Ring-like structures of frequency domains of wavelets *

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Abstract. It is well known that the global frequency domain Ω of any orthonormal wavelet has a hole which contains the origin, viz. the frequency domain Ω possesses a ring-like structure $\Omega = S \setminus S_*$ $(0 \in S_* \subset S)$. We show that under some weak conditions, the set S and the hole S_* are determined uniquely by Ω, where the size of the hole S_* satisfies $0 \in \frac{S}{4} \subset S_* \subset \frac{S}{2}$ and the union of $4\pi\nu-$translations $(\nu \in \mathbb{Z}^d)$ of S is the whole space $\mathbb{R}^d$. Meanwhile, we give the corresponding converse theorem. We also show an interesting result: there is no orthonormal wavelet whose global frequency domain is the difference set of two balls. Finally, in order to explain our theory, we construct various global frequency domains and explain a general method of the construction of a wavelet with a given frequency domain.

Key words: global frequency domain, orthonormal wavelet, regular set

1. Introduction

A set $\Psi = \{\psi_\mu\}_1^{2^d-1}$ of functions is called an orthonormal wavelet if their integral translations and dyadic dilations form an orthonormal basis of $L^2(\mathbb{R}^d)$. The orthonormal wavelets have widely applications in signal and image processing. The support of the Fourier transform of a function is called the frequency domain of this function. For an orthonormal wavelet $\Psi = \{\psi_\mu\}_1^{2^d-1}$, the union of frequency domains of all ψ_μ is called the global frequency domain of the orthonormal wavelet Ψ.

The orthonormal wavelets with compact global frequency domains have an important property: their frequency domains have a hole which contains the origin. Precisely say, if the global frequency domain of an orthonormal wavelet $\{\psi_\mu\}_1^{2^d-1}$ is bounded and for each μ, the Fourier transform of ψ_μ satisfies some weak

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conditions, then there exists a neighborhood D of the origin such that for each μ, $\widehat{\psi}_\mu(\omega) = 0,\ \omega \in D$ [1,2,6]. From this, we see that the global frequency domain of an orthonormal wavelet possesses a ring-like structure. For example, for the tensor product Shannon's wavelet, its global frequency domain is $\Omega = [-2\pi, 2\pi]^d \setminus (-\pi, \pi)^d$; for the tensor product Meyer's wavelet, the global frequency domain is $\Omega = [-\frac{8\pi}{3}, \frac{8\pi}{3}]^d \setminus (-\frac{2\pi}{3}, \frac{2\pi}{3})^d$ [5,6]. Further researches on holes of frequency domains show [2,4] that for an orthonormal wavelet $\psi \in L^2(\mathbb{R})$, if its frequency domain $\Omega \subset [-\frac{8\pi}{3},\ \frac{8\pi}{3}]$, then $\widehat{\psi}_\mu(\omega) = 0,\ \omega \in [-\frac{2\pi}{3},\ \frac{2\pi}{3}]$; if $\widehat{\psi}(\omega) = 0,\ \omega \in (-\pi,\ \pi)$, then ψ is a Shannon wavelet.

For an orthonormal wavelet, the size of the global frequency domain Ω and the size of hole D which contains the origin are described as follows [6].

(i) If $[A,\ B]^d$ is the smallest cube containing Ω, then $B - A \geq 4\pi$.

(ii) If $(a,\ b)^d$ is the largest cube which lies in D, then $b - a \leq 2\pi$.

From the tensor product Shannon wavelet, we know that the above conclusions (i) and (ii) cannot be improved. In general, there is no estimate of the lower bound of $b - a$. For any $\varepsilon > 0$, there exists an orthonormal wavelet $\psi \in L^2(\mathbb{R})$ such that the intersection of its frequency domain Ω and $[-\epsilon, \epsilon]$ is a set with positive measure [2].

In this paper, we discuss ring-like structures of global frequency domains of orthonormal wavelets and determine the size and the position of the holes in global frequency domains. Precisely say, we show that if the global frequency domains Ω of orthonormal wavelets possess the following ring-like structures

$$\Omega = S \setminus S_* \quad (0 \in S_* \subset S)$$

and the origin is an interior point of the hole S_*, then under some weak conditions, we have

(i) the set S can be determined by Ω: $S = \overline{\bigcup_{m \geq 0}} 2^{-m}\Omega$;

(ii) the hole S_* can be determined by S: $S_* \simeq \frac{S}{2} \setminus \bigcup_{\nu \in \mathbb{Z}^d \setminus \{0\}} (\frac{S}{2} + 2\nu\pi)$;

(iii) the size of the hole S_* satisfies $\frac{S}{4} \subset S_* \subset \frac{S}{2}$;

(iv) the union of the $4\nu\pi-$translations of S is just the whole space $\mathbb{R}^d$ $(\nu \in \mathbb{Z}^d)$.

As an example, for Meyer's wavelet, its frequency domain is $\Omega = S \setminus S_*$, where $S = [-\frac{8\pi}{3}, \frac{8\pi}{3}]$ and $S_* = [-\frac{2\pi}{3}, \frac{2\pi}{3}]$ satisfy the conditions (i)-(iv).

Conversely, if S and S_* satisfy (ii)-(iv), then there exists an orthonormal wavelet Ψ with the global frequency domain $\Omega = S \setminus S_*$.

Furthermore we research the case that S is a convex region in the frequency domain $\Omega = S \setminus S_*$. When S is a cuboid, the hole S_* must be a cuboid. For ball, we show an interesting result: there is no orthonormal wavelet whose global frequency domain is the difference set of two balls.

We present several examples at the end of this paper. From these examples, we see that according to our theory, one can construct orthonormal wavelets with various global frequency domains. In Example 5.5, we explain a general method of the construction of a wavelet with a given frequency domain.

This paper is organized as follows. In Section 2, we recall some known concepts and results. In Section 3, we introduce the concepts of regular sets and the kernels of sets, and then we characterize ring-like structures of global frequency domains of orthonormal wavelets. In Section 4, we give many corollaries and converse theorems. In Section 5, we present several examples to explain our theory.

2. Preliminaries

Throughout this paper, the closure, the boundary, and the interior of a set $E \subset \mathbb{R}^d$ are denoted by $\overline{E}$, ∂E, and E^o, respectively. The diameter of E is defined as $Diam\, E = \sup\{|x-y|;\ x,\ y \in E\}$. Let E be a closed region. If a point $\omega_0 \in E$ is such that $\omega_0 + \lambda(\omega - \omega_0) \in E$ $(0 \le \lambda \le 1)$ for any $\omega \in E$, then E is called a star region with the center ω_0. If, for arbitrary two points $\omega_1,\ \omega_2 \in E,\ \omega_1 + \lambda(\omega_2 - \omega_1) \in E$ $(0 \le \lambda \le 1)$, then E is called a convex region. For a set $E \subset \mathbb{R}^d$, $a \in \mathbb{R}^d$, and $\lambda \in \mathbb{R}$, denote

$$E + a := \{\omega + a :\ \omega \in E\}, \qquad \lambda E := \{\lambda\omega,\ \omega \in E\}, \qquad E + 2\pi\mathbb{Z}^d = \bigcup_{\nu \in \mathbb{Z}^d} (E + 2\pi\nu),$$

and $\nu \ne 0$ means $\nu \in \mathbb{Z}^d \setminus \{0\}$. For two sets $E,\ F \subset \mathbb{R}^d,\ E \supset F$ means $|F \setminus E| = 0$; $E \simeq F$ means the measure $|E \setminus F| = |F \setminus E| = 0$, where $|\cdot|$ is the Lebesgue measure.

The support of a function f on $\mathbb{R}^d$ is defined as $\mathrm{supp} f = \overline{\{t \in \mathbb{R}^d \mid f(t) \neq 0\}}$. We always assume $f(t) \neq 0$ for *a.e.* $t \in \mathrm{supp} f$. In this paper, for convenience, the notation " a.e. " is always omitted. Denote the Fourier transform of $f \in L^2(\mathbb{R}^d)$ by $\widehat{f}$: $\widehat{f}(\omega) = \int_{\mathbb{R}^d} f(t)e^{-i(t\cdot\omega)}\mathrm{d}t$, $\omega \in \mathbb{R}^d$. The support of $\widehat{f}$ is called the frequency domain of f.

Below, we will recall some known concepts and results in the wavelet analysis.

The set of functions $\Psi = \{\psi_\mu\}_1^{2^d-1}$ in the space $L^2(\mathbb{R}^d)$ is called an orthonormal wavelet if the system $\{2^{\frac{md}{2}}\psi_\mu(2^m \cdot -n)\}_{m\in\mathbb{Z},\, n\in\mathbb{Z}^d,\, \mu=1,\ldots,2^d-1}$ is an orthonormal basis for $L^2(\mathbb{R}^d)$. The global frequency domain Ω of an orthonormal wavelet $\Psi = \{\psi_\mu\}_1^{2^d-1}$ is defined as the union of frequency domains of all ψ_μ, i.e.,

$$\Omega = \bigcup_{\mu=1}^{2^d-1} \mathrm{supp}\, \widehat{\psi}_\mu.$$

Most of orthonormal wavelets are generated by multiresolution analyses.

A multiresolution analysis (MRA) consists of a sequence of closed subspaces $\{V_m\}_{m\in\mathbb{Z}}$ of $L^2(\mathbb{R}^d)$ satisfying

(i) $V_m \subset V_{m+1}$ $(m \in \mathbb{Z})$, (ii) $\overline{\bigcup_{m\in\mathbb{Z}} V_m} = L^2(\mathbb{R}^d)$, (iii) $\bigcap_{m\in\mathbb{Z}} V_m = \{0\}$,

(iv) $f \in V_m \quad \leftrightarrow \quad f(2\cdot) \in V_{m+1}$ $(m \in \mathbb{Z})$,

(v) there exists a $\varphi \in V_0$ such that $\{\varphi(\cdot - n)\}_{n\in\mathbb{Z}^d}$ is an orthonormal basis of V_0.

The function φ in (v) is called a scaling function [5,6].

For an orthonormal wavelet $\Psi = \{\psi_\mu\}_1^{2^d-1}$, if there exists a MRA such that $\{\psi_\mu(\cdot - n)\}_{n\in\mathbb{Z}^d,\, \mu=1,\ldots,2^d-1}$ is an orthonormal basis of the space W_0 ($V_0 \bigoplus W_0 = V_1$), then Ψ is called a MRA wavelet, where $\bigoplus$ is the orthogonal sum.

It is well-known that there exist many non-MRA orthonormal wavelets, for example, Journe wavelet [6]. The relation between MRA and orthonormal wavelet is as follows.

Lemma 2.1 [8]. Let $\Psi = \{\psi_\mu\}_1^{2^d-1}$ be an orthonormal wavelet with the bounded global frequency domain Ω. Denote $F = \overline{\bigcup_{m\geq 0} 2^{-m}\Omega}$. If, for each μ, $|\widehat{\psi}_\mu|$ is continuous at every point on $\partial F \bigcap \partial\Omega$, then Ψ is a MRA wavelet.

In the paper [8], we proved Lemma 2.1 in the case $d = 1$. For $d > 1$, the argument of Lemma 2.1 is similar.

Lemma 2.2 [6]. Let $\Psi = \{\psi_\mu\}_1^{2^d-1}$ be a MRA wavelet derived by a scaling function φ. Then

$$|\widehat{\varphi}(\omega)|^2 = \sum_{\mu=1}^{2^d-1} \sum_{m\geq 1} |\widehat{\psi}_\mu(2^m\omega)|^2, \quad \omega \in \mathbb{R}^d.$$

In [9], we gave a characterization of the frequency domains of scaling functions.

Lemma 2.3 [9]. Let G be a bounded closed set. Then there is a scaling function φ with $\operatorname{supp}\widehat{\varphi} = G$ if and only if (i) $G \subset 2G$, (ii) $\bigcup_{m\in\mathbb{Z}} 2^m G \simeq \mathbb{R}^d$, (iii) $G + 2\pi\mathbb{Z}^d \simeq \mathbb{R}^d$, and

$$\text{(iv) } (G \setminus \frac{G}{2}) \bigcap (\frac{G}{2} + 2\pi\nu) \simeq \emptyset \ (\nu \in \mathbb{Z}^d).$$

We also need the following lemma. It gives a necessary condition of orthonormal wavelets.

Lemma 2.4 [6]. Let $\Psi = \{\psi_\mu\}_1^{2^d-1}$ be an orthonormal wavelet. Then

$$\sum_{\mu=1}^{2^d-1} \sum_{m\in\mathbb{Z}} |\widehat{\psi}_\mu(2^m\omega)|^2 = 1, \quad \omega \in \mathbb{R}^d \setminus \{0\}.$$

Based on these four lemmas, we will characterize the ring-like structures of global frequency domains of orthonormal wavelets in Section 3.

3. Ring-like structure of global frequency domains of orthonormal wavelets

We first introduce the following concepts of regular sets and kernels of sets.

Definition 3.1. Let a bounded closed set E be a union of finitely many disjoint closed regions of $\mathbb{R}^d$. If E satisfies the following two conditions

$$\text{(i) } 0 \in \frac{E}{2} \subset E^o \qquad \text{and} \qquad \text{(ii) } \omega_0 + 8\pi\nu \notin \partial E \quad \text{for any } \omega_0 \in \partial E \text{ and } \nu \neq 0,$$

then E is called a regular set of $\mathbb{R}^d$, where $\nu \neq 0$ means $\nu \in \mathbb{Z}^d \setminus \{0\}$.

For example, the set $E_1 = [-4\pi, 2\pi] \bigcup [\frac{10\pi}{4}, \frac{15\pi}{4}] \bigcup [6\pi, 7\pi]$ is a regular set of $\mathbb{R}$; the set $E_2 = [-2\pi,\ 2\pi]^2 \bigcup ([\frac{5\pi}{2},\ 3\pi] \times [-2\pi,\ 2\pi])$ is a regular set of $\mathbb{R}^2$. Suppose that a closed star region F of $\mathbb{R}^d$ with the center $\omega = 0$ satisfies the conditions $0 \in F^o$ and $Diam\, F < 8\pi$. Then F is a regular set of $\mathbb{R}^d$.

Definition 3.2. Let E be a closed set of $\mathbb{R}^d$. The set

$$E_{ker} := E^o \setminus \bigcup_{\nu \neq 0} (E + 2\pi\nu)$$

is called the kernel of E, i.e., $E_{ker} = \{\omega \in E^o : \ \omega + 2\pi\nu \notin E, \ \nu \neq 0\}$.

Suppose that B is a closed ball of $\mathbb{R}^d$ with the radius r. We can check that if $r \geq 2\pi$, then $B_{ker} = \emptyset$; if $r \leq \pi$, then $B_{ker} = B^o$; if $\pi < r < 2\pi$ and $d > 1$, then B_{ker} is a simply connected region but it is not a ball. Suppose that E is a cube and $E = [a, \ b]^d$. We can check that if $b - a \geq 4\pi$, then $E_{ker} = \emptyset$; if $b - a \leq 2\pi$, then $E_{ker} = E^o$; if $2\pi < b - a < 4\pi$, then $E_{ker} = (b - 2\pi, \ 2\pi + a)^d$.

Now we characterize the ring-like structures of global frequency domains of orthonormal wavelets.

Theorem 3.3. Let $\Psi = \{\psi_\mu\}_1^{2^d-1}$ be an orthonormal wavelet with the global frequency domain $\Omega \simeq S \backslash S_*^o$ ($0 \in S_*^o \subset S$), where S is a regular set and $0 \in \frac{S_*^o}{2} \subset S_*$. Again, let for each μ, $|\widehat{\psi}_\mu|$ be continuous at every point on ∂S. Then

$$\text{(i) } S = \overline{\bigcup_{m \geq 0} 2^{-m}\Omega}, \quad \text{(ii) } S_* \simeq \left(\frac{S}{2}\right)_{ker}, \quad \text{(iii) } \frac{S}{4} \subset S_* \subset \frac{S}{2}, \quad \text{(iv) } S + 4\pi\mathbb{Z}^d \simeq \mathbb{R}^d. \tag{3.1}$$

Remark. In (3.1), (i) shows that S is determined uniquely by Ω. (ii) shows that the hole S_* is determined uniquely by S. (iii) shows that the relative size of the hole S_* in S. (iv) shows that the union of the $4\pi\nu-$translations of S covers the space $\mathbb{R}^d$ ($\nu \in \mathbb{Z}^d$).

By $S + 4\pi\mathbb{Z}^d \simeq \mathbb{R}^d$, we conclude a known result: if a cube $[A, B]^d \supset \Omega$, then $B - A \geq 4\pi$ [6]. By the hole $S_* \simeq \left(\frac{S}{2}\right)_{ker}$ and the definition of the kernel, we conclude other known result: if a cube $(a, b)^d \subset S_*$, then $b - a \leq 2\pi$ [6].

Proof of Theorem 3.3. The proof is divided into six steps as follows.

Step 1. We prove $\overline{\bigcup_{m \geq 0} 2^{-m}\Omega} = S$.

Since $\Psi = \{\psi_\mu\}_1^{2^d-1}$ is an orthonormal wavelet with global frequency domain Ω, by Lemma 2.4, we have

$$\operatorname{supp} \left(\sum_{\mu=1}^{2^d-1} \sum_{m \in \mathbb{Z}} |\widehat{\psi}_\mu(2^m\omega)|^2 \right) = \mathbb{R}^d. \tag{3.2}$$

Again, by $\text{supp}\,|\widehat{\psi}_\mu(2^m\omega)|^2 = 2^{-m}\text{supp}\,\widehat{\psi}_\mu$ and the global frequency domain $\Omega = \bigcup_{\mu=1}^{2^d-1} \text{supp}\,\widehat{\psi}_\mu$, we conclude

$$\overline{\bigcup_{m\in Z}} 2^{-m}\Omega = \mathbb{R}^d. \tag{3.3}$$

First, we claim $S_* \subset \frac{S}{2}$.

Let $E = S_* \setminus \frac{S}{2}$. Since $E \subset S_*$, by the assumption: $\frac{S_*}{2} \subset S_*$, we obtain $2^m E \subset 2^m S_* \subset S_*$ $(m \le 0)$. From this and $\Omega = S \setminus S_*$, we have $2^m E \bigcap \Omega = \emptyset$ $(m \le 0)$. Furthermore,

$$E \bigcap 2^{-m}\Omega = \emptyset \quad (m \le 0). \tag{3.4}$$

Since S is a regular set, for $m \ge 1$, we have $S \subset 2^{m-1}S$. Since $E \bigcap \frac{S}{2} = \emptyset$, we have $2^m E \bigcap 2^{m-1}S = \emptyset$. From this, we obtain $2^m E \bigcap S = \emptyset$, so $2^m E \bigcap \Omega = \emptyset$. Furthermore, $E \bigcap 2^{-m}\Omega = \emptyset$ $(m \ge 1)$. Combining this with (3.4), we have $E \bigcap \left(\overline{\bigcup_{m\in Z}} 2^{-m}\Omega \right) = \emptyset$. Again, by (3.3), we have $E = \emptyset$. So we obtain the claim

$$S_* \subset \frac{S}{2}. \tag{3.5}$$

From (3.5) and $\Omega = S \setminus S_*$, we conclude that

$$S \setminus \frac{S}{2} \subset \Omega \subset S. \tag{3.6}$$

Furthermore, we have $(2^{-m}S \setminus 2^{-m-1}S) \subset 2^{-m}\Omega$ $(m \ge 0)$. Again, since S is bounded and $0 \in \frac{S}{2} \subset S^o$, we obtain $\bigcap_{m\ge 0} 2^{-m}S = \{0\}$ and

$$S \setminus \{0\} = S \setminus \bigcap_{m\ge 0} 2^{-m}S = \bigcup_{m\ge 0} (2^{-m}S \setminus 2^{-m-1}S) \subset \bigcup_{m\ge 0} 2^{-m}\Omega.$$

Since $2^{-m}\Omega \subset 2^{-m}S \subset S$ $(m \ge 0)$, we have $S \setminus \{0\} \subset \bigcup_{m\ge 0} 2^{-m}\Omega \subset S$. So

$$\overline{\bigcup_{m\ge 0}} 2^{-m}\Omega = S. \tag{3.7}$$

Step 2. We prove $\partial S = \partial S \bigcap \partial\Omega$.

Since S is a regular set, by (3.5), we have $\partial S_* \subset \frac{S}{2} \subset S^o$, so $\partial S_* \bigcap \partial S = \emptyset$. From this, we know that the distance between ∂S and ∂S_* is greater than zero, i.e.,

$$\delta^* = \text{dist}(\partial S,\ \partial S_*) > 0. \tag{3.8}$$

Since S is a closed set and $\Omega = S \setminus S_*^o$, we know that Ω is a closed set. Now we claim $\partial\Omega = \partial S \bigcup \partial S_*$.

(i) First we prove that $\partial\Omega \subset (\partial S \bigcup \partial S_*)$.

Let $\omega_0 \in \partial\Omega$. By (3.8), we know that $\mathrm{dist}(\omega_0,\ \partial S) \geq \frac{\delta^*}{2}$ or $\mathrm{dist}(\omega_0,\ \partial S_*) \geq \frac{\delta^*}{2}$.

In the case $\mathrm{dist}(\omega_0,\ \partial S) \geq \frac{\delta^*}{2}$. For any ball $B(\omega_0,\ \delta)$ $(\delta < \frac{\delta^*}{2})$, by $\omega_0 \in \partial\Omega \subset S$, we have $B(\omega_0,\ \delta) \subset S$. Since $\omega_0 \in \partial\Omega$, there exists two points ω_1 and ω_2 in $B(\omega_0,\ \delta)$ such that $\omega_1 \in \Omega^o$ and $\omega_2 \notin \Omega$. From $\omega_1 \in \Omega^o$ and $\Omega = S \setminus S_*^o$, we have $\omega_1 \notin S_*^o$. From $\omega_2 \notin \Omega$, we have $\omega_2 \notin S$ or $\omega_2 \in S_*^o$. Again by $B(\omega_0,\ \delta) \subset S$, we get $\omega_2 \in S_*^o$, so $\omega_0 \in \partial S_*$.

In the case $\mathrm{dist}(\omega_0,\ \partial S_*) \geq \frac{\delta^*}{2}$. By $\omega_0 \in \partial\Omega \subset \Omega$ and $\Omega = S \setminus S_*^o$, we have $\omega_0 \notin S_*^o$. So any ball $B(\omega_0,\ \delta)$ $(\delta < \frac{\delta^*}{2})$ satisfies $B(\omega_0,\ \delta) \bigcap S_* = \emptyset$. Since $\omega_0 \in \partial\Omega$, there exist two points ω_1 and ω_2 in $B(\omega_0,\ \delta)$ such that $\omega_1 \in \Omega^o$ and $\omega_2 \notin \Omega$. From $\omega_1 \in \Omega^o$ and $\Omega \subset S$, we have $\omega_1 \in S^o$. From $\omega_2 \notin \Omega$ and $\Omega = S \setminus S_*^o$, we have $\omega_2 \notin S$ or $\omega_2 \in S_*^o$. Again by $B(\omega_0,\ \delta) \bigcap S_* = \emptyset$, we have $\omega_2 \notin S$, so $\omega_0 \in \partial S$.

Now we have concluded that if $\omega_0 \in \partial\Omega$, then $\omega_0 \in \partial S$ or $\omega_0 \in \partial S_*$, i.e., $\partial\Omega \subset (\partial S \bigcup \partial S_*)$.

(ii) Conversely, we prove that $\partial\Omega \supset (\partial S \bigcup \partial S_*)$.

Let $\omega_0 \in \partial S$. For any ball $B(\omega_0,\ \delta)$ $(\delta < \delta^*)$, by (3.8), we have $B(\omega_0,\ \delta) \bigcap S_* = \emptyset$. Since $\omega_0 \in \partial S$, there exist two points ω_1 and ω_2 in $B(\omega_0,\ \delta)$ such that $\omega_1 \in S^o$, $\omega_2 \notin S$, so $\omega_2 \notin \Omega$. From $\omega_1 \in S^o$ and $B(\omega_0,\ \delta) \bigcap S_* = \emptyset$, we have $\omega_1 \in \Omega^o$, so $\omega_0 \in \partial\Omega$.

Let $\omega_0 \in \partial S_*$. For any ball $B(\omega_0,\ \delta)$ $(\delta < \delta^*)$, by (3.8) and $\omega_0 \in S$, we have $B(\omega_0,\ \delta) \subset S$. Since $\omega_0 \in \partial S_*$, there exists two points ω_1 and ω_2 in $B(\omega_0,\ \delta)$ such that $\omega_1 \in S_*^o$ and $\omega_2 \notin S_*$. Since $\omega_2 \notin S_*$ and $B(\omega_0,\ \delta) \subset S$, we have $\omega_2 \in S \setminus S_*^o = \Omega$. By $\omega_1 \in S_*^o$, we have $\omega_1 \notin \Omega$, so $\omega_0 \in \partial\Omega$.

Therefore, when $\omega_0 \in \partial S$ or $\omega_0 \in \partial S_*$, we have $\omega_0 \in \partial\Omega$, i.e., $\partial\Omega \supset (\partial S \bigcup \partial S_*)$.

By (i) and (ii), we get the claim $\partial\Omega = \partial S \bigcup \partial S_*$.

By this claim and $\partial S \bigcap \partial S_* = \emptyset$, we get

$$\partial S \bigcap \partial\Omega = \partial S. \tag{3.9}$$

Step 3. Let $G = \frac{S}{2}$. We prove that $(G^o \setminus \frac{G}{2}) \bigcap (\frac{G^o}{2} + 2\pi\nu) = \emptyset,\ \nu \in \mathbb{Z}^d$.

By (3.7) and (3.9), we have $\partial S=\partial(\overline{\bigcup_{m\geq 0}2^{-m}\Omega})\bigcap\partial\Omega$. From this and the assumption that for each μ, $|\widehat{\psi}_\mu|$ is continuous on ∂S, by Lemma 2.1, we conclude that Ψ is a MRA wavelet.

By Lemma 2.2, the corresponding scaling function φ satisfies $|\widehat{\varphi}(\omega)|^2=\sum_{m\geq 1}\sum_{\mu=1}^{2^d-1}|\widehat{\psi}_\mu(2^m\omega)|^2$. This implies

$$\mathrm{supp}\,\widehat{\varphi}=\overline{\bigcup_{m\geq 1}}\bigcup_{\mu=1}^{2^d-1}\mathrm{supp}\,\widehat{\psi}_\mu(2^m\cdot).$$

However,

$$\bigcup_{\mu=1}^{2^d-1}\mathrm{supp}\,\widehat{\psi}_\mu(2^m\cdot)=2^{-m}\bigcup_{\mu=1}^{2^d-1}\mathrm{supp}\widehat{\psi}_\mu=2^{-m}\Omega.$$

So $\mathrm{supp}\widehat{\varphi}=\overline{\bigcup_{m\geq 1}2^{-m}\Omega}=\frac{1}{2}\overline{\bigcup_{m\geq 0}2^{-m}\Omega}$. Therefore, by (3.7), we obtain

$$\mathrm{supp}\,\widehat{\varphi}=\frac{S}{2}=G. \tag{3.10}$$

Since S is a regular set and $G=\frac{S}{2}$, we see that

$$\frac{G}{2}\subset G^o,\quad \omega_0+4\pi\nu\notin\partial G\quad(\omega_0\in\partial G,\ \nu\neq 0). \tag{3.11}$$

These properties are used in the following argument.

Since φ is a scaling function with the frequency domain G, by Lemma 2.3, we have

$$(G\setminus\frac{G}{2})\bigcap(\frac{G}{2}+2\pi\nu)\simeq\emptyset,\quad \nu\in\mathbb{Z}^d.$$

Noticing that $G=\frac{S}{2}$ and S is a union of finitely many disjoint closed regions, we know that $|\partial G|=0$. So

$$(G^o\setminus\frac{G}{2})\bigcap(\frac{G^o}{2}+2\pi\nu)\simeq\emptyset,\quad \nu\in\mathbb{Z}^d.$$

Since the set of the left-hand side of this formula is an open set, the following precisely equality holds

$$(G^o\setminus\frac{G}{2})\bigcap(\frac{G^o}{2}+2\pi\nu)=\emptyset,\quad \nu\in\mathbb{Z}^d. \tag{3.12}$$

Step 4. We further prove that

$$\frac{G^o}{2}\bigcap(\frac{G^o}{2}+2\pi\nu)=\emptyset,\quad \nu\neq 0. \tag{3.13}$$

If (3.13) is not valid, then there exist two points $\xi_1,\ \xi_2 \in \frac{G^o}{2}$ and $\xi_1 - \xi_2 = 2\pi\nu_0$ for some $\nu_0 \neq 0$. Let L be a straight line passing though the points ξ_1 and ξ_2. Let points ω_1 and ω_2 be the nearest points of intersection of line L and the boundary $\partial(\frac{G}{2})$ away from ξ_1 and ξ_2, respectively. Again by (3.11), we have $\omega_1,\ \omega_2 \in \big(G^o \bigcap \partial(\frac{G}{2}) \bigcap L\big)$.

Denote the distance between two points x and y by $\rho(x,\ y)$. Without loss of generality, we assume

$$\rho(\omega_1,\ \xi_1) \leq \rho(\omega_2,\ \xi_2).$$

(a) Suppose that $\rho(\omega_1,\ \xi_1) < \rho(\omega_2,\ \xi_2)$. Since $\omega_1 \in (G^o \bigcap \partial(\frac{G}{2}) \bigcap L)$, we can take ξ_1^* near ω_1 such that

$$\xi_1^* \in G^o, \quad \xi_1^* \notin \frac{G}{2}, \quad \xi_1^* \in L, \quad \text{and} \quad \rho(\omega_1,\ \xi_1^*) < \rho(\omega_2,\ \xi_2) - \rho(\omega_1,\ \xi_1).$$

Let $\xi_2^* = \xi_1^* - 2\pi\nu_0$. By $\xi_2 = \xi_1 - 2\pi\nu_0$, we have

$$\rho(\xi_2^*,\ \xi_2) = \rho(\xi_1^*,\ \xi_1) \leq \rho(\xi_1,\ \omega_1) + \rho(\omega_1,\ \xi_1^*) < \rho(\omega_2,\ \xi_2).$$

Again, since $\xi_2 \in \frac{G^o}{2}$ and ω_2 is a nearest point of intersection of L and $\partial(\frac{G}{2})$ away from ξ_2, we have $\xi_2^* \in \frac{G^o}{2}$. From this and $\xi_1^* \in G^o \setminus \frac{G}{2},\ \xi_2^* + 2\pi\nu_0 = \xi_1^*$, we have

$$\xi_1 \in ((G^o \setminus \frac{G}{2}) \bigcap (\frac{G^o}{2} + 2\pi\nu_0)).$$

This is contrary to (3.12).

(b) Suppose that $\rho(\omega_1,\ \xi_1) = \rho(\omega_2,\ \xi_2)$. Let $\xi_2^0 = \omega_1 - 2\pi\nu_0$. Now we prove $\xi_2^0 \notin \partial(\frac{G}{2})$.

In fact, if $\xi_2^0 \in \partial(\frac{G}{2})$, then $2\xi_2^0 \in \partial G$. However, $2\xi_2^0 + 4\pi\nu_0 = 2\omega_1 \in \partial G$, this is contrary to (3.11).

By $\xi_2 = \xi_1 - 2\pi\nu_0$, we have

$$\rho(\xi_2^0,\ \xi_2) = \rho(\omega_1,\ \xi_1) = \rho(\omega_2,\ \xi_2).$$

Again, by $\xi_2 \in \frac{G^o}{2}$ and the definition of ω_2, we have $\xi_2^0 \in \frac{G}{2}$. From this and $\xi_2^0 \notin \partial(\frac{G}{2})$, we have $\xi_2^0 \in \frac{G^o}{2}$.

Since $\omega_1 \in (G^o \bigcap \partial(\frac{G}{2}) \bigcap L)$ and $\omega_1 - 2\pi\nu_0 = \xi_2^o \in \frac{G^o}{2}$, we know that there exists a point $\xi_1^0 \in G^o \setminus \frac{G}{2}$ near ω_1 such that $\xi_1^0 - 2\pi\nu_0 \in \frac{G^o}{2}$, i.e.,

$$\xi_1^0 \in \left((G^o \setminus \frac{G}{2}) \bigcap (\frac{G^o}{2} + 2\pi\nu_0) \right).$$

This is also contrary to (3.12). So (3.13) holds.

Step 5. We will prove $\Omega \simeq S \setminus (\frac{S}{2})_{ker}$.

From (3.12) and (3.13), we obtain $G^o \bigcap \big(\frac{G^o}{2} + 2\pi\nu \big) = \emptyset,\ \nu \neq 0$. By $|\partial G| = 0$, we obtain

$$G \bigcap (\frac{G}{2} + 2\pi\nu) \simeq \emptyset, \qquad \nu \neq 0.$$

Again by (3.10), we have $\widehat{\varphi}(\omega + 2\pi\nu) = 0$ $(\nu \neq 0)$ for $\omega \in \frac{G}{2}$. Combining this with a necessary condition of the scaling functions [3]:

$$\sum_{\nu \in \mathbb{Z}^d} |\widehat{\varphi}(\omega + 2\pi\nu)|^2 = 1, \qquad \omega \in \mathbb{R}^d, \tag{3.14}$$

we obtain

$$|\widehat{\varphi}(\omega)| = 1, \quad \omega \in \frac{G}{2}. \tag{3.15}$$

Since $G_{ker} := \{\omega \in G^o : \ \ \omega + 2\pi\nu \notin G,\ \nu \neq 0\}$, by $\mathrm{supp}\,\widehat{\varphi} = G$ and (3.14), we obtain

$$G_{ker} \simeq \{\omega \in \mathbb{R}^d : \quad |\widehat{\varphi}(\omega)| = 1\}. \tag{3.16}$$

From this and (3.15), we have

$$\frac{G}{2} \subset G_{ker} \subset G. \tag{3.17}$$

Below we prove $\Omega = S \setminus G_{ker}$.

Since $\mathbb{R}^d = (\mathbb{R}^d \setminus 2G) \bigcup (2G \setminus G) \bigcup (G \setminus G_{ker}) \bigcup G_{ker}$, we have

$$\Omega = \mathbb{R}^d \bigcap \Omega = \Big((\mathbb{R}^d \setminus 2G) \bigcap \Omega \Big) \bigcup \Big((2G \setminus G) \bigcap \Omega \Big) \bigcup \Big((G \setminus G_{ker}) \bigcap \Omega \Big) \bigcup \Big(G_{ker} \bigcap \Omega \Big). \tag{3.18}$$

By Lemma 2.2, we see that $|\widehat{\varphi}(\frac{\omega}{2})|^2 - |\widehat{\varphi}(\omega)|^2 = \sum\limits_{\mu=1}^{2^d-1} |\widehat{\psi}_\mu(\omega)|^2,\ \omega \in \mathbb{R}^d$. So we have

$$\mathrm{supp}\ \Big(|\widehat{\varphi}(\frac{\omega}{2})|^2 - |\widehat{\varphi}(\omega)|^2 \Big) = \bigcup_{\mu=1}^{2^d-1} \mathrm{supp}\,\widehat{\psi}_\mu = \Omega.$$

This implies that $\omega \in \Omega$ if and only if $|\widehat{\varphi}(\frac{\omega}{2})| \neq |\widehat{\varphi}(\omega)|$. Based on this claim, we have

(α) For $\omega \in (\mathbb{R}^d \setminus 2G)$, by $G \subset 2G$ and $\mathrm{supp}\widehat{\varphi} = G$, we see that $\widehat{\varphi}(\omega) = \widehat{\varphi}(\frac{\omega}{2}) = 0$, so, $\omega \notin \Omega$, i.e.,

$$(\mathbb{R}^d \setminus 2G) \bigcap \Omega \simeq \emptyset.$$

(β) For $\omega \in (2G \setminus G)$, by $\operatorname{supp} \widehat{\varphi} = G$, we obtain $\widehat{\varphi}(\omega) = 0$ and $\widehat{\varphi}(\frac{\omega}{2}) \neq 0$, so $\omega \in \Omega$, i.e.,

$$(2G \setminus G) \bigcap \Omega \simeq 2G \setminus G.$$

(γ) For $\omega \in G \setminus G_{ker}$, by (3.15), we have $|\widehat{\varphi}(\frac{\omega}{2})| = 1$. By (3.16), we have $|\widehat{\varphi}(\omega)| \neq 1$, so $\omega \in \Omega$, i.e.,

$$(G \setminus G_{ker}) \bigcap \Omega \simeq G \setminus G_{ker}.$$

(δ) For $\omega \in G_{ker}$, by (3.16), we have $|\widehat{\varphi}(\omega)| = 1$. From (3.17), it follows that $\frac{G_{ker}}{2} \subset G_{ker}$, further, $|\widehat{\varphi}(\frac{\omega}{2})| = 1$, so $\omega \notin \Omega$, i.e.,

$$G_{ker} \bigcap \Omega \simeq \emptyset.$$

Combining (3.18) with (α)-(δ), noticing that $G \subset 2G$ and $G_{ker} \subset G$, we obtain

$$\Omega \simeq (2G \setminus G) \bigcup (G \setminus G_{ker}) \simeq 2G \setminus G_{ker}.$$

Again by $G = \frac{S}{2}$, we have $\Omega \simeq S \setminus (\frac{S}{2})_{ker}$.

Step 6. We prove (3.1).

By (3.7), we get (i). By the assumption, we have $\Omega \simeq S \setminus S_*$ and $S_* \subset S$. Again, by $\Omega \simeq S \setminus (\frac{S}{2})_{ker}$ and $(\frac{S}{2})_{ker} \subset \frac{S}{2} \subset S$, we get (ii). By (3.17) and $G = \frac{S}{2}$, we get (iii). By (3.10), we know that $\frac{S}{2}$ is the frequency domain of the scaling function φ. Using Lemma 2.3, we have $\frac{S}{2} + 2\pi\mathbb{Z}^d \simeq \mathbb{R}^d$. So (iv) holds. Theorem 3.3 is proved. $\square$

4. Some corollaries and converse theorems

In this section, as the corollaries of Theorem 3.3, we discuss a special case that the set S is a convex region in the ring-like structure $\Omega = S \setminus S_*^o$ of the frequency domain of an orthonormal wavelet. On the other hand, we give two converse theorems of Theorem 3.3.

4.1. Some corollaries

If S is a bounded convex region and $0 \in S^o$, then $\frac{S}{2} \subset S^o$, but S may not be a regular set. However, if, in Theorem 3.3, we replace the condition "S is a regular set" by the condition "S is a bounded convex region", then Theorem 3.3 is still valid. This is because the condition $\omega_0 + 8\pi\nu \notin \partial S$ ($\omega_0 \in \partial S,\ \nu \neq 0$) is only used in argument of Formula (3.13). However when S is a bounded convex region, $G = \frac{S}{2}$ is also a bounded convex region. In this case, (3.12) implies clearly (3.13) and this argument does not need the condition $\omega_0 + 8\pi\nu \notin \partial S$ ($\omega_0 \in \partial S,\ \nu \neq 0$). So we have

Corollary 4.1. In Theorem 3.3, if the condition "S is a regular set" is replaced by the condition "S is a bounded convex region", then the conclusion (3.1) is still valid.

In particular, we discuss the case that " S is a cuboid ".

Corollary 4.2. Let $S = \prod_{i=1}^{d} [a_i,\ b_i]$, and let $\Psi = \{\psi_\mu\}_1^{2^d-1}$ be an orthonormal wavelet with global frequency domain $\Omega \simeq S \setminus S_*^o$ ($0 \in S_*^o \subset S$) and for each μ, $|\widehat{\psi}_\mu|$ is continuous on ∂S. If the set S_* satisfies $0 \in \frac{S_*^o}{2} \subset S_*$, then

(i) S_* is also a cuboid and $S_* = \prod_{i=1}^{d} (\frac{b_i}{2} - 2\pi,\ \frac{a_i}{2} + 2\pi)$.

(ii) For each i, we have $-4\pi + \frac{b_i}{2} \leq a_i < 0, \quad 0 < b_i \leq 4\pi + \frac{a_i}{2}$, and $4\pi \leq b_i - a_i \leq \frac{16\pi}{3}$.

Proof. Since S is a bounded convex region, by Corollary 4.1, we know that (3.1) holds. By $S + 4\pi\mathbb{Z}^d \simeq \mathbb{R}^d$ and $S = \prod_{i=1}^{d} [a_i,\ b_i]$, it follows that $b_i - a_i \geq 4\pi$ for each i. By the definition of the kernel and $S_* = (\frac{S}{2})_{ker}$, we get (i). By $\frac{S}{4} \subset S_* \subset \frac{S}{2}$ and $0 \in S^o$, we have

$$\frac{b_i}{4} \leq \frac{a_i}{2} + 2\pi \leq \frac{b_i}{2}, \quad \frac{a_i}{2} \leq \frac{b_i}{2} - 2\pi \leq \frac{a_i}{4}, \quad a_i < 0 < b_i \quad (i = 1, ..., d).$$

We get (ii). Corollary 4.2 is proved. □

It is natural to ask whether there exists an orthonormal wavelet Ψ with the global frequency domain $\Omega = S \setminus S_*$ ($0 \in S_*^o \subset S$), where S and S_* are both balls.

Corollary 4.3. For $d > 1$, there is no orthonormal wavelet $\Psi = \{\psi_\mu\}_1^{2^d-1}$ with the global frequency domain $\Omega \simeq S \setminus S_*^o$ ($0 \in S_*^o \subset S$), where S_* and S are both balls and $|\widehat{\psi}_\mu|$ is continuous on ∂S for each μ.

Proof. Suppose that there exists an orthonormal wavelet Ψ such that the above conditions hold. Using

Corollary 4.1, we can obtain (3.1). From $S+4\pi\mathbb{Z}^d \simeq \mathbb{R}^d$, we know that $Diam\, S > 4\pi$. So the radius of the ball $\frac{1}{2}S$ is greater than π. Furthermore, by the definition of the kernel, we know that $S_* = (\frac{S}{2})_{ker}$ is not a ball in $\mathbb{R}^d$. This is a contradiction. Corollary 4.3 is proved. $\square$

4.2. Converse theorems

Now we discuss the converse theorems of Theorem 3.3. We first prove the following:

Theorem 4.4. Suppose that S is a bounded closed set and $0 \in S^o$, $\frac{S}{2} \subset S$. Again if the set S satisfies the conditions $\frac{S}{4} \subset \left(\frac{S}{2}\right)_{ker}$ and $S+4\pi\mathbb{Z}^d \simeq \mathbb{R}^d$, then there exists an orthonormal wavelet Ψ with the global frequency domain $\Omega \simeq S \setminus (\frac{S}{2})_{ker}$.

Proof. Let $G = \frac{S}{2}$. We will check that G satisfies four conditions of Lemma 2.3.

By the definition of the kernel, we know that $(\frac{S}{2})_{ker} = \frac{S^o}{2} \setminus \bigcup\limits_{\nu \neq 0} (\frac{S}{2} + 2\pi\nu)$. This implies that

$$(\frac{S}{2})_{ker} \bigcap (\frac{S}{2} + 2\pi\nu) = \emptyset, \quad \nu \neq 0. \tag{4.1}$$

By the known condition $\frac{S}{4} \subset \left(\frac{S}{2}\right)_{ker}$ and $G = \frac{S}{2}$, we have $\frac{G}{2} \subset (\frac{S}{2})_{ker}$. Again, by (4.1), $\frac{G}{2} \bigcap (G+2\pi\nu) = \emptyset$ $(\nu \neq 0)$. Furthermore, we have $G \bigcap (\frac{G}{2} + 2\pi\nu) = 0$, $\nu \neq 0$. This implies that

$$(G \setminus \frac{G}{2}) \bigcap (\frac{G}{2} + 2\pi\nu) = 0, \quad \nu \in \mathbb{Z}^d. \tag{4.2}$$

Since $0 \in S^o$, we know that $\omega = 0$ is an interior point of G. So there exists a ball B with the center $\omega = 0$ such that $B \subset G$. From $\bigcup\limits_{m \in \mathbb{Z}} 2^m B = \mathbb{R}^d$, we have

$$\bigcup_{m \in \mathbb{Z}} 2^m G = \mathbb{R}^d. \tag{4.3}$$

By $\frac{S}{2} \subset S$ and $S + 4\pi\mathbb{Z}^d \simeq \mathbb{R}^d$, we have $G \subset 2G$ and $G + 2\pi\mathbb{Z}^d \simeq \mathbb{R}^d$. From this and (4.2)-(4.3), by using Lemma 2.3, we know that there exists a scaling function φ with $\operatorname{supp} \widehat{\varphi} = G = \frac{S}{2}$. Let Ψ be an orthonormal wavelet generated by the scaling function φ. Noticing that $G \bigcap (\frac{G}{2} + 2\pi\nu) = 0$ $(\nu \neq 0)$, similar to the arguments of Step 5 in Theorem 3.3, we conclude that the global frequency domain of Ψ is $\Omega \simeq S \setminus (\frac{S}{2})_{ker}$. Theorem 4.4 is proved. $\square$

From Theorem 4.4, we get the converse theorem of Theorem 3.3 immediately.

Theorem 4.5. Let S be a regular set. If the sets S and S_* satisfy the conditions (ii)-(iv) in (3.1), then there exists an orthonormal wavelet Ψ with the global frequency domain $\Omega = S \setminus S_*^o$.

Observing the argument of Theorem 3.3, we see that in Step 1 and Step 2, we have proved that

$$S = \overline{\bigcup_{m \geq 0} 2^{-m}\Omega} \qquad \text{and} \qquad \partial S = \partial S \bigcap \partial\Omega.$$

Based on these two conclusions, in Step 3, Step 4, and Step 5, we derived the conclusions (ii)-(iv) in (3.1) by using the conditions: " S is a regular set and each $|\widehat{\psi}_\mu|$ is continuous on ∂S ". So we get another converse theorem.

Theorem 4.6. Let $\Psi = \{\psi_\mu\}_1^{2^d-1}$ be an orthonormal wavelet with the global frequency domain Ω. Denote $S = \overline{\bigcup_{m \geq 0} 2^{-m}\Omega}$. If S is a regular set, $\partial S = \partial S \bigcap \partial\Omega$, and for each μ, $|\widehat{\psi}_\mu|$ is continuous on ∂S, then the global frequency domain has the form

$$\Omega \simeq S \setminus (\frac{S}{2})_{ker}, \qquad \text{where} \qquad \frac{S}{4} \subset (\frac{S}{2})_{ker} \subset \frac{S}{2}, \quad S + 4\pi\mathbb{Z}^d \simeq \mathbb{R}^d.$$

As an example, it is easy to check that the tensor product Meyer's wavelet satisfies the condition of Theorem 4.6. In fact, its Fourier transform is infinitely differentiable and its global frequency domain is $\Omega = [-\frac{8\pi}{3}, \frac{8\pi}{3}]^d \setminus (-\frac{2\pi}{3}, \frac{2\pi}{3})^d$. By $\bigcup_{m \geq 0} 2^{-m}\Omega = [-\frac{8\pi}{3}, \frac{8\pi}{3}]^d \setminus \{0\}$, we have $S = \overline{\bigcup_{m \geq 0} 2^{-m}\Omega} = [-\frac{8\pi}{3}, \frac{8\pi}{3}]^d$. So S is a regular set. Again by $\partial\Omega = \partial\left([-\frac{8\pi}{3}, \frac{8\pi}{3}]^d\right) \bigcup \partial\left([-\frac{2\pi}{3}, \frac{2\pi}{3}]^d\right)$, we have $\partial S = \partial\Omega \bigcap \partial S$.

5. Examples

In this section, we will construct various global frequency domains and explain a method of the construction of an orthonormal wavelet with a given frequency domain.

From Theorem 4.4, we see that if a bounded closed set S satisfies the conditions

$$0 \in \frac{S}{4} \subset (\frac{S}{2})_{ker} \subset \frac{S}{2} \subset S^o \quad \text{and} \quad S + 4\pi\mathbb{Z}^d \simeq \mathbb{R}^d,$$

then $\Omega = S \setminus (\frac{S}{2})_{ker}$ is a global frequency domain of some orthonormal wavelet. In the following examples, we start from a union S of closed regions satisfying the above conditions. Removing a hole $(\frac{S}{2})_{ker}$ in the set S, we obtain $S \setminus (\frac{S}{2})_{ker}$ which must be the global frequency domain of some orthonormal wavelet.

Example 5.1. Let S be a union of two rectangles of $\mathbb{R}^2$:

$$S = [-2\pi,\ 2\pi]^2 \bigcup \Big(\ [\,\frac{5\pi}{2},\ 3\pi\,] \times [-2\pi,\ 2\pi] \ \Big).$$

Then we have $0 \in \frac{S}{2} \subset S^o$ and $S + 4\pi\mathbb{Z}^2 = \mathbb{R}^2$. A direct calculation shows that the kernel

$$(\frac{S}{2})_{ker} = (-\pi,\ \pi)^2 \setminus (\,[\,-\frac{3\pi}{4},\ -\frac{\pi}{2}\,] \times [-\pi,\ \pi]\,) = \Big(\,(-\pi,\ -\frac{3\pi}{4}\,) \times (-\pi,\ \pi) \Big) \bigcup \Big((-\frac{\pi}{2},\ \pi) \times (-\pi,\ \pi) \Big)$$

is a union of two rectangles where the rectangle $(-\frac{\pi}{2},\ \pi) \times (-\pi,\ \pi)$ contains the origin. From this we get $\frac{S}{4} \subset (\frac{S}{2})_{ker}$. Therefore S satisfies the conditions of Theorem 4.4. By using Theorem 4.4, there exists an orthonormal wavelet $\Psi = \{\psi_\mu\}_1^3$ with the global frequency domain $\Omega_1 = S \setminus (\frac{S}{2})_{ker}$. Furthermore, we have

$$\Omega_1 = \Big([-2\pi,\ 2\pi]^2 \setminus (\frac{S}{2})_{ker} \Big) \bigcup \Big([\frac{5\pi}{2},\ 3\pi] \times [-2\pi,\ 2\pi] \Big) =: \Omega_1' \bigcup \Omega_1'',$$

where Ω_1' is a triply connected region and Ω_1'' is a rectangle.

Example 5.2. Denote a disk with the center $\omega_0 = (3\pi, \frac{3\pi}{2})$ and the radius $\frac{\pi}{4}$ by $B(\omega_0,\ \frac{\pi}{4})$. Let S be a union of the square $[-2\pi,\ 2\pi]^2$ and the closed disk $\overline{B}(\omega_0,\ \frac{\pi}{4})$. Then $0 \in S^o$, $\frac{S}{2} \subset S$, and $S + 4\pi\mathbb{Z}^2 = \mathbb{R}^2$. A direct calculation shows that the kernel

$$(\frac{S}{2})_{ker} = (-\pi,\ \pi)^2 \setminus \overline{B}\left(\frac{\omega_0}{2} - 2\pi(1,0),\ \frac{\pi}{8} \right)$$

is a doubly connected region and $\frac{S}{4} \subset (\frac{S}{2})_{ker}$. From this, we see that S satisfies the conditions of Theorem 4.4. So there exists an orthonormal wavelet Ψ with the global frequency domain

$$\Omega_2 = S \setminus (\frac{S}{2})_{ker} = \big([-2\pi,\ 2\pi]^2 \setminus (-\pi,\ \pi)^2 \big) \bigcup \Big(\overline{B}(\frac{\omega_0}{2} - 2\pi(1,0),\ \frac{\pi}{8}) \bigcup \overline{B}(\omega_0,\ \frac{\pi}{4}) \Big) =: \Omega_2' \bigcup \Omega_2'',$$

where Ω_2' is the difference set of two squares and Ω_2'' is a union of two disks.

Example 5.3. Let $S \subset \mathbb{R}^2$ be a trapezoid with two curved sides:

$$S = \left\{ (x,y) \in \mathbb{R}^2 : \quad -2\pi \le x \le 2\pi, \quad f(x) \le y \le f(x) + \frac{14\pi}{3} \right\},$$

where f is a continuous function and satisfies $-\frac{7\pi}{3} \le f(x) \le -2\pi$ $(-2\pi \le x \le 2\pi)$. Then

$$\frac{S}{2} = \left\{ (x,\, y) \in \mathbb{R}^2 : \quad -\pi \le x \le \pi, \quad \frac{f(2x)}{2} \le y \le \frac{f(2x)}{2} + \frac{7\pi}{3} \right\}.$$

From $-\frac{7\pi}{3} \le f(x) \le -2\pi$ $(-2\pi \le x \le 2\pi)$, we have

$$\frac{S}{2} \subset [-\pi,\ \pi] \times [-\frac{7\pi}{6},\ \frac{4\pi}{3}] \quad \text{and} \quad [-2\pi,\ 2\pi] \times [-2\pi,\ \frac{7\pi}{3}] \subset S.$$

So $0 \in S^o$ and $\frac{S}{2} \subset S$. A calculation shows that the kernel of $\frac{1}{2}S$ is

$$(\frac{S}{2})_{ker} = \left\{ (x,y) \in \mathbb{R}^2 : \quad -\pi < x < \pi, \ \frac{f(2x)}{2} + \frac{\pi}{3} < y < \frac{f(2x)}{2} + 2\pi \right\}.$$

Again, by the definition, we have

$$\frac{S}{4} \subset \left([-\frac{\pi}{2},\ \frac{\pi}{2}] \times [-\frac{7\pi}{12},\ \frac{4\pi}{6}] \right) \subset \left([-\frac{\pi}{2},\ \frac{\pi}{2}] \times [-\frac{8\pi}{12},\ \frac{5\pi}{6}] \right) \subset (\frac{S}{2})_{ker}.$$

Since $-\frac{7\pi}{3} \le f(x) \le -2\pi$ $(-2\pi \le x \le 2\pi)$, we have $\frac{7\pi}{3} \le f(x) + \frac{14\pi}{3} \le \frac{8\pi}{3}$ $(-2\pi \le x \le 2\pi)$. This implies $S \supset [-2\pi, 2\pi] \times [-2\pi, \frac{7\pi}{3}] \supset [-2\pi, 2\pi]^2$. So $S + 4\pi\mathbb{Z}^2 = \mathbb{R}^2$. By Theorem 4.4, we see that there exists an orthonormal wavelet $\Psi = \{\psi_\mu\}_1^3$ such that its global frequency domain is in the doubly connected domain $\Omega_3 = S \setminus (\frac{S}{2})_{ker}$.

Example 5.4. Let S be a union of three closed intervals $\left[-\frac{5\pi}{2},\ -2\pi\right]$, $\left[-\frac{7\pi}{4},\ \frac{7\pi}{4}\right]$, and $[2\pi,\ \frac{5\pi}{2}]$. Then

$$\frac{S}{2} = \left[-\frac{5\pi}{4},\ -\pi\right] \bigcup \left[-\frac{7\pi}{8},\ \frac{7\pi}{8}\right] \bigcup \left[\pi,\ \frac{5\pi}{4}\right].$$

So $0 \in S^o$, $\frac{S}{2} \subset S$, and $S + 4\pi\mathbb{Z} = \mathbb{R}$. It is easy to check that the kernel $(\frac{S}{2})_{ker} = (-\frac{9\pi}{8},\ -\pi) \bigcup (-\frac{3\pi}{4},\ \frac{3\pi}{4}) \bigcup (\pi,\ \frac{9\pi}{8})$ satisfies $\frac{1}{4}S \subset (-\frac{3\pi}{4},\ \frac{3\pi}{4}) \subset (\frac{S}{2})_{ker}$. By Theorem 4.4, there exists an orthonormal wavelet ψ with the frequency domain

$$\Omega_4 = S \setminus (\frac{S}{2})_{ker} = [-\frac{5\pi}{2},\ -2\pi] \bigcup [-\frac{7\pi}{4},\ -\frac{9\pi}{8}] \bigcup [-\pi,\ -\frac{3\pi}{4}] \bigcup [\frac{3\pi}{4},\ \pi] \bigcup [\frac{9\pi}{8},\ \frac{7\pi}{4}] \bigcup [2\pi,\ \frac{5\pi}{2}].$$

Now we present a typical example to explain a method of the construction of an orthonormal wavelet with a given frequency domain.

Example 5.5. Define a simply connected closed region $S \subset \mathbb{R}^2$ by $S = [-2\pi,\ 2\pi]^2 \bigcup \left(\bigcup_{\nu=1}^{4} S_\nu \right)$, where

$$S_1 := \left\{ (x,y) \in \mathbb{R}^2 | \ -2\pi \le x \le 2\pi, \ \ 2\pi - \sqrt{20\pi^2 - x^2} \le y \le -2\pi \right\},$$

$$S_2 := \left\{ (x,y) \in \mathbb{R}^2 | \ \ 2\pi - \sqrt{20\pi^2 - y^2} \le x \le -2\pi, \ \ -2\pi \le y \le 2\pi \right\},$$

$$S_3 := S_1 + 4\pi(0,1), \qquad\qquad S_4 := S_2 + 4\pi(1,0). \tag{5.1}$$

We see that each S_i is a region whose boundary is a side of the square $[-2\pi, 2\pi]^2$ and a circular arc through two ends of this side.

(i) Check that S satisfies the conditions of Theorem 4.4.

Since $0 \in S^o$ and S is a closed star region with the center $\omega = 0$, we know that $\frac{S}{2} \subset S$.

A direct calculation shows that

$$(\frac{S}{2})_{ker} = \frac{S^o}{2} \setminus \bigcup_{\nu=1}^{4} E_\nu \quad \text{and} \quad \bigcup_{\nu=1}^{4} E_\nu^o \subset \frac{S^o}{2}, \tag{5.2}$$

where $E_1 := \{ \ (x,y) \in \mathbb{R}^2 \ | \ -\pi \le x \le \pi, \quad \pi - \sqrt{5\pi^2 - x^2} \le y \le -3\pi + \sqrt{5\pi^2 - x^2} \ \}$

$$E_2 := \{ \ (x,y) \in \mathbb{R}^2 \ | \ \ \pi - \sqrt{5\pi^2 - y^2} \le x \le -3\pi + \sqrt{5\pi^2 - y^2}, \quad -\pi \le y \le \pi \}$$

$$E_3 := E_1 + 2\pi(0,1), \qquad\qquad E_4 := E_2 + 2\pi(1,0).$$

We see easily that each E_i is a region whose boundary is two circular arcs through two vertexes of a side of the square $[-\pi, \pi]^2$. By (5.1) and (5.2), we know that $(\frac{S}{2})_{ker}$ is a simply connected region and $S + 4\pi\mathbb{Z}^2 = \mathbb{R}^2$. By the definition of S, we know that

$$\frac{S}{4} \subset [\frac{(1-\sqrt{5})\pi}{2}, \ \frac{(\sqrt{5}-1)\pi}{2}]^2 \subset [(\sqrt{5}-3)\pi, \ (3-\sqrt{5})\pi]^2 \subset (\frac{S}{2})_{ker}. \tag{5.3}$$

By Theorem 4.4, we know that there exists an orthonormal wavelet Ψ with the global frequency domain $\Omega = S \setminus (\frac{S}{2})_{ker}$. Now we discuss how to construct such a wavelet Ψ.

(ii) Construct a scaling function φ such that $\operatorname{supp}\widehat{\varphi} = \frac{S}{2}$.

Define $\nu(x,y)$ as a harmonic function satisfying $\frac{\partial^2\nu}{\partial x^2} + \frac{\partial^2\nu}{\partial y^2} = 0$ $((x,y) \in E_1^o)$ with boundary conditions

$$\begin{aligned} &\nu(x,y) = 0, \qquad -\pi \le x \le \pi, \quad y = -3\pi + \sqrt{5\pi^2 - x^2}, \\ &\nu(x,y) = \frac{\pi}{2}, \qquad -\pi \le x \le \pi, \quad y = \pi - \sqrt{5\pi^2 - x^2}. \end{aligned} \tag{5.4}$$

By (5.2), we have $\mathbb{R}^2 = (\frac{S}{2})_{ker} \bigcup \left(\bigcup_{\nu=1}^4 E_\nu \right) \bigcup \left(\mathbb{R}^2 \setminus \frac{S}{2} \right)$.

Define φ such that

$$\widehat{\varphi}(\omega) = 1 \quad (\omega \in (\frac{S}{2})_{ker}), \qquad \widehat{\varphi}(\omega) = 0 \quad (\omega \in \mathbb{R}^2 \setminus \frac{S}{2}), \tag{5.5}$$

and

$$\widehat{\varphi}(\omega) = \begin{cases} \cos\nu(x,\ y), & \omega \in E_1, \\ \cos\nu(y,\ x), & \omega \in E_2, \\ \sin\nu(x,\ y-2\pi), & \omega \in E_3, \\ \sin\nu(y,\ x-2\pi), & \omega \in E_4, \end{cases} \qquad \text{where } \omega = (x,\ y). \tag{5.6}$$

Here $\nu(x,y)$ has been defined on E_1. By the definition, we know that for $(x,\ y) \in E_2$, we have $(y,\ x) \in E_1$; for $(x,\ y) \in E_3$, we have $(x,\ y-2\pi) \in E_1$; for $(x,\ y) \in E_4$, we have $(x-2\pi,\ y) \in E_2$, so $(y,\ x-2\pi) \in E_1$. Hence $\varphi(x,\ y)$ is well-defined on $\bigcup_{\nu=1}^4 E_\nu$. From this and (5.4)-(5.6), we know that $\sum_{k\in\mathbb{Z}^2} |\widehat{\varphi}(\omega + 2\pi k)|^2 = 1,\ \ \omega \in \mathbb{R}^2$ and $\widehat{\varphi}(\omega)$ is a continuous function in $\mathbb{R}^2$ except four points $(-\pi,\ -\pi),\ (-\pi,\ \pi),\ (\pi,\ -\pi)$, and $(\pi,\ \pi)$.

Define $H_0(\omega)$ as

$$H_0(\omega) = \begin{cases} 0, & \omega \in \frac{S}{2} \setminus \frac{S}{4}, \\ \widehat{\varphi}(2\omega), & \omega \in \frac{S}{4}, \end{cases} \qquad H_0(\omega + 2\pi\nu) = H_0(\omega) \quad (\omega \in \mathbb{R}^2,\ \ \nu \in \mathbb{Z}^2). \tag{5.7}$$

Since $\frac{S}{4} \subset [-\pi,\pi]^2 \subset \frac{S}{2}$, H_0 is well-defined.

By (5.4), we have $\frac{S}{4} \subset (\frac{S}{2})_{ker}$. Again by (5.5), we have $\widehat{\varphi}(\omega) = 1$ $(\omega \in \frac{S}{2})$. From this and (5.7), we obtain that $\widehat{\varphi}(2\omega) = H_0(\omega)\widehat{\varphi}(\omega),\ \ \omega \in \mathbb{R}^2$ and H_0 is a bounded, $2\pi\mathbb{Z}^2-$periodic function. Again, since $\widehat{\varphi}$ is continuous at $\omega = 0$ and $\widehat{\varphi}(0) = 1$, by a known result [3], we know that φ is a scaling function. Since $\frac{S}{2} = (\frac{S}{2})_{ker} \bigcup (\bigcup_{\nu=1}^4 E_\nu)$, by (5.5) and (5.6), we have $\operatorname{supp}\widehat{\varphi} = \frac{S}{2}$.

(iii) Construct an orthonormal wavelet Ψ with the global frequency domain $\Omega = S \setminus (\frac{S}{2})_{ker}$.

Let the scaling function φ and the filter H_0 be stated in (5.5)-(5.7). By a known result [7], we obtain a corresponding orthonormal wavelet $\Psi = \{\psi_\mu\}_1^3$ satisfying

$$\widehat{\psi}_\mu(\omega) = H_\mu\left(\frac{\omega}{2}\right)\widehat{\varphi}\left(\frac{\omega}{2}\right) \quad (\mu = 1, 2, 3), \tag{5.8}$$

where
$$H_1(\omega) = e^{i\omega_2} H_0(\omega + \pi(0, 1)),$$

$$H_2(\omega) = e^{i(\omega_1+\omega_2)} H_0(\omega + \pi(1, 0)),$$

$$H_3(\omega) = e^{i\omega_1} H_0(\omega + \pi(1, 1)) \quad (\omega = (\omega_1,\ \omega_2) \in \mathbb{R}^2).$$

Since $\operatorname{supp}\widehat{\varphi} = \frac{S}{2}$ and $\operatorname{supp} H_0 = \frac{S}{4} + 2\pi\mathbb{Z}^2$, by (5.1) and (5.8), we obtain

$$\operatorname{supp} H_1 = \frac{S}{4} - \pi(0, 1) + 2\pi\mathbb{Z}^2$$

$$\operatorname{supp}\widehat{\psi}_1 = 2\left(\operatorname{supp} H_1 \bigcap \operatorname{supp}\widehat{\varphi}\right) = \left(\frac{S}{2} - 2\pi(0, 1) + 4\pi\mathbb{Z}^2\right)\bigcap S = A\bigcap S,$$

where $A = \left(\,(\frac{S}{2} + 2\pi(0, -1))\bigcup(\frac{S}{2} + 2\pi(0, 1))\,\right)$.

Similarly, we have $\operatorname{supp}\widehat{\psi}_2 = B\bigcap S$ and $\operatorname{supp}\widehat{\psi}_3 = C\bigcap S$, where

$$B = (\frac{S}{2} + 2\pi(-1, 0))\bigcup(\frac{S}{2} + 2\pi(1, 0)),$$

$$C = (\frac{S}{2} + 2\pi(-1, -1))\bigcup(\frac{S}{2} + 2\pi(1, 1))\bigcup(\frac{S}{2} + 2\pi(-1, 1))\bigcup(\frac{S}{2} + 2\pi(1, -1)).$$

Hence the global frequency domain of Ψ is

$$\Omega = \bigcup_{\mu=1}^{3} \operatorname{supp}\widehat{\psi}_\mu = (A\bigcup B\bigcup C)\bigcap S. \tag{5.9}$$

Since $\frac{S}{2} \supset [-\pi,\ \pi]^2$, we have $(A\bigcup B\bigcup C) \supset (\,[-3\pi,\ 3\pi]^2 \setminus (-\pi,\ \pi)^2\,)$. Again, by $(A\bigcup B) \supset (\,\bigcup_{\nu=1}^{4} E_\nu\,)$ and $S \subset [-3\pi,\ 3\pi]^2$, we obtain $(\,(A\bigcup B\bigcup C)\bigcap S\,) \supset (\,(S \setminus (-\pi,\ \pi)^2)\bigcup(\,\bigcup_{\nu=1}^{4} E_\nu\,)\,)$. Again, by (5.2) and $S \supset \frac{S^o}{2} \supset (-\pi,\ \pi)^2 \supset (\frac{S}{2})_{ker}$, we have

$$\Omega = \left((A\bigcup B\bigcup C)\bigcap S\right) \supset \left((S \setminus (-\pi,\ \pi)^2)\bigcup(\frac{S^o}{2} \setminus (\frac{S}{2})_{ker})\right) \supset S \setminus (\frac{S}{2})_{ker}. \tag{5.10}$$

On the other hand, since $\Omega \subset S$ and $(A \bigcup B \bigcup C) \bigcap (\frac{S}{2})_{ker} \simeq \emptyset$, we have $\Omega \subset S \setminus (\frac{S}{2})_{ker}$. From this, by (5.9) and (5.10), we get $\Omega \simeq S \setminus (\frac{S}{2})_{ker}$.

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