Existance theorem and minimal cardinality of UEP framelets and MEP bi-framelets *

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Abstract Based on multiresolution analysis (MRA) structures combined with the unitary extension principle (UEP), many frame wavelets were constructed, which are called UEP framelets. The aim of this letter is to derive general properties of UEP framelets based on the spectrum of the center space of the underlying MRA structures. We first give the existence theorem, that is, we give a necessary and sufficient condition that an MRA structure can derive UEP framelets. Second, we present a split trick that each mother function can be split into several functions such that the set consisting of these functions is still a UEP framelet. Third, we determine the minimal cardinality of UEP framelets. Finally, we directly construct UEP framelets with the minimal cardinality. Based on a pair of multiresolution analysis (MRA) structures, when their spectra intersect, we can always construct a pair of dual frame wavelets using mixed extension principle (MEP). This pair of dual frame wavelets are called a pair of MEP bi-framelets. We also give the split trick and find out the minimal cardinality of such MEP bi-framelets.

Keywords: UEP framelet, MEP bi-framelet, spectrum, cardinality

1. Introduction

It is well known that multiresolution analysis (MRA) is one of the most important tools for constructing and analyzing wavelets. Orthogonal MRAs can derive orthonormal wavelets [7,22,23], while frame MRAs can

*The research of the first author was partially supported by Fundamental Research Funds for the Central Universities (Key Program), National Key Science Program for Global Change Research No.2010CB950504, 863 project No. 2010AA012305, NSFC No.41076125, and Polar Climate and Environment Key Laboratory. The research of the second author was partially supported by ONR grants: N00014-09-1-0041, N00014-09-1-0318, N00014-12-1-0117.

derive semi-orthogonal frame wavelets [1,2,18–21]. Recently, based on a kind of MRA structures, many authors constructed tight frame wavelets, which are called UEP framelets, by using the unitary extension principle (UEP) [6,8–12,14–30]. The characteristic of this type of MRA structure is that the integer-translations of the scaling function form a Bessel sequence. In the theory of MRA structures $\{V_m,\, \varphi\}$ of $L^2(\mathbb{R}^d)$, the *spectrum* (or the spectral support) Ω of the central space V_0, which is defined by $\Omega := \operatorname{supp} \sum_{n\in\mathbb{R}^d} |\widehat{\varphi}(\omega + 2n\pi)|^2$, plays a key role. We rewrite the spectrum in the form

$$\Omega = \operatorname{supp}\widehat{\varphi} + 2\pi\mathbb{Z}^d, \qquad \text{where} \quad \widehat{\varphi} \text{ is the Fourier transform of } \varphi.$$

In this letter we will derive general properties of UEP framelets based on the spectrum of the center space. For convenience, we explain our main idea in the setting of dyadic dilation as follows.

Let $\{V_m,\ \varphi\}$ be an MRA structure with the spectrum Ω. We consider the vector

$$u_0(\omega) = \big(\, (\tau_0\, \chi_\Omega)(\omega + \pi\nu)\, \big)_{\nu\in\{0,1\}^d}, \tag{1.1}$$

where τ_0 is a refinement filter and χ_Ω is the characteristic function of the spectrum Ω, and $\{0,1\}^d$ are the set of vertices of the cube $[0,1]^d$.

First, we will give the existence theorem of UEP framelets. This theorem provides a necessary and sufficient condition under which an MRA structure can derive a UEP framelet system. This condition is that the norm of $u_0(\omega)$ satisfies $\|u_0(\omega)\|_{l^2} \le 1$ for *a.e.* $\omega \in \mathbb{R}^d$.

Second, we will derive a split trick for UEP framelets and show that if $\{\psi_\mu\}_1^r$ form a UEP framelet, then each ψ_μ can be split into several functions such that the set consisting of these functions is still a UEP framelet derived by the same MRA structure. For example, we may only split ψ_r into two functions $\psi_r^{(1)}$ and $\psi_r^{(2)}$ such that $\psi_r = \psi_r^{(1)} + \psi_r^{(2)}$ and $\{\psi_1, ..., \psi_{r-1}, \psi_r^{(1)}, \psi_r^{(2)}\}$ still forms a UEP framelet. This split trick demonstrates the intrinsic redundancy of UEP framelets.

Third, in order to determine the minimal cardinality σ of a UEP framelet, we consider the sequence of the translations of the spectrum Ω:

$$\{\Omega - \pi\nu\}_{\nu \in \{0,1\}^d} \tag{1.2}$$

which contains 2^d sets. If a point ω lies in γ sets of 2^d sets in (1.2), we say $\omega \in \Omega_\gamma$, i.e.,

$$\Omega_\gamma = \left\{ \omega \in \mathbb{R}^d, \quad \sum_{\nu \in \{0,1\}^d} \chi_\Omega(\omega + \pi\nu) = \gamma \right\}.$$

Based on this definition of Ω_γ, we will derive the following result.

If $\lambda \in \mathbb{Z}_+$ is such that the measure $|\Omega_\lambda| > 0$ and $|\Omega_\gamma| = 0$ $(\gamma > \lambda)$, then the minimal value σ of the cardinality for these UEP framelets derived by the MRA structure $\{V_m, \varphi\}$ satisfies

$$\sigma = \lambda \qquad \text{or} \qquad \sigma = \lambda - 1.$$

If $u_0(\omega)$ in (1.1) is the unit vector for $\omega \in \Omega_\lambda$, then $\sigma = \lambda - 1$. Otherwise, $\sigma = \lambda$.

Finally, we will directly construct a UEP framelet with the minimal cardinality σ. From this and the above split trick, we will deduce that the set of cardinalities of all UEP framelets can be written in the form

$$[\sigma, \infty) \bigcap \mathbb{Z}_+.$$

For a pair of MRA structures, using the so-called mixed extension principle (MEP), many pairs of dual frame wavelets known as MEP bi-framelets have been constructed, which are a generalization of quasi-biorthogonal frame wavelets [18]. In Section 8, we will show that any pair of MRA structures can derive pairs of MEP bi-framelets if their spectra intersect. We will also derive the corresponding split trick. We would like to note that in [31] we also studied the split trick of quasi-biorthogonal frame wavelets, but the results obtained in [31] are strictly for the quasi-biorthogonal frame wavelets and different from those obtained in this letter, which are for the UEP framelets and the MEP bi-framelets. Finally, we will discuss the minimal cardinality of mother functions, and directly construct a pair of MEP bi-framelets with the minimal cardinality. Since MEP bi-framelets are a generalization of quasi-biorthogonal frame wavelets, an estimate of lower bounds of cardinalities in [31] can be viewed as a corollary of Theorem 8.3 in this letter. More importantly, [31] did not show that this lower bound is in fact the same as the minimal cardinality.

2. MRA structures and extension principles

Let S be a $d \times d$ matrix whose entries are integer-valued and whose eigenvalues all have modulus greater than 1. Let S^* be the transposed matrix of S and let G , G^* be quotient groups $\mathbb{Z}^d/S\mathbb{Z}^d$, $Z^d/S^*\mathbb{Z}^d$, respectively. The orders of both G and G^* are $\rho = |\det S|$.

Let $\{V_m\}$ be a sequence of subspaces of $L^2(\mathbb{R}^d)$ such that

$$V_m \subset V_{m+1} \ (m \in \mathbb{Z}), \quad \overline{\bigcup_{m\in\mathbb{Z}} V_m} = L^2(\mathbb{R}^d), \quad \text{and}$$

$$f \in V_m \leftrightarrow f(S\cdot) \in V_{m+1} \ (m \in \mathbb{Z}).$$

If there exists a function $\varphi \in V_0$, $\lim_{\omega\to 0} \widehat{\varphi}(\omega) = 1$ such that $V_0 = \overline{\text{span}}\{\varphi(\cdot - n)\}_{n\in\mathbb{Z}^d}$ and $\{\varphi(\cdot - n)\}_{n\in\mathbb{Z}^d}$ is a Bessel sequence, then $\{V_m, \varphi\}$ is said to be an *MRA structure* [11].

We rewrite extension principles of [11] with the spectrum for frame wavelets and pairs of dual frame wavelets into matrix versions. See [1–5, 13] for the concepts of general frames and wavelet frames.

For convenience, we introduce the following notation. For a set E, denote

$$E + 2\pi\mathbb{Z}^d = \bigcup_{n\in\mathbb{Z}^d} (E + 2n\pi).$$

If a set E satisfies $E + 2\pi\mathbb{Z}^d = E$, then E is said to be a $2\pi\mathbb{Z}^d$-periodic set. Define its measure $|E|$ as the measure of the set $E\bigcap[-\pi, \pi]^d$. Denote by χ_E the characteristic function of the set E and by $\widehat{f}$ the Fourier transform of a function f. We say $\tau \in L^\infty(\mathbb{T}^d)$ if τ is a bounded $2\pi\mathbb{Z}^d$-periodic function.

Let $\{V_m, \varphi\}$ be an MRA structure. Then the following set

$$\Omega = \text{supp}\,\widehat{\varphi} + 2\pi\mathbb{Z}^d = \bigcup_{n\in\mathbb{Z}^d} (\text{supp}\,\widehat{\varphi} + 2n\pi)$$

is called the *spectrum* of the central space V_0.

(i) Let $\tau_0 \subset L^\infty(\mathbb{T}^d)$ be the refinement filter corresponding to the scaling function φ and $\{\tau_\mu\}_1^r \subset L^\infty(\mathbb{T}^d)$. Denote the $(r+1)\times\rho$ matrix

$$M_r(\omega) := (\,(\tau_\mu \chi_\Omega)(\omega + 2\pi S^{*-1}\nu)\,)_{\mu=0,\ldots,r;\ \nu\in G^*} \tag{2.1}$$

and let $M_r^*(\omega)$ be the conjugate and transposed matrix of $M_r(\omega)$. Define $\{\psi_\mu\}_1^r$ as $\widehat{\psi}_\mu(S^*\cdot) = \tau_\mu \widehat{\varphi}$ $(\mu = 1, ..., r)$. If $M_r(\omega)$ satisfies

$$M_r^*(\omega)\, M_r(\omega) = \operatorname{diag}\,(\chi_\Omega(\omega + 2\pi S^{*-1}\nu))_{\nu\in G^*} \tag{2.2}$$

for a.e. $\omega \in \mathbb{R}^d$, then $\{\psi_\mu\}_1^r$ is a tight frame wavelet which is called a *UEP framelet with cardinality* r.

(ii) Let $\{V_m,\ \varphi\}$ and $\{\widetilde{V}_m,\ \widetilde{\varphi}\}$ be a pair of MRA structures and

$$\Omega' = \operatorname{supp}\widehat{\varphi} + 2\pi\mathbb{Z}^d, \qquad \Omega'' = \operatorname{supp}\widehat{\widetilde{\varphi}} + 2\pi\mathbb{Z}^d.$$

We always assume that $\Omega = \Omega' \bigcap \Omega''$ is a set with a positive measure. Then Ω is called the spectrum of this pair of MRA structures.

Let τ_0 and $\widetilde{\tau}_0$ be refinement filters corresponding to φ and $\widetilde{\varphi}$, respectively, and $\{\tau_\mu,\ \widetilde{\tau}_\mu\}_1^r \subset L^\infty(\mathbb{T}^d) \times L^\infty(\mathbb{T}^d)$. Denote $\widetilde{M}_r(\omega) := (\,(\widetilde{\tau}_\mu\chi_\Omega)(\omega + 2\pi S^{*-1}\nu)\,)_{\mu=0,\ldots,r;\ \nu\in G^*}$. Define $\{\psi_\mu,\ \widetilde{\psi}_\mu\}_1^r$ as $\widehat{\psi}_\mu(S^*\cdot) = \tau_\mu\,\widehat{\varphi}$, $\widehat{\widetilde{\psi}}_\mu(S^*\cdot) = \widetilde{\tau}_\mu\ \widehat{\widetilde{\varphi}}$ $(\mu = 1, ..., r)$. If a pair of matrices $M_r(\omega)$ and $\widetilde{M}_r(\omega)$ satisfies

$$M_r^*(\omega)\widetilde{M}_r(\omega) = \operatorname{diag}\,(\chi_\Omega(\omega + 2\pi S^{*-1}\nu))_{\nu\in G^*} \tag{2.3}$$

for a.e. $\omega \in \mathbb{R}^d$, then $\{\psi_\mu,\ \widetilde{\psi}_\mu\}_1^r$ form a pair of dual frame wavelets [11] whenever the wavelet systems $\{\psi_{\mu,m,n}\}$ and $\{\widetilde{\psi}_{\mu,m,n}\}$ are both Bessel sequences. Such a pair of dual frame wavelets $\{\psi_\mu,\ \widetilde{\psi}_\mu\}_1^r$ is called *a pair of MEP bi-framelets with cardinality* r.

3. Some lemmas

Since the dilation matrix S is a matrix with the integer entries and $G^* = \mathbb{Z}^d/S^*\mathbb{Z}^d$, for a fixed $\nu \in \mathbb{Z}^d$, when $l \in G^*$, we have

$$l + \nu = S^*\alpha_l + \beta_l, \qquad (\alpha_l \in \mathbb{Z}^d,\ \beta_l \in G^*) \tag{3.1}$$

and $\beta_l = \beta(l)$ maps G^* to G^*, one-to-one. The following two lemmas are proved easily.

Lemma 3.1. If (2.2) holds at a point ω_0, then for each $\nu \in G^*$, (2.2) holds at the point $\omega_\nu = \omega_0 + 2\pi S^{*-1}\nu$.

Define a $2\pi\mathbb{Z}^d$-periodic set

$$N := S^{*-1}[-2\pi,\ 0]^d + 2\pi\mathbb{Z}^d. \tag{3.2}$$

Lemma 3.2. Let N be stated as (3.2). Then $\mathbb{R}^d = \bigcup_{l\in G^*}(N + 2\pi S^{*-1}l)$ and this is a disjoint union.

Denote by I_m the identity matrix of order m.

Lemma 3.3. Let the l^2-norm of the vector $(a_1, ..., a_m)$ be less than or equal to 1. Then there exists a $(m+1)\times m$ matrix P such that its first row is $(a_1, ..., a_m)$ and $P^*P = I_m$.

Proof. Take a_{m+1} such that $(a_1, ..., a_m, a_{m+1})$ is a unit vector. Taking a unitary matrix $B = (b_{ij})_{i,j=1,\ldots,m+1}$ such that $b_{1,j} = a_j$ $(j = 1, ..., m+1)$, we have $B^*B = I_{m+1}$. Let P be a matrix consisting of the first m columns of B. We then get $P^*P = I_m$. $\square$

Lemma 3.4. For two m-dimensional vectors $(a_1^1, ..., a_m^1)$ and $(a_1^2, ..., a_m^2)$,

(i) there exist always two $(m+1)\times m$ matrices P and Q such that their first rows are $(a_1^1, ..., a_m^1)$ and $(a_1^2, ..., a_m^2)$, respectively, and $P^*Q = I_m$;

(ii) if the inner product of these two vectors is equal to 1, then there exist two square matrices P and Q of order m such that their first rows are $(a_1^1, ..., a_m^1)$ and $(a_1^2, ..., a_m^2)$, respectively, and $P^*Q = I_m$.

Proof. The proof of (ii) is obvious. Below we only prove (i).

Take a_{m+1}^1 and a_{m+1}^2 such that

$$\sum_{k=1}^{m+1} a_k^1\,\overline{a_k^2} = 1.$$

By (ii), there exist square matrices $B_1 = (b_{ij}^1)_{i,j=1,\ldots,m+1}$ and $B_2 = (b_{ij}^2)_{i,j=1,\ldots,m+1}$ such that

$$b_{1j}^1 = a_j^1, \quad b_{1j}^2 = a_j^2 \qquad (j = 1, ..., m+1)$$

and $B_1^*B_2 = I_{m+1}$. Forming two $(m+1)\times m$ matrices as

$$P = (b_{ij}^1)_{i=1,\ldots,m+1;\ j=1,\ldots,m} \quad \text{and} \quad Q = (b_{ij}^2)_{i=1,\ldots,m+1;\ j=1,\ldots,m},$$

we get $P^*Q = I_m$. So (i) is proved. $\Box$

4. The existence theorem of UEP framelets

We derive a necessary and sufficient condition that an MRA structure can derive UEP framelets.

Theorem 4.1. Let $\{V_m,\ \varphi\}$ be an MRA structure with the spectrum Ω and $\tau_0 \in L^\infty(\mathbb{T}^d)$ be the refinement filter. Then this MRA structure can derive UEP framelets if and only if

$$\sum_{\nu\in G^*} |(\tau_0\chi_\Omega)(\omega + 2\pi S^{*-1}\nu)|^2 \le 1 \qquad a.e. \quad \omega \in \mathbb{R}^d. \tag{4.1}$$

Proof. "Only if" part:

If $\{\psi_\mu\}_1^r$ is a UEP framelet derived by MRA structure, then, by the definition, there exist wavelet filters $\{\tau_\mu\}_1^r \subset L^\infty(\mathbb{T}^d)$ such that $\widehat{\psi}_\mu(S^*\cdot) = \tau_\mu\widehat{\varphi}\ (\mu = 1, ..., r)$ and (2.2) holds. Therefore, for a.e. $\omega \in \mathbb{R}^d$, if $\{k_j\}_0^{L-1} \subset G^*$ is such that

$$\omega + 2\pi S^{*-1}k_j \in \Omega \ (j = 0, 1, ..., L-1), \qquad \omega + 2\pi S^{*-1}\nu \notin \Omega \ (\nu \in G^*\backslash\{k_j\}_0^{L-1}), \tag{4.2}$$

the k_0th,...,k_{L-1}th column vectors of the matrix $M_r(\omega)$ are L orthogonal $(r+1)$-dimensional unit vectors. So $r+1 \ge L$. We add $(r+1-L)$ column vectors such that the obtained matrix P is a unitary matrix of order $r+1$. Therefore, the first row of P:

$$\big(\tau_0(\omega + 2\pi S^{*-1}k_0),\ \tau_0(\omega + 2\pi S^{*-1}k_1),\ \cdots,\ \tau_0(\omega + 2\pi S^{*-1}k_{L-1}),\ a_L(\omega),\ a_{L+1}(\omega),\ \cdots,\ a_r(\omega)\big)$$

is a unit vector, where $a_i(\omega)$ is the first entry of ith column of the matrix P $(i = L, ..., r)$. Therefore, for a.e. $\omega \in \mathbb{R}^d$, we have

$$\sum_{j=0}^{L-1} |\tau_0(\omega + 2\pi S^{*-1}k_j)|^2 \le 1.$$

From this, with (4.2), we get (4.1).

"If" part: If (4.1) holds, then the l^2-norm of the ρ-dimensional vector

$$u_0(\omega) = \{(\tau_0\chi_\Omega)(\omega + 2\pi S^{*-1}\nu)\}_{\nu\in G^*}$$

is less or equal to 1 for a.e. $\omega \in \mathbb{R}^d$, where $\rho = |\det S|$ is the order of S^*. We will use the unitary extension principle to construct a frame wavelet with cardinality ρ.

Since $\tau_0\chi_\Omega \in L^\infty(\mathbb{T}^d)$, by Lemma 3.3, we may take a $(\rho+1)\times\rho$ matrix whose entries are bounded and $2\pi\mathbb{Z}^d$-periodic functions on $\mathbb{R}^d$: $P(\omega) = (p_{\mu,\nu}(\omega))_{\mu=0,\ldots,\rho;\ \nu\in G^*}$ such that its first row is $u_0(\omega)$ and

$$P^*(\omega)P(\omega) = I_\rho \qquad (\text{a.e. } \omega \in \mathbb{R}^d), \tag{4.3}$$

where I_ρ is the identity matrix of order ρ.

We define wavelet filters $\tau_\mu(\omega)$ $(\mu = 1, ..., \rho)$ by

$$\tau_\mu(\omega + 2\pi S^{*-1}\nu) = p_{\mu,\,\nu}(\omega) \qquad \text{for} \quad \omega \in N,\ \ \nu \in G^*,$$

where the set N is stated as in (3.2). Then we have well defined $\{\tau_\mu\}_1^\rho$ on $\mathbb{R}^d$ (see Lemma 3.2). From this, we deduce by (4.3) that for $\omega \in N$,

$$\begin{aligned}&\sum_{\mu=0}^{\rho}(\tau_\mu\,\chi_\Omega)(\omega + 2\pi S^{*-1}\,\nu_1)(\overline{\tau}_\mu\,\chi_\Omega)(\omega + 2\pi S^{*-1}\,\nu_2)\\ = &\left(\sum_{\mu=0}^{\rho} p_{\mu,\nu_1}(\omega)\,\overline{p}_{\mu,\nu_2}(\omega)\right)\chi_\Omega(\omega + 2\pi S^{*-1}\,\nu_1)\,\chi_\Omega(\omega + 2\pi S^{*-1}\,\nu_2) = \delta_{\nu_1,\nu_2}\,\chi_\Omega(\omega + 2\pi S^{*-1}\nu_1),\end{aligned}$$

where $\delta_{\nu_1,\nu_2} = 0$ $(\nu_1 \neq \nu_2)$ and $\delta_{\nu_1,\nu_2} = 1$ $(\nu_1 = \nu_2)$. Therefore, for $\omega \in N$, the wavelet filters $\{\tau_\mu\}_1^\rho$ satisfy (2.2) with $r = \rho$. Again, by Lemmas 3.1 and 3.2, we know that (2.2) holds with $r = \rho$ for a.e. $\omega \in \mathbb{R}^d$. Let $\widehat{\psi}_\mu(S\cdot) = \tau_\mu\widehat{\varphi}$ $(\mu = 1, ..., \rho)$. Then $\{\psi_\mu\}_1^\rho$ is a UEP framelet with cardinality ρ. $\square$

For an MRA structure $\{V_m,\ \varphi\}$, its refinement filter τ_0 is not determined uniquely by the scaling function φ, but $\tau_0\chi_\Omega$ is determined uniquely by φ.

5. A split trick for UEP framelets

From Theorem 4.1 and its proof, we know that for an MRA structure, if the condition (4.1) holds, then it can derive a UEP framelet with ρ mother functions. In this section, we will further show that UEP framelets possess the following important property.

Theorem 5.1. Let $\{V_m,\ \varphi\}$ be an MRA structure with spectrum Ω and $\{\psi_\mu\}_1^r$ be a UEP framelet derived by it. Then we may split ψ_r into two functions $\psi_r = \psi_r^{(1)} + \psi_r^{(2)}$ such that $\{\psi_1, ..., \psi_{r-1}, \psi_r^{(1)}, \psi_r^{(2)}\}$ is still a UEP framelet derived by this MRA structure.

Proof. **Step 1.** Split ψ_r into $\psi_r = \psi_r^{(1)} + \psi_r^{(2)}$ as follows.

Denote

$$A = \left(\bigcup_{k \in G^*} (\operatorname{supp} \widehat{\psi}_r(S^* \cdot) + 2\pi S^{*-1} k) \right) + 2\pi \mathbb{Z}^d \tag{5.1}$$

and $\widetilde{A} = A \bigcap N$, where N is stated as in (3.2). Take a $2\pi\mathbb{Z}^d-$periodic set $J \subset N$ such that

$$|J \bigcap \widetilde{A}| > 0 \qquad \text{and} \qquad |(N \setminus J) \bigcap \widetilde{A}| > 0. \tag{5.2}$$

Define

$$I_1 = \bigcup_{k \in G^*} (J + 2\pi S^{*-1} k), \qquad I_2 = \bigcup_{k \in G^*} ((N \setminus J) + 2\pi S^{*-1} k). \tag{5.3}$$

By Lemma 3.2 and $J \subset N$, we have $I_1 + I_2 = \mathbb{R}^d$.

Now we prove that

$$|I_1 \bigcap \operatorname{supp} \widehat{\psi}_r(S^* \cdot)| > 0 \qquad \text{and} \qquad |I_2 \bigcap \operatorname{supp} \widehat{\psi}_r(S^* \cdot)| > 0. \tag{5.4}$$

By (5.3), we get that for $\nu \in \mathbb{Z}^d$

$$I_1 + 2\pi S^{*-1} \nu = \bigcup_{k \in G^*} (J + 2\pi S^{*-1}(k + \nu)).$$

Since $k + \nu = S^* n + l$ $(n \in \mathbb{Z}^d;\ l \in G^*)$ and $J + 2\pi \mathbb{Z}^d = J$, by (5.3), we obtain that

$$I_1 + 2\pi S^{*-1} \nu = \bigcup_{l \in G^*} (J + 2\pi n + 2\pi S^{*-1} l) = \bigcup_{l \in G^*} (J + 2\pi S^{*-1} l) = I_1.$$

Similarly, we get $I_2 + 2\pi S^{*-1} \nu = I_2$. Hence

$$I_i + 2\pi S^{*-1} \nu = I_i \qquad (\nu \in \mathbb{Z}^d;\ \ i = 1, 2). \tag{5.5}$$

We assume that (5.4) is not valid. Then either $|I_1 \bigcap \operatorname{supp} \widehat{\psi}_r(S^* \cdot)| = 0$ or $|I_2 \bigcap \operatorname{supp} \widehat{\psi}_r(S^* \cdot)| = 0$.

If $|I_1 \bigcap \operatorname{supp} \widehat{\psi}_r(S^* \cdot)| = 0$, by (5.1) and (5.5), we have $|I_1 \bigcap A| = 0$. By $J \subset I_1$ and $\widetilde{A} \subset A$, we get $|J \bigcap \widetilde{A}| = 0$. This is contrary to the first formula in (5.2). If $|I_2 \bigcap \operatorname{supp} \widehat{\psi}_r(S^* \cdot)| = 0$, similarly, we have $|(N \setminus J) \bigcap \widetilde{A}| = 0$. This is contrary to the second formula in (5.2). Therefore, (5.4) holds.

By (5.5), we know that the characteristic function χ_{I_i} satisfies

$$\chi_{I_i}(\cdot + 2\pi S^{*-1} \nu) = \chi_{I_i} \qquad (\nu \in \mathbb{Z}^d, \ i = 1, 2). \tag{5.6}$$

Now we define

$$\tau_r^{(1)} = \tau_r \chi_{I_1}, \qquad \tau_r^{(2)} = \tau_r \chi_{I_2}, \tag{5.7}$$

where τ_r is the wavelet filter of ψ_r. Then $\tau_r^{(1)}, \ \tau_r^{(2)} \in L^\infty(\mathbb{T}^d)$. Again, define $\psi_r^{(1)}$ and $\psi_r^{(2)}$ by

$$\widehat{\psi}_r^{(1)}(S^* \cdot) = \tau_r^{(1)} \widehat{\varphi}, \qquad \widehat{\psi}_r^{(2)}(S^* \cdot) = \tau_r^{(2)} \widehat{\varphi}.$$

Since $\widehat{\psi}_r(S^* \cdot) = \tau_r \, \widehat{\varphi}$ and $I_1 \bigcup I_2 = \mathbb{R}^d$, we get

$$\widehat{\psi}_r^{(1)}(S^* \cdot) + \widehat{\psi}_r^{(2)}(S^* \cdot) = (\tau_r^{(1)} + \tau_r^{(2)}) \, \widehat{\varphi} = \tau_r \, \chi_{I_1 \bigcup I_2} \, \widehat{\varphi} = \tau_r \, \widehat{\varphi} = \widehat{\psi}_r(S^* \cdot).$$

So $\psi_r = \psi_r^{(1)} + \psi_r^{(2)}$. From (5.4), we know that $\psi_r^{(1)}$ and $\psi_r^{(2)}$ are both nonzero functions. In fact, from $\widehat{\psi}_r^{(i)}(S^* \cdot) = \widehat{\psi}_r(S^* \cdot) \chi_{I_i}$, we get

$$|\operatorname{supp} \widehat{\psi}_r^{(i)}(S^* \cdot)| = |\operatorname{supp} \widehat{\psi}_r(S^* \cdot) \bigcap I_i| > 0 \qquad (i = 1, 2).$$

Step 2. We prove that $\{\tau_1, ..., \tau_{r-1}, \ \tau_r^{(1)}, \ \tau_r^{(2)}\}$ satisfy (2.2).

Since $\{\psi_\mu\}_1^r$ is a UEP framelet, $\{\tau_\mu\}_1^r$ satisfy (2.2). So we have

$$\left(\sum_{\mu=0}^{r} \tau_\mu(\omega + 2\pi S^{*-1} \nu) \overline{\tau}_\mu(\omega + 2\pi S^{*-1} l) \right) \chi_\Omega(\omega + 2\pi S^{*-1} \nu) \chi_\Omega(\omega + 2\pi S^{*-1} l)$$
$$= \delta_{\nu, l} \, \chi_\Omega(\omega + 2\pi S^{*-1} \nu) \quad (\nu, \ l \in G^*). \tag{5.8}$$

By (5.6), (5.7), and $I_1 \cup I_2 = \mathbb{R}^d$, we have

$$\sum_{i=1}^{2} \tau_r^{(i)}(\omega + 2\pi S^{*-1} \nu) \overline{\tau}_r^{(i)}(\omega + 2\pi S^{*-1} l)$$

$$= \tau_r(\omega + 2\pi S^{*-1}\nu)\,\overline{\tau}_r(\omega + 2\pi S^{*-1}l)\,\chi_{I_1 \bigcup I_2}(\omega) = \tau_r(\omega + 2\pi S^{*-1}\nu)\overline{\tau}_r(\omega + 2\pi S^{*-1}l).$$

From this and (5.8), we see that the filters $\{\tau_1, ..., \tau_{r-1},\ \tau_r^{(1)},\ \tau_r^{(2)}\}$ satisfy (2.2). Therefore, $\{\psi_1, ..., \psi_{r-1},\ \psi_r^{(1)},\ \psi_r^{(2)}\}$ is a UEP framelet with cardinality $r+1$, derived by the same MRA structure. $\square$

6. The minimal cardinality of UEP framelets

Let $\{V_m,\ \varphi\}$ be an MRA structure with the spectrum Ω and the refinement filter τ_0. We always assume that (4.1) holds. In this section we will determine the minimal cardinality σ of UEP framelets derived by this MRA structure.

Define the test set

$$F := \left\{ \omega \in \mathbb{R}^d, \quad \sum_{\nu \in G^*} |(\tau_0 \chi_\Omega)(\omega + 2\pi S^{*-1}\nu)|^2 = 1 \right\}. \tag{6.1}$$

Clearly, $\omega \in F$ means that the first row of the matrix $M_r(\omega)$ (see (2.1)) is a unit vector.

Definition 6.1. Let $\{V_m, \varphi\}$ be an MRA structure with the spectrum Ω. Consider the set

$$\{\Omega - 2\pi S^{*-1}\nu\}_{\nu \in G^*} = \left\{ \Omega - 2\pi S^{*-1}\nu_0, \quad \Omega - 2\pi S^{*-1}\nu_1, \quad \cdots, \quad \Omega - 2\pi S^{*-1}\nu_{\rho-1} \right\},$$

where $G^* = \mathbb{Z}^d / S^* \mathbb{Z}^d$ is identified with $\{\nu_0, \nu_1, ..., \nu_{\rho-1}\}$. If a point ω lies in γ sets of 2^d sets in $\{\Omega - 2\pi S^{*-1}\nu\}_{\nu \in G^*}$, we say $\omega \in \Omega_\gamma$, i.e.,

$$\Omega_\gamma = \left\{ \omega \in \mathbb{R}^d, \quad \sum_{i=0}^{\rho-1} \chi_\Omega(\omega + 2\pi S^{*-1}\nu_i) = \gamma \right\}.$$

Now, define

$$\lambda := \max\left\{ \gamma \in \mathbb{Z}^+ : \quad |\Omega_\gamma| > 0 \right\},$$

which we call the *order* of this MRA structure.

Theorem 6.2. Let $\{V_m, \varphi\}$ be an MRA structure with the spectrum Ω and can derive UEP framelets. If the order of this MRA structure is λ, then the minimal cardinality of these UEP framelets is $\sigma = \lambda$ or $\sigma = \lambda - 1$. Moreover, $\sigma = \lambda - 1$ if and only if $\Omega_\lambda \subset F$, where sets Ω_λ and F are stated in Definition 6.1 and (6.1).

Proof. Let $\{\psi_\mu\}_1^r$ be a UEP framelet with cardinality r, derived by the MRA structure.

Step 1. We prove that the cardinality $r \geq \lambda - 1$.

Since $\{V_m,\ \varphi\}$ is an MRA structure of order λ, by Definition 6.1, we have $|\Omega_\lambda| > 0$. Denote the wavelet filters of the framelet $\{\psi_\mu\}_1^r$ by $\{\tau_\mu\}_1^r$. Then we know that $\{\tau_\mu\}_1^r$ satisfy

$$M_r^*(\omega)M_r(\omega) = \mathrm{diag}\,(\chi_\Omega(\omega + 2\pi S^{*-1}\nu))_{\nu\in G^*} \qquad (\text{a.e.}\, \omega \in \mathbb{R}^d), \tag{6.2}$$

where $M_r(\omega)$ is stated in (2.1). By the definition of Ω_λ, we know that for $\omega \in \Omega_\lambda$, there exist exactly λ nonzero entries in the diagonal matrix $\mathrm{diag}\,(\chi_\Omega(\omega + 2\pi S^{*-1}\nu))_{\nu\in G^*}$. This implies that the rank of the matrix $M_r(\omega)$ is greater than or equal to λ on Ω_λ. Since $|\Omega_\lambda| > 0$ and $M_r(\omega)$ is a $(r+1)\times\rho$ matrix, we obtain $r \geq \lambda - 1$.

Step 2. We prove that if the cardinality $r = \lambda - 1$, then $\Omega_\lambda \subset F$.

For $\omega \in \Omega_\lambda$, by Definition 6.1, we deduce that there exists $\{k_j\}_{j=0}^{\lambda-1} \subset G^*$ such that

$$\chi_\Omega(\omega + 2\pi S^{*-1}k_j) = 1, \quad j = 0, ..., \lambda - 1, \qquad \chi_\Omega(\omega + 2\pi S^{*-1}\nu) = 0, \quad \nu \in G^*\backslash\{k_j\}_{j=0}^{\lambda-1}. \tag{6.3}$$

Suppose $r = \lambda - 1$. Then for $\omega \in \Omega_\lambda$, the k_0th, ..., $k_{\lambda-1}$th column vectors of the matrix $M_{\lambda-1}(\omega)$ (see (2.1)) form a square matrix: $D(\omega) = ((\tau_\mu\chi_\Omega)(\omega + 2\pi S^{*-1}k_j))_{\mu,j=0,\ldots,\lambda-1}$. By (6.2) and (6.3), we have

$$D^*(\omega)D(\omega) = I_\lambda \ (\omega \in \Omega_\lambda). \tag{6.4}$$

Since $D(\omega)$ is a square matrix, by (6.4), we know that $D(\omega)$ is a unitary matrix on Ω_λ. This implies that the first row of the matrix $D(\omega)$ is the unit vector on Ω_λ, i.e.,

$$\sum_{j=0}^{\lambda-1}(|\tau_0|^2\chi_\Omega)(\omega + 2\pi S^{*-1}k_j) = 1 \qquad (\omega \in \Omega_\lambda).$$

By (6.3), we have $\sum\limits_{\nu\in G^*}(|\tau_0|^2\chi_\Omega)(\omega + 2\pi S^{*-1}\nu) = 1$ $(\omega \in \Omega_\lambda)$. By (6.1), we know that $\Omega_\lambda \subset F$.

In Section 7, we will construct a UEP framelet with the cardinality λ in the case of $\Omega_\lambda \not\subset F$, and will construct a UEP framelet with the cardinality $\lambda-1$ in the case of $\Omega_\lambda \subset F$. This completes the proof of Theorem 6.2. $\Box$

Corollary 6.3. For an MRA structure which can derive UEP framelets, the relationship among the order λ, the minimal cardinality σ, and $\rho = |\det S|$ is

$$1 \leq \sigma \leq \lambda \leq \rho.$$

Example 6.4. Let φ be a function and $\widehat{\varphi} = \chi_{[-\pi/2,\ \pi/2]^d}$. Denote $V_m = \overline{\mathrm{span}}\{\varphi(2^m \cdot -n)\}_{n\in\mathbb{Z}^d}$ $(m \in \mathbb{Z})$. Then $\{V_m,\ \varphi\}$ is an MRA structure with dilation matrix $S = 2\,I_d$. The spectrum of V_0 is

$$\Omega = \operatorname{supp}\widehat{\varphi} + 2\pi\mathbb{Z}^d = [-\pi/2,\ \pi/2]^d + 2\pi\mathbb{Z}^d$$

and its associated sets $\Omega_1 = \mathbb{R}^d$, $\Omega_\gamma = 0$ $(\gamma \neq 1)$. So it is an MRA structure of order 1. The refinement filter $\tau_0 = \chi_{[-\pi/4,\ \pi/4]^d + 2\pi\mathbb{Z}^d}$ and by (6.1), the test set is

$$F = \bigcup_{\nu=\{0,1\}^d} \left([-\pi/4,\ \pi/4]^d + \pi\nu \right) + 2\pi\mathbb{Z}^d.$$

Clearly, $\Omega_1 \not\subset F$. By Theorem 6.1, we know that the minimal cardinality is 1. In fact, denote

$$\tau = \chi_{(\,[-\pi/2,\ \pi/2]^d \setminus (-\pi/4,\ \pi/4)^d\,) + 2\pi\mathbb{Z}^d}$$

and define $\psi \in L^2(\mathbb{R}^d)$ by $\widehat{\psi} = \tau\left(\frac{\cdot}{2}\right)\, \widehat{\varphi}\left(\frac{\cdot}{2}\right)$. We can directly check that ψ is a UEP framelet with the cardinality 1. In other words, in this example, we have $\rho = |\det S| = 2^d$ and $\sigma = \lambda = 1$.

7. The construction of the UEP framelets with the minimal cardinality

Let $\{V_m, \varphi\}$ be an MRA structure with the spectrum Ω. In Section 6, we defined the associated set Ω_γ of the spectrum Ω. To construct framelets with the minimal cardinality, we now give the decomposition of the associated set Ω_γ.

Definition 7.1. For each $\rho-$dimensional vector $\alpha = (\alpha_\nu)_{\nu \in G^*} \in \{0,1\}^\rho$, define the set

$$A_\alpha := \left\{ \omega \in \mathbb{R}^d \mid \chi_\Omega(\omega + 2\pi S^{*-1}\nu) = \alpha_\nu \quad (\nu \in G^*) \right\}.$$

Denote the number of nonzero components of the vector α by n_α and denote all nonzero components of the vector α by $\{\alpha_{k_j}\}_{j=0}^{n_\alpha-1}$. By Definition 7.1, for $\omega \in A_\alpha$, we have

$$\chi_\Omega(\omega + 2\pi S^{*-1}k_j) = 1 \qquad (j = 0, ..., n_\alpha - 1), \tag{7.1}$$

$$\chi_\Omega(\omega + 2\pi S^{*-1}\nu) = 0 \qquad (\nu \in G^*\backslash\{k_j\}_0^{n_\alpha-1}). \tag{7.2}$$

Using Definitions 6.1 and 7.1, let us decompose the set Ω_γ as $\Omega_\gamma = \bigcup\limits_{n_\alpha=\gamma} A_\alpha$ (a disjoint union).

Denote $N := S^{*-1}[-2\pi,\ 0)^d + 2\pi\mathbb{Z}^d$ as in (3.2). For each $\alpha \in \{0,1\}^\rho$, define

$$A_\alpha^0 := A_\alpha \bigcap N.$$

Lemma 7.2. Let $\{V_m,\ \varphi\}$ be an MRA structure of order λ. The following decomposition of $\mathbb{R}^d$ holds:

$$\mathbb{R}^d = \bigcup_{n_\alpha \le \lambda} \bigcup_{\nu \in G^*} (A_\alpha^0 + 2\pi S^{*-1}\nu) \tag{7.3}$$

and this is a disjoint union, where n_α is the number of nonzero components of the vector α.

Proof. By Definition 6.1 and the decomposition of Ω_γ, we have

$$\mathbb{R}^d = \bigcup_{\gamma=0}^{\lambda} \Omega_\gamma = \bigcup_{\gamma=0}^{\lambda} \bigcup_{n_\alpha=\gamma} A_\alpha = \bigcup_{n_\alpha \le \lambda} A_\alpha,$$

and so

$$N = \bigcup_{n_\alpha \le \lambda} (A_\alpha \bigcap N) = \bigcup_{n_\alpha \le \lambda} A_\alpha^0.$$

Furthermore, $N + 2\pi S^{*-1}\,\nu = \bigcup\limits_{n_\alpha \le \lambda} (A_\alpha^0 + 2\pi S^{*-1}\,\nu)$. From this and Lemma 3.2, we get (7.3).

For $\nu_1 \ne \nu_2$, by Lemma 3.2, we get

$$\left(A_\alpha^0 + 2\pi S^{*-1}\nu_1 \right) \bigcap \left(A_\alpha^0 + 2\pi S^{*-1}\nu_2 \right) \subset \left(N + 2\pi S^{*-1}\nu_1 \right) \bigcap \left(N + 2\pi S^{*-1}\nu_2 \right) = \emptyset.$$

For $\nu_1 = \nu_2$ and $\alpha \ne \beta$, since $A_\alpha^0 \bigcap A_\beta^0 = \emptyset$, we get

$$\left(A_\alpha^0 + 2\pi S^{*-1}\nu_1 \right) \bigcap \left(A_\beta^0 + 2\pi S^{*-1}\nu_2 \right) = \emptyset.$$

Therefore, (7.3) is a disjoint union. $\square$

Below we construct a UEP framelet with the minimal cardinality σ.

We always assume the MRA structure $\{V_m, \varphi\}$ can derive UEP framelets, so the l^2-norm $\| u_0(\omega) \|_{l^2} \leq 1$, where the vector $u_0(\omega) = ((\tau_0 \chi_\Omega)(\omega + 2\pi S^{*-1}\nu))_{\nu \in G^*}$. Let the associated set Ω_λ and the test set F be stated in Definition 6.1 and (6.1), respectively. We have the following two cases.

(i) The case $\Omega_\lambda \not\subset F$. We will construct a UEP framelet with the cardinality λ.

By Lemma 7.2, we only need to define wavelet filters $\{\tau_\mu(\omega)\}_1^\lambda$ on each set $A_\alpha^0 + 2\pi S^{*-1}\nu$ ($n_\alpha \leq \lambda, \nu \in G^*$).

Since $A_\alpha^0 \subset A_\alpha$, by (7.2) and the assumption condition $\| u_0(\omega) \|_{l_2} \leq 1$, we know that for $\omega \in A_\alpha^0$, the l^2-norm of the vector $b_\alpha(\omega) := \left((\tau_0 \chi_\Omega)(\omega + 2\pi S^{*-1} k_j) \right)_{j=0}^{n_\alpha - 1}$ is less than and equal to 1. Again since A_α^0 is a $2\pi\mathbb{Z}^d$-periodic set, by Lemma 3.3, we may take a $(n_\alpha + 1) \times n_\alpha$ matrix of bounded, $2\pi\mathbb{Z}^d$-periodic functions on the set A_α^0

$$D_\alpha(\omega) = (d_{\mu,j}^\alpha(\omega))_{\mu,j} \qquad (\mu = 0, ..., n_\alpha, \;\; j = 0, 1, ..., n_\alpha - 1)$$

such that its first row is $b_\alpha(\omega)$ and

$$D_\alpha^*(\omega) D_\alpha(\omega) = I_{n_\alpha} \quad \text{on} \;\; A_\alpha^0. \tag{7.4}$$

Define $\tau_\mu(\omega)$ ($\mu = 1, ..., n_\alpha$) on each set $A_\alpha^0 + 2\pi S^{*-1}\nu \qquad (n_\alpha \leq \lambda, \; \nu \in G^*)$ by

$$\tau_\mu(\omega + 2\pi S^{*-1}\nu) = \begin{cases} d_{\mu,j}^\alpha(\omega), & \omega \in A_\alpha^0 \;\; (\nu = k_j, \;\; j = 0, ..., n_\alpha - 1), \\ \\ 0, & \omega \in A_\alpha^0 \;\; (\nu \in G^* \backslash \{k_j\}_0^{n_\alpha - 1}) \end{cases} \tag{7.5}$$

and for $\mu = n_\alpha + 1, ..., \lambda$,

$$\tau_\mu(\omega + 2\pi S^{*-1}\nu) = 0, \qquad \omega \in A_\alpha^0 \quad (\nu \in G^*).$$

By Lemma 7.2, we know that $\{\tau_\mu\}_1^\lambda$ is well defined on $\mathbb{R}^d$ and $\{\tau_\mu\}_1^\lambda \subset L^\infty(T^d)$.

For $\omega \in A_\alpha^0$. By (7.1) and (7.5), we obtain

$$d_{\mu,j}^\alpha(\omega) = (\tau_\mu \chi_\Omega)(\omega + 2\pi S^{*-1} k_j) \qquad (\mu = 0, ..., n_\alpha, j = 0, ..., n_\alpha - 1).$$

From this and (7.4), we deduce that (2.2) holds with $r = \lambda$ for each $\omega \in A_\alpha^0$. Finally, by Lemmas 3.1 and 7.2, we know that $\{\tau_\mu\}_1^\lambda$ satisfy the condition of the unitary extension principle. Therefore, defining $\{\psi_\mu\}_1^\lambda$ by

$\widehat{\psi}_\mu(S^*\cdot) = \tau_\mu \widehat{\varphi}$ $(\mu = 1, ..., \lambda)$, $\{\psi_\mu\}_1^\lambda$ is a UEP framelet with the cardinality λ. Since $\Omega_\lambda \not\subset F$, by Theorem 6.2, we know that it is a UEP framelet with minimal cardinality.

(ii) The case $\Omega_\lambda \subset F$. We will construct a UEP framelet with cardinality $\lambda - 1$.

For $\alpha \in \{0,1\}^d$ satisfying $n_\alpha < \lambda$, we define the filters $\{\tau_\mu\}_1^{\lambda-1}$ on each set $A_\alpha^0 + 2\pi S^{*-1}\nu$ $(\nu \in G^*)$ as in the case (i).

For $\alpha \in \{0,1\}^d$ satisfying $n_\alpha = \lambda$ and $\nu \in G^*$, we define $\{\tau_\mu\}_1^{\lambda-1}$ on each set $A_\alpha^0 + 2\pi S^{*-1}\nu$ as follows. By $A_\alpha^0 \subset A_\alpha$ and $n_\alpha = \lambda$, we have $A_\alpha^0 \subset \Omega_\lambda \subset F$, so the λ-dimensional vector $((\tau_0\chi_\Omega)(\omega + 2\pi S^{*-1}k_j))_{j=0,1,...,\lambda-1}$ is the unit vector for $\omega \in A_\alpha^0$. We take a λ order orthogonal matrix of bounded, $2\pi\mathbb{Z}^d$-periodic functions on the set A_α^0,

$$C_\alpha(\omega) = (c_{\mu,j}^\alpha(\omega))_{\mu,j} \qquad (\mu, j = 0, 1, ..., \lambda - 1),$$

where $c_{0,j}^\alpha(\omega) = (\tau_0\chi_\Omega)(\omega + 2\pi S^{*-1}k_j)$ $(j = 0, ..., \lambda - 1)$.

Define $\tau_\mu(\omega)$ $(\mu = 1, ..., \lambda - 1)$ on each set $A_\alpha^0 + 2\pi S^{*-1}\nu$ by

$$\tau_\mu(\omega + 2\pi S^{*-1}\nu) = \begin{cases} c_{\mu,j}^\alpha(\omega), & \omega \in A_\alpha^0 \ \ (\nu = k_j, \ \ j = 0, ..., \lambda - 1), \\ \\ 0, & \omega \in A_\alpha^0 \ \ (\nu \in G^* \backslash \{k_j\}_0^{\lambda-1}). \end{cases}$$

We have defined $\{\tau_\mu\}_1^{\lambda-1}$ on $\mathbb{R}^d$. Again let $\{\psi_\mu\}_1^{\lambda-1}$ be such that $\widehat{\psi}_\mu(S^*\cdot) = \tau_\mu\widehat{\varphi}$ $(\mu = 1, 2, ..., \lambda - 1)$. Using the argument similar to (i), we deduce $\{\psi_\mu\}_1^{\lambda-1}$ is a UEP framelet. By Theorem 6.2 and $\Omega_\lambda \subset F$, we know that it is a UEP framelet with the minimal cardinality.

8. MEP bi-framelets

Let $\{V_m, \ \varphi\}$ and $\{\widetilde{V}_m, \ \widetilde{\varphi}\}$ be a pair of MRA structures. Define their spectrum Ω by

$$\Omega = \Omega' \bigcap \Omega'',$$

where Ω' and Ω'' are the spectra of the center space of MRA structures $\{V_m, \varphi\}$ and $\{\widetilde{V}_m, \widetilde{\varphi}\}$, respectively, i.e.,

$$\Omega' = \operatorname{supp} \widehat{\varphi} + 2\pi\mathbb{Z}^d \qquad \text{and} \qquad \Omega'' = \operatorname{supp} \widehat{\widetilde{\varphi}} + 2\pi\mathbb{Z}^d.$$

We always assume that $|\Omega' \bigcap \Omega''| > 0$. Similarly to Definition 6.1, based on the above spectrum Ω, we define the associated set Ω_γ and the order λ of this pair of MRA structures.

From Lemma 3.4, using the similar argument of "If part" in Theorem 4.1, we obtain the following existence theorem in which there is no additional condition on the refinement filters τ_0 and $\widetilde{\tau}_0$. This is a quite different case from the UEP framelets.

Theorem 8.1. Any pair of MRA structures $\{V_m,\ \varphi\}$ and $\{\widetilde{V}_m,\ \widetilde{\varphi}\}$ with the spectrum Ω may derive a pair of bounded filters $\{\tau_\mu, \widetilde{\tau}_\mu\}_1^\rho$ satisfying (2.3), where $\rho = |\mathrm{det} S|$. Let $\widehat{\psi}_\mu(S^* \cdot) = \tau_\mu \widehat{\varphi}$ and $\widehat{\widetilde{\psi}}_\mu(S^* \cdot) = \widetilde{\tau}_\mu \widehat{\widetilde{\varphi}}$ $(\mu = 1, ..., \rho)$. If the wavelet systems $\{\psi_{\mu,m,n}\}$ and $\{\widetilde{\psi}_{\mu,m,n}\}$ are both Bessel sequences, then $\{\psi_\mu,\ \widetilde{\psi}_\mu\}_1^\rho$ are a pair of MEP bi-framelets.

Below we also derive a split trick for MEP bi-framelets.

Theorem 8.2. Let $\{V_m,\ \varphi\}$ and $\{\widetilde{V}_m,\ \widetilde{\varphi}\}$ be a pair of MRA structures with the spectrum Ω. If $\{\psi_\mu,\ \widetilde{\psi}_\mu\}_1^r$ are a pair of MEP bi-framelets derived by them, then both ψ_r and $\widetilde{\psi}_r$ can be split into two functions, respectively

$$\psi_r = \psi_r^{(1)} + \psi_r^{(2)}, \qquad \widetilde{\psi}_r = \widetilde{\psi}_r^{(1)} + \widetilde{\psi}_r^{(2)},$$

such that $\{\psi_1, ..., \psi_{r-1},\ \psi_r^{(1)},\ \psi_r^{(2)}\}$ and $\{\widetilde{\psi}_1, ..., \widetilde{\psi}_{r-1},\ \widetilde{\psi}_r^{(1)},\ \widetilde{\psi}_r^{(2)}\}$ are still a pair of MEP bi-framelets derived by this pair of MRA structures.

Denote the test set

$$F_d := \left\{ \omega \in \mathbb{R}^d; \quad \sum_{\nu \in G^*} (\tau_0 \overline{\widetilde{\tau}}_0 \chi_\Omega)(\omega + 2\pi S^{*-1}\nu) = 1 \right\}.$$

Theorem 8.3. Let $\{V_m,\ \varphi\}$ and $\{\widetilde{V}_m,\ \widetilde{\varphi}\}$ be a pair of MRA structures with the spectrum Ω and the order λ. Then the minimal cardinality of pairs of MEP bi-framelets derived by it

$$h = \begin{cases} \lambda - 1 & \text{if } \Omega_\lambda \subset F_d, \\ \\ \lambda & \text{if } \Omega_\lambda \not\subset F_d. \end{cases}$$

Since the argument of this theorem is similar to that of Theorem 6.2, we omit its proof.

Let us now construct an MEP bi-framelets with the minimal cardinality h. Suppose that $\Omega_\lambda \not\subset F_d$. Then, by Theorem 8.3, the minimal cardinality $h=\lambda$. Below we give the construction of a pair of MEP bi-framelets $\{\psi_\mu,\ \widetilde{\psi}_\mu\}_1^\lambda$ with the minimal cardinality λ.

By Lemma 7.2, we only need to define wavelet filters $\{\tau_\mu,\ \widetilde{\tau}_\mu\}_1^\lambda$ on each set $A_\alpha^0+2\pi S^{*-1}\nu$. By (7.2), we know that for $\omega\in A_\alpha^0,\ \chi_\Omega(\omega+2\pi S^{*-1}\nu)=0$ $(\nu\in(G^*\setminus\{k_j\}_0^{n_\alpha-1}))$. Denote two n_α-dimensional vectors:

$$b_\alpha(\omega):=\left(\ (\tau_0\chi_\Omega)(\omega+2\pi S^{*-1}k_j)\ \right)_{j=0}^{n_\alpha-1},$$

$$\widetilde{b}_\alpha(\omega):=\left(\ (\widetilde{\tau}_0\chi_\Omega)(\omega+2\pi S^{*-1}k_j)\ \right)_{j=0}^{n_\alpha-1}.$$

By Lemma 3.4 (i), we may take two $(n_\alpha+1)\times n_\alpha$ matrices of bounded, $2\pi\mathbb{Z}^d$-periodic functions on each set A_α^0 $(n_\alpha\le\lambda)$

$$B_\alpha(\omega)=(b_{\mu,j}^\alpha(\omega))_{\mu,j},\qquad \widetilde{B}_\alpha(\omega)=(\widetilde{b}_{\mu,j}^\alpha(\omega))_{\mu,j}\qquad (\mu=0,...,n_\alpha,\ \ j=0,1,...,n_\alpha-1)$$

such that the first rows of the matrices $B_\alpha(\omega)$ and $\widetilde{B}_\alpha(\omega)$ are $b_\alpha(\omega)$ and $\widetilde{b}_\alpha(\omega)$, respectively, and

$$B_\alpha^*(\omega)\widetilde{B}_\alpha(\omega)=I_{n_\alpha}\qquad \text{on }\ A_\alpha^0.$$

Now define $\tau_\mu(\omega)$ $(\mu=1,...,\lambda)$ on each set $A_\alpha^0+2\pi S^{*-1}\nu$. For $\mu=1,2,...,n_\alpha$,

$$\tau_\mu(\omega+2\pi S^{*-1}\nu)=\begin{cases} b_{\mu,j}^\alpha(\omega), & \omega\in A_\alpha^0\ \ (\nu=k_j,\ \ j=0,...,n_\alpha-1),\\ \\ 0, & \omega\in A_\alpha^0\ \ (\nu\in G^*\backslash\{k_j\}_0^{n_\alpha-1})\end{cases}$$

and for $\mu=n_\alpha+1,...,\lambda$,

$$\tau_\mu(\omega+2\pi S^{*-1}\nu)=0,\qquad \omega\in A_\alpha^0\quad(\nu\in G^*).$$

Define $\widetilde{\tau}_\mu(\omega)$ $(\mu=1,...,\lambda)$ on each set on $A_\alpha^0+2\pi S^{*-1}\nu$. For $\mu=1,2,...,n_\alpha$,

$$\widetilde{\tau}_\mu(\omega+2\pi S^{*-1}\nu)=\begin{cases} \widetilde{b}_{\mu,j}^\alpha(\omega), & \omega\in A_\alpha^0\ \ (\nu=k_j,\ j=0,...,n_\alpha-1),\\ \\ 0, & \omega\in A_\alpha^0\ \ (\nu\in G^*\backslash\{k_j\}_0^{n_\alpha-1})\end{cases}$$

and for $\mu=n_\alpha+1,...,\lambda$,

$$\widetilde{\tau}_\mu(\omega+2\pi S^{*-1}\nu)=0,\qquad \omega\in A_\alpha^0\quad(\nu\in G^*).$$

By Lemma 7.2, we have defined $\{\tau_\mu\}_1^\lambda$ and $\{\widetilde{\tau}_\mu\}_1^\lambda$ on $\mathbb{R}^d$. We easily check that the wavelet filters $\{\tau_\mu\}_1^\lambda$ and $\{\widetilde{\tau}_\mu\}_1^\lambda$ satisfy (2.3). By the mixed extension principle, we know that if we define $\{\psi_\mu,\ \widetilde{\psi}_\mu\}_1^\lambda$ as $\widehat{\psi}_\mu(S^*\cdot) = \tau_\mu\widehat{\varphi},\ \widehat{\widetilde{\psi}}_\mu(S^*\cdot) = \widetilde{\tau}_\mu\widehat{\widetilde{\varphi}}$, then $\{\psi_\mu,\ \widetilde{\psi}_\mu\}_1^\lambda$ is a pair of MEP bi-framelets with the minimal cardinality λ, whenever the wavelet systems $\{\psi_{\mu,m,n}\}$ and $\{\widetilde{\psi}_{\mu,m,n}\}$ are both Bessel sequences.

Suppose that $\Omega_\lambda \subset F_d$. Using Lemma 3.4 (ii), we can also construct a pair of MEP bi-framelets with the minimal cardinality $\lambda - 1$. Due to the similarity of the arguments, we omit the details here.

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