Constructions of periodic wavelet frames using extension principles*

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Abstract. Since the extension principles of constructing wavelet frames were presented, a lot of symmetric and compactly supported wavelet frames with high vanishing moments have been constructed. However the problem of constructing periodic wavelet frames with the help of extension principles is open. In this paper, we will construct tight periodic wavelet frames using the unitary extension principle and construct pairs of dual periodic wavelet frames using the mixed extension principle.

Key words periodic wavelet frame, dual frame, extension principle, multiresolution analysis

1. Introduction

In recent years, some researches have shown that it is often more convenient to work with wavelet frames than wavelet orthonormal bases in noise reduction and image compression [1]. The research of wavelet frames is a hot point in wavelet analysis. A series of important results have been given [1]-[20].

B. Han [10] gave a characterization of tight wavelet frames of $L^2(\mathbb{R})$: For $\psi \in L^2(\mathbb{R})$, the affine system $\{\psi_{m,n}\}_{m,n\in\mathbb{Z}}$ is a tight wavelet frame with bound 1 if and only if

$$\sum_{m\in\mathbb{Z}} |\widehat{\psi}(2^m\omega)|^2 = 1 \quad \text{and} \quad \sum_{m=0}^{\infty} \widehat{\psi}(2^m\omega)\overline{\widehat{\psi}}(2^m(\omega + (4k+2)\pi)) = 0 \quad (k \in \mathbb{Z}).$$

Ron and Shen [18] presented the unitary extension principle of constructing tight wavelet frames: If H_0 is a refinement filter and $H_1, ..., H_l$ are wavelet filters, and

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$$\sum_{i=0}^{l} H_i(\omega)\overline{H_i}(\omega+\nu\pi) = \begin{cases} 1, & \nu = 0, \\ 0, & \text{otherwise} \end{cases} \qquad \text{for } \nu \in \{0,1\}^d, \tag{1.1}$$

then the corresponding affine system is a tight frame for $L^2(\mathbb{R}^d)$ with bound 1. In [19], replacing $H_i(\omega)\overline{H}_i(\omega+\nu\pi)$ in (1.1) by $H_i(\omega)\overline{\widetilde{H}_i}(\omega+\nu\pi)$, where $\widetilde{H}_0, ..., \widetilde{H}_l$ are dual filters, they presented further the mixed extension principle to constructing dual wavelet frames.

These methods of constructions of wavelet frames are generalized from one-dimension to high-dimension, from tight frames to dual frames, from a single scaling function to a scaling function vector. More importantly, based on these approaches to constructing wavelet frames, a lot of symmetric and compactly supported wavelet frames with high vanishing moments are constructed (See [3]-[7], [11]-[12]. [17]-[20]).

It is well-known that under some decay conditions one uses the method of periodization to construct periodic wavelet bases with the help of wavelet bases. We use the method of periodization to construct periodic wavelet frames with the help of wavelet frames. In [20], starting from a band-passing function ψ satisfying $\operatorname{supp}\widehat{\psi} \subset [-\pi, -\epsilon]\bigcup[\epsilon, \pi]$, $\epsilon > 0$, we constructed a pair of dual periodic wavelet frames. However the problem of constructing general periodic wavelet frames with the help of extension principles is still open. In this paper, deleting the strong condition $\operatorname{supp}\widehat{\psi} \subset [-\pi, -\epsilon]\bigcup[\epsilon, \pi]$ in [20], we construct general periodic wavelet frames using extension principles, precisely say, we prove that under some decay condition, the periodization of any wavelet frame constructed by the unitrary extension principle is a periodic wavelet frame, the periodization of any pair of dual wavelet frames constructed by the mixed extension principle is a pair of dual periodic wavelet frames.

This paper is organized as follows. In section 2, we recall the concepts of frames and extension principles. In Section 3, we state main results: the constructions of periodic wavelet frames and pairs of dual periodic wavelet frames. In Section 4, we give the proof of main results. Finally, in Section 5, we present an example to explain our theory.

2. Preliminaries

We recall some known concepts and results.

In this paper, we denote the inner product and the norm in $L^2([0,1]^d)$ by $(\cdot,\cdot)$ and $\|\cdot\|$, the inner product and the norm in $L^2(\mathbb{R}^d)$ by $(\cdot,\cdot)_{L^2(\mathbb{R}^d)}$ and $\|\cdot\|_{L^2(\mathbb{R}^d)}$, respectively. We denote the set of vertexes of the cube $[0,1]^d$ by $\{0,1\}^d$. For a set E of $\mathbb{R}^d$, a point $b\in\mathbb{R}^d$, and a number $c\in\mathbb{R}$, denote

$$E+b=\{x+b,\ x\in E\},\qquad cE=\{cx,\quad x\in E\}.$$

Denote the characteristic function of a set E by $\mathcal{X}_E$. For $t,\ s\in\mathbb{R}^d$, denote the inner product and the norm by $t\cdot s$ and $|t|,\ |s|$, respectively. For convenience, we denote

$$D_m=[0,\ 2^m-1]^d\bigcap\mathbb{Z}^d,\qquad m\in\mathbb{Z}^+\bigcup\{0\}.$$

For a matrix Ω, denote its conjugate transposed matrix by Ω^*, denote the unit matrix by I.

2.1. Frames

Frames are generalization of orthonormal bases. Let $\mathcal{H}$ be a separable Hilbert space and $\{h_n\}_1^\infty$ a sequence in $\mathcal{H}$. If, there is a $B>0$ such that

$$\sum_{n=1}^{\infty}|(f,\ h_n)_{\mathcal{H}}|^2\le B\parallel f\parallel_{\mathcal{H}}^2\quad\forall\ f\in\mathcal{H},$$

then $\{h_n\}_1^\infty$ is called a Bessel sequence. If there exist two positive constants $A,\ B$ such that

$$A\parallel f\parallel_{\mathcal{H}}^2\le\sum_{n=1}^{\infty}|(f,h_n)_{\mathcal{H}}|^2\le B\parallel f\parallel_{\mathcal{H}}^2\quad\forall\ f\in\mathcal{H},$$

then $\{h_n\}_1^\infty$ is called a frame for $\mathcal{H}$ and $A,\ B$ frame bounds. If $A=B$, then it is called a tight frame. Let $\{h_n\}_1^\infty$ and $\{\widetilde{h}_n\}_1^\infty$ be two frames for $\mathcal{H}$. If, for any $f\in\mathcal{H}$,

$$f=\sum_{n=1}^{\infty}(f,\widetilde{h}_n)_{\mathcal{H}}h_n=\sum_{n=1}^{\infty}(f,h_n)_{\mathcal{H}}\widetilde{h}_n,$$

then $\{h_n,\ \widetilde{h}_n\}_1^\infty$ is called a pair of dual frames for $\mathcal{H}$.

Proposition 2.1 [2]. Let $\{h_n\}_1^\infty$ and $\{\widetilde{h}_n\}_1^\infty$ be two Bessel sequences in Hilbert space $\mathcal{H}$. Then $\{h_n,\ \widetilde{h}_n\}_1^\infty$ is a pair of dual frames if and only if there exists a dense set H_0 of $\mathcal{H}$, such that

$$\sum_{n=1}^{\infty}(f,\ h_n)_{\mathcal{H}}\overline{(g,\ \widetilde{h}_n)_{\mathcal{H}}} = (f,\ g)_{\mathcal{H}} \qquad \forall\ f,\ g \in H_0.$$

Wavelet frames are generalization of wavelet orthonomal bases. Let $\{\psi_\mu\}_1^l \subset L^2(\mathbb{R}^d)$ and

$$\psi_{\mu,m,n} := 2^{\frac{md}{2}}\psi_\mu(2^m\cdot -n), \quad \mu = 1,...,l;\ \ m \in \mathbb{Z};\ \ n \in \mathbb{Z}^d.$$

If the affine system $\{\psi_{\mu,m,n}\}$ is a frame for $L^2(\mathbb{R}^d)$, then $\{\psi_{\mu,m,n}\}$ is called a wavelet frame. If two wavelet frames $\{\psi_{\mu,m,n}\}$ and $\{\widetilde{\psi}_{\mu,m,n}\}$ are a pair of dual frames for $L^2(\mathbb{R}^d)$, then they are called a pair of dual wavelet frames.

2.2. Various Parseval identities

For $f \in L^1(\mathbb{R}^d)$, define the Fourier transform as

$$\widehat{f}(\omega) = \int_{\mathbb{R}^d} f(t)\, e^{-it\cdot\omega}\, \mathrm{d}\omega.$$

If $f,\ g \in L^2(\mathbb{R}^d)$, then the Parseval identity $(f,\ g)_{L^2(\mathbb{R}^d)} = \frac{1}{(2\pi)^d}\,(\widehat{f},\ \widehat{g})_{L^2(\mathbb{R}^d)}$ holds.

Let $f \in L^2([0,1]^d)$. Define the Fourier coefficients as

$$c_n(f) = \int_{[0,1]^d} f(t)\, e^{-2\pi i n\cdot t}\, \mathrm{d}t.$$

If $f,\ g \in L^2([0,1]^d)$, then the Parseval identity $(f,\ g) = \sum_{k\in\mathbb{Z}^d} c_k(f)\,\overline{c_k}(g)$ holds.

For convenience, we denote $D_m^\gamma = [0,\ 2^m-1]^d \bigcap \mathbb{Z}^d - \gamma,\ \gamma \in \mathbb{Z}^d$. For any sequence $\alpha = \{\alpha_k\}_{k\in D_m^\gamma}$, define the discrete Fourier transform as

$$(\mathcal{F}\alpha)(n) = \sum_{k\in D_m^\gamma} \alpha_k\, e^{-\frac{2\pi i}{2^m}\,k\cdot n}, \qquad n \in D_m^\gamma.$$

If the sequences $\alpha = \{\alpha_k\}_{k\in D_m^\gamma}$ and $\beta = \{\beta_k\}_{k\in D_m^\gamma}$, then the following Parseval identity holds:

$$\sum_{k\in D_m^\gamma} \alpha_k\, \overline{\beta}_k = 2^{-md} \sum_{n\in D_m^\gamma} (\mathcal{F}\alpha)(n)\overline{(\mathcal{F}\beta)(n)}.$$

2.3. Extension principles

Now we recall extension principles of constructing wavelet frames.

Definition 2.2 [19]. If $\varphi \in L^2(\mathbb{R}^d)$ satisfies

(i) $\widehat{\varphi}$ is continuous at the origin and $\widehat{\varphi}(0) = 1$,

(ii) there exists an $M > 0$ such that $\sum\limits_{k\in\mathbb{Z}^d} |\widehat{\varphi}(\omega + 2k\pi)|^2 \leq M$, $a.e.\ \omega \in \mathbb{R}^d$,

(iii) $\widehat{\varphi}(2\omega) = H_0(\omega)\widehat{\varphi}(\omega)$, where H_0 is a $2\pi\mathbb{Z}^d-$periodic bounded function,

then we call φ a **scaling function**.

Proposition 2.3 (unitrary extension principle) [18]. Let φ be a scaling function and H_0 the corresponding refinement filter. For each $\mu = 1, ..., l$, let H_μ be a $2\pi\mathbb{Z}^d-$periodic bounded function. Define ψ_μ by $\widehat{\psi}_\mu(2\omega) = H_\mu(\omega)\widehat{\varphi}(\omega)$. If the matrix

$$\Omega = (H_\mu(\omega - \nu\pi))_{\mu=0,\dots,l;\, \nu\in\{0,1\}^d} \tag{2.1}$$

satisfies

$$\Omega^*\Omega = I \qquad a.e.,$$

then the affine system $\{\psi_{\mu,m,n}\}_{\mu=1,\dots,l;\, m\in\mathbb{Z};\, n\in\mathbb{Z}^d}$ is a tight wavelet frame for $L^2(\mathbb{R}^d)$ with bound 1, where Ω^* is the conjugate transposed matrix and I is the unit matrix.

Proposition 2.4 (mixed extension principle) [19]. Let φ and $\widetilde{\varphi}$ be two scaling functions, and H_0, $\widetilde{H}_0$ two corresponding refinement filters. For each $\mu = 1, ..., l$, let H_μ and $\widetilde{H}_\mu$ be $2\pi\mathbb{Z}^d-$periodic bounded functions. Define ψ_μ, $\widetilde{\psi}_\mu$ as

$$\widehat{\psi}_\mu(2\omega) = H_\mu(\omega)\, \widehat{\varphi}(\omega), \qquad \widehat{\widetilde{\psi}}_\mu(2\omega) = H_\mu(\omega)\, \widehat{\widetilde{\varphi}}(\omega).$$

If both $\{\psi_{\mu,m,n}\}_{\mu=1,\dots,l;\, m\in\mathbb{Z};\, n\in\mathbb{Z}^d}$ and $\{\widetilde{\psi}_{\mu,m,n}\}_{\mu=1,\dots,l;\, m\in\mathbb{Z};\, n\in\mathbb{Z}^d}$ are Bessel sequences, and the matrices

$$\Omega = (\, H_\mu(\omega - \nu\pi)\,)_{\mu=0,\dots,l;\ \nu\in\{0,1\}^d} \qquad \text{and} \qquad \widetilde{\Omega} = \Big(\, \widetilde{H}_\mu(\omega - \nu\pi)\,\Big)_{\mu=0,\dots,l;\ \nu\in\{0,1\}^d} \tag{2.2}$$

satisfy

$$\Omega^*\widetilde{\Omega} = I \qquad a.e.,$$

then $\{\psi_{\mu,m,n},\ \widetilde{\psi}_{\mu,m,n}\}_{\mu=1,\dots,l;\, m\in\mathbb{Z},\, n\in\mathbb{Z}^d}$ is a pair of dual wavelet frames for $L^2(\mathbb{R}^d)$.

2.4. Periodization

The periodization of wavelet frames is based on the following proposition.

Proposition 2.5 [2]. Let $f \in L^1(\mathbb{R}^d)$. Then

(i) the series $\sum\limits_{k\in\mathbb{Z}^d} f(t+k)$ is absolutely convergent for $a.e.\ t \in \mathbb{R}^d$;

(ii) $F(t) = \sum\limits_{k\in\mathbb{Z}^d} f(t+k) \in L^1([0,1]^d)$;

(iii) $\displaystyle\int\limits_{[0,1]^d} F(t)\, e^{-2\pi i k\cdot t}\, \mathrm{d}t = \widehat{f}(2\pi k),\ \ k \in \mathbb{Z}^d.$

Notation 2.6. If $f \in L^1(\mathbb{R}^d)$, then we define

$$f^{per}(t) = \sum_{l\in\mathbb{Z}^d} f(t+l).$$

From Proposition 2.5, we know that f^{per} is well-defined and it is a $\mathbb{Z}^d-$periodic local integrable function.

3. Main results

First, we present an approach to constructing periodic wavelet frames with the help of the unitary extension principle.

Theorem 3.1. Let φ be a scaling function, and let $\Psi = \{\psi_\mu\}_1^l \subset L^2(\mathbb{R}^d)$ satisfy

$$\widehat{\psi}_\mu(2\omega) = H_\mu(\omega)\widehat{\varphi}(\omega), \quad \mu = 0, ..., l, \quad \omega \in \mathbb{R}^d \qquad (\psi_0 = \varphi),$$

where each H_μ $(\mu = 0, ..., l)$ is a $2\pi\mathbb{Z}^d-$periodic bounded function. Let the matrix Ω be stated in (2.1). If the equality of matrices

$$\Omega^*\Omega = I$$

holds, and there is a $\epsilon > 0$ such that for each $\mu \geq 0$, $\psi_\mu(t) = O((1+|t|)^{-(1+\epsilon)d})$ $(\psi_0 = \varphi)$, then the system

$$\Psi^{per} = \left\{ 1, \quad \psi^{per}_{\mu,m,n}\ \ (\mu = 1, ..., l;\ \ m \geq 0;,\ \ n \in D_m) \right\}$$

is a tight frame for $L^2([0,1]^d)$ with bound 1, where $D_m=[0,\ 2^m-1]^d\bigcap\mathbb{Z}^d$.

Next, we present an approach of constructing pairs of dual periodic wavelet frames with the help of the mixed extension principle.

Theorem 3.2. Let φ and $\widetilde{\varphi}$ be two scaling functions, and let $\{\psi_\mu\}_1^l\subset L^2(\mathbb{R}^d),\ \{\widetilde{\psi}_\mu\}_1^l\subset L^2(\mathbb{R}^d)$ satisfy

$$\widehat{\psi}_\mu(2\omega)=H_\mu(\omega)\,\widehat{\varphi}(\omega),\qquad \widehat{\widetilde{\psi}}_\mu(2\omega)=\widetilde{H}_\mu(\omega)\,\widehat{\widetilde{\varphi}}(\omega),\quad \mu=0,...,l,\quad \omega\in\mathbb{R}^d\qquad (\psi_0=\varphi,\ \widetilde{\psi}_0=\widetilde{\varphi}),$$

where each H_μ and $\widetilde{H}_\mu$ $(\mu=0,...,l)$ are $2\pi\mathbb{Z}^d-$periodic bounded functions. Let matrices Ω and $\widetilde{\Omega}$ be stated as in (2.2). If both $\{\psi_{\mu,m,n}\}_{\mu=1,...,l;\,m\in\mathbb{Z};\,n\in\mathbb{Z}^d}$ and $\{\widetilde{\psi}_{\mu,m,n}\}_{\mu=1,...,l;\,m\in\mathbb{Z};\,n\in\mathbb{Z}^d}$ are Bessel sequences for $L^2(\mathbb{R}^d)$, and the following equality of matrixes holds:

$$\Omega^*\widetilde{\Omega}=I, \tag{3.1}$$

and there exists $\epsilon>0$ such that for each $\mu\geq 0$,

$$\psi_\mu(t)=O((1+|t|)^{-(1+\epsilon)d}),\qquad \widetilde{\psi}_\mu(t)=O((1+|t|)^{-(1+\epsilon)d})\qquad (\psi_0=\varphi,\ \widetilde{\psi}_0=\widetilde{\varphi}), \tag{3.2}$$

then $\{\Psi^{per},\ \widetilde{\Psi}^{per}\}$ is a pair of dual frames for $L^2([0,1]^d)$, where

$$\Psi^{per}=\left\{1,\quad \psi^{per}_{\mu,m,n}\quad (\mu=1,...,l;\quad m\geq 0;\quad n\in D_m\ )\right\},$$

$$\widetilde{\Psi}^{per}=\left\{1,\quad \widetilde{\psi}^{per}_{\mu,m,n}\quad (\mu=1,...,l;\quad m\geq 0;\quad n\in D_m\ )\right\}.$$

Under the assumptions of Theorem 3.1, we let $\widetilde{\varphi}=\varphi$ and $\widetilde{H}_\mu=H_\mu$. Then we have $\Omega=\widetilde{\Omega}$ and

$$\Omega^*\widetilde{\Omega}=I,\qquad \Psi^{per}=\widetilde{\Psi}^{per}.$$

Using Theorem 3.2, we deduce that $\{\Psi^{per},\ \Psi^{per}\}$ is a pair of dual frames for $L^2([0,1]^d)$. So, by the definition, Ψ^{per} is a tight frame for $L^2([0,1]^d)$ with bound 1, i.e., Theorem 3.1 is proved. So we only need prove Theorem 3.2.

4. Proof of Theorem 3.2

By Proposition 2.1, we know that in order to prove $\{\Psi^{per},\ \widetilde{\Psi}^{per}\}$ is a pair of dual frames for $L^2([0,1]^d)$, we only need to prove the following:

(i) Ψ^{per} and $\widetilde{\Psi}^{per}$ are both Bessel sequences for $L^2([0,1]^d)$;

(ii) for any trigonometric polynomials f and g,

$$(f,\ g) = (f,\ 1)(g,\ 1)^- + \sum_{\mu=1}^{l} \sum_{m\geq 0} \sum_{n\in D_m} (f,\ \psi^{per}_{\mu,m,n})(g,\ \widetilde{\psi}^{per}_{\mu,m,n})^-.$$

We prove (i) and (ii) in the next Lemmas 4.1 and 4.2, respectively.

Lemma 4.1. The sequences Ψ^{per} and $\widetilde{\Psi}^{per}$ are both Bessel sequences for $L^2([0,1]^d)$.

Proof. For convenience, we let

$$T_\psi(f) = \sum_{\mu=1}^{l} \sum_{m\geq 0} \sum_{n\in D_m} |(f,\ \psi^{per}_{\mu,m,n})|^2,$$

$$T_{\widetilde{\psi}}(f) = \sum_{\mu=1}^{l} \sum_{m\geq 0} \sum_{n\in D_m} |(f,\ \widetilde{\psi}^{per}_{\mu,m,n})|^2.$$

In order to prove that Ψ^{per} and $\widetilde{\Psi}^{per}$ are both Bessel sequences, we need to prove that there exist $B,\ \widetilde{B} > 0$ such that for any $f \in L^2([0,1]^d)$, we have

$$T_\psi(f) + |(f,1)|^2 \leq B \parallel f \parallel^2,$$

$$T_{\widetilde{\psi}}(f) + |(f,1)|^2 \leq \widetilde{B} \parallel f \parallel^2 .$$

By the definition of $\psi^{per}_{\mu,m,n}$, we have

$$T_\psi(f) \leq \sum_{\mu=1}^{l} \sum_{m\geq 0} \sum_{n\in D_m} \left(\sum_{k\in\mathbb{Z}^d} |(f,\ \psi_{\mu,m,n}(\cdot+k))| \right)^2 .$$

However,

$$\begin{aligned} \left(\sum_{k\in\mathbb{Z}^d} |(f,\ \psi_{\mu,m,n}(\cdot+k))| \right)^2 \ &\leq\ 2\left(\sum_{k\in B(0,\,3\sqrt{d})} |(f,\ \psi_{\mu,m,n}(\cdot+k))| \right)^2 \\ &+\ 2\left(\sum_{k\notin B(0,\,3\sqrt{d})} |(f,\ \psi_{\mu,m,n}(\cdot+k))| \right)^2 =: 2I_1 + 2I_2, \end{aligned}$$

where $B(0,\ r)$ is the ball with center 0 and radius r. So we have

$$T_\psi(f) \leq 2\sum_{\mu=1}^{l} \sum_{m\geq 0} \sum_{n\in D_m} (I_1 + I_2) =: 2J_1 + 2J_2,$$

where

$$J_1=\sum_{\mu=1}^{l}\sum_{m\geq 0}\sum_{n\in D_m} I_1=\sum_{\mu=1}^{l}\sum_{m\geq 0}\sum_{n\in D_m}\left(\sum_{k\in B(0,\,3\sqrt{d})} |\int\limits_{[0,1]^d} f(t)\overline{\psi}_{\mu,m,n}(t+k)\,\mathrm{d}t\,|\right)^2 .$$

Since the number of integral points in the ball $B(0,\,3\sqrt{d}\,)$ is finite, we deduce

$$\left(\sum_{k\in B(0,\,3\sqrt{d})} |\int\limits_{[0,1]^d} f(t)\overline{\psi}_{\mu,m,n}(t+k)\,\mathrm{d}t\,|\right)^2 = O(1)\sum_{k\in B(0,\,3\sqrt{d}\,)} |\int\limits_{[0,1]^d} f(t)\overline{\psi}_{\mu,m,n}(t+k)\,\mathrm{d}t\,|^2,$$

where the bound of $O(1)$ only depends on d. Again, by

$$\psi_{\mu,m,n}(t+k)=\psi_{\mu,m,n-2^m k}(t),$$

we have

$$J_1=O(1)\sum_{\mu=1}^{l}\sum_{m\in\mathbb{Z}}\sum_{n\in\mathbb{Z}^d} |\int\limits_{[0,1]^d} f(t)\overline{\psi}_{\mu,m,n}(t)\,\mathrm{d}t\,|^2 = O(1)\sum_{\mu=1}^{l}\sum_{m\in\mathbb{Z}}\sum_{n\in\mathbb{Z}^d} \left|\,(f\mathcal{X}_{[0,1]^d},\ \ \psi_{\mu,m,n})_{L^2(\mathbb{R}^d)}\,\right|^2 .$$

By the assumptions, we know that $\{\psi_{\mu,m,n},\ \ \mu=1,...,l;\ m\in\mathbb{Z};\ n\in\mathbb{Z}^d\}$ is a Bessel sequence for $L^2(\mathbb{R}^d)$, so we deduce that there exists $B_1>0$ such that

$$J_1\leq B_1 \parallel f\mathcal{X}_{[0,1]^d}\parallel^2_{L^2(\mathbb{R}^d)}= B_1\parallel f\parallel^2 .$$

Now we compute J_2.

$$J_2=\sum_{\mu=1}^{l}\sum_{m\geq 0}\sum_{n\in D_m}\left(\sum_{k\notin B(0,\,3\sqrt{d})} |(f,\ \ \psi_{\mu,m,n}(\cdot+k))|\right)^2 . \tag{4.1}$$

We extend $f(t)$ from $[0,1]^d$ to a $\mathbb{Z}^d-$periodic function. So, by the Schwarz inequality, we have

$$|(f,\ \psi_{\mu,m,n}(\cdot+k))| = \left|\ \int\limits_{[0,1]^d-k} f(t)\,\overline{\psi}_{\mu,m,n}(t)\,\mathrm{d}t\ \right| \leq\parallel f\parallel\left(\ \int\limits_{[0,1]^d-k} |\psi_{\mu,m,n}(t)|^2\,\mathrm{d}t\right)^{\frac{1}{2}} . \tag{4.2}$$

By (3.2), we have

$$|\psi_{\mu,m,n}(t)|^2 = O(2^{dm})(1+|2^m t-n|)^{-2(1+\epsilon)d}.$$

For $n\in D_m$ and $t\in[0,1]^d-k,\ k\notin B(0,\,3\sqrt{d})$, we have

$$|t|\geq|k|-\sqrt{d}\geq 3\sqrt{d}-\sqrt{d}=2\sqrt{d}\quad\text{and}\quad |n|\leq 2^m\sqrt{d},$$

This implies that

$$|2^m t-n|\geq 2^m|t|-|n|\geq 2^m\sqrt{d}\quad\text{and}\quad |2^m t-n|\geq 2^{m-1}|t|\geq 2^{m-1}(|k|-\sqrt{d})\geq 2^{m-2}|k|.$$

So we get

$$\begin{aligned}\int\limits_{[0,1]^d-k}|\psi_{\mu,m,n}(t)|^2\,\mathrm{d}t &= O(2^{dm})\int\limits_{[0,1]^d-k}(1+|2^m t-n|)^{-2(1+\epsilon)d}\,\mathrm{d}t\\ &= O(2^{(1-\epsilon)d\,m})\int\limits_{[0,1]^d-k}(1+|2^m t-n|)^{-(2+\epsilon)d}\,\mathrm{d}t\\ &= O(2^{(1-\epsilon)d\,m})(1+2^{m-2}|k|)^{-(2+\epsilon)d}.\end{aligned}$$

By (4.2), we deduce that for $n\in D_m,\ k\notin B(0,\,3\sqrt{d})$,

$$|(f,\ \psi_{\mu,m,n}(\cdot+k))|=O(2^{\frac{1}{2}(1-\epsilon)d\,m})\parallel f\parallel\cdot(1+2^{m-2}|k|)^{-(1+\frac{\epsilon}{2})\,d}.$$

Again, since

$$\sum_{k\notin B(0,\,3\sqrt{d})}\frac{1}{(1+2^{m-2}|k|)^{(1+\frac{\epsilon}{2})d}}=O\left(\int\limits_{\mathbb{R}^d}\frac{\mathrm{d}t}{(1+2^{m-2}|t|)^{d+\frac{\epsilon d}{2}}}\right)=O(2^{-md})$$

we have

$$\sum_{k\notin B(0,3\sqrt{d})}|(f,\ \ \psi_{\mu,m,n}(\cdot+k))|=O\left(2^{-\frac{1}{2}(1+\epsilon)d\,m}\right)\parallel f\parallel.$$

Hence, by (4.1), we have

$$J_2=O(\parallel f\parallel^2)\sum_{m\geq 0}\ \sum_{n\in D_m}2^{-(1+\epsilon)d\,m}=O(\parallel f\parallel^2)\sum_{m\geq 0}2^{-\epsilon\,d\,m}=O(\parallel f\parallel^2),$$

So there exists a $B_2>0$ such that $|J_2|\leq B_2\parallel f\parallel^2$.

Therefore, we have

$$T_\psi(f)+|(f,\ 1)|^2\leq(2B_1+2B_2+1)\parallel f\parallel^2=:B\parallel f\parallel^2.$$

Similarly, we have $T_{\widetilde{\psi}}(f)+|(f,\ 1)|^2\leq\widetilde{B}\parallel f\parallel^2$. Lemma 4.1 is proved.

Lemma 4.2. Let f and g be trigonometric polynomials, i.e.,

$$f(t)=\sum_{n\in\mathbb{Z}^d}c_n(f)\,e^{2\pi in\cdot t}\quad\text{and}\quad g(t)=\sum_{n\in\mathbb{Z}^d}c_n(g)\,e^{2\pi in\cdot t},$$

where the sequences $\{c_n(f)\}$ and $\{c_n(g)\}$ have only finitely many nonzero terms. Then the following formula holds

$$(f,g) = (f,1)(g,\ 1)^- + \sum_{\mu=1}^{l} \sum_{m\geq 0} \sum_{n\in D_m} (f,\ \psi^{per}_{\mu,m,n})(g,\ \widetilde{\psi}^{per}_{\mu,m,n})^-. \tag{4.3}$$

Proof. This argument is divided into four steps.

Step 1. We prove that $\varphi^{per}(t) = \widetilde{\varphi}^{per}(t) = 1$.

Since $\varphi \in L^1(\mathbb{R}^d)$, we know that $\widehat{\varphi}$ is continuous and $\lim\limits_{|\omega|\to\infty} \widehat{\varphi}(\omega) = 0$. By a known result [9, 13], we have $\widehat{\varphi}(2\pi\alpha) = 0$ for any $\alpha \in \mathbb{Z}^d \setminus \{0\}$. Again, by Proposition 2.5, we have

$$\varphi^{per}(t) = \sum_{s\in\mathbb{Z}^d} \varphi(t+s) = \sum_{\nu\in\mathbb{Z}^d} \widehat{\varphi}(2\pi\nu)\, e^{i2\pi\nu\cdot t} = 1.$$

Similarly, we have $\widetilde{\varphi}^{per}(t) = 1$.

Step 2. Now we rearrange and rewrite the following series:

$$\sum_{n\in D_m} (f,\ \varphi^{per}_{m,n})(g,\ \widetilde{\varphi}^{per}_{m,n})^- \quad \text{and} \quad \sum_{n\in D_m} (f,\ \psi^{per}_{m,n})(g,\ \widetilde{\psi}^{per}_{m,n})^-.$$

First, we deduce that

$$\begin{aligned}(f,\ \varphi^{per}_{m,n})(g,\ \widetilde{\varphi}^{per}_{m,n})^- &= \left(\sum_{k\in\mathbb{Z}^d} (f,\ \varphi_{m,n}(\cdot+k))\right)\left(\sum_{s\in\mathbb{Z}^d} (g,\ \widetilde{\varphi}_{m,n}(\cdot+s))^-\right) \\ &= \sum_{s\in\mathbb{Z}^d} \sum_{k\in\mathbb{Z}^d} (f,\ \varphi_{m,n}(\cdot+k))(g,\ \widetilde{\varphi}_{m,n}(\cdot+s))^-\end{aligned}$$

Since $\varphi(t) = O((1+|t|)^{-(1+\epsilon)d})$, $\widetilde{\varphi}(t) = O((1+|t|)^{-(1+\epsilon)d})$, we deduce that the last series in this formula is absolutely convergent. From this, we know that the above operation is reasonable, and the last series of the above equation can be rearranged, so we have

$$(f,\ \varphi^{per}_{m,n})(g,\ \widetilde{\varphi}^{per}_{m,n})^- = \sum_{s\in\mathbb{Z}^d} \sum_{k\in\mathbb{Z}^d} (f,\ \varphi_{m,n}(\cdot+k))(g,\ \widetilde{\varphi}_{m,n}(\cdot+k+s))^-. \tag{4.4}$$

We define

$$F_s(t) = f(t)\mathcal{X}_{[0,1]^d - s}, \qquad G_s(t) = g(t)\mathcal{X}_{[0,1]^d - s}, \tag{4.5}$$

where $s \in \mathbb{Z}^d$ and $\mathcal{X}_{[0,1]^d - s}$ is the characteristic function of the set $[0,1]^d - s$. Since f and g are $\mathbb{Z}^d-$periodic functions, by (4.4) and (4.5), we have

$$\begin{aligned}
(f,\ \varphi_{m,n}^{per})(g,\ \widetilde{\varphi}_{m,n}^{per})^- &= \sum_{s\in\mathbb{Z}^d}\sum_{k\in\mathbb{Z}^d}\left(\int\limits_{[0,1]^d} f(t)\overline{\varphi}_{m,n}(t+k)\,\mathrm{d}t\right)\left(\int\limits_{[0,1]^d} g(t)\overline{\widetilde{\varphi}}_{m,n}(t+k+s)\,\mathrm{d}t\right)^- \\
&= \sum_{s\in\mathbb{Z}^d}\sum_{k\in\mathbb{Z}^d}\left(\int\limits_{[0,1]^d} f(t)\overline{\varphi}_{m,n}(t+k)\,\mathrm{d}t\right)\left(\int\limits_{[0,1]^d-s} g(t)\overline{\widetilde{\varphi}}_{m,n}(t+k)\,\mathrm{d}t\right)^- \qquad (4.6)\\
&= \sum_{s\in\mathbb{Z}^d}\sum_{k\in\mathbb{Z}^d}\left(\int\limits_{\mathbb{R}^d} F_0(t)\overline{\varphi}_{m,n}(t+k)\,\mathrm{d}t\right)\left(\int\limits_{\mathbb{R}^d} G_s(t)\overline{\widetilde{\varphi}}_{m,n}(t+k)\,\mathrm{d}t\right)^-.
\end{aligned}$$

Since $\varphi_{m,n}(t+k) = \varphi_{m,n-2^m k}(t)$, by (4.6), we have

$$\begin{aligned}
\sum_{n\in D_m}(f,\ \varphi_{m,n}^{per})(g,\ \widetilde{\varphi}_{m,n}^{per})^- &= \sum_{s\in\mathbb{Z}^d}\ \sum_{n\in D_m}\ \sum_{k\in\mathbb{Z}^d}(F_0,\ \varphi_{m,n-2^m k})_{L^2(\mathbb{R}^d)}(G_s,\ \widetilde{\varphi}_{m,n-2^m k})^-_{L^2(\mathbb{R}^d)} \\
&= \sum_{s\in\mathbb{Z}^d}\ \sum_{k\in\mathbb{Z}^d}(F_0,\ \varphi_{m,k})_{L^2(\mathbb{R}^d)}(G_s,\ \widetilde{\varphi}_{m,k})^-_{L^2(\mathbb{R}^d)}.
\end{aligned} \qquad (4.7)$$

Similarly, for each $\mu \geq 0$, we have

$$\sum_{n\in D_m}(f,\ \psi_{\mu,m,n}^{per})(g,\ \widetilde{\psi}_{\mu,m,n}^{per})^- = \sum_{s\in\mathbb{Z}^d}\ \sum_{k\in\mathbb{Z}^d}(F_0,\ \psi_{\mu,m,k})_{L^2(\mathbb{R}^d)}(G_s,\ \widetilde{\psi}_{\mu,m,k})^-_{L^2(\mathbb{R}^d)}. \qquad (4.8)$$

Step 3. Prove that for any $m \geq 0$, we have

$$(f,\ 1)(g,\ 1)^- + \sum_{\mu=1}^{l}\ \sum_{j=0}^{m}\ \sum_{n\in D_j}(f,\ \psi_{\mu,j,n}^{per})(g,\ \widetilde{\psi}_{\mu,j,n}^{per})^- = \sum_{n\in D_{m+1}}(f,\ \varphi_{m+1,\,n}^{per})(g,\ \widetilde{\varphi}_{m+1,\,n}^{per})^-.$$

We consider the sum of the right side in (4.8) over $\mu = 0, ..., l$,

$$P_m := \sum_{\mu=0}^{l}\ \sum_{s\in\mathbb{Z}^d}\ \sum_{k\in\mathbb{Z}^d}(F_0,\ \psi_{\mu,m,k})_{L^2(\mathbb{R}^d)}(G_s,\ \widetilde{\psi}_{\mu,m,k})^-_{L^2(\mathbb{R}^d)}. \qquad (4.9)$$

By the Parseval identity of the Fourier transform and $\widehat{\psi}_\mu(2\omega) = H_\mu(\omega)\,\widehat{\varphi}(\omega)$, we deduce that

$$(F_0,\ \psi_{\mu,m,k})_{L^2(\mathbb{R}^d)} = 2^{-\frac{md}{2}}\,(2\pi)^{-d}\int\limits_{\mathbb{R}^d}\widehat{F}_0(\omega)\,\overline{\widehat{\psi}}_\mu(2^{-m}\omega)\,e^{-2^{-m}ik\cdot\omega}\,\mathrm{d}\omega$$

$$= 2^{-\frac{md}{2}}\,(2\pi)^{-d}\int\limits_{\mathbb{R}^d}\widehat{F}_0(\omega)\,\overline{\widehat{\varphi}}(2^{-m-1}\omega)\,\overline{H}_\mu(2^{-m-1}\omega)\,e^{-2^{-m}ik\cdot\omega}\,\mathrm{d}\omega$$

$$= 2^{-\frac{md}{2}}(2\pi)^{-d} \int\limits_{[-2^m\pi,\ 2^m\pi]^d} \sum_{\alpha\in\mathbb{Z}^d} \widehat{F}_0(\omega+2^{m+1}\pi\alpha)\,\overline{\widehat{\varphi}}(2^{-m-1}\omega+\pi\alpha)\overline{H}_\mu(2^{-m-1}\omega+\pi\alpha)\,e^{-2^{-m}ik\cdot\omega}\,\mathrm{d}\omega.$$

Since H_μ is bounded, we have

$$\int\limits_{[-2^m\pi,2^m\pi]^d} \left(\sum_{\alpha\in\mathbb{Z}^d} |\widehat{F}_0(\omega+2^{m+1}\pi\alpha)\,\overline{\widehat{\varphi}}(2^{-m-1}\omega+\pi\alpha)\,\overline{H}_\mu(2^{-m-1}\omega+\pi\alpha)| \right) \mathrm{d}\omega$$

$$\leq \| H_\mu \|_{L^\infty(\mathbb{R}^d)} \int\limits_{\mathbb{R}^d} |\widehat{F}_0(\omega)\,\overline{\widehat{\varphi}}(2^{-m-1}\omega)|\,\mathrm{d}\omega < \infty.$$

So, the exchange of the integral and the summation is reasonable in the above formula, again, by periodicity of H_μ, we deduce that

$$(F_0,\ \psi_{\mu,m,k})_{L^2(\mathbb{R}^d)} = 2^{-\frac{md}{2}}(2\pi)^{-d} \int\limits_{[-2^m\pi,\ 2^m\pi]^d} \{ \sum_{\alpha'\in\mathbb{Z}^d} \sum_{\nu\in\{0,1\}^d} \widehat{F}_0(\omega+2^{m+1}\pi(2\alpha'+\nu))\,\overline{\widehat{\varphi}}(2^{-m-1}\omega+\pi(2\alpha'+\nu))$$
$$\cdot\ \overline{H}_\mu(2^{-m-1}\omega+\pi\nu)\,\}\,e^{-2^{-m}ik\cdot\omega}\,\mathrm{d}\omega. \tag{4.10}$$

Similarly, we have

$$(G_s,\ \widetilde{\psi}_{\mu,m,k})_{L^2(\mathbb{R}^d)} = 2^{-\frac{md}{2}}(2\pi)^{-d} \int\limits_{[-2^m\pi,\ 2^m\pi]^d} \{ \sum_{\beta'\in\mathbb{Z}^d} \sum_{\tau\in\{0,1\}^d} \widehat{G}_s(\omega+2^{m+1}\pi(2\beta'+\tau))\,\overline{\widehat{\widetilde{\varphi}}}(2^{-m-1}\omega+\pi(2\beta'+\tau))$$
$$\cdot\ \overline{\widetilde{H}}_\mu(2^{-m-1}\omega+\pi\tau)\}\,e^{-2^{-m}ik\cdot\omega}\,\mathrm{d}\omega. \tag{4.11}$$

Since $\{(2^{m+1}\pi)^{-\frac{d}{2}}\,e^{-2^{-m}ik\cdot\omega}\}_{m\in\mathbb{Z}}$ is an orthonormal basis for $L^2([-2^m\pi, 2^m\pi]^d)$, by (4.10) and (4.11), using the Parseval identity of the Fourier series, we have

$$\sum_{k\in\mathbb{Z}^d} (F_0,\ \psi_{\mu,m,k})_{L^2(\mathbb{R}^d)} (G_s,\ \widetilde{\psi}_{\mu,m,k})^{-}_{L^2(\mathbb{R}^d)}$$

$$= (2\pi)^{-d} \int\limits_{[-2^m\pi,2^m\pi]^d} \left\{ \sum_{\alpha'\in\mathbb{Z}^d} \sum_{\nu\in\{0,1\}^d} \widehat{F}_0(\omega+2^{m+1}\pi(2\alpha'+\nu))\ \overline{\widehat{\varphi}}(2^{-m-1}\omega+\pi(2\alpha'+\nu))\,\overline{H}_\mu(2^{-m-1}\omega+\pi\nu) \right\}$$
$$\bullet \left\{ \sum_{\beta'\in\mathbb{Z}^d} \sum_{\tau\in\{0,1\}^d} \overline{\widehat{G}}_s(\omega+2^{m+1}\pi(2\beta'+\tau))\,\widehat{\widetilde{\varphi}}(2^{-m-1}\omega+\pi(2\alpha'+\nu))\,\widetilde{H}_\mu(2^{-m-1}\nu+\pi\nu) \right\}^{-} \mathrm{d}\omega.$$

Again, by (4.9), we deduce that

$$P_m = (2\pi)^{-d} \sum_{s\in\mathbb{Z}^d} \int\limits_{[-2^m\pi,\, 2^m\pi]^d} \sum_{\alpha',\,\beta'\in\mathbb{Z}^d} \sum_{\nu,\,\tau\in\{0,1\}^d} \widehat{F}_0(\omega + 2^{m+1}\pi(2\alpha'+\nu))\overline{\widehat{G}}_s(\omega + 2^{m+1}\pi(2\beta'+\tau))$$

$$\overline{\widehat{\varphi}}(2^{-m-1}\omega + \pi(2\alpha'+\nu))\,\widehat{\widetilde{\varphi}}(2^{-m-1}\omega + \pi(2\beta'+\tau))\,\left(\sum_{\mu=0}^{l} \overline{H}_\mu(2^{-m-1}\omega+\pi\nu)\,\widetilde{H}_\mu(2^{-m-1}\omega+\pi\tau)\right)\mathrm{d}\omega$$

By the assumption $\Omega^*\widetilde{\Omega} = I$, for each μ, we have

$$\sum_{\mu=0}^{l} \overline{H}_\mu(2^{-m-1}\omega+\pi\nu)\,\widetilde{H}_\mu(2^{-m-1}\omega+\pi\tau) = \begin{cases} 1, & \nu = \tau, \\ 0, & \nu \neq \tau, \end{cases}$$

Again, letting

$$\begin{aligned} u_m(\omega) &= \sum_{\alpha\in\mathbb{Z}^d} \widehat{F}_0(\omega + 2^{m+2}\pi\alpha)\,\overline{\widehat{\varphi}}(2^{-m-1}(\omega+2^{m+2}\pi\alpha)), \\ v_{m,s}(\omega) &= \sum_{\beta\in\mathbb{Z}^d} \overline{\widehat{G}}_s(\omega + 2^{m+2}\pi\beta)\,\widehat{\widetilde{\varphi}}(2^{-m-1}(\omega+2^{m+2}\pi\beta)), \end{aligned} \tag{4.12}$$

we conclude that

$$\begin{aligned} P_m &= (2\pi)^{-d} \sum_{s\in\mathbb{Z}^d} \sum_{\nu\in\{0,1\}^d} \int\limits_{[-2^m\pi,2^m\pi]^d} u_m(\omega+2^{m+1}\pi\nu)\overline{v_{m,s}(\omega+2^{m+1}\pi\nu)}\,\mathrm{d}\omega \\ &= (2\pi)^{-d} \sum_{s\in\mathbb{Z}^d} \sum_{\nu\in\{0,1\}^d} \int\limits_{[-2^m\pi,2^m\pi]-2^{m+1}\pi\nu} u_m(\omega)\,\overline{v_{m,s}(\omega)}\,\mathrm{d}\omega, \end{aligned}$$

Noticing that $u_m(\omega)$ and $v_{m,s}(\omega)$ are $2^{m+2}\pi\mathbb{Z}^d-$periodic functions and

$$\bigcup_{\nu\in\{0,1\}^d} \left([-2^m\pi,\ 2^m\pi]^d - 2^{m+1}\pi\nu\right) = [-3\cdot 2^m\pi,\ 2^m\pi]^d,$$

we have

$$P_m = (2\pi)^{-d} \sum_{s\in\mathbb{Z}^d} \int\limits_{[-2^{m+1}\pi,\ 2^{m+1}\pi]^d} u_m(\omega)\overline{v_{m,s}(\omega)}\,\mathrm{d}\omega. \tag{4.13}$$

Using the Parseval identity of the Fourier series, we obtain that

$$\int\limits_{[-2^{m+1}\pi,2^{m+1}\pi]^d} u_m(\omega)\overline{v}_{m,s}(\omega)\,\mathrm{d}\omega$$

$$= 2^{-(m+1)d}\,(2\pi)^{-d} \sum_{k\in\mathbb{Z}^d} \Big(\int\limits_{[-2^{m+1}\pi,2^{m+1}\pi]^d} u_m(\omega)\, e^{-2^{-m-1} ik\cdot\omega}\,\mathrm{d}\omega\ \Big)$$

$$\bullet\ \Big(\int\limits_{[-2^{m+1}\pi,\ 2^{m+1}\pi]^d} v_{m,s}(\omega)\, e^{-2^{-m-1} ik\cdot\omega}\,\mathrm{d}\omega\ \Big)^{-}$$

$$= 2^{-(m+1)d}\,(2\pi)^{-d} \sum_{k\in\mathbb{Z}^d} \Big(\int\limits_{\mathbb{R}^d} \widehat{F}_0(\omega)\,\overline{\widehat{\varphi}}(2^{-m-1}\omega)\, e^{-2^{-m-1} ik\cdot\omega}\,\mathrm{d}\omega\ \Big)$$

$$\bullet\ \Big(\int\limits_{\mathbb{R}^d} \widehat{G}_s(\omega)\,\overline{\widehat{\widetilde{\varphi}}}(2^{-m-1}\omega)\, e^{-2^{-m-1} ik\cdot\omega}\,\mathrm{d}\omega\ \Big)^{-}.$$

Again, by the Parseval identity of the Fourier transform and (4.13), we have

$$P_m = \sum_{s\in\mathbb{Z}^d} \sum_{k\in\mathbb{Z}^d} (F_0,\ \varphi_{m+1,k})_{L^2(\mathbb{R}^d)} (G_s,\ \widetilde{\varphi}_{m+1,k})^{-}_{L^2(\mathbb{R}^d)}. \tag{4.14}$$

By (4.9) and (4.14), it follows that

$$\begin{aligned} &\sum_{s\in\mathbb{Z}^d} \sum_{k\in\mathbb{Z}^d} (F_0,\ \varphi_{m+1,k})_{L^2(\mathbb{R}^d)} (G_s,\ \widetilde{\varphi}_{m+1,k})^{-}_{L^2(\mathbb{R}^d)} \\ &= \sum_{s\in\mathbb{Z}^d} \sum_{k\in\mathbb{Z}^d} (F_0,\ \varphi_{m,k})_{L^2(\mathbb{R}^d)} (G_s,\ \widetilde{\varphi}_{m,k})^{-}_{L^2(\mathbb{R}^d)} \\ &+ \sum_{\mu=1}^{l} \sum_{s\in\mathbb{Z}^d} \sum_{k\in\mathbb{Z}^d} (F_0,\ \psi_{\mu,m,k})_{L^2(\mathbb{R}^d)} (G_s,\ \widetilde{\psi}_{\mu,m,k})^{-}_{L^2(\mathbb{R}^d)}. \end{aligned} \tag{4.15}$$

In the last equality of (4.15), we use $\psi_{0,m,k} = \varphi_{m,k}$ and $\widetilde{\psi}_{0,m,k} = \widetilde{\varphi}_{m,k}$.

By (4.15), (4.7), and (4.8), we have

$$\sum_{n\in D_{m+1}} (f,\ \varphi^{per}_{m+1,n})(g,\ \widetilde{\varphi}^{per}_{m+1,n})^{-} = \sum_{n\in D_m} (f,\ \varphi^{per}_{m,n})(g,\ \widetilde{\varphi}^{per}_{m,n})^{-} + \sum_{\mu=1}^{l} \sum_{n\in D_m} (f,\ \psi^{per}_{\mu,m,n})(g,\ \psi^{per}_{\mu,m,n})^{-}.$$

By $\varphi^{per}_{0,0}(t) = \widetilde{\varphi}^{per}_{0,0}(t) = 1$ and $D_0 = \{0\}$, we know that when $m = 0$, we have

$$(f,\ 1)(g,\ 1)^{-} + \sum_{\mu=1}^{l} (f,\ \psi^{per}_{\mu,0,0})(g,\ \widetilde{\psi}^{per}_{\mu,0,0})^{-} = \sum_{n\in D_1} (f,\ \varphi^{per}_{1,n})(g,\ \widetilde{\varphi}^{per}_{1,n})^{-}.$$

In general, for any trigonometric polynomials f and g and $m \geq 0$, we have

$$(f,\ 1)(g,\ 1)^{-} + \sum_{\mu=1}^{l} \sum_{j=0}^{m} \sum_{n\in D_j} (f,\ \psi^{per}_{\mu,j,n})(g,\ \widetilde{\psi}^{per}_{\mu,j,n})^{-} = \sum_{n\in D_{m+1}} (f,\ \varphi^{per}_{m+1,\,n})(g,\ \widetilde{\varphi}^{per}_{m+1,\,n})^{-}. \tag{4.16}$$

Step 4. For any trigonometric polynomials f and g, we have

$$S_m := \sum_{n\in D_m} (f,\ \varphi_{m,n}^{per})(g,\ \widetilde{\varphi}_{m,n}^{per})^- \to (f,\ g), \qquad m\to\infty. \tag{4.17}$$

In fact, since $\varphi_{m,n+2^m}^{per} = \varphi_{m,n}^{per},\ \ \widetilde{\varphi}_{m,n+2^m}^{per} = \widetilde{\varphi}_{m,n}^{per}$, we have

$$\sum_{n\in D_m} (f,\ \varphi_{m,n}^{per})(g,\ \widetilde{\varphi}_{m,n}^{per})^- = \sum_{n\in D_m^*} (f,\ \varphi_{m,n}^{per})(g,\ \widetilde{\varphi}_{m,n}^{per})^-,$$

where $D_m = [0,\ 2^m-1]^d$ and $D_m^* = [-2^{m-1},\ 2^{m-1}-1]^d$. Since f and g are both trigonometric polynomials, there exists $m_0 \in \mathbb{Z}^+$ such that

$$f(t) = \sum_{k\in\mathbb{Z}^d} c_k(f)\, e^{2\pi i k\cdot t}, \qquad g(t) = \sum_{k\in\mathbb{Z}^d} c_k(g)\, e^{2\pi i k,\cdot t}, \quad \text{where} \quad c_k(f) = c_k(g) = 0\ \ (k\notin D_{m_0}^*). \tag{4.18}$$

Again, let

$$\varphi_{m,n}^{per}(t) = \sum_{k\in\mathbb{Z}^d} d_k\, e^{2\pi i k\cdot t}, \qquad \widetilde{\varphi}_{m,n}^{per}(t) = \sum_{k\in\mathbb{Z}^d} \widetilde{d}_k\, e^{2\pi i k\cdot t}.$$

By Proposition 2.5, we have

$$d_k = (\varphi_{m,n})^{\Lambda}(2\pi k) = 2^{-\frac{md}{2}}\, \widehat{\varphi}(\frac{2\pi k}{2^m})\, e^{-\frac{2\pi i}{2^m}\,(k\cdot n)},$$

$$\widetilde{d}_k = 2^{-\frac{md}{2}}\, \widehat{\widetilde{\varphi}}(\frac{2\pi k}{2^m})\, e^{-\frac{2\pi i}{2^m}\,(k\cdot n)}.$$

Since $c_k(f) = c_k(g) = 0,\ k\notin D_{m_0}$, by the Parseval identity of the Fourier series, we have

$$(f,\ \varphi_{m,n}^{per}) = \sum_{k\in D_m^*} c_k(f)\overline{d}_k = \sum_{k\in D_m^*} \left(c_k(f) 2^{-\frac{md}{2}}\, \widehat{\varphi}(\frac{2\pi k}{2^m}) \right) e^{\frac{2\pi i}{2^m}\,(k\cdot n)}, \quad m\ge m_0.$$

Similarly, we have

$$(g,\ \widetilde{\varphi}_{m,n}^{per}) = \sum_{k\in D_m^*} c_k(g)\, 2^{-\frac{md}{2}}\, \widehat{\widetilde{\varphi}}(\frac{2\pi k}{2^m})\, e^{\frac{2\pi i}{2^m}\,(k\cdot n)}, \quad m\ge m_0.$$

Using the Parseval identity of the discrete Fourier transform, we obtain that for $m > m_0$,

$$S_m := \sum_{n\in D_m^*} (f,\ \varphi_{m,n}^{per})(g,\ \widetilde{\varphi}_{m,n}^{per})^- = \sum_{k\in D_m^*} c_k(f)\, \overline{c}_k(g)\, \widehat{\varphi}(\frac{2\pi k}{2^m})\, \overline{\widehat{\widetilde{\varphi}}}(\frac{2\pi k}{2^m}).$$

Again, by (4.18), we have $c_k(f) = c_k(g) = 0$ ($k \notin D^*_{m_0}$), so, for $m > m_0$,

$$S_m = \sum_{k \in D^*_{m_0}} c_k(f)\, \overline{c}_k(g)\, \widehat{\varphi}(\frac{2\pi k}{2^m})\, \overline{\widehat{\widetilde{\varphi}}}(\frac{2\pi k}{2^m}).$$

Since $\lim\limits_{\omega \to 0} \widehat{\varphi}(\omega) = \lim\limits_{\omega \to 0} \widehat{\widetilde{\varphi}}(\omega) = 1$, we have

$$\lim_{m \to \infty} S_m = \sum_{k \in D^*_{m_0}} c_k(f)\overline{c}_k(g) = \sum_{k \in \mathbb{Z}^d} c_k(f)\overline{c}_k(g).$$

Again, by the Parseval identity of the Fourier series, we get (4.17).

From (4.16) and (4.17), we deduce that for any trigonometric polynomials f and g, we have

$$(f,\ 1)(g,\ 1)^- + \sum_{\mu=1}^{l} \sum_{j=0}^{\infty} \sum_{n \in D_j} (f,\ \psi^{per}_{\mu,j,n})(g,\ \widetilde{\psi}^{per}_{\mu,j,n})^- = (f,\ g),$$

i.e., (4.3) holds. Lemma 4.2 is proved.

Proof of Theorem 3.2. By Lemma 4.1, we know that the sequences Ψ^{per} and $\widetilde{\Psi}^{per}$ are both Bessel sequences. By Lemma 4.2, we know that for any trigonometric polynomials f and g, (4.3) holds. Again, since the set of trigonometric polynomials is dense in $L^2([0,1]^d)$, using Proposition 2.1, we know that $\{\Psi^{per},\ \widetilde{\Psi}^{per}\}$ is a pair of dual frames for $L^2([0,1]^d)$. Theorem 3.2 is proved.

5. Example

We start from a known example [18] of wavelet frames to construct a pair of dual periodic wavelet frames. Let $\tau \in \mathbb{Z}^+$ and the scaling function φ satisfy

$$\widehat{\varphi}(\omega) = \left(\frac{\sin \frac{\omega}{2}}{\frac{\omega}{2}} \right)^{2\tau}, \qquad \omega \in \mathbb{R}.$$

Then $\widehat{\varphi}(2\omega) = H_0(\omega)\, \widehat{\varphi}(\omega)$, where $H_0(\omega) = \cos^{2\tau} \frac{\omega}{2}$. Let

$$H_\mu(\omega) = \sqrt{C^\mu_{2\tau}}\, \sin^\mu \frac{\omega}{2}\, \cos^{2\tau-\mu} \frac{\omega}{2}, \qquad 1 \le \mu \le 2\tau,$$

where $C^\mu_{2\tau} = \frac{(2\tau)!}{\mu!(2\tau-\mu)!}$. Then $H_\mu(\omega)$ is a $2\pi-$periodic bounded function and the matrix

$$\Omega = (H_\mu(\omega - \nu\pi))_{\mu=1,\dots,2\tau;\ \nu=0,1}$$

satisfy $\Omega^*\Omega = I$. For each $\mu = 1, ..., 2\tau$, define ψ_μ as

$$\widehat{\psi}_\mu(2\omega) = H_\mu(\omega)\,\widehat{\varphi}(\omega) = i^\mu \,\sqrt{C_{2\tau}^{\mu}}\,\frac{\cos^{2\tau-\mu}\,\frac{\omega}{4}\,\sin^{2\tau+\mu}\,\frac{\omega}{4}}{(\frac{\omega}{4})^{2\tau}}. \tag{5.1}$$

A known result [18] shows that $\{\psi_{\mu,j,k}\}_{\mu=1,...,2\tau;\, j,k\in\mathbb{Z}}$ is a tight frame for $L^2(\mathbb{R})$ with bound 1.

By Theorem 3.1, we know that $\{1,\ \psi^{per}_{\mu,j,k};\ \ \mu = 1, ..., 2\tau;\ j \geq 0,\ k = 0, 1, ..., 2^j - 1\}$ is a tight frame for $L^2([0, 1])$with bound 1. By Proposition 2.5, we have

$$\psi^{per}_{\mu,j,k}(t) = \sum_{\nu\in\mathbb{Z}} \widehat{\psi}_{\mu,j,k}(2\pi\nu)\, e^{i2\pi\nu t}. \tag{5.2}$$

Since ψ_μ is a spline of degree $(2\tau - 1)$, $\psi^{per}_{\mu,j,k}$ is a periodic spline function with period 1. By (5.1) and (5.2), we know that the Fourier coefficients

$$\begin{aligned}\widehat{\psi}_{\mu,j,k}(2\pi\nu) &= 2^{\frac{jd}{2}}\ \widehat{\psi}_\mu(2^{-j+1}\pi\nu)\, e^{-2\pi ik\nu/2^j}\\ &= 2^{\frac{jd}{2}}\, i^\mu\, \sqrt{C^{\mu}_{2\tau}}\ \frac{(\cos 2^{-j-2}\pi\nu)^{2\tau-\mu}\,(\sin 2^{-j-2}\pi\nu)^{2\tau+\mu}}{(2^{-j-2}\pi\nu)^{2\tau}}\ e^{-2^{-j+1}\pi ik\nu}.\end{aligned}$$

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