i_1 i_2 \cdots i_k} &= \omega_{i_1 \cdots i_k, i} v^i - \sum_{r=1}^{k} \omega_{i_1 \cdots i_{r-1} i i_{r+1} \cdots i_k, i_r} v^i \\ &\quad + \omega_{i i_2 \cdots i_k, i_1} v^i - \sum_{r=2}^{k} \omega_{i i_2 \cdots i_{r-1} i_1 i_{r+1} \cdots i_k, i_r} v^i \\ &\quad + \omega_{i i_2 \cdots i_k} v^i_{,i_1} - \sum_{r=2}^{k} \omega_{i i_2 \cdots i_{r-1} i_1 i_{r+1} \cdots i_k} v^i_{,i_r}\end{aligned}$$

Moreover, since one has $(-1)^{r-2+r-1} = (-1)^{2r-3} = -1$, we can write

$$\omega_{i i_2 \cdots i_{r-1} i_1 i_{r+1} \cdots i_k} = -\,\omega_{i_1 i_2 \cdots i_{r-1} i i_{r+1} \cdots i_k}.$$

We thus find

$$\omega_{i i_2 \cdots i_k, i_1} v^i - \sum_{r=2}^{k} \omega_{i i_2 \cdots i_{r-1} i_1 i_{r+1} \cdots i_k, i_r} v^i = \sum_{r=1}^{k} \omega_{i_1 \cdots i_{r-1} i i_{r+1} \cdots i_k, i_r} v^i$$

and see, consequently, that the second line above cancels the second term in the first line. If we arrange as well the last line in the similar way, we finally conclude that

$$\begin{aligned}\Omega_{i_1 i_2 \cdots i_k} &= \omega_{i_1 \cdots i_k, i} v^i + \sum_{r=1}^{k} \omega_{i_1 i_2 \cdots i_{r-1} i i_{r+1} \cdots i_k} v^i_{,i_r} \\ &= (\pounds_V \omega)_{i_1 i_2 \cdots i_k}.\end{aligned}$$

Thus for any form $\omega \in \Lambda(M)$, the Cartan magic formula

$$\pounds_V \omega = \mathbf{i}_V(d\omega) + d\mathbf{i}_V(\omega) \tag{5.11.5}$$

becomes valid. In operator form, we can express this relation as follows

$$\pounds_V = \mathbf{i}_V \circ d + d \circ \mathbf{i}_V : \Lambda^k(M) \to \Lambda^k(M),\ 0 \le k \le m. \qquad \square$$

We now consider a form $\omega \in \Lambda^k(M)$ and vector fields $U, V \in T(M)$ and let us calculate the form $\pounds_U\big(\mathbf{i}_V(\omega)\big) \in \Lambda^{k-1}(M)$. Since we have

$$\begin{aligned}\omega &= \frac{1}{k!}\,\omega_{i_1 \cdots i_k} dx^{i_1} \wedge \cdots \wedge dx^{i_k}, \\ \mathbf{i}_V(\omega) &= \frac{1}{(k-1)!}\,\omega_{j i_2 \cdots i_k} v^j\, dx^{i_2} \wedge \cdots \wedge dx^{i_k}\end{aligned}$$

we obtain from (5.11.4) that

$$\pounds_U\big(\mathbf{i}_V(\omega)\big) = \frac{1}{(k-1)!}\Big[\omega_{ji_2\cdots i_k,i}v^j u^i + \omega_{ji_2\cdots i_k}v^j_{,i}u^i + \sum_{r=2}^{k}\omega_{ji_2\cdots i_{r-1}ii_{r+1}\cdots i_k}v^j u^i_{,i_r}\Big]dx^{i_2}\wedge\cdots\wedge dx^{i_k}.$$

By adding and subtracting the terms $\omega_{ji_2\cdots i_k}v^i u^j_{,i}$ to the coefficients within brackets above and changing dummy indices appropriately we cast this expression into the equivalent form given below

$$\begin{aligned}\pounds_U\big(\mathbf{i}_V(\omega)\big) &= \frac{1}{(k-1)!}\,\omega_{ji_2\cdots i_k}(u^i v^j_{,i} - v^i u^j_{,i})\,dx^{i_2}\wedge\cdots\wedge dx^{i_k}\\ &\quad+\frac{1}{(k-1)!}\Big[\omega_{ji_2\cdots i_k,i}u^i + \omega_{ii_2\cdots i_k}u^i_{,j}\\ &\quad+\sum_{r=2}^{k}\omega_{ji_2\cdots i_{r-1}ii_{r+1}\cdots i_k}u^i_{,i_r}\Big]v^j\,dx^{i_2}\wedge\cdots\wedge dx^{i_k}\\ &=\frac{1}{(k-1)!}\,\omega_{ji_2\cdots i_k}w^j\,dx^{i_2}\wedge\cdots\wedge dx^{i_k} + \frac{1}{(k-1)!}\Big[\omega_{ji_2\cdots i_k,i}u^i\\ &\quad+\sum_{r=1}^{k}\omega_{i_1i_2\cdots i_{r-1}ii_{r+1}\cdots i_k}u^i_{,i_r}\Big]v^{i_1}\,dx^{i_2}\wedge\cdots\wedge dx^{i_k}\end{aligned}$$

where

$$w^j = (u^i v^j_{,i} - v^i u^j_{,i}) = [U,V]^j = (\pounds_U V)^j$$

are components of the Lie derivative of the vector field V with respect to the vector field U whereas the expression within brackets are nothing but the coefficients of the Lie derivative of the form ω with respect to the vector field U. Consequently, the above expression is now transformed into

$$\begin{aligned}\pounds_U\big(\mathbf{i}_V(\omega)\big) &= \frac{1}{(k-1)!}\,\omega_{ii_2\cdots i_k}(\pounds_U V)^i\,dx^{i_2}\wedge\cdots\wedge dx^{i_k}\\ &\quad+\frac{1}{(k-1)!}\,(\pounds_U\omega)_{ii_2\cdots i_k}v^i\,dx^{i_2}\wedge\cdots\wedge dx^{i_k},\end{aligned}$$

Thus for any form $\omega\in\Lambda(M)$, we obtain

$$\pounds_U\big(\mathbf{i}_V(\omega)\big) = \mathbf{i}_{\pounds_U V}(\omega) + \mathbf{i}_V\big(\pounds_U(\omega)\big). \tag{5.11.6}$$

Hence, we realised that we have managed to establish the following connection between the operators of Lie derivative and interior product

$$\mathbf{i}_{£_U V} = \mathbf{i}_{[U,V]} = £_U \circ \mathbf{i}_V - \mathbf{i}_V \circ £_U = [£_U, \mathbf{i}_V]. \quad (5.11.7)$$

Since the interior product with zero vector vanishes, if $[U, V] = 0$ or $U = V$, namely, if vectors are commutative, then (5.11.7) yields

$$£_U \circ \mathbf{i}_V = \mathbf{i}_V \circ £_U \;\text{ or }\; £_V \circ \mathbf{i}_V = \mathbf{i}_V \circ £_V. \quad (5.11.8)$$

This means that *the operators* $£_U$ *and* $\mathbf{i}_V$ *or* $£_V$ *and* $\mathbf{i}_U$ *commute if vector fields* U *and* V *are commutative*.

Let us apply (5.11.5) to the form $d\omega$. Since $d^2 = 0$, we get

$$£_V\, d\omega = \mathbf{i}_V(d^2\omega) + d\mathbf{i}_V(d\omega) = d\mathbf{i}_V(d\omega) = d\big(£_V\, \omega - d\mathbf{i}_V(\omega)\big) = d£_V\, \omega.$$

This equality is valid for every form. We thus conclude that

$$£_V \circ d = d \circ £_V. \quad (5.11.9)$$

Hence, *the operators* $£_V$ *and* d *commute*.

Let us take $f \in \Lambda^0(M)$ and $V \in T(M)$. If we pay attention to the relations (5.4.7), we deduce that the Lie derivative of a form ω with respect to the vector fV is found to be

$$\begin{aligned} £_{fV}\omega &= \mathbf{i}_{fV}(d\omega) + d\mathbf{i}_{fV}(\omega) = f\mathbf{i}_V(d\omega) + d\big(f\mathbf{i}_V(\omega)\big) \quad (5.11.10)\\ &= f\mathbf{i}_V(d\omega) + df \wedge \mathbf{i}_V(\omega) + fd\mathbf{i}_V(\omega)\\ &= f£_V\, \omega + df \wedge \mathbf{i}_V(\omega). \end{aligned}$$

We immediately see due to (5.4.7) and (5.11.5) that

$$£_{U+V}\, \omega = £_U\, \omega + £_V\, \omega \quad\text{or}\quad £_{U+V} = £_U + £_V. \quad (5.11.11)$$

But, if only $f = c =$ *constant*, then we get $£_{cV}\omega = c£_V\omega$. In this case, it is clear that the addition and scalar multiplication of Lie operators are again Lie operators. Therefore, Lie operators form a linear vector space over $\mathbb{R}$.

Next, we would like to discuss the action of the operator $£_{[U,V]}$, where $U, V \in T(M)$, on a form $\omega \in \Lambda(M)$. In view of (5.11.6), we can write

$$\begin{aligned} \mathbf{i}_{£_U V}(d\omega) &= £_U\left(\mathbf{i}_V(d\omega)\right) - \mathbf{i}_V\left(£_U(d\omega)\right),\\ \mathbf{i}_{£_U V}(\omega) &= £_U\left(\mathbf{i}_V(\omega)\right) - \mathbf{i}_V\left(£_U(\omega)\right). \end{aligned}$$

Let us then introduce these expressions into the Cartan formula

$$£_{[U,V]}\omega = £_{£_U V}\omega = \mathbf{i}_{£_U V}(d\omega) + d\mathbf{i}_{£_U V}(\omega).$$

If we note that the operators $£_U$ and d commute, we reach to the following relation

$$\begin{aligned}
£_{[U,V]}\omega &= £_U\big(\mathbf{i}_V(d\omega)\big) - \mathbf{i}_V\big(£_U(d\omega)\big) + d£_U\big(\mathbf{i}_V(\omega)\big) - d\mathbf{i}_V\big(£_U(\omega)\big) \\
&= £_U\big(\mathbf{i}_V(d\omega)\big) - \mathbf{i}_V\big(d£_U(\omega)\big) + £_U d\big(\mathbf{i}_V(\omega)\big) - d\mathbf{i}_V\big(£_U(\omega)\big) \\
&= £_U\big(\mathbf{i}_V(d\omega) + d\big(\mathbf{i}_V(\omega)\big) - \big[\mathbf{i}_V\big(d£_U(\omega) + d\mathbf{i}_V\big(£_U(\omega)\big)\big] \\
&= £_U£_V\omega - £_V£_U\omega \\
&= (£_U£_V - £_V£_U)\omega.
\end{aligned}$$

Since this relation will be satisfied for all $\omega \in \Lambda(M)$, we get the operator identity given below [*see* (2.10.17)]

$$£_{[U,V]} = £_U£_V - £_V£_U = [£_U, £_V]. \tag{5.11.12}$$

We now assume that an involutive distribution $\mathcal{D} \subseteq T(M)$ is prescribed by linearly independent vector fields $V_\alpha \in T(M), \alpha = 1, \dots, r \le m$ satisfying the conditions

$$[V_\alpha, V_\beta] = c^\gamma_{\alpha\beta} V_\gamma$$

Let us now associate a Lie operator $£_{V_\alpha}$ to each vector V_α. Then, it follows from (5.11.12) and (5.11.10) that

$$[£_{V_\alpha}, £_{V_\beta}]\,\omega = £_{[V_\alpha, V_\beta]}\,\omega = £_{c^\gamma_{\alpha\beta}V_\gamma}\,\omega = c^\gamma_{\alpha\beta}\,£_{V_\gamma}\omega + dc^\gamma_{\alpha\beta} \wedge \mathbf{i}_{V_\gamma}(\omega)$$

for any $\omega \in \Lambda(M)$ so that we obtain

$$[£_{V_\alpha}, £_{V_\beta}] = c^\gamma_{\alpha\beta}\,£_{V_\gamma} + dc^\gamma_{\alpha\beta} \wedge \mathbf{i}_{V_\gamma}. \tag{5.11.13}$$

Hence, if only the coefficients $c^\gamma_{\alpha\beta}$ are constants, then we are able to write $[£_{V_\alpha}, £_{V_\beta}] = c^\gamma_{\alpha\beta}\,£_{V_\gamma}$. Only in this situation, the operators $£_{V_\alpha}, \alpha = 1, \dots, r$ constitute as well a Lie algebra of operators on the exterior algebra and $c^\gamma_{\alpha\beta}$ becomes structure constants of that algebra. We know that this Lie algebra generates a r-parameter Lie group [*see p.* 191].

We now consider a flow $e^{tV} : M \to M$ on a manifold M generated by a vector field V and the pull-back $\omega^*(t) = (e^{tV})^*\omega$ of a form $\omega \in \Lambda(M)$. The derivative of the form $\omega^*(t)$ with respect to the parameter t can be evaluated as

$$\begin{aligned}
\frac{d\omega^*(t)}{dt} &= \lim_{\tau\to 0}\frac{\omega^*(t+\tau) - \omega^*(t)}{\tau} = \lim_{\tau\to 0}\frac{(e^{(t+\tau)V})^*\omega - (e^{tV})^*\omega}{\tau} \\
&= \lim_{\tau\to 0}\frac{(e^{\tau V}\circ e^{tV})^*\omega - (e^{tV})^*\omega}{\tau}
\end{aligned}$$

$$\begin{aligned}&= \lim_{\tau\to 0}\frac{(e^{\tau V})^*\circ(e^{tV})^*\omega - (e^{tV})^*\omega}{\tau}\\ &= \lim_{\tau\to 0}\frac{(e^{\tau V})^* - I}{\tau}(e^{tV})^*\omega\\ &= \pounds_V\omega^*(t).\end{aligned}$$

We have seen earlier that the formal solution of this ordinary differential equation under the initial condition $\omega^*(p;0)=\omega(p)$ is given by

$$\begin{aligned}\omega^*(p;t) &= e^{t\pounds_V}\omega(p) \qquad (5.11.14)\\ &= \omega + t\pounds_V\omega + \frac{t^2}{2!}\pounds_V^2\omega + \cdots + \frac{t^n}{n!}\pounds_V^n\omega + \cdots\end{aligned}$$

The above relation implies that we can write $\omega^* = (e^{tV})^*\omega = e^{t\pounds_V}\omega$ for all forms $\omega\in\Lambda(M)$. Therefore, we formally arrive at the result $(e^{tV})^* = e^{t\pounds_V}$. If $\omega^*(p;t)=\omega(p)$ for all t, we say that the form ω remains ***invariant*** under the flow generated by the vector field V. Evidently, (5.11.14) implies that $\pounds_V\omega = 0$ is the necessary and sufficient condition for ω to be invariant.

Let us now suppose that a submodule $\mathcal{L}$ of $\Lambda(M)$ has the following property: $\omega^* = (e^{tV})^*\omega\in\mathcal{L}$ for every form $\omega\in\mathcal{L}$ under the flow e^{tV} generated by a vector field V. We then say that $\mathcal{L}$ is ***stable*** or ***invariant*** submodule under the ***Lie transport*** with respect to the vector field V. It is quite clear that $\mathcal{L}$ is stable if and only if one has $\pounds_V\omega\in\mathcal{L}$ for every form $\omega\in\mathcal{L}$. We symbolically depict this property as $\pounds_V\mathcal{L}\subset\mathcal{L}$. In fact, let us first assume that $\pounds_V\omega\in\mathcal{L}$ for all $\omega\in\mathcal{L}$. We then obtain $\pounds_V(\pounds_V\omega)=\pounds_V^2\omega\in\mathcal{L}$ and similarly $\pounds_V^n\omega\in\mathcal{L}$ for all $n\in\mathbb{N}$. Since $\mathcal{L}$ is a submodule, (5.11.14) implies that $\omega^*\in\mathcal{L}$. Conversely, let us suppose that $\omega^*\in\mathcal{L}$ or all $\omega\in\mathcal{L}$. Since $\omega^*-\omega\in\mathcal{L}$ and t is an arbitrary parameter, we deduce from (5.11.14) that the conditions $\pounds_V\omega\in\mathcal{L},\ \pounds_V^2\omega\in\mathcal{L},\ \ldots,\ \pounds_V^n\omega\in\mathcal{L},\ \ldots$ must be satisfied for all $\omega\in\mathcal{L}$. These conditions are automatically satisfied when $\pounds_V\omega\in\mathcal{L}$. We see that *if a submodule $\mathcal{L}$ of $\Lambda(M)$ is stable under a vector field V, then it is not possible for a form $\omega\in\mathcal{L}$ to escape from that submodule through the action of the Lie derivative*.

Theorem 5.11.2. *The subalgebra $\mathcal{C}(M)$ of closed forms and the subalgebra $\mathcal{E}(M)$ of exact forms of the exterior algebra $\Lambda(M)$ are stable under every vector field $V\in T(M)$.*

If $\omega\in\mathcal{C}(M)$, then $d\omega=0$. Hence, for all vector fields we get $d\pounds_V\omega = \pounds_V d\omega = 0$ and $\pounds_V\omega\in\mathcal{C}(M)$. In like fashion, if $\omega\in\mathcal{E}(M)$, then there is a form $\sigma\in\Lambda(M)$ such that $\omega=d\sigma$. We thus obtain

$$\pounds_V\omega = \pounds_V d\sigma = d\pounds_V\sigma$$

implying that $\pounds_V\omega \in \mathcal{E}(M)$. □

Example 5.11.1. We want to calculate the Lie derivative of the volume form $\mu \in \Lambda^m(M^m)$ given by (5.9.14). Since $d\mu = 0$, we get

$$\pounds_V\mu = \mathbf{i}_V(d\mu) + d\mathbf{i}_V(\mu) = d\mathbf{i}_V(\mu).$$

On recalling (5.5.9) and the exterior derivatives of top down generated bases given on p. 279, it follows from $\mathbf{i}_V(\mu) = v^i\mu_i$ that

$$\begin{aligned}\pounds_V\mu &= v^i_{,j}\,dx^j \wedge \mu_i + v^i d\mu_i = v^i_{,j}\,\delta^j_i\mu + v^i\frac{(\sqrt{g}\,)_{,i}}{\sqrt{g}}\mu \\ &= \Big(v^i_{,i} + v^i\frac{(\sqrt{g}\,)_{,i}}{\sqrt{g}}\Big)\mu = v^i_{;i}\mu.\end{aligned}$$

Thus the volume form μ is invariant under divergenceless, or *solenoidal*, vector fields satisfying the condition $v^i_{;i} = 0$.

As another example, let us calculate the Lie derivatives of the basis forms $\mu_i \in \Lambda^{m-1}(M)$. Since we can write

$$\pounds_V\mu_i = d(v^j\mu_{ji}) + v^j\mathbf{i}_{\partial_j}(d\mu_i) = v^j_{,k}dx^k \wedge \mu_{ji} + v^j d\mu_{ji} + v^j\frac{(\sqrt{g}\,)_{,i}}{\sqrt{g}}\mu_j$$

on taking notice of relations

$$\begin{aligned}dx^k \wedge \mu_{ji} &= \delta^k_j\mu_i - \delta^k_i\mu_j, \\ d\mu_{ji} &= \frac{(\sqrt{g}\,)_{,k}}{\sqrt{g}}\delta^{lk}_{ij}\mu_l = \frac{(\sqrt{g}\,)_{,j}}{\sqrt{g}}\mu_i - \frac{(\sqrt{g}\,)_{,i}}{\sqrt{g}}\mu_j\end{aligned}$$

we finally get the result

$$\begin{aligned}\pounds_V\mu_i &= v^j_{,j}\mu_i - v^j_{,i}\mu_j + v^j\frac{(\sqrt{g}\,)_{,j}}{\sqrt{g}}\mu_i - v^j\frac{(\sqrt{g}\,)_{,i}}{\sqrt{g}}\mu_j + v^j\frac{(\sqrt{g}\,)_{,i}}{\sqrt{g}}\mu_j \\ &= \Big(v^j_{,j} + v^j\frac{(\sqrt{g}\,)_{,j}}{\sqrt{g}}\Big)\mu_i - v^j_{,i}\mu_j = v^j_{;j}\mu_i - v^j_{,i}\mu_j = (v^k_{;k}\,\delta^j_i - v^j_{,i})\mu_j.\end{aligned}$$

Thus the forms μ_i are invariant under vector fields satisfying the relation $v^j_{,i} = v^k_{;k}\,\delta^j_i$. On contracting this expression, we obtain

$$v^k_{,k} = mv^k_{;k} \quad \text{and} \quad mv^j_{,i} = v^k_{,k}\,\delta^j_i.$$ ■

We are now ready to evaluate the Lie derivative of any tensor if we take notice of the relations $(2.10.5)_2$ and (5.11.3) and recall that Lie derivative of tensor products verify the Leibniz rule as emphasised in (4.3.5). Let a tensor field $\mathcal{T} \in \mathfrak{T}(M)^k_l$ be designated by

$$\mathcal{T} = t^{i_1\cdots i_k}_{j_1\cdots j_l}\frac{\partial}{\partial x^{i_1}}\otimes\cdots\otimes\frac{\partial}{\partial x^{i_k}}\otimes dx^{j_1}\otimes\cdots\otimes dx^{j_l}.$$

The Lie derivative of this tensor with respect to a vector field V can then be expressed as

$$\begin{aligned}\pounds_V\mathcal{T} &= (\pounds_V t^{i_1\cdots i_k}_{j_1\cdots j_l})\,\frac{\partial}{\partial x^{i_1}}\otimes\cdots\otimes\frac{\partial}{\partial x^{i_k}}\otimes dx^{j_1}\otimes\cdots\otimes dx^{j_l}\\ &\quad+\sum_{r=1}^{k} t^{i_1\cdots i_k}_{j_1\cdots j_l}\frac{\partial}{\partial x^{i_1}}\otimes\cdots\otimes\pounds_V(\frac{\partial}{\partial x^{i_r}})\otimes\cdots\otimes\frac{\partial}{\partial x^{i_k}}\otimes dx^{j_1}\otimes\cdots\otimes dx^{j_l}\\ &\quad+\sum_{r=1}^{l} t^{i_1\cdots i_k}_{j_1\cdots j_l}\frac{\partial}{\partial x^{i_1}}\otimes\cdots\otimes\frac{\partial}{\partial x^{i_k}}\otimes dx^{j_1}\otimes\cdots\otimes\pounds_V(dx^{j_r})\otimes\cdots\otimes dx^{j_l}.\end{aligned}$$

We thus obtain

$$\begin{aligned}\pounds_V\mathcal{T} &= t^{i_1\cdots i_k}_{j_1\cdots j_l,i}v^i\,\frac{\partial}{\partial x^{i_1}}\otimes\cdots\otimes\frac{\partial}{\partial x^{i_k}}\otimes dx^{j_1}\otimes\cdots\otimes dx^{j_l} \qquad (5.11.15)\\ &\quad-\sum_{r=1}^{k} t^{i_1\cdots i_r\cdots i_k}_{j_1\cdots j_l}v^i_{,i_r}\frac{\partial}{\partial x^{i_1}}\otimes\cdots\otimes\frac{\partial}{\partial x^{i}}\otimes\cdots\otimes\frac{\partial}{\partial x^{i_k}}\otimes dx^{j_1}\otimes\cdots\otimes dx^{j_l}\\ &\quad+\sum_{r=1}^{l} t^{i_1\cdots i_k}_{j_1\cdots j_r\cdots j_l}v^{j_r}_{,j}\frac{\partial}{\partial x^{i_1}}\otimes\cdots\otimes\frac{\partial}{\partial x^{i_k}}\otimes dx^{j_1}\otimes\cdots\otimes dx^{j}\otimes\cdots\otimes dx^{j_l}\\ &= (\pounds_V\mathcal{T})^{i_1\cdots i_k}_{j_1\cdots j_l}\,\frac{\partial}{\partial x^{i_1}}\otimes\cdots\otimes\frac{\partial}{\partial x^{i_k}}\otimes dx^{j_1}\otimes\cdots\otimes dx^{j_l}\end{aligned}$$

where the components of the tensor $\pounds_V\mathcal{T}$ are given by

$$\begin{aligned}(\pounds_V\mathcal{T})^{i_1\cdots i_k}_{j_1\cdots j_l} &= t^{i_1\cdots i_k}_{j_1\cdots j_l,i}v^i - \sum_{r=1}^{k} t^{i_1\cdots i_{r-1}ii_{r+1}\cdots i_k}_{j_1\cdots j_l}v^{i_r}_{,i} \qquad (5.11.16)\\ &\quad+\sum_{r=1}^{l} t^{i_1\cdots i_k}_{j_1\cdots j_{r-1}jj_{r+1}\cdots j_l}v^{j}_{,j_r}.\end{aligned}$$

Let M and N be smooth manifolds and $\phi : M \to N$ be a smooth mapping. Let us consider a form $\omega \in \Lambda(N)$ and a vector field $V \in T(M)$. Let us calculate the Lie derivative of the form ω with respect to the vector field $V^* = \phi_* V \in T(N)$:

$$\pounds_{\phi_* V}\,\omega = \mathbf{i}_{\phi_* V}(d\omega) + d\mathbf{i}_{\phi_* V}(\omega) \in \Lambda(N).$$

We then pull the above form back to $\Lambda(M)$. On making use of (5.7.6) and Theorem 5.8.2, we can write

$$\phi^*\pounds_{\phi_*V}\,\omega = \phi^*\mathbf{i}_{\phi_*V}(d\omega) + \phi^* d\,\mathbf{i}_{\phi_*V}(\omega) = \mathbf{i}_V\big(d(\phi^*\omega)\big) + d\mathbf{i}_V(\phi^*\omega).$$

Therefore, for all forms $\omega \in \Lambda(N)$ we are led to

$$\phi^*\pounds_{\phi_*V}(\omega) = \pounds_V(\phi^*\omega) \in \Lambda(M) \tag{5.11.17}$$

and, consequently, to the relation

$$\phi^* \circ \pounds_{\phi_*V} = \phi^* \circ \pounds_{V^*} = \pounds_V \circ \phi^*. \tag{5.11.18}$$

5.12. ISOVECTOR FIELDS OF IDEALS

Let $\mathcal{I}$ be an ideal of the exterior algebra $\Lambda(M)$. If this ideal is stable under the flow generated by a vector field $V \in T(M)$, namely, if $\pounds_V\omega \in \mathcal{I}$ for every $\omega \in \mathcal{I}$ so that $\pounds_V\mathcal{I} \subset \mathcal{I}$, in other words, if the ideal $\mathcal{I}$ becomes *invariant* under the flow generated by V, then this vector field is called an ***isovector field*** of the ideal $\mathcal{I}$.

Theorem 5.12.1. *Let $\mathcal{I}(\omega^\alpha)$ be the ideal generated by the forms $\omega^\alpha \in \Lambda(M), \alpha = 1, \dots, r$. A vector field $V \in T(M)$ is an isovector field of $\mathcal{I}$ if and only if $\pounds_V\omega^\alpha \in \mathcal{I}$ for every generator ω^α of the ideal.*

If V is an isovector, then one has $\pounds_V\omega \in \mathcal{I}$ for every form $\omega \in \mathcal{I}$ so the generators ω^α must also fulfil the condition $\pounds_V\omega^\alpha \in \mathcal{I}$. This means that there exist appropriate forms $\lambda^\alpha_\beta \in \Lambda(M)$ such that $\pounds_V\omega^\alpha = \lambda^\alpha_\beta \wedge \omega^\beta$. Conversely, let us assume that $\pounds_V\omega^\alpha \in \mathcal{I}$ so that the relations $\pounds_V\omega^\alpha = \lambda^\alpha_\beta \wedge \omega^\beta$ are satisfied. Because of the restriction $deg\,(\lambda^\alpha_\beta) = deg\,(\omega^\alpha) - deg\,(\omega^\beta) \geq 0$, the forms whose degrees higher than that of ω^α cannot take place in the above sum. If $\omega \in \mathcal{I}$, then we can find forms $\gamma_\alpha \in \Lambda(M)$ so that we are able to write $\omega = \gamma_\alpha \wedge \omega^\alpha$. Therefore, we obtain

$$\pounds_V\omega = (\pounds_V\gamma_\alpha) \wedge \omega^\alpha + \gamma_\alpha \wedge \pounds_V\omega^\alpha = (\pounds_V\gamma_\alpha + \gamma_\beta \wedge \lambda^\beta_\alpha) \wedge \omega^\alpha$$

implying that $\pounds_V\omega \in \mathcal{I}$. □

If the ideal $\mathcal{I}$ is generated by forms of the same degree, then the vector field V is an isovector field of that ideal if we can find smooth functions $\lambda^\alpha_\beta \in \Lambda^0(M)$ that enable us to write $\pounds_V\omega^\alpha = \lambda^\alpha_\beta\,\omega^\beta$.

Theorem 5.12.2. *Isovectors of an ideal $\mathcal{I}$ of the exterior algebra $\Lambda(M)$ constitute a Lie algebra that is a subalgebra of the module $\mathfrak{V}(M)$.*

It is easy to show that isovectors form a subspace of the linear vector space $\mathfrak{V}(M)$ over the field of real numbers $\mathbb{R}$. Let V_1 and V_2 be two isovectors. We thus have $\sigma_1 = £_{V_1}\omega \in \mathcal{I}$ and $\sigma_2 = £_{V_2}\omega \in \mathcal{I}$ for every form $\omega \in \mathcal{I}$. On the other hand, (5.11.11) allows us to write $£_{V_1+V_2}\omega = £_{V_1}\omega + £_{V_2}\omega = \sigma_1 + \sigma_2 \in \mathcal{I}$. Hence, $V_1 + V_2$ is an isovector as well. Similarly, Let V be an isovector so that one finds $\sigma = £_V\omega \in \mathcal{I}$ for all $\omega \in \mathcal{I}$. For all functions $f \in \Lambda^0(M)$, the expression (5.11.10) yields

$$£_{fV}\,\omega = f£_V\,\omega + df \wedge \mathbf{i}_V(\omega) = f\sigma + df \wedge \mathbf{i}_V(\omega).$$

Hence, if only $df = 0$, that is, $f =$ *constant*, then one gets $£_{fV}\,\omega = f\sigma \in \mathcal{I}$. Accordingly, isovectors form a vector space only over $\mathbb{R}$. If U and V are isovectors, then (5.11.12) leads to $£_{[U,V]}\omega = £_U£_V\omega - £_V£_U\omega \in \mathcal{I}$ for all $\omega \in \mathcal{I}$ which means that the Lie product $[U, V]$ is also an isovector. Thus, isovectors constitute a Lie algebra over $\mathbb{R}$. □

The, say, r-dimensional Lie algebra formed by isovectors is, of course, determined by linearly independent vectors $V_\alpha, \alpha = 1, \dots, r \leq m$ and there exist *structure constants* $c^\gamma_{\alpha\beta}$ so that the conditions $[V_\alpha, V_\beta] = c^\gamma_{\alpha\beta}V_\gamma$ hold. Then, on recalling Sec. 3.8, we reach to the conclusion that isovectors generate an r-parameter Lie transformation group on the manifold M and the ideal $\mathcal{I}$ remains invariant under this mapping. In other words, a flow generated by an isovector transforms a form in the ideal to another form also in the ideal.

Theorem 5.12.3. *If the vector field $V \in T(M)$ is an isovector field of an ideal $\mathcal{I}(\omega^\alpha)$ of the exterior algebra $\Lambda(M)$, then it is also an isovector field of its closure $\overline{\mathcal{I}}(\omega^\alpha, d\omega^\alpha)$.*

If V is an isovector field of the ideal $\mathcal{I}$, then there are appropriate forms $\lambda^\alpha_\beta \in \Lambda(M)$ such that one is able to write $£_V\omega^\alpha = \lambda^\alpha_\beta \wedge \omega^\beta$. Employing this relation, we get

$$£_V d\omega^\alpha = d£_V\omega^\alpha = d\lambda^\alpha_\beta \wedge \omega^\beta + (-1)^{deg\,(\lambda^\alpha_\beta)}\lambda^\alpha_\beta \wedge d\omega^\beta.$$

We consider a form $\sigma \in \overline{\mathcal{I}}$ that can be written as $\sigma = \gamma_\alpha \wedge \omega^\alpha + \Gamma_\alpha \wedge d\omega^\alpha$. Hence, we obtain

$$\begin{aligned}£_V\sigma &= £_V\gamma_\alpha \wedge \omega^\alpha + \gamma_\alpha \wedge £_V\omega^\alpha + £_V\Gamma_\alpha \wedge d\omega^\alpha + \Gamma_\alpha \wedge £_V d\omega^\alpha\\ &= (£_V\gamma_\alpha + \gamma_\beta \wedge \lambda^\beta_\alpha + \Gamma_\beta \wedge d\lambda^\beta_\alpha) \wedge \omega^\alpha\\ &\quad + (£_V\Gamma_\alpha + (-1)^{deg\,(\lambda^\beta_\alpha)}\Gamma_\beta \wedge \lambda^\beta_\alpha) \wedge d\omega^\alpha \in \overline{\mathcal{I}}\end{aligned}$$

This expression means that V is also an isovector of the closure $\overline{\mathcal{I}}$ of the ideal $\mathcal{I}$. □

Evidently, this theorem does not imply that isovectors of the ideals $\mathcal{I}$ and $\overline{\mathcal{I}}$ are the same. Some isovectors of the closed ideal $\overline{\mathcal{I}}$ may not belong to the set of isovectors of the ideal $\mathcal{I}$. This situation will be remedied to some extent by the following theorem.

Theorem 5.12.4. *If an ideal $\mathcal{I}(\omega^\alpha)$ is generated by forms of the same degree, then isovectors of the ideals $\mathcal{I}$ and $\overline{\mathcal{I}}$ are coincident.*

We have demonstrated in Theorem 5.12.3 that isovectors of $\mathcal{I}$ are also isovectors of $\overline{\mathcal{I}}$. In order prove the present theorem, we have to show that the converse statement is also true. If V is an isovector of $\overline{\mathcal{I}}$, then there are suitable forms λ^α_β and Λ^α_β so that we can write

$$\pounds_V\omega^\alpha = \lambda^\alpha_\beta \wedge \omega^\beta + \Lambda^\alpha_\beta \wedge d\omega^\beta$$

whence we deduce that

$$\pounds_V d\omega^\alpha = d\pounds_V\omega^\alpha = d\lambda^\alpha_\beta \wedge \omega^\beta + \big((-1)^{deg\,(\lambda^\alpha_\beta)}\lambda^\alpha_\beta + d\Lambda^\alpha_\beta\big) \wedge d\omega^\beta.$$

However, if all forms ω^α possess the same degree, say k, then the degree of all forms $d\omega^\alpha$ is $k+1$ implying that we have to take $\Lambda^\alpha_\beta = 0$ and $\lambda^\alpha_\beta \in \Lambda^0(M)$. In this case, the above relations reduce to

$$\pounds_V\omega^\alpha = \lambda^\alpha_\beta\omega^\beta, \;\; \pounds_V d\omega^\alpha = d\lambda^\alpha_\beta \wedge \omega^\beta + \lambda^\alpha_\beta\, d\omega^\beta$$

from which we conclude that an isovector V of the ideal $\overline{\mathcal{I}}$ is also an isovector of the ideal $\mathcal{I}$. □

The following theorem provides a somewhat simplified approach to evaluate isovectors of an ideal.

Theorem 5.12.5. *Let $\mathcal{I}(\omega^\alpha)$ be an ideal of $\Lambda(M)$ generated by forms $\omega^\alpha, \alpha = 1, \dots, r$ whose degrees satisfy the condition $deg\,\omega^\alpha < k$. We then consider forms $\sigma^a, a = 1, \dots, s$ such that $deg\,\sigma^a \geq k$. A vector field V is an isovector of the ideal $\mathcal{I}(\omega^\alpha, \sigma^a)$ if and only if*
(i) it is an isovector of the ideal $\mathcal{I}(\omega^\alpha)$,
(ii) $\pounds_V\sigma^a \in \mathcal{I}(\omega^\alpha, \sigma^a)$.

Let us first assume that the vector field V is an isovector of the ideal $\mathcal{I}(\omega^\alpha)$ so that one has $\pounds_V\omega^\alpha = \lambda^\alpha_\beta \wedge \omega^\beta$. We further assume that $\pounds_V\sigma^a = \lambda^a_\alpha \wedge \omega^\alpha + \lambda^a_b \wedge \sigma^b$. If $\omega \in \mathcal{I}(\omega^\alpha, \sigma^a)$, then $\omega = \gamma_\alpha \wedge \omega^\alpha + \gamma_a \wedge \sigma^a$ and its Lie derivative with respect to V is found to be

$$\begin{aligned}\pounds_V\omega &= \pounds_V\gamma_\alpha \wedge \omega^\alpha + \gamma_\alpha \wedge \pounds_V\omega^\alpha + \pounds_V\gamma_a \wedge \sigma^a + \gamma_a \wedge \pounds_V\sigma^a \\ &= (\pounds_V\gamma_\alpha + \gamma_\beta \wedge \lambda^\beta_\alpha + \gamma_a \wedge \lambda^a_\alpha) \wedge \omega^\alpha \\ &\quad + (\pounds_V\gamma_a + \gamma_b \wedge \lambda^b_a) \wedge \sigma^a \in \mathcal{I}(\omega^\alpha, \sigma^a).\end{aligned}$$

Hence V is an isovector of the ideal $\mathcal{I}(\omega^\alpha, \sigma^a)$. Conversely, let us suppose that V is an isovector of the ideal $\mathcal{I}(\omega^\alpha, \sigma^a)$ implying that $\pounds_V\omega \in \mathcal{I}(\omega^\alpha, \sigma^a)$ for all $\omega \in \mathcal{I}(\omega^\alpha, \sigma^a)$. Hence, the above relation requires that the condition

$\gamma_\alpha \wedge \pounds_V \omega^\alpha + \gamma_a \wedge \pounds_V \sigma^a \in \mathcal{I}(\omega^\alpha, \sigma^a)$ must hold. This last expression should be valid of course for all forms ω in the ideal $\mathcal{I}(\omega^\alpha, \sigma^a)$, and consequently, for all forms $\gamma_\alpha, \gamma_a \in \Lambda(M)$ implying that we must have $\pounds_V \omega^\alpha \in \mathcal{I}(\omega^\alpha, \sigma^a)$ and $\pounds_V \sigma^a \in \mathcal{I}(\omega^\alpha, \sigma^a)$. We thus conclude that there must be suitable forms $\lambda^\alpha_\beta, \lambda^\alpha_a, \lambda^a_\alpha, \lambda^a_b$ so that we can write

$$\pounds_V \omega^\alpha = \lambda^\alpha_\beta \wedge \omega^\beta + \lambda^\alpha_a \wedge \sigma^a, \;\; \pounds_V \sigma^a = \lambda^a_\alpha \wedge \omega^\alpha + \lambda^a_b \wedge \sigma^b.$$

But, due to the restrictions $deg\, \omega^\alpha < k$ and $deg\, \sigma^a \geq k$, we get $\lambda^\alpha_a = 0$ and we find that $\pounds_V \omega^\alpha = \lambda^\alpha_\beta \wedge \omega^\beta$. Thus V must also be an isovector of the ideal $\mathcal{I}(\omega^\alpha)$. □

Based on the Theorem (5.12.5), we may propose quite an effective method to determine isovector fields of an ideal generated by forms of different degrees. Let us arrange the generators of the ideal according to increasing degrees and collate all forms of the same degree into a set so that let us write $\mathcal{I}(\omega^\alpha, \sigma^a, \gamma^A, \dots)$. The degrees of the forms in each set $\{\omega^\alpha\}$, $\{\sigma^a\}$, $\{\gamma^A\}, \dots$ are the same and they are ordered as follows: $deg\, \omega^\alpha < deg\, \sigma^a < deg\, \gamma^A < \cdots$. In this case, in order to determine the isovector fields, we have to ensure that the conditions

$$\pounds_V \omega^\alpha \in \mathcal{I}(\omega^\alpha), \;\; \pounds_V \sigma^a \in \mathcal{I}(\omega^\alpha, \sigma^a), \;\; \pounds_V \gamma^A \in \mathcal{I}(\omega^\alpha, \sigma^a, \gamma^A), \; \dots$$

are satisfied. Since we deal with a lesser number of forms in each set with uniform degrees, calculations turn out to be relatively simpler. Besides, if degrees in two sets differ just 1, and if some generators in one set happen to be exterior derivatives of some forms in the other set, then we can disregard these generators in view of Theorem 5.12.4.

Example 5.12.1. Let us determine the isovector fields of the ideal $\mathcal{I}(\omega^1)$ of the exterior algebra $\Lambda(\mathbb{R}^3)$ generated by $\omega^1 = x\,dy + y\,dz$. We denote a vector field by $V = v^x \partial_x + v^y \partial_y + v^z \partial_z$. We have to show that there exists a function $\lambda \in \Lambda^0(\mathbb{R}^3)$ such that $\pounds_V \omega^1 = \lambda\, \omega^1$. Let us write $d\omega^1 = dx \wedge dy + dy \wedge dz$, $\mathbf{i}_V(d\omega^1) = -v^y dx + (v^x - v^z)dy + v^y dz$ and $\mathbf{i}_V(\omega^1) = xv^y + yv^z = F(x, y, z)$. We thus obtain

$$\pounds_V \omega^1 = (F_x - v^y)dx + (F_y + v^x - v^z)dy + (F_z + v^y)dz = \lambda x\,dy + \lambda y\,dz$$

yielding $F_x - v^y = 0$, $F_y + v^x - v^z = \lambda x$ and $F_z + v^y = \lambda y$. Solution of these equations gives $\lambda = (F_x + F_z)/y$ and the isovector field specified by an arbitrary function F becomes

$$V_F = \frac{1}{y}(F + xF_z - yF_y)\frac{\partial}{\partial x} + F_x \frac{\partial}{\partial y} + \frac{1}{y}(F - xF_x)\frac{\partial}{\partial z}.$$

If the isovector fields produced by functions F and G are denoted by V_F and V_G, then their Lie product must be given by $[V_F, V_G] = V_H$. It is rather straightforward to verify that the function $H(x, y, z)$ is obtainable as

$$H = F_x G_y - G_x F_y + \frac{1}{y}(FG_x - GF_x + FG_z - GF_z) + \frac{x}{y}(F_z G_x - G_z F_x).$$

It is plainly seen that isovectors of the ideal $\mathcal{I}(\omega^1)$ constitute an infinite dimensional Lie algebra. ■

We have the following theorem if some of the isovectors of an ideal of $\Lambda(M)$ are also characteristic vectors of the same ideal.

Theorem 5.12.6. *If some of the isovectors of an ideal $\mathcal{I}$ are at the same time characteristic vectors of this ideal, then they form a Lie subalgebra of the Lie algebra of isovectors.*

If U and V are isovectors of an ideal $\mathcal{I}$, then we have $£_U\omega, £_V\omega \in \mathcal{I}$ for all $\omega \in \mathcal{I}$. If these vectors are also characteristic vectors of $\mathcal{I}$, they must satisfy $\mathbf{i}_U(\omega)$, $\mathbf{i}_V(\omega) \in \mathcal{I}$. On making use of (5.11.7), we get

$$\mathbf{i}_{[U,V]}(\omega) = £_U\big(\mathbf{i}_V(\omega)\big) - \mathbf{i}_V\big(£_U(\omega)\big) \in \mathcal{I}.$$

That means that the Lie product $[U, V]$ which is known to be an isovector is also a characteristic vector of the ideal. Therefore, such a subset of isovectors that are also the characteristic vectors of $\mathcal{I}$, is closed under the Lie product, that is, it is a Lie subalgebra. □

We can reach to a more interesting result in closed ideals.

Theorem 5.12.7. *If an ideal $\mathcal{I}$ of $\Lambda(M)$ is closed, then the subspace formed by its isovectors contains the characteristic subspace $\mathcal{S}(\mathcal{I})$.*

Let us assume that the ideal $\mathcal{I}$ is generated by forms $\omega^1, \omega^2, \dots,$ $\omega^r \in \Lambda(M)$ of various degrees. Since $\mathcal{I}$ is closed, then there are suitable forms $\lambda^\alpha_\beta \in \Lambda(M), \alpha, \beta = 1, \dots, r$ such that $d\omega^\alpha = \lambda^\alpha_\beta \wedge \omega^\beta$. On the other hand, if $V \in \mathcal{S}(\mathcal{I})$, then there exist appropriate forms $\mu^\alpha_\beta \in \Lambda(M)$ such that $\mathbf{i}_V(\omega^\alpha) = \mu^\alpha_\beta \wedge \omega^\beta$. Hence, according to $(5.4.1)_4$ we find that

$$\begin{aligned}\mathbf{i}_V(d\omega^\alpha) &= \mathbf{i}_V(\lambda^\alpha_\beta) \wedge \omega^\beta + (-1)^{deg(\lambda^\alpha_\beta)} \lambda^\alpha_\beta \wedge \mathbf{i}_V(\omega^\beta)\\ &= \big[\mathbf{i}_V(\lambda^\alpha_\beta) + (-1)^{deg(\lambda^\alpha_\gamma)} \lambda^\alpha_\gamma \wedge \mu^\gamma_\beta)\big] \wedge \omega^\beta \in \mathcal{I}.\end{aligned}$$

But the exterior derivative of the form $\mathbf{i}_V(\omega^\alpha)$ gives

$$\begin{aligned}d\mathbf{i}_V(\omega^\alpha) &= d\mu^\alpha_\beta \wedge \omega^\beta + (-1)^{deg(\mu^\alpha_\beta)} \mu^\alpha_\beta \wedge d\omega^\beta\\ &= \big[d\mu^\alpha_\beta + (-1)^{deg(\mu^\alpha_\gamma)} \mu^\alpha_\gamma \wedge \lambda^\gamma_\beta)\big] \wedge \omega^\beta \in \mathcal{I}\end{aligned}$$

from which we deduce that

$$\pounds_V \omega^\alpha = \mathbf{i}_V(d\omega^\alpha) + d\mathbf{i}_V(\omega^\alpha) \in \mathcal{I}.$$

Then Theorem 5.12.1 states that the characteristic vector V is also an isovector of the closed ideal $\mathcal{I}$, that is, the characteristic subspace of the ideal $\mathcal{I}$ belongs to the subspace generated by isovectors of this closed ideal. □

When we combine this theorem with Theorem 5.12.6 we arrive immediately at the following result: *characteristic vectors of a closed ideal constitute a Lie algebra*. However, we have to stress the fact that the converse of Theorem 5.12.7 is in general not true, i.e., all isovectors of a closed ideal are not necessarily characteristic vectors of this ideal.

5.13. EXTERIOR SYSTEMS AND THEIR SOLUTIONS

We have seen in *p.* 258 how we can engender a nontrivial, $r \geq k$ dimensional solution of an exterior equation $\omega = 0$ where $\omega \in \Lambda^k(M)$. We shall now explore the notion of exterior equations in a more general context.

Let us consider a set $\{\omega^\alpha, \alpha = 1, \ldots, N\}$ of forms that might be of different degrees. We specify an r-dimensional submanifold S by the mapping $\phi : S \to M$. *If we get* $\phi^*\omega^\alpha = 0, \alpha = 1, \ldots, N$, *namely, if the mapping* $\phi^* : \Lambda(M) \to \Lambda(S)$ *annihilates the forms* $\{\omega^\alpha\}$ *then the mapping* ϕ, *in other words*, *the submanifold* S *is said to be a solution of the system of exterior equations* $\{\omega^\alpha = 0, \alpha = 1, \ldots, N\}$. A submanifold whose dimension is less than the lowest degree of the forms ω^α is of course a trivial solution of the exterior system. Let us now take the ideal $\mathcal{I}(\omega^\alpha)$ into consideration. The mapping ϕ will be the solution of every form $\omega \in \mathcal{I}(\omega^\alpha)$ as well. In fact, if we write $\omega = \lambda_\alpha \wedge \omega^\alpha$, we find from (5.7.4) that $\phi^*\omega = \phi^*\lambda_\alpha \wedge \phi^*\omega^\alpha = 0.$ Conversely, we can easily demonstrate that the forms annihilated on a submanifold S prescribed by a mapping $\phi : S \to M$, or amounting to the same thing, all forms which annihilates the subbundle $T(S) \subset T(M)$ constitute an ideal of the exterior algebra $\Lambda(M)$. Let us consider the pull-back mapping $\phi^* : \Lambda(M) \to \Lambda(S)$ induced by the mapping ϕ. All forms annihilated on the submanifold S satisfy the relation $\phi^*\omega = 0$. We denote the set of all forms ω such that $\phi^*\omega = 0$ by $\mathcal{I} \subset \Lambda(M)$. If $\omega_1, \omega_2 \in \mathcal{I}$ are two forms with the same degree, then we have $\phi^*(\omega_1 + \omega_2) = \phi^*(\omega_1) + \phi^*(\omega_2) = 0$ implying that $\omega_1 + \omega_2 \in \mathcal{I}$. Similarly, if $\omega \in \mathcal{I}$ and $\gamma \in \Lambda(M)$ is an arbitrary form, then $\phi^*(\gamma \wedge \omega) = \phi^*(\gamma) \wedge \phi^*(\omega) = 0$ which means that $\gamma \wedge \omega \in \mathcal{I}$. Hence, $\mathcal{I}$ is an ideal of the exterior algebra.

If all forms of the exterior algebra $\Lambda(M)$ that are annihilated by every solution of exterior equations $\{\omega^\alpha = 0\}$ belong to the ideal $\mathcal{I}(\omega^\alpha)$ generated by forms ω^α, then $\mathcal{I}$ is called a ***complete ideal***.

Theorem 5.13.1. *An ideal of the exterior algebra* $\Lambda(M)$ *generated by* *linearly independent* 1*-forms is complete.*

Let us assume that the ideal is generated by linearly independent forms $\omega^\alpha \in \Lambda^1(M)$, $\alpha = 1, \ldots, N \le m$. As we have mentioned above, the solutions of the exterior equations $\{\omega^\alpha = 0\}$ annihilate every form within the ideal. We now suppose that solutions of the system $\{\omega^\alpha = 0\}$ annihilate a form $\omega \in \Lambda(M)$ as well. By adding suitable linearly independent 1-forms, we can determine a basis of $T^*(M)$ as follows: $\omega^1, \ldots, \omega^N,\ \omega^{N+1}, \ldots, \omega^m$. The form ω can now be constructed as a combination of exterior products of these forms. However, we have assumed that $\omega = 0$ whenever $\omega^1 = \cdots = \omega^N = 0$. Therefore, at least one of the factors $\omega^1, \ldots, \omega^N$ must be present in each term. Hence, we conclude that ω is expressible as

$$\omega = \lambda_1 \wedge \omega^1 + \lambda_2 \wedge \omega^2 + \cdots + \lambda_N \wedge \omega^N \in \mathcal{I}(\omega^\alpha) \qquad \square$$

Let us next consider two exterior systems $\{\omega^\alpha, \alpha = 1, \ldots, N_1\}$ and $\{\sigma^a, a = 1, \ldots, N_2\}$, and the ideals $\mathcal{I}_1 = \mathcal{I}(\omega^\alpha)$ and $\mathcal{I}_2 = \mathcal{I}(\sigma^a)$ generated by them. If these ideals are equal, namely, if they satisfy the relations $\mathcal{I}_1 \subseteq \mathcal{I}_2$ and $\mathcal{I}_2 \subseteq \mathcal{I}_1$, we say that these two exterior systems are *algebraically equivalent*. In this situation, there are appropriate forms λ^α_a and Λ^a_α so that we can write $\omega^\alpha = \lambda^\alpha_a \wedge \sigma^a$ and $\sigma^a = \Lambda^a_\alpha \wedge \omega^\alpha$.

Example 5.13.1. Let us consider a system of exterior equations of the exterior algebra $\Lambda(\mathbb{R}^4)$ specified by the forms $\omega^1 = dx^1 \wedge dx^3$, $\omega^2 = dx^1 \wedge dx^4$, $\omega^3 = dx^1 \wedge dx^2 - dx^3 \wedge dx^4$. A 2-dimensional submanifold of $\mathbb{R}^4$ is determined by the mapping $x^i = \phi^i(u^1, u^2), 1 \le i \le 4$. We now impose the condition that this mapping must satisfy

$$\begin{aligned}&\phi^*\omega^1 = \phi^1_{,\alpha}\phi^3_{,\beta}\, du^\alpha \wedge du^\beta = 0, \quad \phi^*\omega^2 = \phi^1_{,\alpha}\phi^4_{,\beta}\, du^\alpha \wedge du^\beta = 0,\\ &\phi^*\omega^3 = (\phi^1_{,\alpha}\phi^2_{,\beta} - \phi^3_{,\alpha}\phi^4_{,\beta})\, du^\alpha \wedge du^\beta = 0,\ \alpha, \beta = 1, 2.\end{aligned}$$

We immediately discover a solution by just inspection as $\phi^1 =$ *constant* and $\phi^3 =$ *constant*. We then consider the form $\omega = dx^1 \wedge dx^2$. We find that $\phi^*\omega = \phi^1_{,\alpha}\phi^2_{,\beta}\, du^\alpha \wedge du^\beta = 0$. But, we realise at once that this form does not belong to the ideal $\mathcal{I}(\omega^1, \omega^2, \omega^3)$. Hence, this ideal is not complete. ■

Certain significant properties of ideals of the exterior algebra can be discussed by means of Lie derivatives. An effective tool implementing this approach is provided by the following Cartan theorem.

Theorem 5.13.2 (The Cartan Theorem). *Let $\mathcal{I}$ be an ideal of the exterior algebra $\Lambda(M)$ and let $\mathcal{S}(\mathcal{I}) \subset T(M)$ be the characteristic subspace of constant dimension of this ideal. If $\mathcal{I}$ is a closed ideal, then the subspace $\mathcal{S}(\mathcal{I})$ is an involutive distribution of $T(M)$.*

We know that the characteristic subspace of the ideal $\mathcal{I}$ is defined by $\mathcal{S}(\mathcal{I}) = \{V \in T(M) : \mathbf{i}_V(\mathcal{I}) \subseteq \mathcal{I}\}$. Since we have assumed that $\mathcal{S}$ has the same dimension, say, r at every point of the manifold M, the characteristic subspace is spanned by r linearly independent vector fields $V_\alpha \in T(M), \alpha = 1, \dots, r$. It follows from (5.11.7) that

$$\begin{aligned}\mathbf{i}_{[U,V]}(\omega) &= £_U\big(\mathbf{i}_V(\omega)\big) - \mathbf{i}_V\big(£_U(\omega)\big)\\ &= \mathbf{i}_U[d\big(\mathbf{i}_V(\omega)\big)] + d[\mathbf{i}_U\big(\mathbf{i}_V(\omega)\big)] - \mathbf{i}_V\big(\mathbf{i}_U(d\omega)\big) - \mathbf{i}_V[d\big(\mathbf{i}_U(\omega)\big)]\end{aligned}$$

for all $\omega \in \Lambda(M)$ and $U, V \in T(M)$. Thus we obtain

$$\begin{aligned}&\mathbf{i}_{[V_\alpha,V_\beta]}(\omega) =\\ &\qquad \mathbf{i}_{V_\alpha}[d\big(\mathbf{i}_{V_\beta}(\omega)\big)] + d[\mathbf{i}_{V_\alpha}\big(\mathbf{i}_{V_\beta}(\omega)\big)] - \mathbf{i}_{V_\beta}\big(\mathbf{i}_{V_\alpha}(d\omega)\big) - \mathbf{i}_{V_\beta}[d\big(\mathbf{i}_{V_\alpha}(\omega)\big)]\end{aligned}$$

for all $\omega \in \mathcal{I}$ and $V_\alpha, V_\beta \in \mathcal{S}$. Since V_α and V_β are characteristic vectors of the closed ideal, we can write $\mathbf{i}_{V_\alpha}(\mathcal{I}) \subseteq \mathcal{I}, \alpha = 1, \dots, r$ and $d\mathcal{I} \subseteq \mathcal{I}$. This implies that each term in the right hand side of the above expression is in the ideal. Hence, we get $\mathbf{i}_{[V_\alpha,V_\beta]}(\omega) \in \mathcal{I}$ for all $\omega \in \mathcal{I}$. This amounts to say that $[V_\alpha, V_\beta] \in \mathcal{S}$. In other words, the characteristic subspace is closed under the Lie product. Thus $\mathcal{S}$ is an involutive distribution. Therefore, the characteristic vector fields of a closed ideal engender a smooth r-dimensional submanifold of M. □

Let us now consider the exterior system $D_r = \{\omega^\alpha\}$ comprised of r linearly independent 1-forms. The exterior equations $\omega^\alpha = 0, \alpha = 1, \dots, r$ constitute a ***Pfaff system*** [German mathematician Johann Friedrich Pfaff (1765-1825)]. According to Theorem 5.13.1, the ideal $\mathcal{I}(D_r)$ generated by these forms is complete. The exterior system D_r is ***completely integrable*** if it is annihilated on every one of the $(m-r)$-dimensional submanifolds prescribed by equations of the form

$$g^\alpha(\mathbf{x}) = c^\alpha,\ \alpha = 1, \dots, r$$

with r parameter. c^α are arbitrary real constants. Since $\mathcal{I}(D_r)$ is a complete ideal, all forms annihilated by those submanifolds, which are called ***characteristic manifolds***, must belong to this ideal.

Theorem 5.13.3. *An exterior system D_r is completely integrable if and only if it is possible to find a regular $r \times r$ matrix function $\mathbf{A}(\mathbf{x})$ and r independent functions $g^\alpha(\mathbf{x})$ such that the following relations are valid*:

$$\omega^\alpha = A^\alpha_\beta\, dg^\beta,\ \alpha, \beta = 1, \dots, r,\ \mathbf{A}(\mathbf{x}) = [A^\alpha_\beta(\mathbf{x})]. \tag{5.13.1}$$

If the forms $\{\omega^\alpha\}$ are given by the relations (5.13.1), when $g^\beta = c^\beta =$ *constant* we find $dg^\beta = 0$ and consequently $\omega^\alpha = 0$. Thus the exterior system D_r is completely integrable. Conversely, let us assume that the exterior system D_r is completely integrable. Hence, there are r independent functions $g^\alpha(\mathbf{x})$ and the ideal $\mathcal{I}(D_r)$ is annihilated by hypersurfaces $g^\alpha(\mathbf{x}) = c^\alpha$. Next, we form the ideal $\mathcal{I}(dg^\alpha)$ by the forms $dg^\alpha \in \Lambda^1(M)$. Since $dg^\alpha = 0$ on these hypersurfaces, this ideal is also annihilated by them. Because of the fact that both ideals are complete, we arrive at the result $\mathcal{I}(D_r) = \mathcal{I}(dg^\alpha)$. This implies that there are functions $A^\alpha_\beta \in \Lambda^0(M)$ such that $\omega^\alpha = A^\alpha_\beta\, dg^\beta$. The forms ω^α and dg^α are linearly independent. Therefore, we ought to have $\omega^1 \wedge \cdots \wedge \omega^r \neq 0$ and $dg^1 \wedge \cdots \wedge dg^r \neq 0$. Thus, the relation

$$\omega^1 \wedge \cdots \wedge \omega^r = \det(A^\alpha_\beta)\, dg^1 \wedge \cdots \wedge dg^r \neq 0$$

requires that $\det(A^\alpha_\beta) \neq 0$. □

If we calculate the exterior derivative of the expression (5.13.1), we get

$$d\omega^\alpha = dA^\alpha_\beta \wedge dg^\beta = (A^{-1})^\beta_\gamma\, dA^\alpha_\beta \wedge \omega^\gamma \in \mathcal{I}(D_r),$$

Hence, *if the exterior system D_r is completely integrable, then the ideal $\mathcal{I}(D_r)$ must be closed.* That the converse proposition is also true is provided by the following theorem referred to Frobenius.

Theorem 5.13.4 (The Frobenius Theorem). *An exterior system D_r is completely integrable if and only if the ideal $\mathcal{I}(D_r)$ generated by r linearly independent 1-forms $\{\omega^\alpha\}$ is closed, that is, if $d\mathcal{I}(D_r) \subseteq \mathcal{I}(D_r)$ or if there exist r^2 forms $\Gamma^\alpha_\beta \in \Lambda^1(M)$ such that the relations $d\omega^\alpha = \Gamma^\alpha_\beta \wedge \omega^\beta$ are satisfied or if we verify that $d\omega^\alpha \wedge \omega^1 \wedge \cdots \wedge \omega^r = 0$ for $\alpha = 1, \dots, r$.*

We have already seen that the ideal $\mathcal{I}(D_r)$ will be closed if the exterior system D_r is completely integrable. Let us assume, this time, that the ideal $\mathcal{I}(D_r)$ is closed. We know that the dimension of the characteristic subspace $\mathcal{S}(D_r)$ of this ideal is $m - r$ [*see* Theorem 5.6.2]. Let the linearly independent vectors $U_a, a = r+1, \dots, m$ be a basis of that subspace. According to the Cartan theorem 5.13.2, $\mathcal{S}(D_r)$ is an involutive distribution, i.e., there are functions $c^c_{ab} \in \Lambda^0(M)$ such that $[U_a, U_b] = c^c_{ab} U_c$. In this situation, we can choose, as we have done in Theorem 2.11.1, a new basis set as vectors V_a, $a = r+1, \dots, m$ of $\mathcal{S}(D_r)$ such that $[V_a, V_b] = 0$. We shall now show that this property guaranties the existence of independent functions $g^\alpha(\mathbf{x}), \alpha = 1, \dots, r$ satisfying the relations $V_a(g^\alpha) = \mathbf{i}_{V_a}(dg^\alpha) = 0$. To this end, we look for the solutions of the system of differential equations $V_a(f) = 0$. On repeating our approach in Sec. 2.11, we start with $V_{r+1}(f) = 0$. It is known that the *independent solutions* of the first order partial differential equation

$$v^i_{r+1}(\mathbf{x}) \frac{\partial f}{\partial x^i} = 0$$

can be determined through the method of characteristics as follows

$$h^1(\mathbf{x}) = C^1,\ h^2(\mathbf{x}) = C^2,\ \ldots\ ,h^{m-1}(\mathbf{x}) = C^{m-1}$$

where $C^1, \ldots, C^{m-1}$ are constants. We then find

$$0 = \frac{\partial h^{\mathfrak{a}}}{\partial x^i}\frac{dx^i}{dt} = v^i_{r+1}(\mathbf{x})\frac{\partial h^{\mathfrak{a}}}{\partial x^i} = V_{r+1}(h^{\mathfrak{a}})$$

where $\mathfrak{a} = 1, \ldots, m-1$. We thus write

$$\begin{aligned}
V_{r+1}(f) &= v^1_{r+1}(\mathbf{x})\frac{\partial f}{\partial x^1} + \cdots + v^m_{r+1}(\mathbf{x})\frac{\partial f}{\partial x^m} = 0\\
V_{r+1}(h^1) &= v^1_{r+1}(\mathbf{x})\frac{\partial h^1}{\partial x^1} + \cdots + v^m_{r+1}(\mathbf{x})\frac{\partial h^1}{\partial x^m} = 0\\
&\vdots\\
V_{r+1}(h^{m-1}) &= v^1_{r+1}(\mathbf{x})\frac{\partial h^{m-1}}{\partial x^1} + \cdots + v^m_{r+1}(\mathbf{x})\frac{\partial h^{m-1}}{\partial x^m} = 0.
\end{aligned}$$

Since $V_{r+1} \neq 0$, it is only possible to find a nontrivial solution to this homogeneous system of equations if the Jacobian, or the functional determinant, of the functions $f, h^1, \ldots,\ h^{m-1}$ vanishes

$$\frac{\partial(f, h^1, \ldots h^{m-1})}{\partial(x^1, x^2, \ldots, x^m)} = 0.$$

It is known that the general solution of the foregoing equation is

$$f = f(h^1, h^2, \ldots, h^{m-1}). \tag{5.13.2}$$

In the second step, let us apply the operator V_{r+2} on the function (5.13.2) to obtain

$$0 = V_{r+2}(f) = v^i_{r+2}\frac{\partial f}{\partial x^i} = v^i_{r+2}\frac{\partial f}{\partial h^{\mathfrak{a}}}\frac{\partial h^{\mathfrak{a}}}{\partial x^i} = V_{r+2}(h^{\mathfrak{a}})\frac{\partial f}{\partial h^{\mathfrak{a}}}. \tag{5.13.3}$$

On the other hand, because of the relation $V_{r+1}V_{r+2} = V_{r+2}V_{r+1}$, we find

$$V_{r+1}\big(V_{r+2}(h^{\mathfrak{a}})\big) = V_{r+2}\big(V_{r+1}(h^{\mathfrak{a}})\big) = V_{r+2}(0) = 0$$

which means that the functions $V_{r+2}(h^{\mathfrak{a}})$ become solutions of the equation $V_{r+1}(u) = 0$. We can thus write as in (5.13.2)

$$V_{r+2}(h^{\mathfrak{a}}) = H^{\mathfrak{a}}(h^1, h^2, \ldots, h^{m-1})$$

and the equation (5.13.3) takes the form

$$H^{\mathfrak{a}}(h^{\mathfrak{b}})\frac{\partial f}{\partial h^{\mathfrak{a}}} = 0, \ \ \mathfrak{a}, \mathfrak{b} = 1, \dots, m-1.$$

Hence, the number of independent variables reduces to $m-1$ from m. By repeating the same procedure as above we obtain $f = f(k^1, k^2, \dots, k^{m-2})$ where $k^s = k^s(h^1, h^2, \dots, h^{m-1}), s = 1, \dots, m-2$. On applying the operators $V_{r+1}, \dots, V_m$, respectively, on the function f, we see that f is dependent on $m-(m-r) = r$ independent functions $g^\alpha \in \Lambda^0(M), \alpha = 1, \dots, r$ as follows:

$$f = f(g^1, g^2, \dots, g^r). \tag{5.13.4}$$

The functions g^α are clearly determined by successively solving a sequence of ordinary differential equations with ever decreasing number of dependent variables. We can then write

$$V_a(f) = v_a^i \frac{\partial f}{\partial g^\alpha}\frac{\partial g^\alpha}{\partial x^i} = V_a(g^\alpha)\frac{\partial f}{\partial g^\alpha} = 0, \ a = r+1, \dots, m.$$

This relation would of course be valid for all functions in the form (5.13.4). If we choose $f = g^\beta$, we find

$$V_a(g^\alpha)\,\delta_\alpha^\beta = V_a(g^\beta) = 0, \ a = r+1, \dots, m, \beta = 1, \dots, r$$

implying that

$$V_a(g^\alpha) = \mathbf{i}_{V_a}(dg^\alpha) = 0. \tag{5.13.5}$$

Since the functions g^α are independent, the forms $dg^\alpha \in \Lambda^1(M)$ must be linearly independent so that one gets $\Omega = dg^1 \wedge \cdots \wedge dg^r \neq 0$. According to Theorem 5.6.1 the relations (5.13.5) express the fact that the vectors $\{V_a\}$ are also characteristic vectors of the ideal $\mathcal{I}(dg^\alpha)$. We can now readily prove that $\mathcal{I}(D_r) = \mathcal{I}(\omega^\alpha) \subseteq \mathcal{I}(dg^\alpha)$. Let us assume that one of the generators of the ideal $\mathcal{I}(\omega^\alpha)$, say ω^α, does not belong to the ideal $\mathcal{I}(dg^\alpha)$. On referring to the statement on *p.* 249, we are thus compelled to assume that $\mathbf{i}_{V_a}(\omega^\alpha) \neq 0$. However, V_a is a characteristic vector of the ideal $\mathcal{I}(\omega^\alpha)$ as well and the condition $\mathbf{i}_{V_a}(\omega^\alpha) = 0$ must be satisfied. In order to remove this contradiction, we have to take $\omega^\alpha \in \mathcal{I}(dg^\alpha)$. Hence, all generators of $\mathcal{I}(\omega^\alpha)$ must belong to $\mathcal{I}(dg^\alpha)$. This means that $\mathcal{I}(\omega^\alpha) \subseteq \mathcal{I}(dg^\alpha)$. Therefore, there exists a regular matrix $[A_\beta^\alpha(\mathbf{x})]$ such that the relations $\omega^\alpha = A_\beta^\alpha\, dg^\beta$ are to be satisfied. Thus, the exterior system is completely integrable. □

We have to pay attention to the fact that the functions g^α and the matrix $[A_\beta^\alpha]$ cannot be determined uniquely. Provided that the functions

$h^\alpha = h^\alpha(g^1, g^2, \dots, g^r)$ are so chosen that the condition det $(\partial h^\alpha/\partial g^\beta) \neq 0$ is satisfied, the forms dh^α become linearly independent and we find that

$$\mathbf{i}_{V_a}(dh^\alpha) = \mathbf{i}_{V_a}\Big(\frac{\partial h^\alpha}{\partial g^\beta} dg^\beta\Big) = \frac{\partial h^\alpha}{\partial g^\beta}\mathbf{i}_{V_a}(dg^\beta) = 0.$$

Hence, we can write $\omega^\alpha = B^\alpha_\beta \, dh^\beta$. But, it is easily verified that the relation

$$A^\alpha_\beta = B^\alpha_\gamma \frac{\partial h^\gamma}{\partial g^\beta}$$

must be satisfied.

The generalisation of the Frobenius theorem to ideals generated by forms of diverse degrees is given below.

Theorem 5.13.5. *Let $\mathcal{I}$ be a closed ideal of the exterior algebra $\Lambda(M)$ generated by forms of various degrees. If the dimension of the characteristic subspace $\mathcal{S}(\mathcal{I})$ of $\mathcal{I}$ is $m - r$, then there exist r functionally independent functions $g^\alpha(\mathbf{x}) \in \Lambda^0(M), \alpha = 1, \dots, r$ and the ideal $\mathcal{I}$ is contained in the closed ideal generated by forms $dg^\alpha \in \Lambda^1(M), \alpha = 1, \dots, r$.*

Since $\mathcal{I}$ is closed, its characteristic subspace is an involutive distribution in view of Theorem 5.13.2. Hence, the characteristic basis vectors $V_a = v^i_a \partial_i, a = r+1, \dots, m$ can be so chosen that $[V_a, V_b] = 0$. Thereby, following the path leading to Theorem 5.13.4 we can determine independent functions $g^\alpha(\mathbf{x}), \alpha = 1, \dots, r$ satisfying the relations $V_a(g^\alpha) = \mathbf{i}_{V_a}(dg^\alpha) = 0$. Let $\mathcal{J}(dg^\alpha)$ denote the completely integrable closed ideal generated by forms $dg^\alpha \in \Lambda^1(M)$. Then Theorem 5.6.4 implies that $\mathcal{I} \subseteq \mathcal{J}(dg^\alpha)$. Since the ideal $\mathcal{J}(dg^\alpha)$ is generated by 1-forms, it is the largest ideal admitting $\mathcal{S}(\mathcal{I})$ as its characteristic subspace. In this case, if $\omega \in \mathcal{I}$ then there are suitable forms $\gamma_\alpha \in \Lambda(M)$ so that one is able to write $\omega = \gamma_\alpha \wedge dg^\alpha$. Consequently, if we introduce $(m-r)$-dimensional *characteristic submanifolds* prescribed by the relations $g^\alpha(\mathbf{x}) = c^\alpha =$ *constant*, $\alpha = 1, \dots, r$ obtained through integration of the following sets of ordinary differential equations

$$\frac{dx^1}{v^1_a} = \frac{dx^2}{v^2_a} = \cdots = \frac{dx^m}{v^m_a} \quad \text{or} \quad \frac{dx^i}{dt} = v^i_a; \;\; a = r+1, \dots, m$$

which determine the integral curves of characteristic vector fields, then it is quite clear that those manifolds are also a solution of the ideal $\mathcal{I}$. □

It is now obvious that a solution of a closed ideal $\mathcal{I}$ provided by Theorem 5.13.5 corresponds to a solution determined by maximal number of independent functions g^α. Hence, this approach cannot usually reveal all solutions of the ideal $\mathcal{I}$. It might be quite possible that there exist submanifolds annihilating the ideal $\mathcal{I}$ whose dimensions are larger than $m - r$ so that they can be determined by means of a smaller amount of functions, but not solving the ideal $\mathcal{J}$. However, it is impossible to offer a systematic approach based on the above procedure to access such kinds of solutions corresponding, most probably, to much more realistic situations. Unfortunately, we can frequently produce only rather trivial solutions by applying Theorem 5.13.5.

Example 5.13.2. We build an ideal of the exterior algebra $\Lambda(\mathbb{R}^4)$ by forms $\omega^1 = dx^1 + x^2\,dx^3 + dx^4 \in \Lambda^1(\mathbb{R}^4)$, $\omega^2 = x^2\,dx^2 \wedge dx^3 \in \Lambda^2(\mathbb{R}^4)$. Since $d\omega^1 = dx^2 \wedge dx^3 = \omega^2/x^2$ and $\mathrm{d}\omega^2 = 0$, the ideal $\mathcal{I}(\omega^1, \omega^2)$ is closed and its characteristic vectors must satisfy the relations

$$\begin{aligned}
&\mathbf{i}_V(\omega^1) = v^1 + x^2\,v^3 + v^4 = 0\\
&\mathbf{i}_V(\omega^2) = x^2\,(v^2 dx^3 - \, v^3 dx^2) = \lambda(dx^1 + x^2\,dx^3 + dx^4)
\end{aligned}$$

from which we obtain

$$\lambda = v^2 = 0,\ v^3 = 0,\ v^4 = \,-v^1$$

and $V = v^1(\partial_1 - \partial_4)$. Thus the basis vector of 1-dimensional characteristic subspace can be chosen as $V_4 = \partial_1 - \partial_4$. Therefore, the solution of the differential equation

$$V_4(f) = \frac{\partial f}{\partial x^1} - \frac{\partial f}{\partial x^4} = 0$$

yields $f = f(x^1 + x^4, x^2, x^3)$ and we have $g^1 = x^1 + x^4$, $g^2 = x^2, g^3 = x^3$. Hence, 1-dimensional solution submanifolds are determined by $x^1 + x^4 = c^1$, $x^2 = c^2, x^3 = c^3$. We immediately observe that if we define the forms $dg^1 = dx^1 + dx^4$, $dg^2 = dx^2$, $dg^3 = dx^3$ we can write $\omega^1 = dg^1 + x^2\,dg^3$, $\omega^2 = x^2\,dg^2 \wedge dg^3$ meaning that $\mathcal{I}(\omega^1, \omega^2) \subset \mathcal{J}(dg^1, dg^2, dg^3)$. However, we can easily check that $\mathcal{I} \neq \mathcal{J}$. For instance, forms like $g(\mathbf{x})\,dg^2$ does not belong to $\mathcal{I}(\omega^1, \omega^2)$.

Let us now search for a larger, say 2-dimensional solution submanifold of the same ideal. We designate the mapping $\phi : \mathbb{R}^2 \to \mathbb{R}^4$ by functions $x^i = f^i(x, y), i = 1, \ldots, 4$. The exterior equations

$$\begin{aligned}
&\phi^*\omega^1 = \Big(\frac{\partial f^1}{\partial x} + f^2\frac{\partial f^3}{\partial x} + \frac{\partial f^4}{\partial x}\Big)\,dx + \Big(\frac{\partial f^1}{\partial y} + f^2\frac{\partial f^3}{\partial y} + \frac{\partial f^4}{\partial y}\Big)\,dy = 0\\
&\phi^*\omega^2 = f^2\Big(\frac{\partial f^2}{\partial x}\frac{\partial f^3}{\partial y} - \frac{\partial f^2}{\partial y}\frac{\partial f^3}{\partial x}\Big)\,dx \wedge dy = 0
\end{aligned}$$

can only be satisfied if we choose the functions f^i as solutions of the first order partial differential equations

$$\frac{\partial f^1}{\partial x} + f^2\frac{\partial f^3}{\partial x} + \frac{\partial f^4}{\partial x} = 0,\ \frac{\partial f^1}{\partial y} + f^2\frac{\partial f^3}{\partial y} + \frac{\partial f^4}{\partial y} = 0,$$
$$f^2\Big(\frac{\partial f^2}{\partial x}\frac{\partial f^3}{\partial y} - \frac{\partial f^2}{\partial y}\frac{\partial f^3}{\partial x}\Big) = 0.$$

For a simple example, we choose to take $f^2 = 0$. Then the solution is easily found to be

$$f^1 = f(x,y),\ f^2 = 0,\ f^3 = g(x,y),\ f^4 = c - f(x,y)$$

where$f$ and g are arbitrary functions and c is an arbitrary constant. ■

We know that if the ideal $\mathcal{I}(\omega^\alpha)$ is not closed, then a closed ideal containing $\mathcal{I}$ is its closure $\overline{\mathcal{I}}(\omega^\alpha, d\omega^\alpha)$.

Theorem 5.13.6. *Let an ideal of the exterior algebra $\Lambda(M)$ be $\mathcal{I}$ and its closure be $\overline{\mathcal{I}} = \mathcal{I} \cup d\mathcal{I}$. If a mapping $\phi : S \to M$ is a solution of the ideal $\mathcal{I}$, then it is likewise a solution of its closure $\overline{\mathcal{I}}$.*

When $\omega \in \overline{\mathcal{I}}$, we have $\omega, d\omega \in \overline{\mathcal{I}}$. If $\phi^*\omega = 0$, then we get $\phi^*(d\omega) = d(\phi^*\omega) = 0$ according to Theorem 5.8.2. Thus the ideal $\overline{\mathcal{I}}$ is also annihilated under this mapping. In other words, characteristic manifolds of an ideal $\mathcal{I}$ and characteristic manifolds of its closure are the same. □

Theorem 5.13.5 and 5.13.6 help us to specify some solutions of a system of exterior equations generating an ideal that is not closed by means of characteristic manifolds. Let us suppose that the dimension of the characteristic subspace $\mathcal{S}(\overline{\mathcal{I}})$ of the closure $\overline{\mathcal{I}}$ of the ideal $\mathcal{I}$ is $m - r$. Then we can find in the usual way functions $g^\alpha(\mathbf{x}) \in \Lambda^0(M), \alpha = 1, \dots, r$ enabling us to write $\mathcal{I} \subset \overline{\mathcal{I}} \subseteq \mathcal{J}(dg^\alpha)$. Hence the equations $g^\alpha(\mathbf{x}) = c^\alpha$ produce $(m - r)$-dimensional characteristic manifolds annihilating the ideal $\mathcal{I}$. But, since $d\mathcal{I} \not\subset \mathcal{I}$, we are required to enlarge the ideal in order to close it, and consequently, to reduce the dimension of the characteristic subspace. Thus, we are compelled to keep the completely integrable system, in which the ideal $\mathcal{I}$ is embedded, larger than it was necessary.

Even if an ideal $\mathcal{I}$ is not closed, it can be placed into a completely integrable system if its characteristic subspace is 1-dimensional because of the fact that such a subspace constitutes trivially a Lie algebra.

Example 5.13.3. We construct an ideal of the exterior algebra $\Lambda(\mathbb{R}^4)$ by the forms $\omega^1 = dx^1 - x^2\,dx^3,\ \omega^2 = x^4\,dx^1 \wedge dx^3 - x^1 dx^2 \wedge dx^4$. We then have

$$d\omega^1 = -\,dx^2 \wedge dx^3,\ d\omega^2 = \Big(\frac{dx^1}{x^1} + \frac{dx^4}{x^4}\Big) \wedge \omega^2.$$

We obviously get $d\omega^2 \in \mathcal{I}(\omega^1, \omega^2)$. However, we can easily verify that we find $d\omega^1 \notin \mathcal{I}(\omega^1, \omega^2)$. Hence, the closure of $\mathcal{I}$ is $\overline{\mathcal{I}}(\omega^1, \omega^2, d\omega^1)$. Thus, the characteristic subspace of $\overline{\mathcal{I}}$ is prescribed by the equations

$$\begin{aligned} v^1 - x^2 v^3 &= 0 \\ x^4(v^1 dx^3 - v^3 dx^1) - x^1(v^2 dx^4 - v^4 dx^2) &= \lambda(dx^1 - x^2\, dx^3) \\ -(v^2 dx^3 - v^3 dx^2) &= \mu(dx^1 - x^2\, dx^3) \end{aligned}$$

whose solution yields $\lambda = \mu = 0$ and $v^1 = v^2 = v^3 = v^4 = 0$. We thus obtain $V = 0$ so that the characteristic subspace is the zero space. The ideal generated by functions $g^i = x^i$ is just $\mathcal{J}(dx^i) = \Lambda(\mathbb{R}^4)$. Hence, we can only get trivial information about the solution. On the other hand, the characteristic subspace of $\mathcal{I}$ is prescribed by the equations

$$\begin{aligned} v^1 - x^2 v^3 &= 0, \\ x^4(v^1 dx^3 - v^3 dx^1) - x^1(v^2 dx^4 - v^4 dx^2) &= \lambda(dx^1 - x^2\, dx^3) \end{aligned}$$

whose solution is $\lambda = -x^4 v^3$ and $v^1 = x^2 v^3$, $v^2 = v^4 = 0$. Thus the characteristic subspace is 1-dimensional and is spanned by the vector $V_4 = x^2\partial_1 + \partial_3$. The solution of the partial differential equation $V_4(f) = 0$ is readily obtained as $f = f(g^1, g^2, g^3)$ where we define $g^1 = x^1 - x^2x^3$, $g^2 = x^2$, $g^3 = x^4$. In this case, we can write $\mathcal{I}(\omega^1, \omega^2) \subset \mathcal{J}(dg^1, dg^2, dg^3)$. Indeed, the relations

$$\begin{aligned} \omega^1 &= dg^1 + x^3 dg^2, \\ \omega^2 &= -\frac{x^4}{x^2}\, dx^1 \wedge dg^1 - \frac{x^3x^4}{x^2}\, dx^1 \wedge dg^2 - x^1 dg^2 \wedge dg^3 \end{aligned}$$

can easily be verified. ■

If we have managed to determine a resolvent mapping for an ideal, new resolvent mappings may be created via an isovector field of that ideal.

Theorem 5.13.7. *Let $\mathcal{I}$ be an ideal of the exterior algebra $\Lambda(M)$ and $\phi : S \to M$ be a resolvent mapping for that ideal. If the vector field V is an isovector field of the ideal, then the flow generated by V transforms ϕ into a 1-parameter family of resolvent mappings.*

If $\phi : S \to M$ is a resolvent mapping, then we have $\omega|_S = \phi^*\omega = 0$ for all $\omega \in \mathcal{I}$. If we further assume that V is an isovector, this implies that $£_V\omega \in \mathcal{I}$ for all $\omega \in \mathcal{I}$. We denote the flow $\psi_V(t) : M \to M$ generated by the isovector field V by $\psi_V(t)(p) = e^{tV}(p)$ and define the 1-parameter family of mappings $\phi_V(t) : S \to M$ as follows

$$\phi_V(t) = \psi_V(t) \circ \phi = e^{tV} \circ \phi.$$

On utilising (5.11.14), we obtain

$$\phi_V(t)^*\omega = (e^{tV} \circ \phi)^*\omega = \big[\phi^* \circ (e^{tV})^*\big]\omega = \phi^* \circ (e^{tV})^*\omega$$
$$= \phi^*\omega^* = \phi^*(e^{t\pounds_V}\omega)$$

for all $\omega \in \mathcal{I}$. However, due to the relation $e^{t\pounds_V}\omega \in \mathcal{I}$ we find $\phi_V(t)^*\omega = 0$. Therefore, each member of the 1-parameter family of mappings $\phi_V(t)$ is also a solution of the ideal $\mathcal{I}$. □

Example 5.13.4. We have already determined the isovector fields of the ideal $\mathcal{I}(x\,dy + y\,dz)$ of the exterior algebra $\Lambda(\mathbb{R}^3)$ in Example 5.12.1. For a tangible example, let us choose $F = xz$. In this case the components of the isovector field become

$$v^x = \frac{x(x+z)}{y}, \quad v^y = z, \quad v^z = 0.$$

The flow created by this vector field is found as the solution of the ordinary differential equations

$$\frac{d\overline{x}}{dt} = \frac{\overline{x}(\overline{x}+\overline{z})}{\overline{y}}, \quad \frac{d\overline{y}}{dt} = \overline{z}, \quad \frac{d\overline{z}}{dt} = 0$$

under the initial conditions $\overline{x}(0) = x, \overline{y}(0) = y, \overline{z}(0) = z$. Hence, the mapping $\psi_V(t)$ is determined by

$$\overline{x}(t) = \frac{x(y+zt)}{y-xt}, \quad \overline{y}(t) = zt + y, \quad \overline{z}(t) = z.$$

We shall now look for a 1-dimensional solution of the exterior equation $\omega = x\,dy + y\,dz = 0$ in the form $x = \phi^1(u), y = \phi^2(u), z = \phi^3(u)$. Then $\phi^*\omega = 0$ ends up in the equation

$$\phi^1\frac{d\phi^2}{du} + \phi^2\frac{d\phi^3}{du} = 0.$$

In this situation, the family of resolvent mappings $\phi_V(t) = \psi_V(t) \circ \phi$ is designated by

$$\phi_V^1(u;t) = \phi^1(u)\frac{\phi^2(u) + t\phi^3(u)}{\phi^2(u) - t\phi^1(u)},$$
$$\phi_V^2(u;t) = \phi^2(u) + t\phi^3(u), \qquad \phi_V^3(u;t) = \phi^3(u).$$

The mapping described by $x = \phi_V^1(u;t)$, $y = \phi_V^2(u;t)$, $z = \phi_V^3(u;t)$ is also a solution of the exterior equation $\omega = 0$ for each t. In fact, if we insert these relation into that equation, we obtain

$$\phi_V^1 \frac{d\phi_V^2}{du} + \phi_V^2 \frac{d\phi_V^3}{du} = \frac{\phi^2 + t\phi^3}{\phi^2 - t\phi^1}\left(\phi^1 \frac{d\phi^2}{du} + \phi^2 \frac{d\phi^3}{du}\right) = 0.$$

As a simple example, let us take $\phi^1 = -2c_1u^2, \phi^2 = c_2u, \phi^3 = c_1u^2$ where c_1 and c_2 are constants. The new family of solutions is then found to be

$$\phi_V^1 = -\frac{2c_1(c_2 + c_1tu)u^2}{c_2 + 2c_1tu}, \ \ \phi_V^2 = (c_2 + c_1tu)u, \ \ \phi_V^3 = c_1u^2. \qquad \blacksquare$$

5.14. FORMS DEFINED ON A LIE GROUP

Let G be a finite m-dimensional Lie group. We denote the exterior algebra on this smooth manifold by $\Lambda(G)$. We consider the left and right translations L_g and R_g on G defined by (3.3.1) and (3.3.2), respectively. These diffeomorphisms give rise to the mappings $L_g^* : \Lambda(G) \to \Lambda(G)$ and $R_g^* : \Lambda(G) \to \Lambda(G)$. If a form $\omega \in \Lambda^k(G)$ satisfies the relation

$$L_g^*\omega(g*h) = \omega(h) \ \text{ or } \ L_g^*\omega = \omega \tag{5.14.1}$$

for all $g, h \in G$, it is called a ***left-invariant form***. Because of the equality $L_g^{-1} = L_{g^{-1}}$, we infer that $(L_g^{-1})^* = L_{g^{-1}}^*$. Hence, it follows from (5.14.1) that we obtain

$$\omega(g*h) = L_{g^{-1}}^*\omega(h) \tag{5.14.2}$$

for a left-invariant form ω and for all $g, h \in G$. If we take $h = e$, (5.14.2) leads to

$$\omega(g) = L_{g^{-1}}^*\omega(e) \tag{5.14.3}$$

for all $g \in G$. Consequently, all left-invariant k-forms are generated by forms $\omega(e) \in \Lambda^k(G)$ defined on the tensor product $\bigotimes_k T_e^*(G)$ at the identity element $e \in G$. Thus, left-invariant 1-forms are produced by 1-forms in the dual space $T_e^*(G)$. Since the dimension of the vector space $T_e^*(G)$ is m, then there are exactly m linearly independent left-invariant 1-forms and the entire left-invariant 1-forms are expressible as their linear combinations. If we denote a basis of $T_e^*(G)$ by $\omega^1, \omega^2, \ldots, \omega^m$, we can then express a form $\omega(e) \in \Lambda^k(G)$ as follows

$$\omega(e) = \frac{1}{k!}\,\omega_{i_1i_2\cdots i_k}\omega^{i_1} \wedge \omega^{i_2} \wedge \cdots \wedge \omega^{i_k}$$

where $\omega_{i_1 i_2 \cdots i_k} \in \mathbb{R}$ are completely antisymmetric *constant* coefficients. According to the relation (5.14.3), any left-invariant k-form is extracted from the foregoing form with constant coefficients. Similarly, right-invariant forms are defined as

$$R_g^* \omega = \omega \tag{5.14.4}$$

for all $g \in G$ and we write $\omega(g) = R_{g^{-1}}^* \omega(e)$. Hence, right-invariant forms are also generated by m linearly independent 1-forms chosen from the dual space $T_e^*(G)$. The relation $L_g \circ R_g = R_g \circ L_g$ [*see* (3.3.3)] leads of course to $R_g^* \circ L_g^* = L_g^* \circ R_g^*$. Therefore, if ω is a left-invariant form we find that

$$L_g^*(R_g^* \omega) = R_g^*(L_g^* \omega) = R_g^* \omega.$$

Thus $R_g^* \omega$ is a left-invariant form. In the same way, If ω is a right-invariant form, then $L_g^* \omega$ turns out to be a right-invariant form.

Theorem 5.14.1. *If ω is a left* (*right*) *invariant form, then $d\omega$ is also a left* (*right*) *invariant form.*

According to Theorem 5.8.2, we obtain

$$L_g^* d\omega = dL_g^* \omega = d\omega$$

for all $g \in G$. Similarly, we get $R_g^* d\omega = d\omega$. □

Theorem 5.14.2. *Let G and H be Lie groups and $\phi : G \to H$ be a Lie group homomorphism. Then the pull-back operator $\phi^* : \Lambda(H) \to \Lambda(G)$ transports the left-invariant forms in H to the left-invariant forms in G.*

Let $\omega \in \Lambda(H)$ be a left-invariant form. Since ϕ is a group homomorphism, we readily obtain

$$L_g^*(\phi^* \omega) = (\phi \circ L_g)^* \omega = (L_{\phi(g)} \circ \phi)^* \omega = \phi^*(L_{\phi(g)}^* \omega) = \phi^* \omega$$

[*see* p. 188]. This implies that the form $\phi^* \omega$ is left-invariant. The same property is also valid for right-invariant forms. □

The Lie algebra $\mathfrak{g}$ of the Lie group G that consist of left-invariant vectors is designated by the tangent space $T_e(G)$ and left-invariant 1-forms are elements of the dual space $T_e^*(G)$. Hence, when we choose a basis V_1, $V_2, \ldots, V_m$ in $\mathfrak{g} = T_e(G)$, we can find a reciprocal basis $\omega^1, \omega^2, \ldots, \omega^m$ in $\mathfrak{g}^* = T_e^*(G)$ such that we get $\omega^i(V_j) = \delta_j^i$.

Theorem 5.14.3. *A form $\omega \in \Lambda^k(G)$ is left-invariant if and only if the function $\omega(V_1, V_2, \ldots, V_k)$ is constant for every k left-invariant vector fields $V_1, V_2, \ldots, V_k$.*

Let ω be a left-invariant k-form. We can thus write $L_g^* \omega(g * h) = \omega(h)$ and $dL_g(V_i|_h) = V_i|_{g*h}$, $i = 1, \ldots, k$. (5.7.1) then leads to

$$L_g^*\omega\big|_{g*h}(V_1\big|_h, \dots, V_k\big|_h) = \omega\big|_{g*h}\big(dL_g(V_1\big|_h), \dots, dL_g(V_k\big|_h)\big) \qquad (5.14.5)$$

from which we obtain

$$\omega\big|_h(V_1\big|_h, \dots, V_k\big|_h) = \omega\big|_{g*h}(V_1\big|_{g*h}, \dots, V_k\big|_{g*h})$$

since $V_1, \dots, V_k$ are left-invariant vectors. If we take $h = e$, then for every $g \in G$ we find that

$$\omega\big|_g(V_1\big|_g, \dots, V_k\big|_g) = \omega\big|_e(V_1\big|_e, \dots, V_k\big|_e) = \textit{constant.} \qquad (5.14.6)$$

Conversely, if the function $\omega(V_1, V_2, \dots, V_k)$ is constant for every k left-invariant vector fields $V_1, V_2, \dots, V_k$, then (5.14.5) yields

$$L_g^*\omega(g)(V_1\big|_e, \dots, V_k\big|_e) = \omega(e)(V_1\big|_e, \dots, V_k\big|_e)$$

whence we deduce that the relation $L_g^*\omega(g) = \omega(e)$, that is, ω is a left-invariant form. □

The left-invariant 1-forms engendering the dual $\mathfrak{g}^*$ of the Lie algebra $\mathfrak{g}$ of the Lie group G are called ***Maurer-Cartan forms*** [German mathematician Ludwig Maurer (1859-1927)]. So Theorem 5.14.3 implies that the function $\omega(V)$ remains constant for fields $\omega \in \mathfrak{g}^*$ and $V \in \mathfrak{g}$.

Theorem 5.14.4. *Let G be a Lie group and $\theta^i \in \mathfrak{g}^*, i = 1, \dots, m$ be a basis for left-invariant 1-forms. In this case, the following Maurer-Cartan structure equations are satisfied*

$$d\theta^k = -\frac{1}{2}\, c_{ij}^k\, \theta^i \wedge \theta^j = -\sum_{1 \le i < j \le m} c_{ij}^k\, \theta^i \wedge \theta^j. \qquad (5.14.7)$$

where $c_{ij}^k = -c_{ji}^k$ are real constants. The constants c_{ij}^k are the same as the structure constants of Lie algebra $\mathfrak{g}$.

According to Theorem 5.14.1, if a basis form θ^k is left-invariant, then its exterior derivative $d\theta^k$ is likewise left-invariant. Therefore, in terms of basis in the dual space $\mathfrak{g}^*$ we can write

$$d\theta^k = -\frac{1}{2}\, b_{ij}^k\, \theta^i \wedge \theta^j, \quad i, j, k = 1, \dots, m$$

with constant coefficients b_{ij}^k. These numbers ought to satisfy naturally the antisymmetry conditions $b_{ij}^k = -b_{ji}^k$. On the other hand, we get

$$0 = d^2\theta^k = -\frac{1}{2}\, b_{ij}^k(d\theta^i \wedge \theta^j - \theta^i \wedge d\theta^j)$$

$$= \frac{1}{4}\, b^k_{ij}(b^i_{lm}\theta^l \wedge \theta^m \wedge \theta^j - b^j_{lm}\theta^i \wedge \theta^l \wedge \theta^m)$$
$$= \frac{1}{4}\, b^k_{ij}b^i_{lm}\theta^l \wedge \theta^m \wedge \theta^j - \frac{1}{4}\, b^k_{ji}b^i_{lm}\theta^l \wedge \theta^m \wedge \theta^j$$
$$= \frac{1}{2}\, b^k_{ij}b^i_{lm}\theta^l \wedge \theta^m \wedge \theta^j = \frac{1}{2}\, b^k_{i[j}b^i_{lm]}\theta^l \wedge \theta^m \wedge \theta^j.$$

Thus the coefficients b^k_{ij} must satisfy the relations

$$\frac{3!}{2} b^k_{i[j}b^i_{lm]} = b^i_{lm}b^k_{ij} + b^i_{mj}b^k_{il} + b^i_{jl}b^k_{im} = 0$$

dictated by the Jacobi identity. Let $V_i \in \mathfrak{g}, i = 1, \dots, m$ be the reciprocal basis of the Lie algebra with respect to the forms θ^i, that is, the relations $\theta^i(V_j) = \delta^i_j, i, j = 1, \dots, m$ are to be satisfied. This basis vectors have to verify the relations $[V_i, V_j] = c^k_{ij}V_k$ where c^k_{ij} are structure constants of the Lie algebra $\mathfrak{g}$ with respect to the basis $\{V_i\}$ [*see* (3.3.9)]. In view of the relation (5.2.6), we can write $b^k_{ij} = -\, d\theta^k(V_i, V_j)$. Consider a 1-form $\omega = \omega_i dx^i$. The value of the form $d\omega = \omega_{i,j}\, dx^j \wedge dx^i$ on vector fields $U, V \in T(M)$ is given by

$$d\omega(U, V) = \omega_{i,j}(u^j v^i - u^i v^j) = (\omega_{i,j} - \omega_{j,i})u^j v^i.$$

On the other hand, the relation

$$U\big(\omega(V)\big) - V\big(\omega(U)\big) = (\omega_{i,j} - \omega_{j,i})u^j v^i + \omega_i(v^i_{,j}u^j - u^i_{,j}v^j)$$

leads immediately to

$$d\omega\,(U, V) = U\big(\omega(V)\big) - V\big(\omega(U)\big) - \omega\,([U, V]). \qquad (5.14.8)$$

Consequently, because of $\theta^k(V_i) = \delta^k_i$, $\theta^k(V_j) = \delta^k_j$ we obtain

$$b^k_{ij} = -\, d\theta^k(V_i, V_j) = \theta^k([V_i, V_j]) = \theta^k(c^l_{ij}V_l) = c^l_{ij}\delta^k_l = c^k_{ij}. \qquad \square$$

We can now prove the following theorem.

Theorem 5.14.5. *The structure constants of an m-dimensional Lie group vanish if and only if it is locally isomorphic to the group $\mathbb{R}^m$.*

(i). Let the Lie group G be isomorphic to the group $\mathbb{R}^m$. We have seen in Example 3.3.1 that the structure constants of $\mathbb{R}^m$ are zero. The relation (3.4.3) then requires that the structure constants of G are also zero so that G becomes an Abelian group.

(ii). Let the structure constants of the Lie group G be zero. Therefore, (5.14.6) gives $d\theta^k = 0, k = 1, \dots, m$. *According to the Poincaré lemma,* *there are m smooth functions $\vartheta^k : U \to \mathbb{R}$ on the domain U of a local chart (U, φ) such that $\theta^k = d\vartheta^k$* [*see p.* 334]. We can choose those functions ϑ^k as coordinate functions. Since the forms θ^k are left-invariant, we obtain

$$L_g^*\theta^k(g*h) = \theta^k(h) = d\vartheta^k(h) = dh^k$$

for all $g, h \in G$. $g^k = \vartheta^k(g), h^k = \vartheta^k(h), k = 1, \dots, m$ are coordinates of g and h. Furthermore, we can readily write $(g*h)^k = \vartheta^k(g*h) = \vartheta^k(L_g h) = \vartheta^k \circ L_g(h) = L_g^k(h)$. Then, on making use of Theorem 5.8.2 we get

$$\begin{aligned} L_g^*\theta^k(g*h) &= L_g^* d\vartheta^k(g*h) = L_g^* dL_g^k(h) = dL_g^* L_g^k(h) \\ &= d\big(L_g^k(h) \circ L_g\big) = dL_g^k(h)\big|_h = \frac{\partial L_g^k(h)}{\partial h^l}\, dh^l. \end{aligned}$$

If we compare the two expressions which we have found for $L_g^*\theta^k(g*h)$, then we deduce that

$$\frac{\partial L_g^k(h)}{\partial h^l}\, dh^l = dh^k \text{ or } \frac{\partial L_g^k(h)}{\partial h^l} = \delta_l^k.$$

It is quite easy to integrate these differential equations to obtain

$$L_g^k(h) = \Theta^k(g) + h^k. \tag{5.14.9}$$

$\Theta^k(g)$ are arbitrary functions. Since the functions ϑ^k are to be determined up to a constant, we can impose the restriction $\vartheta^k(e) = 0, k = 1, \dots, m$ without loss of generality. Because $L_g(e) = g$, we get $L_g^k(e) = \vartheta^k(g) = g^k$ and when we evaluate the expression (5.14.9) for $h = e$, we end up with the relation $\Theta^k(g) = g^k$. Hence, we find that $\vartheta^k(g*h) = L_g^k(h) = g^k + h^k$. Let us next define the smooth function $\boldsymbol{\vartheta} = (\vartheta^1, \dots, \vartheta^m) : U \to \mathbb{R}^m$ and the elements $\mathbf{g} = (g^1, \dots, g^m) \in \mathbb{R}^m$ and $\mathbf{h} = (h^1, \dots, h^m) \in \mathbb{R}^m$. We thus conclude that

$$\boldsymbol{\vartheta}(g*h) = \mathbf{g} + \mathbf{h} = \boldsymbol{\vartheta}(g) + \boldsymbol{\vartheta}(h). \tag{5.14.10}$$

This implies that the Lie group G is locally isomorphic to the group $\mathbb{R}^m$. □

V. EXERCISES

5.1. We define on the manifold $\mathbb{R}^4$ with the coordinate cover (x, y, z, t) the following exterior forms

$$\omega^1 = y \cos t\, dx + e^x\, dy + t\, dz + (y - z)\, dt \in \Lambda^1(\mathbb{R}^4),$$

$$\omega^2 = \tan x\, dx \wedge dz + (y - z^3)\, dx \wedge dt + \sinh z\, dy \wedge dz \in \Lambda^2(\mathbb{R}^4),$$
$$\omega^3 = e^y\, dy \wedge dz \wedge dt - \cos y\, dx \wedge dy \wedge dz + x\ dx \wedge dz \wedge dt \in \Lambda^3(\mathbb{R}^4),$$
$$\omega^4 = (x^2 + t^3)\, dx \wedge dy \wedge dz \wedge dt \in \Lambda^4(\mathbb{R}^4).$$

Evaluate the exterior forms $\omega^1 \wedge \omega^3, \omega^1 \wedge \omega^2 + \omega^3$, $\omega^3 \wedge \omega^1 - \omega^2 \wedge \omega^2 + \omega^4$, $d\omega^2 - \omega^3 + \omega^2 \wedge \omega^1$, $d\omega^1 \wedge \omega^2 + d\omega^1 \wedge d\omega^1$, $d\omega^3 + d(\omega^1 \wedge \omega^2)$. The vector fields $U, V \in T(\mathbb{R}^4)$ are given by

$$U = y\frac{\partial}{\partial x} - z\frac{\partial}{\partial z} + \frac{\partial}{\partial t},\ \ V = x\frac{\partial}{\partial y} - t\frac{\partial}{\partial z}.$$

Find the forms $\mathbf{i}_U\omega^1$, $\mathbf{i}_V\omega^2$, $\mathbf{i}_U\omega^3$, $\mathbf{i}_V\omega^4$, $\mathbf{i}_U(d\omega^1 + \omega^2)$, $\mathbf{i}_V\mathbf{i}_U\omega^4 + \mathbf{i}_V(d\omega^2)$, $d(\mathbf{i}_U\omega^2) + \mathbf{i}_U(d\omega^2)$, $\pounds_V\omega^1$, $\pounds_U\omega^2$, $\pounds_V\omega^3$, $\pounds_U\omega^4$, $\pounds_U\mathbf{i}_V\omega^2 - \mathbf{i}_U\pounds_V\omega$.

5.2. Consider an exterior form $\omega = z\ dx - x\ dy + x\ dz \in \Lambda^1(\mathbb{R}^3)$ and a vector field $V = y\,\partial_x + z\,\partial_y + x\partial_z \in T(\mathbb{R}^3)$. Evaluate the forms $\pounds_V\omega$, $\pounds_V\pounds_V\omega$, $\pounds_V\pounds_V\pounds_V\omega$, $\pounds_V\pounds_V\pounds_V\pounds_V\omega$ and $\exp(t\pounds_V)\,\omega$.

5.3. Determine vector fields $V \in T(\mathbb{R}^4)$ in such a way that they satisfy the relations (a) $\mathbf{i}_V\omega^1 = 0$, (b) $\mathbf{i}_V\omega^2 = 0$, (c) $\mathbf{i}_V\omega^3 = 0$, (d) $\mathbf{i}_V\omega^4 = 0$. This amounts to say that they will be characteristic vectors of those forms. Forms $\omega^1, \omega^2, \omega^3, \omega^4$ are defined in Exercise **5.1**.

5.4. Express the forms $\omega^1, \omega^2, \omega^3, \omega^4$ in Exercise **5.1** in terms of bases induced by the volume form $\mu = dx \wedge dy \wedge dz \wedge dt$.

5.5. Let $\{\theta^i\} \subset T^*(M)$ and $\{V_i\} \subset T(M), i = 1, \dots, m$ be reciprocal basis vectors. Verify the equality

$$\mathbf{i}_{V_i}(\theta^{i_1} \wedge \cdots \wedge \theta^{i_k}) = \begin{cases} 0, \ \text{if}\ i \neq i_r, r = 1, \dots, k \\ (-1)^{r-1}\theta^{i_1} \wedge \cdots \wedge \theta^{i_{r-1}} \wedge \theta^{i_{r+1}} \wedge \cdots \wedge \theta^{i_k}, \text{if}\ i = i_r \end{cases}$$

5.6. We define the mapping $\phi : \mathbb{R}^3 \to \mathbb{R}^4$ by the relations

$$x = u\cos v,\quad y = u\sin v,\quad z = w - 2,\quad t = uw.$$

(a) Find the pulled back forms $\phi^*\omega^1$, $\phi^*\omega^2$, $\phi^*\omega^3$, $\phi^*\omega^4$ [*see* Exercise **5.1**], (b) determine the range $\mathcal{R}(\phi) \subset \mathbb{R}^4$, (c) evaluate the inverse mapping $\phi^{-1} : \mathcal{R}(\phi) \to \mathbb{R}^3$, (d) find the vectors $\phi_*\partial_u$, $\phi_*\partial_v$ and $\phi_*\partial_w$, (d) If $\omega = dx \wedge dy \wedge dz$, then evaluate the forms $\phi^*\big(\mathbf{i}_{\phi_*\partial_u}(\omega)\big)$ and $\phi^*\big(\mathbf{i}_{\phi_*\partial_v}(\omega)\big)$.

5.7. A mapping $\phi : \mathbb{R}^2 \to \mathbb{R}^3$ is described by the relations $u = y^2$, $v = xy$, $w = x^3$. The vector fields $U, V \in T(\mathbb{R}^2)$ and the form $\omega \in \Lambda^2(\mathbb{R}^3)$ are given as follows:

$$U = y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y},\ \ V = -\frac{\partial}{\partial x} + y^2\frac{\partial}{\partial y},\ \ \omega = u\,du \wedge dv - vw\,dv \wedge dw.$$

Evaluate the quantities $\phi^*\omega$, ϕ_*U, ϕ_*V, $(\phi^*\omega)(U, V)$, $\omega(\phi_*U, \phi_*V)$, $\mathbf{i}_U(\phi^*\omega), \mathbf{i}_V(\phi^*\omega)$, $\phi^*(\mathbf{i}_{\phi_*U}\omega)$, $\phi^*(\mathbf{i}_{\phi_*V}\omega)$.

5.8. Determine all mappings $\phi : \mathbb{R}^k \to \mathbb{R}^4, 1 \leq k \leq 4$ satisfying the relations (a)

$\phi^*\omega^1 = 0$, (b) $\phi^*\omega^2 = 0$, (c) $\phi^*\omega^3 = 0$, (d) $\phi^*\omega^4 = 0$ where the forms $\omega^1, \omega^2, \omega^3, \omega^4$ are those given in Exercise **5.1**

5.9. Show that the form $\omega = z\,dx \wedge dy + yz\,dy \wedge dz + y\;dx \wedge dz \in \Lambda^2(\mathbb{R}^3)$ is closed. Determine a form $\Omega \in \Lambda^1(\mathbb{R}^3)$ such that $\omega = d\Omega$.

5.10. We define the isomorphisms $\phi : \mathbb{R}^3 \to \Lambda^1(\mathbb{R}^3)$, $\psi : \Lambda^0(\mathbb{R}^3) \to \Lambda^3(\mathbb{R}^3)$ by

$$\phi(\mathbf{U}) = \omega_{\mathbf{U}} = u_x\,dx + u_y\,dy + u_z\,dz,$$
$$\psi(f) = \omega_f = f\,dx \wedge dy \wedge dz$$

where $\mathbf{U} = (u_x, u_y, u_z) \in \mathbb{R}^3$ and $f \in \Lambda^0(\mathbb{R}^3)$. Verify that (a) $\omega_{\mathbf{U}\cdot\mathbf{V}} = \omega_{\mathbf{U}} \wedge *\omega_{\mathbf{V}} = *\omega_{\mathbf{U}} \wedge \omega_{\mathbf{V}}$, (b) $\omega_{\mathbf{U}\times\mathbf{V}} = *(\omega_{\mathbf{U}} \wedge \omega_{\mathbf{V}})$, (c) $\omega_{\mathrm{div}\,\mathbf{U}} = *\,d*\omega_{\mathbf{U}}$, (d) $\omega_{\mathrm{curl}\,\mathbf{U}} = *\,d\omega_{\mathbf{U}}$ and show that (e) $\mathbf{U}\cdot\mathrm{curl}\,\mathbf{U} = 0$ if $d\omega_{\mathbf{U}} \wedge \omega_{\mathbf{U}} = 0$.

5.11. Verify the following relations in $\mathbb{R}^3$:

(a) $\mathbf{U} \times (\mathbf{V} \times \mathbf{W}) = (\mathbf{U}\cdot\mathbf{W})\mathbf{V} - (\mathbf{U}\cdot\mathbf{V})\mathbf{W}$

(b) $\mathbf{U}\cdot(\mathbf{V}\times\mathbf{W}) = \mathbf{V}\cdot(\mathbf{W}\times\mathbf{U}) = \mathbf{W}\cdot(\mathbf{U}\times\mathbf{V})$

(c) $\boldsymbol{\nabla}(fg) = g\boldsymbol{\nabla}f + f\boldsymbol{\nabla}g$

(d) $\boldsymbol{\nabla}(\mathbf{U}\cdot\mathbf{V}) = \mathbf{U}\times\mathrm{curl}\,\mathbf{V} + \mathbf{V}\times\mathrm{curl}\,\mathbf{U} + (\mathbf{V}\cdot\boldsymbol{\nabla})\mathbf{U} + (\mathbf{U}\cdot\boldsymbol{\nabla})\mathbf{V}$

(e) $\mathrm{div}\,(f\mathbf{U}) = f\,\mathrm{div}\,\mathbf{U} + \mathbf{U}\cdot\boldsymbol{\nabla}f$

(f) $\mathrm{curl}\,(\mathbf{U}\times\mathbf{V}) = (\mathrm{div}\,\mathbf{V})\mathbf{U} - (\mathrm{div}\,\mathbf{U})\mathbf{V} + (\mathbf{V}\cdot\boldsymbol{\nabla})\mathbf{U} - (\mathbf{U}\cdot\boldsymbol{\nabla})\mathbf{V}$

(g) $\mathrm{div}\,(\mathrm{curl}\,\mathbf{U}) = 0$, $\mathrm{curl}\,(\boldsymbol{\nabla}f) = \mathbf{0}$, $\mathrm{div}\,(\boldsymbol{\nabla}f \times \boldsymbol{\nabla}g) = 0$

5.12. If M is a Riemannian manifold and $V_1, V_2 \in T(M)$, show that

$$\mathrm{div}\,[V_1, V_2] = V_1(\mathrm{div}\,V_2) - V_2(\mathrm{div}\,V_1).$$

5.13. Let $\omega \in \Lambda^k(M)$ and $V_0, V_1, \ldots, V_k \in T(M)$. Verify the relation

$$d\omega(V_0, V_1, \ldots, V_k) = \sum_{i=0}^{k}(-1)^i V_i\big(\omega(V_0, V_1, \ldots, V_{i-1}, V_{i+1}, \ldots, V_k)\big)$$
$$+\sum_{0\le i\le j\le k}(-1)^{i+j}\omega([V_i, V_j], V_0, V_1, \ldots, V_{i-1}, V_{i+1}, \ldots, V_{j-1}, V_{j+1}, \ldots, V_k).$$

5.14. Let $\omega \in \Lambda^k(M)$ and $V, V_1, \ldots, V_k \in T(M)$. Verify the relation

$$\pounds_V\big(\omega(V_1, \ldots, V_k)\big) =$$
$$(\pounds_V\omega)(V_1, \ldots, V_k) + \sum_{i=1}^{k}\omega(V_1, \ldots, V_{i-1}, [V, V_i], V_{i+1}, \ldots, V_k).$$

5.15. When $U, V \in T(M)$, verify the validity of the operator identity

$$\pounds_U \circ \mathbf{i}_V - \pounds_V \circ \mathbf{i}_U - \mathbf{i}_{[U,V]} = [d, \mathbf{i}_U \circ \mathbf{i}_V].$$

5.16. Provided that $g \in \Lambda^0(M)$, $dg \neq 0$, show that a function $f \in \Lambda^0(M)$ can be expressed in the form $f(p) = F\big(g(p)\big)$ if it meets the condition $df \wedge dg = 0$. F is a smooth function.

5.17. Let us assume that $g^1, g^2, \ldots, g^r \in \Lambda^0(M)$, $dg^1 \wedge dg^2 \wedge \cdots \wedge dg^r \neq 0$. Show that if a function $f \in \Lambda^0(M)$ satisfies the relation $df \wedge dg^1 \wedge \cdots \wedge dg^r = 0$, then it is expressible in the form $f = F\big(g^1, g^2, \dots, g^r\big)$. F is a smooth function of its arguments.

5.18. Let us assume that $g^1, \dots, g^r \in \Lambda^0(M)$ and $dg^1 \wedge \cdots \wedge dg^r \neq 0$. If we can write for a function $h \in \Lambda^0(M)$ the relation $dh = f_1 dg^1 + \cdots + f_r dg^r$ with functions $f_1, \dots, f_r \in \Lambda^0(M)$, then show that the relations $h = h\big(g^1, \dots, g^r\big)$ and $f_i = \dfrac{\partial h}{\partial g^i}$, $i = 1, \dots, r$ must be valid.

5.19. Show that $d{*}df = -{*}\Delta f = -(\Delta f)\mu$ if $f \in \Lambda^0(M)$.

5.20. Show that $d\big(f{*}(dg)\big) = df \wedge {*}(dg) - (f\Delta g)\mu$ if $f, g \in \Lambda^0(M)$.

5.21. Let $\boldsymbol{\mathcal{V}} = V^1 \wedge \cdots \wedge V^k \in \mathfrak{X}^k(M)$ and $\omega \in \Lambda^{k+l}(M)$. We define the *interior product* of the form ω with $\boldsymbol{\mathcal{V}}$ in such a manner that the following relation would be satisfied for all vectors $V^{k+1}, \dots, V^{k+l} \in \mathfrak{X}(M)$:

$$(\boldsymbol{i}_{\boldsymbol{\mathcal{V}}}\,\omega)(V^{k+1}, \dots, V^{k+l}) = \omega(V^1, \dots, V^k, V^{k+1}, \dots, V^{k+l}).$$

Show that this interior product is well defined and prove the operator equality $\boldsymbol{i}_{\boldsymbol{\mathcal{U}} \wedge \boldsymbol{\mathcal{V}}} = \boldsymbol{i}_{\boldsymbol{\mathcal{U}}} \circ \boldsymbol{i}_{\boldsymbol{\mathcal{V}}}$.

5.22. For $\boldsymbol{\mathcal{U}} \in \mathfrak{X}^k(M)$ and $\boldsymbol{\mathcal{V}} \in \mathfrak{X}^l(M)$ verify the equality

$$\boldsymbol{i}_{\langle \boldsymbol{\mathcal{U}}, \boldsymbol{\mathcal{V}} \rangle}\omega = (-1)^{(k+1)\,l}\boldsymbol{i}_{\boldsymbol{\mathcal{U}}} d\boldsymbol{i}_{\boldsymbol{\mathcal{V}}}\omega + (-1)^k \boldsymbol{i}_{\boldsymbol{\mathcal{V}}} d\boldsymbol{i}_{\boldsymbol{\mathcal{U}}}\omega - \boldsymbol{i}_{\boldsymbol{\mathcal{U}}}\boldsymbol{i}_{\boldsymbol{\mathcal{V}}} d\omega.$$

5.23. Assume that $\boldsymbol{\mathcal{V}} = V^1 \wedge V^2 \in \mathfrak{X}^2(M)$ and $f, g, h \in \Lambda^0(M)$. (a) Show that

$$\boldsymbol{i}_{\boldsymbol{\mathcal{V}}}\,(df \wedge dg \wedge dh) = \boldsymbol{i}_{\boldsymbol{\mathcal{V}}}\,(df \wedge dg)\,dh + \boldsymbol{i}_{\boldsymbol{\mathcal{V}}}\,(dg \wedge dh)\,df + \boldsymbol{i}_{\boldsymbol{\mathcal{V}}}\,(dh \wedge df)\,dg.$$

(b) We define the mapping $\{,\} : \Lambda^0(M) \times \Lambda^0(M) \to \Lambda^0(M)$ by the relation

$$\{f, g\} = \boldsymbol{i}_{\boldsymbol{\mathcal{V}}}\,(df \wedge dg).$$

We also name this mapping [*see* p. 707] as the *Poisson bracket* [French mathematician Siméon Denis Poisson (1781-1840)]. Show that this mapping is bilinear and antisymmetric. Prove the identity

$$\big\{\{f, g\}, h\big\} + \big\{\{g, h\}, f\big\} + \big\{\{h, f\}, g\big\} = \boldsymbol{i}_{\langle \boldsymbol{\mathcal{V}}, \boldsymbol{\mathcal{V}} \rangle}(df \wedge dg \wedge dh)$$

and then demonstrate that the condition $\langle \boldsymbol{\mathcal{V}}, \boldsymbol{\mathcal{V}} \rangle = \mathbf{0}$ should be satisfied [*see* Exercise **4.17**] in order for this bracket to satisfy the Jacobi identity, and consequently, $\Lambda^0(M)$ endowed with the *product* $\{,\}$ to form a Lie algebra. (c) Show further that the bracket satisfies the equality

$$\{f, gh\} = \{f, g\}h + \{f, h\}g.$$

5.24. Let the vectors U and V be characteristic vectors of an exterior form $\omega \in \Lambda(M)$. Show that $\mathbf{i}_{[U,V]}(\omega) = \mathbf{i}_U \circ \mathbf{i}_V(d\omega)$. Thus, prove that characteristic vector fields of a form ω constitute a Lie subalgebra if and only if the condition $\mathbf{i}_U \circ \mathbf{i}_V(d\omega) = 0$ is satisfied for every pair of characteristic vectors U and V.

5.25. Let $\omega^1, \omega^2, \omega^3, \omega^4$ be the forms given in Exercise **5.1**. Determine the characteristic and isovector fields of the ideals $I(\omega^1,\omega^2)$, $I(\omega^1,\omega^2,\omega^3)$, $I(\omega^1,\omega^2,\omega^4)$. Find maximal solutions annihilating these ideals.

5.26. We define the forms $\omega^1,\omega^2 \in \Lambda^1(\mathbb{R}^4)$ as $\omega^1 = y\,dx + z\,dt, \omega^2 = z\,dy - y\,dz$. Show that the ideal $I(\omega^1,\omega^2)$ is closed. Determine its characteristic and isovector fields. Find the maximal solution annihilating this ideal.

5.27. Determine the characteristic subspaces and isovector fields of ideals

$$I(y\,dx + x\,dy\, + y\,dz),$$
$$I\big((1+y^2)\,dx\, + x\,dy, x^3dz\big),$$
$$I(y\,dx + xz\,dy, dy \wedge dz)$$

of $\Lambda(\mathbb{R}^3)$. Find maximal solutions annihilating these ideals.

5.28. M is a Riemannian manifold with a metric tensor $\boldsymbol{\mathcal{G}}$. Show that any submanifold N of M can be made a Riemannian manifold equipped with a metric tensor $\boldsymbol{\mathcal{G}}'$ defined by the relation $\boldsymbol{\mathcal{G}}'(U,V) = \boldsymbol{\mathcal{G}}(U,V)$ for all pair of vectors $U,V \in T(N) \subseteq T(M)$.

5.29. We consider a 4-dimensional manifold M with a coordinate cover $(x^i, f^i : i = 1,2)$ and define the following 1-forms

$$\omega^i = df^i + f^j\alpha^i_j - \beta^i,$$
$$\alpha^i_j = \alpha^i_{jk}dx^k,\;\; \beta^i = \beta^i_j dx^j$$

where $\alpha^i_{jk} = \alpha^i_{jk}(x^1,x^2)$ and $\beta^i_j = \beta^i_j(x^1,x^2)$ are given functions.

(a) Let S be a submanifold with the coordinate cover (x^1,x^2). Show that the requirements $\phi^*\omega^i = 0$ that a resolvent mapping $\phi : S \to M$ must satisfy give rise to the first order partial differential equations

$$\frac{\partial f^i}{\partial x^j} + \alpha^i_{kj} f^k = \beta^i_j$$

determining the functions $f^i = f^i(x^1,x^2)$.

(b) Show that the ideal $\mathcal{I}(\omega^1,\omega^2)$ is closed if only the relations

$$d\alpha^i_j - \alpha^k_j \wedge \alpha^i_k = 0,$$
$$d\,\beta^i - \beta^j \wedge \alpha^i_j = 0$$

are satisfied and these relations conduce to the integrability conditions

$$\frac{\partial \alpha^i_{jn}}{\partial x^m} - \frac{\partial \alpha^i_{jm}}{\partial x^n} + \alpha^i_{kn}\alpha^k_{jm} - \alpha^i_{km}\alpha^k_{jn} = 0,$$
$$\frac{\partial \beta^i_j}{\partial x^k} - \frac{\partial \beta^i_k}{\partial x^j} + \beta^l_j\alpha^i_{lk} - \beta^l_k\alpha^i_{lj} = 0.$$

(c) Show that if the conditions for the ideal $\mathcal{I}(\omega^1,\omega^2)$ to be closed are satisfied, then there exist functions $\Omega^i_j, w^j \in \Lambda^0(M)$ so that one can write

$$\omega^i = \Omega^i_j du^j$$

and solutions of the differential equations are found as

$$u^i(x^1, x^2, f^1, f^2) = \textit{constant}.$$

5.30. G is a Lie group, $\omega \in \Lambda^1(G)$ is a left-invariant form, U and V are left-invariant vector fields. Show that

$$d\omega(U, V) = -\omega([U, V]).$$