Wavelets, their autocorrelation functions, and multiresolution representation of signals

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ABSTRACT

We summarize the properties of the auto-correlation functions of compactly supported wavelets, their connection to iterative interpolation schemes, and the use of these functions for multiresolution analysis of signals. We briefly describe properties of representations using dilations and translations of these auto-correlation functions (the auto-correlation shell) which permit multiresolution analysis of signals.

1. WAVELETS AND THEIR AUTOCORRELATION FUNCTIONS

The auto-correlation functions of compactly supported scaling functions were first studied in the context of the Lagrange iterative interpolation scheme in [6], [5]. Let $\Phi(x)$ be the auto-correlation function,

$$\Phi(x) = \int_{-\infty}^{+\infty} \varphi(y)\varphi(y-x)dy, \tag{1.1}$$

where $\varphi(x)$ is the scaling function which appears in the construction of compactly supported wavelets in [3]. The function $\Phi(x)$ is exactly the "fundamental function" of the symmetric iterative interpolation scheme introduced in [6], [5]. Thus, there is a simple one-to-one correspondence between iterative interpolation schemes and compactly supported wavelets [12], [11].

In particular, the scaling function corresponding to Daubechies's wavelet with two vanishing moments yields the scheme in [6]. In general, the scaling functions corresponding to Daubechies's wavelets with M vanishing moments lead to the iterative interpolation schemes which use the Lagrange polynomials of degree $2M$ [5]. Additional variants of iterative interpolation schemes may be obtained using compactly supported wavelets described in [4].

Let us outline the derivation of the two-scale difference equation for the function $\Phi(x)$. Let $m_0(\xi)$ and $m_1(\xi)$ be the 2π-periodic functions,

$$m_0(\xi) = \frac{1}{\sqrt{2}} \sum_{k=0}^{L-1} h_k \mathrm{e}^{\mathrm{i}k\xi}, \tag{1.2}$$

and

$$m_1(\xi) = \frac{1}{\sqrt{2}} \sum_{k=0}^{L-1} g_k \mathrm{e}^{\mathrm{i}k\xi} = \mathrm{e}^{\mathrm{i}(\xi+\pi)}\overline{m_0(\xi+\pi)}, \tag{1.3}$$

satisfying the quadrature mirror (filter) condition,

$$|m_0(\xi)|^2 + |m_1(\xi)|^2 = 1. \tag{1.4}$$

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If we consider trigonometric polynomial solutions of (1.4), then from from (1.2) and (1.4) follows that

$$|m_0(\xi)|^2 = \frac{1}{2} + \frac{1}{2}\sum_{k=1}^{L/2} a_{2k-1}\cos(2k-1)\xi, \tag{1.5}$$

where $\{a_k\}$ are the auto-correlation coefficients of the filter $H = \{h_k\}_{0\le k\le L-1}$,

$$a_k = 2\sum_{l=0}^{L-1-k} h_l h_{l+k} \qquad \text{for} \quad k = 1,\dots,L-1, \tag{1.6}$$

and

$$a_{2k} = 0 \qquad \text{for} \quad k = 1,\dots,L/2-1. \tag{1.7}$$

Using the two-scale difference equation for the scaling function φ,

$$\varphi(x) = \sqrt{2}\sum_{k=0}^{L-1} h_k\varphi(2x-k), \tag{1.8}$$

it is easy to verify that

$$\Phi(x) = \Phi(2x) + \frac{1}{2}\sum_{l=1}^{L/2} a_{2l-1}\left(\Phi(2x-2l+1) + \Phi(2x+2l-1)\right). \tag{1.9}$$

Introducing the autocorrelation function of the wavelet

$$\Psi(x) = \int_{-\infty}^{+\infty}\psi(y)\psi(y-x)dy, \tag{1.10}$$

where

$$\psi(x) = \sqrt{2}\sum_{k=0}^{L-1} g_k\varphi(2x-k), \tag{1.11}$$

we also have

$$\Psi(x) = \Phi(2x) - \frac{1}{2}\sum_{l=1}^{L/2} a_{2l-1}\left(\Phi(2x-2l+1) + \Phi(2x+2l-1)\right). \tag{1.12}$$

By direct examination of (1.9) and (1.12), it is easy to see that both Φ and Ψ are supported within the interval $[-L+1, L-1]$. Finally, $\Phi(x)$ and $\Psi(x)$ have vanishing moments,

$$\mathcal{M}_\Psi^m = \int_{-\infty}^{+\infty} x^m\Psi(x)dx = 0, \qquad \text{for} \quad 0 \le m \le L, \tag{1.13}$$

$$\mathcal{M}_\Phi^m = \int_{-\infty}^{+\infty} x^m\Phi(x)dx = 0, \qquad \text{for} \quad 1 \le m \le L, \tag{1.14}$$

and

$$\int_{-\infty}^{+\infty}\Phi(x)dx = 1. \tag{1.15}$$

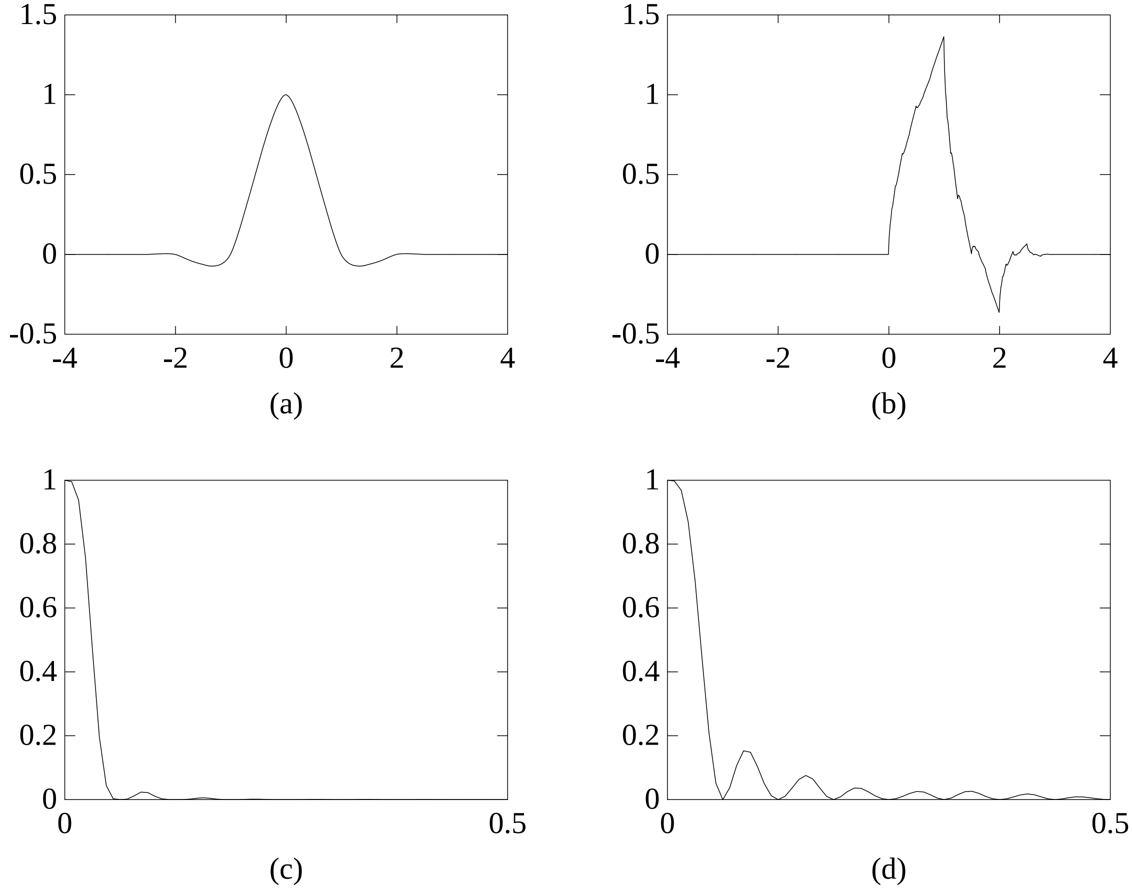

Figure 1: Plots of the auto-correlation function $\Phi(x)$ and the Daubechies's scaling function $\varphi(x)$ with $L = 4$. (a) $\Phi(x)$. (b) $\varphi(x)$. (c) Magnitude of the Fourier transform of $\Phi(x)$. (d) Magnitude of the Fourier transform of $\varphi(x)$.

It is also easy to obtain (see [1]) that even moments of the coefficients a_{2k-1} from (1.6) vanish, namely

$$\sum_{k=1}^{L/2} a_{2k-1}(2k-1)^{2m} = 0 \quad \text{for} \quad 1 \leq m \leq M-1, \tag{1.16}$$

where $M = L/2$ (for wavelets in [3]).

Since L consecutive moments of the auto-correlation function $\Psi(x)$ vanish (1.13), we have

$$\hat{\Psi}(\xi) = O(\xi^L), \tag{1.17}$$

where $\hat{\Psi}(\xi)$ is the Fourier transform of $\Psi(x)$. Thus, $\hat{\Psi}(\xi)$ may be viewed as the symbol of a pseudo-differential operator which behaves like an approximation of the derivative operator $(d/dx)^L$. Therefore, convolution with $\Psi(x)$ behaves essentially like a differential operator in detecting changes of spatial intensity and is designed to act at any desired scale. We display functions $\Phi(x)$, $\varphi(x)$, $\Psi(x)$, $\psi(x)$, and the magnitudes of their Fourier transforms in Figures 1 and 2.

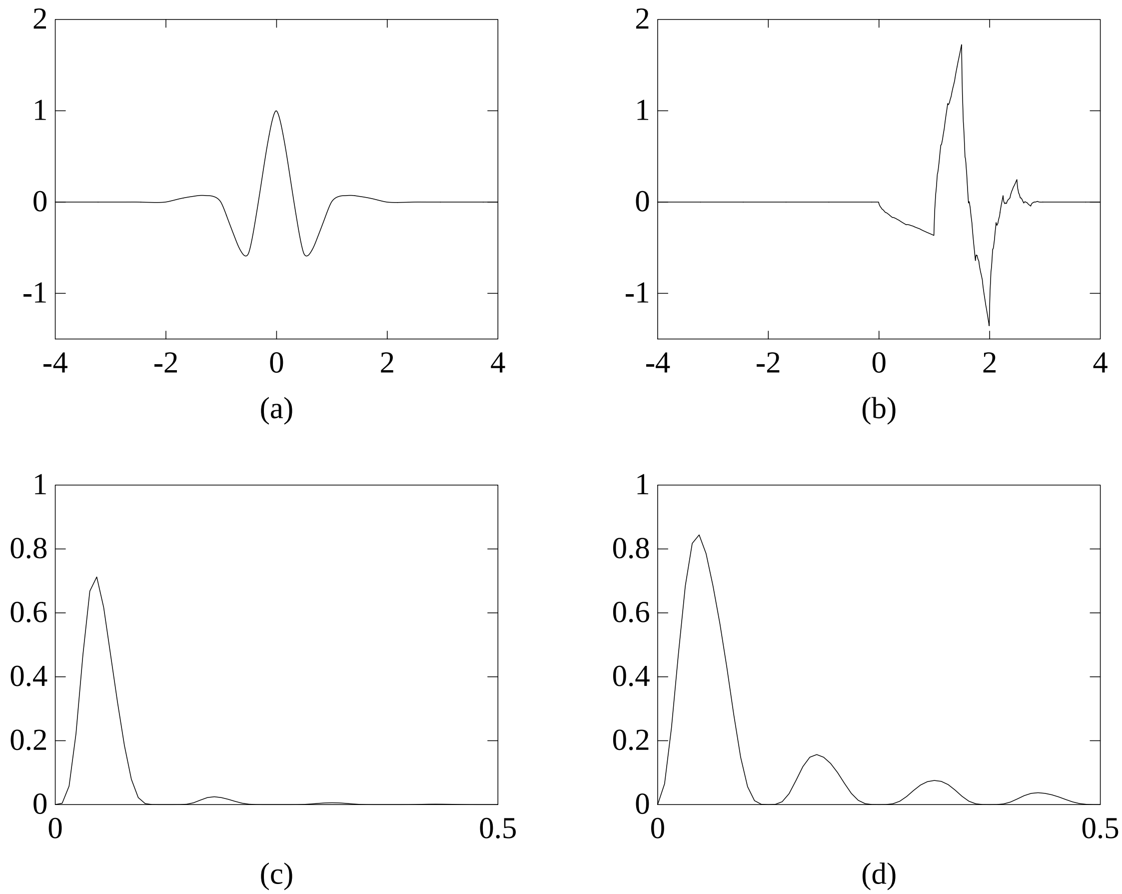

Figure 2: Plots of the auto-correlation function $\Psi(x)$ and the Daubechies's wavelet $\psi(x)$ with two vanishing moments and $L = 4$. (a) $\Psi(x)$. (b) $\psi(x)$. (c) Magnitude of the Fourier transform of $\Psi(x)$. (d) Magnitude of the Fourier transform of $\psi(x)$.

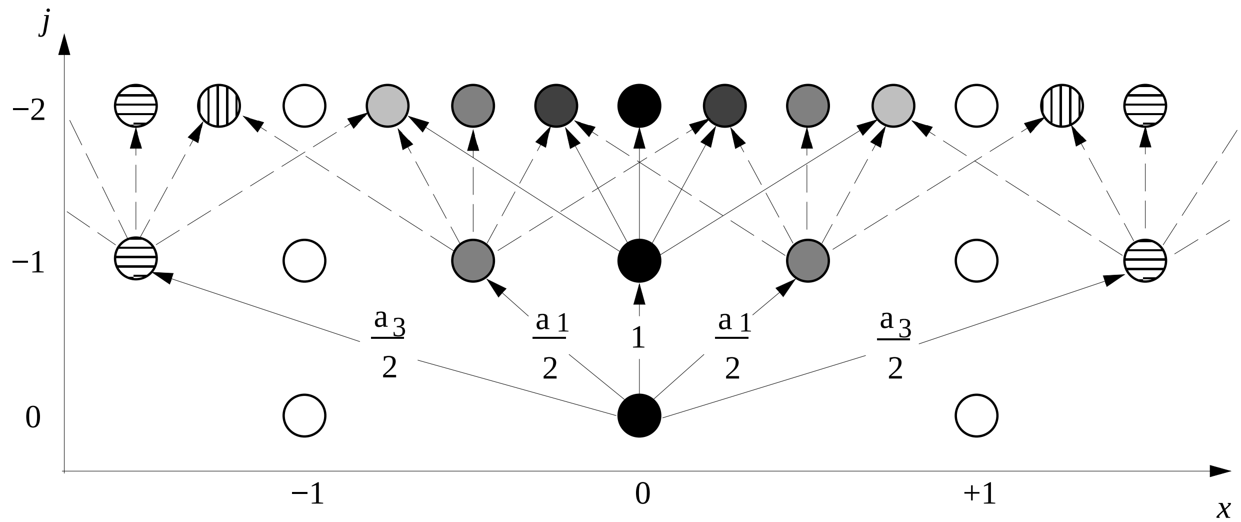

Figure 3: The Lagrange iterative interpolation of the unit impulse sequence with the associated quadrature mirror filter of length $L = 4$, i.e., $a_1 = 9/8$ and $a_3 = -1/8$. Black nodes at $x = 0$ indicate 1 and white nodes at $x = \pm 1$ have value 0. Shaded nodes have values other than 0 or 1. Note that the values of nodes existing at the j-th scale do not change at the $(j-1)$-th scale and higher. The result of repeating this procedure converges to $\Phi(x)$ as $j \to -\infty$.

Let us briefly review the properties of the autocorrelation functions in (1.1) and (1.10) and their relation to interpolation. Following [6] and [5], let us consider the following problem: given values of $f(x)$ on the set B_0, where B_n is the set of dyadic rationals $m/2^n, m = 0, 1, \ldots$, extend f to $B_1, B_2, \ldots$ in an iterative manner. For $x \in B_{n+1} \setminus B_n$, Dubuc in [6] has suggested the following formula to compute the value $f(x)$,

$$f(x) = \frac{9}{16}\left(f(x-h) + f(x+h)\right) - \frac{1}{16}\left(f(x-3h) + f(x+3h)\right), \tag{1.18}$$

where $h = 1/2^{n+1}$.

Figure 3 illustrates a few steps of this iterative process applied to the unit impulse.

This interpolation scheme is generalized futher in [5],

$$f(x) = \sum_{k \in \mathbf{Z}} F(k/2) f(x + kh), \quad \text{for} \quad x \in B_{n+1} \setminus B_n \text{ and } h = 1/2^{n+1}, \tag{1.19}$$

where the coefficients $F(k/2)$ are computed by generating the function satisfying

$$F(x/2) = \sum_{k \in \mathbf{Z}} F(k/2) F(x - k). \tag{1.20}$$

Using the Lagrange polynomials with $L = 2M$ nodes, we have

$$f(x) = \sum_{k=1}^{M} \mathcal{P}^L_{2k-1}(0)\ \left(f(x - (2k-1)h) + f(x + (2k-1)h)\right), \tag{1.21}$$

where $\{\mathcal{P}^L_{2k-1}(x)\}_{-M+1 \le k \le M}$ is a set of the Lagrange polynomials of the degree $L-1$ with nodes $\{-L+$

$1, -L+3, \ldots, L-3, L-1\}$,

$$\mathcal{P}^L_{2k-1}(x) = \prod_{l=-M+1, l\neq k}^{M} \frac{x-(2l-1)}{(2k-1)-(2l-1)}. \tag{1.22}$$

In this case, (1.20) reduces to

$$F_L(x) = F_L(2x) + \sum_{k=1}^{M} \mathcal{P}^L_{2k-1}(0) \ \left(F_L(2x-2k+1) + F_L(2x+2k-1)\right), \tag{1.23}$$

where F_L is the fundamental function of Dubuc and Deslauriers. This special case of (1.19) is called the "Lagrange iterative interpolation." The original Dubuc's scheme (1.18) corresponds to $L=4$ in (1.21).

We have

$$F(x) = \Phi(x), \tag{1.24}$$

where $F(x)$ is the fundamental function defined in (1.20) and $\Phi(x)$ is the auto-correlation function of the scaling function $\varphi(x)$ (see [11]). Using the two-scale difference equation (1.9), we obtain

$$\Phi(k/2) = \Phi(k) + \frac{1}{2}\sum_{l\in\mathbf{N}} a_{2l-1}\left(\Phi(k-2l+1) + \Phi(k+2l-1)\right). \tag{1.25}$$

and, therefore,

$$\Phi(k/2) = a_k/2. \tag{1.26}$$

In other words, the two-scale difference equation for the function Φ in (1.9) may be rewritten as

$$\Phi(x/2) = \sum_{k\in\mathbf{Z}} \Phi(k/2)\Phi(x-k). \tag{1.27}$$

For any polynomial P of degree smaller than L, the Lagrange iterative interpolation of the sequence $f(n) = P(n)$, $n\in\mathbf{Z}$, via (1.21) is precisely the function $f(x) = P(x)$ for any $x\in\mathbf{R}$.

If the number of vanishing moments $M=1$ and $L=2$ (the Haar basis), then we have

$$\Phi_{\text{Haar}}(x) = \begin{cases} 1+x & \text{for } -1\leq x\leq 0, \\ 1-x & \text{for } 0\leq x\leq 1, \\ 0 & \text{otherwise.} \end{cases} \tag{1.28}$$

The interpolation process then corresponds to the linear interpolation.

Using expressions (3.49)–(3.52) of [1], the relation (1.5) may be rewritten as

$$|m_0(\xi)|^2 = \frac{1}{2} + \frac{1}{2}\left[\frac{(2M-1)!}{(M-1)!\ 4^{M-1}}\right]^2 \sum_{k=1}^{M} \frac{(-1)^{k-1}\cos(2k-1)\xi}{(2k-1)\ (M-m)!\ (M+k-1)!}. \tag{1.29}$$

If $M\to\infty$, then

$$|m_0(\xi)|^2 \to \frac{1}{2} + \frac{1}{2}\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{2k-1}\cos(2k-1)\xi, \tag{1.30}$$

which is the Fourier series of the characteristic function of $[-\pi/2, \pi/2]$. This implies that the corresponding auto-correlation function is

$$\Phi_\infty(x) = \mathrm{sinc}(x) = \frac{\sin \pi x}{\pi x}. \tag{1.31}$$

The interpolation process then corresponds to the so-called band-limited interpolation. If the number M of the vanishing moments of the compactly supported wavelets approaches infinity, then (see [4])

$$\varphi_\infty(x) = \mathrm{sinc}(x). \tag{1.32}$$

As a result, we have the following relation,

$$\varphi_\infty(x) = \Phi_\infty(x), \tag{1.33}$$

and

$$\sqrt{2} h_k = \frac{a_k}{2} = \frac{\sin \pi k/2}{\pi k/2} \quad \text{for} \quad k \in \mathbf{Z}. \tag{1.34}$$

Thus, we have a family of the symmetric iterative interpolation schemes parameterized by the number of vanishing moments $1 \leq M < \infty$.

The derivative of the function $f(x)$ in (1.21) is computed via

$$f'(x) = \sum_{k=1}^{L-2} r_k \left(f(x+kh) - f(x-kh) \right), \tag{1.35}$$

where $h = 2^{-n}$, $x \in B_m$, where $m \leq n$, and

$$r_k = \int_{-\infty}^{+\infty} \varphi(x-k) \frac{d}{dx} \varphi(x) dx. \tag{1.36}$$

The coefficients r_k may be computed (see [1]) by solving

$$r_k = 2 \left[r_{2k} + \frac{1}{2} \sum_{l=1}^{L/2} a_{2l-1} (r_{2k-2l+1} + r_{2k+2l-1}) \right], \tag{1.37}$$

and

$$\sum_{k \in \mathbf{Z}} k\, r_k = -1, \tag{1.38}$$

where the coefficients a_{2l-1} are given in (1.6). If the number of vanishing moments of the wavelet $M \geq 2$, then equations (1.37) and (1.38) have a unique solution with a finite number of non-zero r_k, namely, $r_k \neq 0$ for $-L+2 \leq k \leq L-2$ and

$$r_k = -r_{-k}. \tag{1.39}$$

2. MULTIRESOLUTION REPRESENTATION OF SIGNALS

In theory, by analyzing the growth or decay from scale to scale of the coefficients of the orthonormal wavelet expansions, it is possible to estimate the local behavior of signals. However, since the coefficients of the orthonormal wavelet expansions are not shift invariant, redundant representations are being used in order to simplify the analysis of coefficients from scale to scale (see e.g. [8]).

Another difficulty in using the compactly supported orthonormal wavelets for the analysis of signals is their asymmetric shape (see [3]). On one hand, the quadrature mirror filters associated with compactly supported wavelets are of finite size and, therefore, are exact in computer implementations. On the other hand, the symmetric basis functions are preferred since their use simplifies finding zero-crossings (or extrema) corresponding to the locations of edges in images at later stages of processing. There are several approaches for dealing with this problem. The first approach consists in constructing approximately symmetric orthonormal wavelets and gives rise to approximate quadrature mirror filters [9]. The second consists in using biorthogonal bases [2], [13], so that the basis functions may be chosen to be exactly symmetric.

Alternatively, a redundant representation using dilations and translations of the auto-correlation functions of compactly supported wavelets (*the auto-correlation shell*), may be used for signal analysis instead of the wavelets per se [11]. The exact filters for the decomposition are the auto-correlation coefficients of the quadrature mirror filter coefficients of the compactly supported wavelets. The decomposition filters are, therefore, exactly symmetric. The recursive definition of the auto-correlation functions of compactly supported wavelets leads to fast recursive algorithms to generate the multiresolution representations. One of the interesting features of this representation is its convertibility to the orthonormal shell of the corresponding compactly supported wavelets on each scale, independently of other scales. The algorithm for such conversion is discussed in detail in [11].

Representation using the auto-correlation functions of compactly supported wavelets can also be viewed as a way to obtain a continuous-like multiresolution analysis. Another approach to make the connection between continuous and discrete multiresolution analyses is developed in [7], where the starting point is the continuous version of the multiresolution analysis.

Representation using the auto-correlation functions of compactly supported wavelets may also be compared with those using the approximation of the Laplacian of a Gaussian function (the so-called Mexican-hat function) by the Difference of two Gaussian functions (the so-called DOG function) as

$$\frac{d^2}{dx^2}G(x;\sigma) \approx aG(ax;\sigma) - G(x;\sigma), \tag{2.40}$$

where

$$G(x;\sigma) = \frac{1}{\sqrt{2\pi}\sigma}\mathrm{e}^{-x^2/2\sigma^2}, \tag{2.41}$$

and $a = 1.6$ as was suggested in [10]. It follows from (1.9) and (1.12) that

$$\Psi(x) = 2\Phi(2x) - \Phi(x), \tag{2.42}$$

which should be compared with (2.40).

Since the auto-correlation functions of the compactly supported wavelets may be viewed as pseudo-differential operators of the even order, and essentially behave as the derivative operators of the same order, the zero-crossings in this representation correspond to the locations of edges at different scales in the signal. Dubuc's iterative interpolation is naturally associated with such representation and allows us to define zero-crossings for multiresolution representations of discrete signals. By using iterative interpolation, we locate the zero-crossings and compute slopes at these points within the prescribed numerical accuracy. To reconstruct the signal, we set up a linear system where the entries of the matrix are computed from the values of the auto-correlation function and its derivative (at the integer translates of zero-crossings). The original signal is then reconstructed within the prescribed accuracy by solving this linear system. For the details of this algorithm, we refer to [11].

Examples of representation of signals in the auto-correlation shell are presented in Figures 4, 5 and 6.

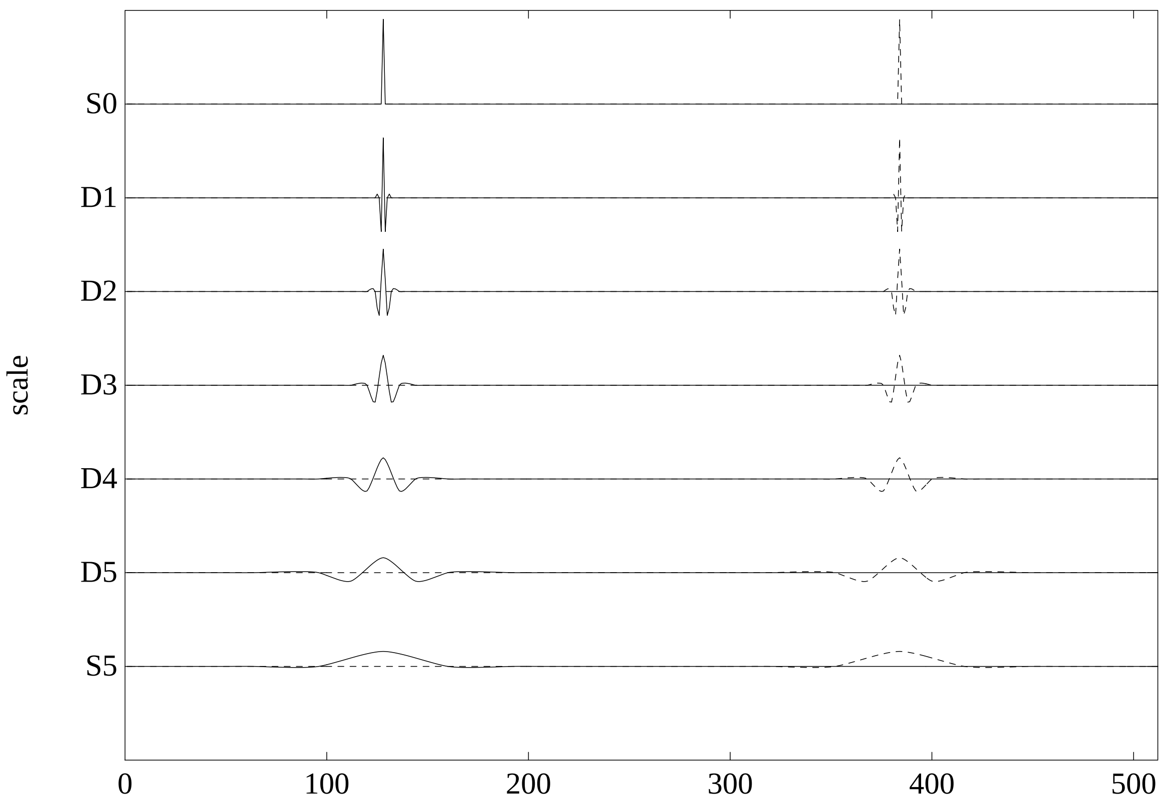

Figure 4: The expansion of two unit impulses in the auto-correlation shell using the auto-correlation functions of the Daubechies's wavelet with $L = 2M = 4$.

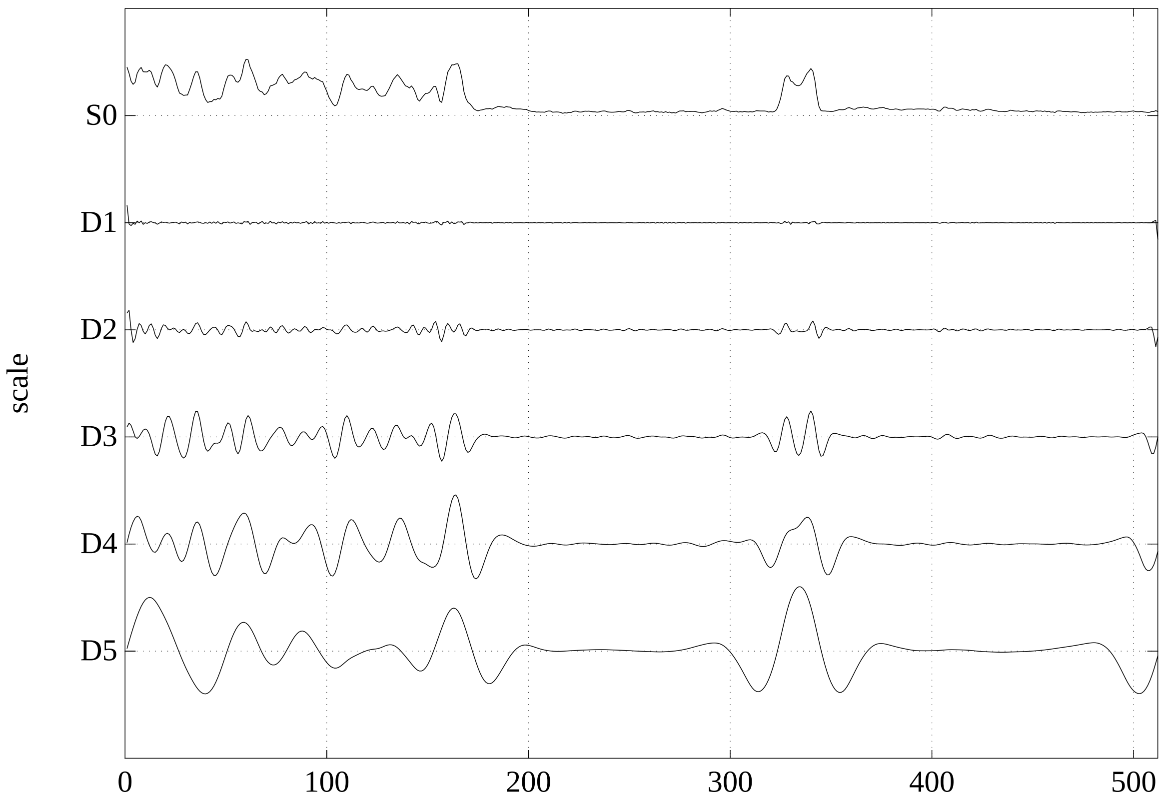

Figure 5: The expansion of the signal in the auto-correlation shell using the auto-correlation functions of the Daubechies's wavelet with $L = 2M = 4$. The top row is the original signal. Note that the locations of edges in the original signal correspond to the zero-crossings in this representation.

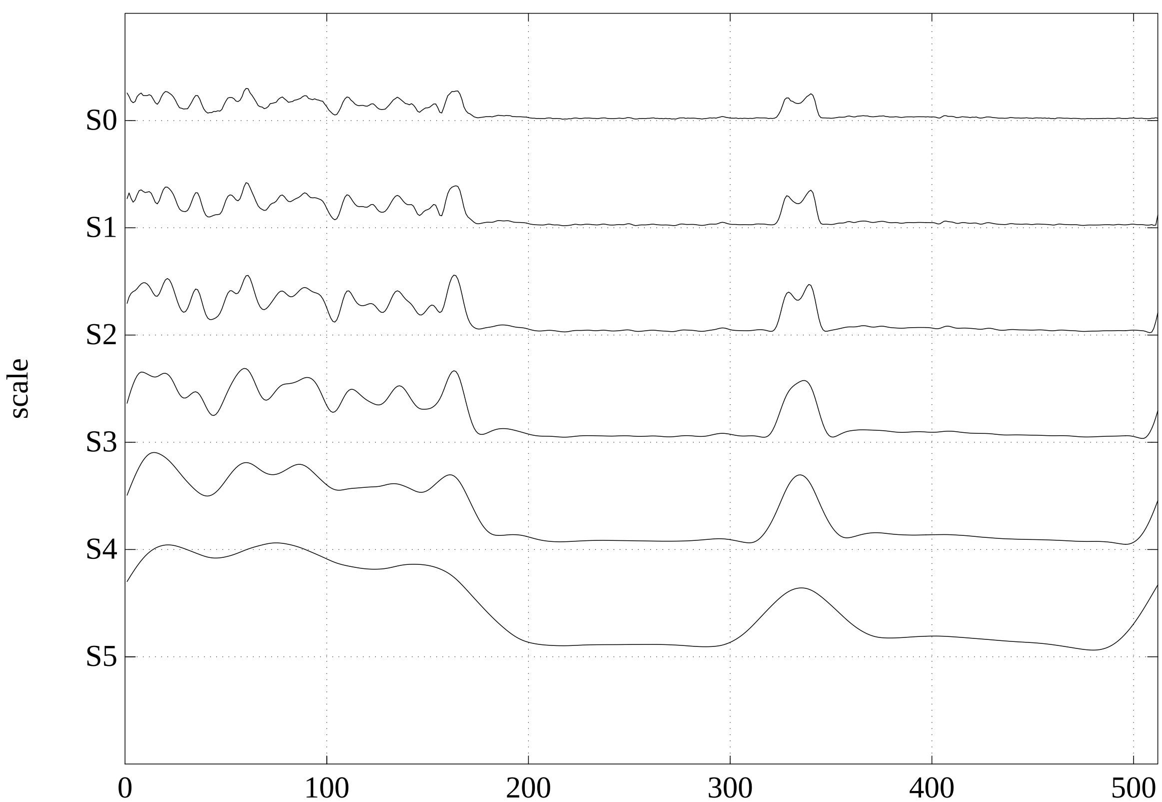

Figure 6: The averages on different scales (the top row is the original signal).

3. ACKNOWLEDGMENTS

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4. REFERENCES

[1] G. Beylkin, "On the representation of operators in bases of compactly supported wavelets," *SIAM J. Numer. Anal.*, Dec. 1992, to appear.

[2] A. Cohen, I. Daubechies, and J.-C. Feauveau, "Biorthogonal bases of compactly supported wavelets," *preprint*, 1990.

[3] I. Daubechies, "Orthonormal bases of compactly supported wavelets," *Comm. Pure and Appl. Math.*, 41:909–996, 1988.

[4] I. Daubechies, "Orthonormal bases of compactly supported wavelets. II. variations on a theme," *preprint*, 1990.

[5] G. Deslauriers and S. Dubuc, "Symmetric iterative interpolation processes," *Constructive Approximation*, 5:49–68, 1989.

[6] S. Dubuc, "Interpolation through an iterative scheme," *J. Math. Anal. and Appl.*, 114:185–204, 1986.

[7] M. Duval-Destin, M.A. Muschietti, and B. Torresani, "Continuous wavelet decompositions, multiresolution and contrast analysis," *SIAM J. Math. Anal.*, 1992. to appear.

[8] R. Kronland-Martinet, J. Morlet, and A. Grossman, "Analysis of sound patterns through wavelet transforms," *J. Pattern Recognition and Artifitial Intell.*, 1(2):273–302, 1987.

[9] S. Mallat, "A theory for multiresolution signal decomposition," *IEEE Trans. Pattern Anal. Machine Intell.*, 11(7):674–693, Jul. 1989.

[10] D. Marr, *Vision*, W.H.Freeman and Company, 1982.

[11] N. Saito and G. Beylkin, "Multiresolution representations using the auto-correlation functions of compactly supported wavelets," Submitted to *IEEE Trans. Signal Processing*, 1992. Also Research Note at Schlumberger-Doll Research, Ridgefield, CT, 1991 and in Proceedings of ICASSP-92, 4:381–384.

[12] M. J. Shensa, "The discrete wavelet transform: wedding the à trous and Mallat algorithms," *IEEE Trans. Signal Processing*, Oct. 1992. to appear.

[13] M. Vetterli and C. Herley, "Wavelets and filter banks: theory and design," *Technical report, Center for Telecommunications Research, Columbia University*, 1990.