On the Localization Behavior of Graph Laplacian Eigenfunctions

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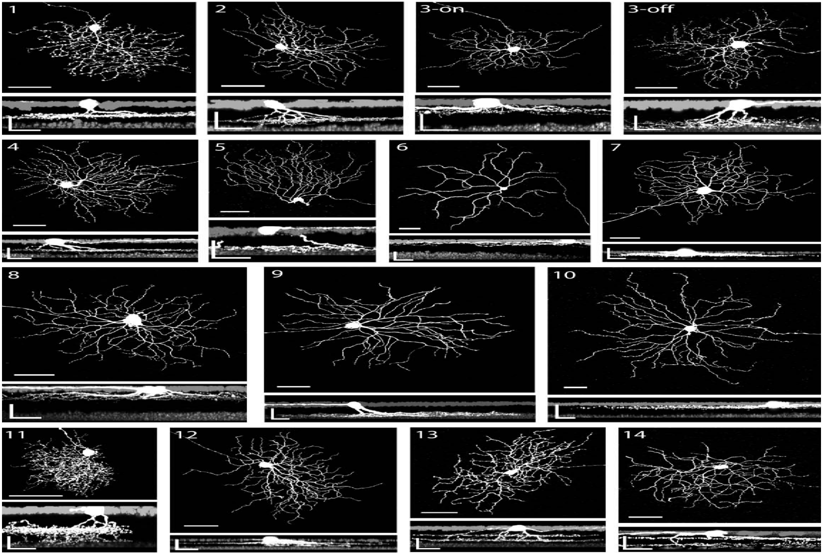
Outline

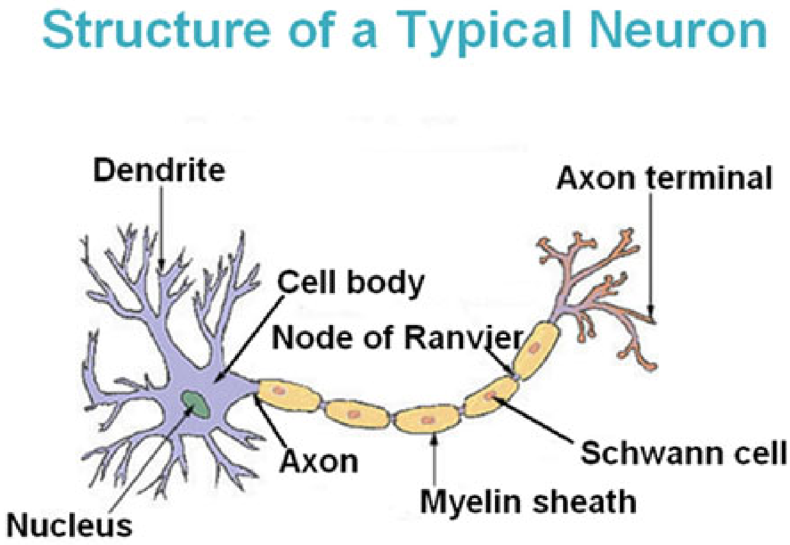
1. Motivations
2. Our Dataset
3. Graph Laplacians
4. Analysis of Starlike Trees
5. Our Conjecture
6. Conclusions & Future Plans
7. References/Acknowledgment

Outline

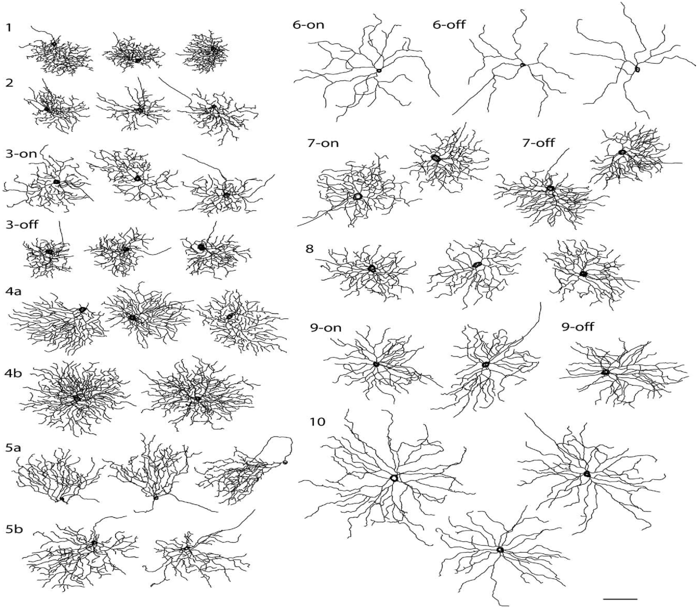
Clustering Mouse Retinal Ganglion Cells . . . 3D Data



A Typical Neuron (from Wikipedia)



Clustering using Features Derived by Neurolucida®



Motivations

We observed an interesting phase-transition or thresholding phenomenon on the behavior of the eigenvalues and eigenfunctions of graph Laplacians defined on trees constructed from actual neuronal dendrites.

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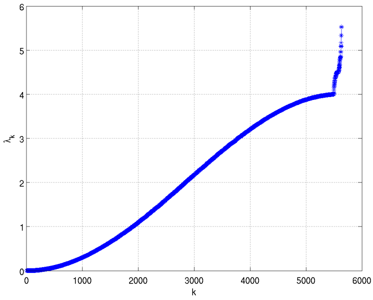

(a) RGC #60

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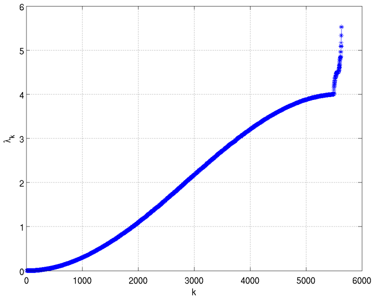

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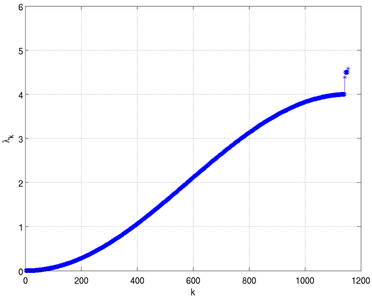

(b) RGC #100

Motivations . . .

We have observed that this value 4 is critical since:

- the eigenfunctions corresponding to the eigenvalues below 4 are semi-global oscillations (like Fourier cosines/sines) over the entire dendrites or one of the dendrite arbors;
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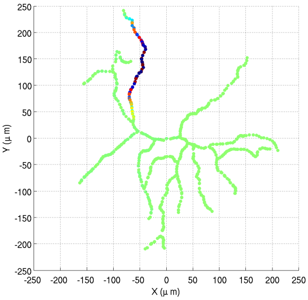

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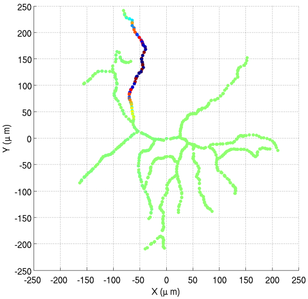

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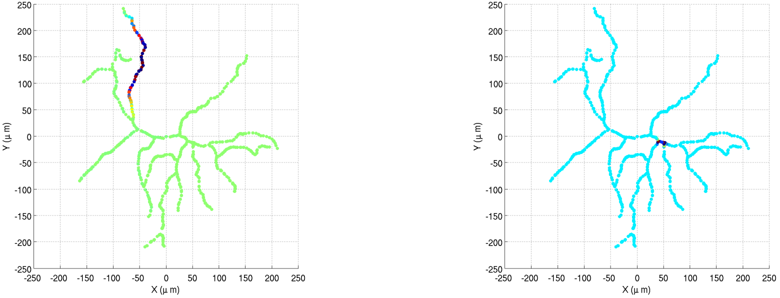
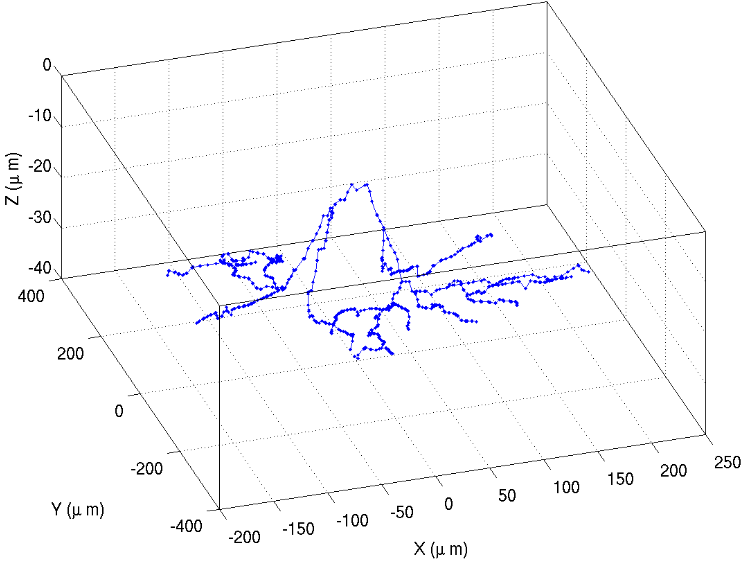


(a) RGC #100; $\lambda_{1141} = 3.9994$ (b) RGC #100; $\lambda_{1142} = 4.3829$

Outline

Our Dataset ⟹ Trees



Outline

Graph Laplacians

The Laplacian matrix (often called the combinatorial Laplacian matrix) of a graph $G = (V, E)$ is defined as

$$
\begin{aligned}
L(G) &:= D(G) - A(G) \\
D(G) &:= \operatorname{diag}(d_{v_1}, \ldots, d_{v_n}) \quad \text{the degree matrix} \\
A(G) &= (a_{ij}) \quad \text{the adjacency matrix where} \\
a_{ij} &:= \begin{cases} 1 & \text{if } v_i \sim v_j; \\ 0 & \text{otherwise.} \end{cases}
\end{aligned}
$$

Let $f \in L^2(V)$. Then

$$L(G)f(u) = d_u f(u) - \sum_{v \sim u} f(v),$$

i.e., this is a generalization of the finite difference approximation to the Laplace operator.

Graph Laplacians . . .

- Let $|V(G)| = n$, and let $0 = \lambda_0(G) \le \lambda_1(G) \le \cdots \le \lambda_{n-1}(G)$ be the sorted eigenvalues of $L(G)$.
- $m_G(\lambda) :=$ the multiplicity of λ.
- Let $I \subset \mathbb{R}$ be an interval of the real line. Then define $m_G(I) := \#\{\lambda_k(G) \in I\}$.
- A vertex of degree 1 is called a pendant vertex; a vertex adjacent to a pendant vertex is called pendant neighbor.
- Let $p(G)$ and $q(G)$ be the number of pendant vertices and that of pendant neighbors, respectively.

A Simple Yet Important Example: A Path Graph

$$L(G) = D(G) - A(G)$$

$$\begin{bmatrix} 1 & -1 & & & & \\ -1 & 2 & -1 & & & \\ & -1 & 2 & -1 & & \\ & & \ddots & \ddots & \ddots & \\ & & & -1 & 2 & -1 \\ & & & & -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & & & & & \\ & 2 & & & & \\ & & 2 & & & \\ & & & \ddots & & \\ & & & & 2 & \\ & & & & & 1 \end{bmatrix} - \begin{bmatrix} 0 & 1 & & & & \\ 1 & 0 & 1 & & & \\ & 1 & 0 & 1 & & \\ & & \ddots & \ddots & \ddots & \\ & & & 1 & 0 & 1 \\ & & & & 1 & 0 \end{bmatrix}$$

The eigenvectors of this matrix are exactly the DCT Type II basis vectors used for the JPEG image compression standard! (See e.g., Strang, SIAM Review, 1999).

- $\lambda_k = 2 - 2\cos(\pi k/n)$, $k = 0, 1, \ldots, n-1$.
- $\phi_k = \left(\cos(\pi k(\ell + \frac{1}{2})/n)\right)_{0 \le \ell < n}$, $k = 0, 1, \ldots, n-1$.

Outline

A Starlike Tree

is a tree where there is only one vertex whose degree is larger than 2.

- Let $S(n_1, n_2, \ldots, n_k)$ be a starlike tree that has $k(\geq 3)$ paths (i.e., branches) emanating from the center vertex v_1.
- Let the ith branch have n_i vertices excluding v_1.
- Let $n_1 \geq n_2 \geq \cdots \geq n_k$.
- The total number of vertices: $n = 1 + \sum_{i=1}^{k} n_i$.

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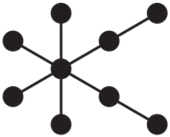

(a) $S(2, 2, 1, 1, 1, 1)$

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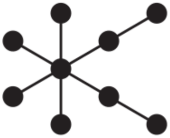

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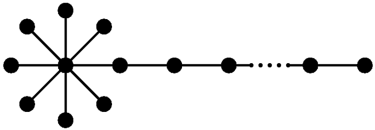


(b) $S(n_1, 1, 1, 1, 1, 1, 1, 1)$ a.k.a. comet

Known Results on Starlike Trees

- We proved (in 2010) the largest eigenvalue for a comet is always larger than 4.
- K. Ch. Das (2007) proved the following results.
  - $\lambda_{\max} = \lambda_{n-1} < k + 1 + \dfrac{1}{k-1}$
  - $2 + 2\cos\left(\dfrac{2\pi}{2n_k + 1}\right) \le \lambda_{n-2} \le 2 + 2\cos\left(\dfrac{2\pi}{2n_1 + 1}\right)$
- On the other hand, Grone and Merris (1994) proved the following lower bound for a general graph G with at least one edge:

$$\lambda_{\max} \ge \max_{1 \le j \le n} d(v_j) + 1.$$

- Hence if G is a starlike tree, the threshold phenomenon with the value 4 is completely explained.

Outline

Our Observation via Numerical Experiments

- Unfortunately, actual dendrite trees are not starlike.
- However, our numerical computations and data analysis indicate that:

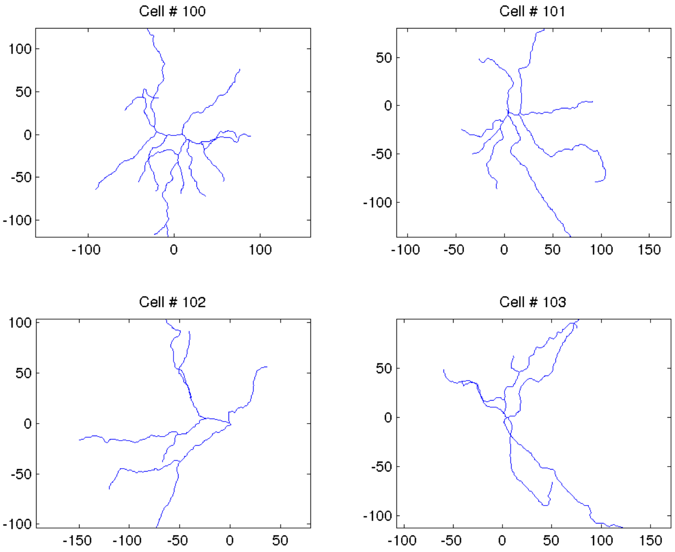
$$0 \leq \frac{\#\{j \in (1, n) \mid d(v_j) \gneqq 2\} - m_G([4, \infty))}{n} \leq 0.047$$

  for each cell where $n = |V(G)|$.
- We can define the starlikeliness $S\ell(T)$ of a given tree $G = T$ as follows:

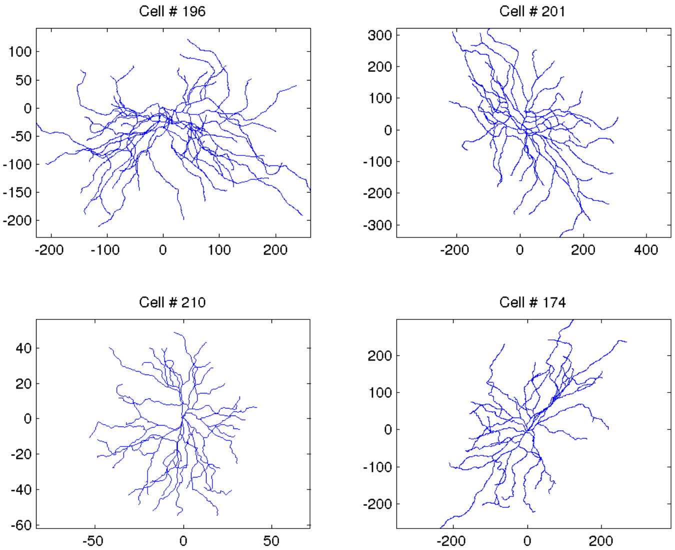
$$S\ell(T) := 1 - \frac{\#\{j \in (1, n) \mid d(v_j) \gneqq 2\} - m_T([4, \infty))}{n}.$$

- We found $S\ell(T) \equiv 1$ for all the dendrites in Cluster 6.

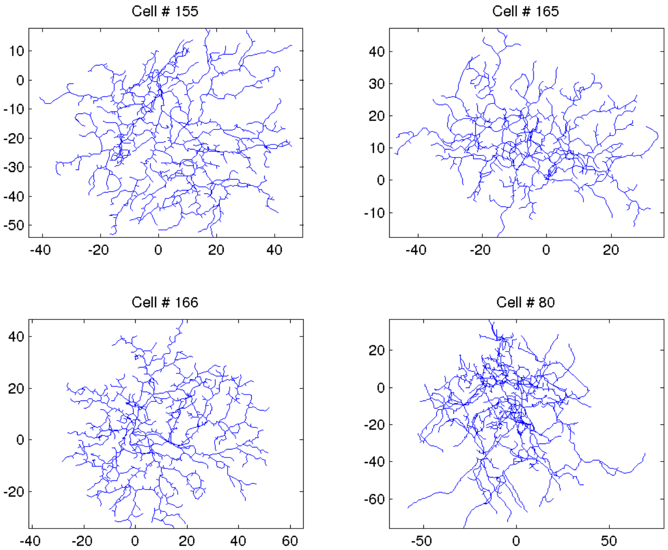
Dendrites with $S\ell(T) = 1$



More dendrites with $S\ell(T) = 1$



Dendrites with $S\ell(T) \lneqq 1$



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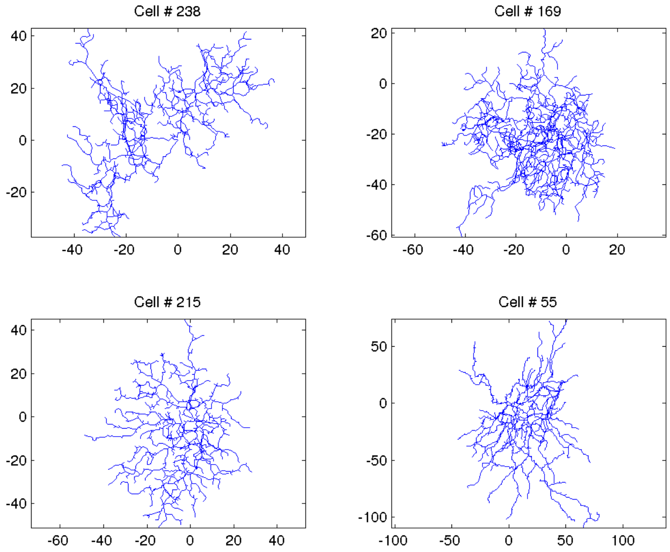
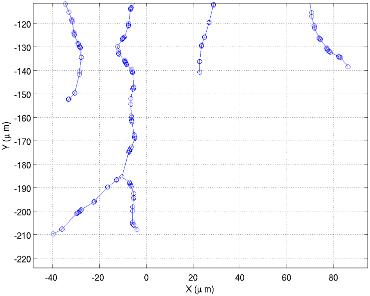
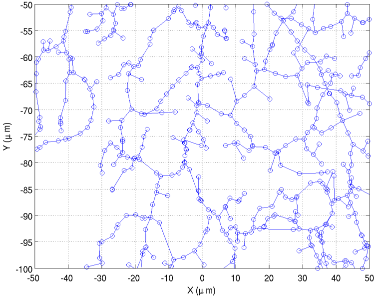


Zoom up



(a) RGC #100; $S\ell(T) = 1$



(b) RGC #155; $S\ell(T) = 0.953 \lneqq 1$

Our Conjecture and Questions

Conjecture

For any tree T of finite volume, we have

$$0 \leq m_T([4, \infty)) \leq \#\{j \in (1, n) \,|\, d(v_j) \gneqq 2\}$$

and each eigenfunction corresponding to $\lambda \geq 4$ has its largest component (in the absolute value) on the vertices whose degree are larger than 2.

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- If so, what kind of trees are they?
- How about simple connected graphs instead of trees?

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Questions

- Is there any tree whose graph Laplacian has an exact eigenvalue 4?
- If so, what kind of trees are they?
- How about simple connected graphs instead of trees? $\Longrightarrow$ Repeated eigenvalues 4 exist for regular lattice graphs with $d > 1$. But the corresponding eigenfunctions are not localized.

Our Conjecture and Questions . . .

- It turned out that Guo proved the following theorem:

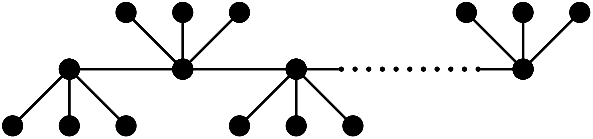
Theorem (Guo 2006)

Let T be a tree with n vertices. Then,

$$\lambda_j(T) \leq \left\lceil \frac{n}{n-j} \right\rceil, \quad j = 0, \ldots, n-1,$$

and the equality holds iff a) $j \neq 0$; b) $n-j$ divides n; and c) T is spanned by $n-j$ vertex disjoint copies of $K_{1,\frac{j}{n-j}}$.

- This implies that if $n = 4m$, there is an eigenvalue exactly equal to 4 at $j = 3m$, i.e., $\lambda_{3m} = 4$, and this tree consists of m copies of $K_{1,3}$:



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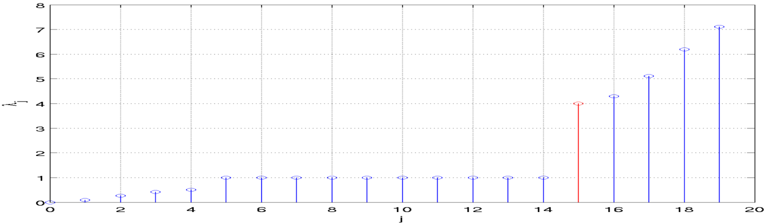
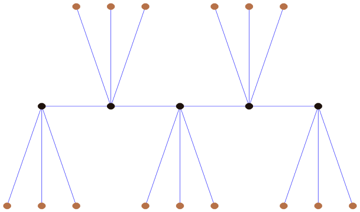
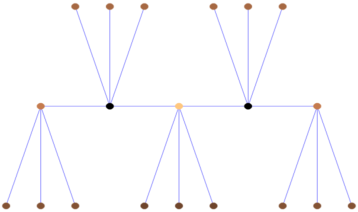


(a) $\{\lambda_j\}_{j=0}^{19}; S\ell(T) = 1$



(b) ϕ_{15}



(c) ϕ_{19}

Outline

Conclusions & Future Plans

- Observed a global-to-local phase transition phenomenon of the eigenvalues and eigenfunctions of such dendrite patterns $\Longrightarrow$ leads to a theorem?
- Further investigate the meaning of the eigenvalue 4.
- How about the weighted graph Laplacians?

Outline

References

- Laplacian Eigenfunction Resource Page http://www.math.ucdavis.edu/~saito/lapeig/ contains
  - All the talk slides of the previous minisymposia "Laplacian Eigenfunctions and Their Applications, " which Mauro Maggioni, Xiaoming Huo, and I organized for ICIAM 2007 (Zürich) and SIAM Imaging Conference 2008 (San Diego); and
  - A Link to the recent workshop on "Laplacian eigenvalues and eigenfunctions: Theory, application, computation," Feb. 2009, at Institute for Pure and Applied Mathematics (IPAM), UCLA
  - My Course Note (elementary) on "Laplacian Eigenfunctions: Theory, Applications, and Computations"
- The following articles are available at http://www.math.ucdavis.edu/~saito/publications/

  - N. Saito and E. Woei: "Analysis of neuronal dendrite patterns using eigenvalues of graph Laplacian," *Japan SIAM Letter*, vol. 1, pp. 13–16, 2009, Invited paper.
  - N. Saito: "Data analysis and representation using eigenfunctions of Laplacian on a general domain," *Applied & Computational Harmonic Analysis*, vol. 25, no. 1, pp. 68–97, 2008.

Acknowledgment

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Thank you very much for your attention!