Polyharmonic Local Cosine Transform for Improving the Reproduction Quality of JPEG-Compressed Images

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Motivations

- Want to improve the quality of images (e.g., less blocking artifacts/visible discontinuities between blocks) reconstructed from the low bit rate JPEG files.

(a) Original: 8 bpp
Motivations

- Want to improve the quality of images (e.g., less blocking artifacts/visible discontinuities between blocks) reconstructed from the low bit rate JPEG files.

(a) Original: 8 bpp

(b) JPEG: 0.162 bpp
Motivations . . .

- Want to develop a local image transform that generates faster decaying expansion coefficients than block DCT used in JPEG and our previous Polyharmonic Local Sine Transform (PHLST) because the faster coefficient decay $\implies$ more efficient compression.
- Want to fully incorporate the *infrastructure* provided by the JPEG standard, e.g., the block DCT algorithm, the quantization method, the file format, etc.
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Review of Fourier Cosine Series

- Let $\Omega = (0, 1)^2 \subset \mathbb{R}^2$ and $f \in C(\overline{\Omega})$ but not periodic: the periodically extended version of $f$ is discontinuous at $\partial \Omega$.

- Then the size of the complex Fourier coefficients $c_k$ of $f$ decay as $O(\|k\|^{-1})$, where $k = (k_1, k_2) \in \mathbb{Z}^2$.

- Instead, expanding $f$ into the Fourier cosine series gives rise to the decay rate $O(\|k\|^{-2})$ because it is equivalent to the complex Fourier series expansion of the extended version of $f$ via even reflection that is continuous at $\partial \Omega$.

- This is one of the main reasons why the JPEG Baseline method adopts Discrete Cosine Transform (DCT) instead of Discrete Fourier Transform (DFT).
We now consider a decomposition \( f = u + v \).

The \( u \) (or polyharmonic) component satisfies Laplace's equation with the Dirichlet boundary condition.

\[
\Delta u = 0 \quad \text{in } \Omega; \quad u = f \quad \text{on } \partial \Omega.
\]

The \( u \) component is solely represented by the boundary values of \( f \) via the fast and highly accurate Dirichlet problem solver of Averbuch, Israeli, & Vozovoi (1998).

The residual \( v = f - u \) vanishes on \( \partial \Omega \) \( \implies \) The Fourier sine coefficients of \( v \) decay as \( O(\|k\|^{-3}) \) for \( v \in C^1(\overline{\Omega}) \).

Review of Polyharmonic Local Sine Transform . . .

Original Signal Supported on [0,1]

After Periodization

After Even Reflection

After Lin Removal+Odd Reflect

DFT Coefficients

DCT Coefficients

LLST Coefficients

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PHLCT Compression

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Want to use DCT for fully utilizing the JPEG infrastructure.
Want coefficients decaying faster than $O(||k||^{-3})$.
To do so, we need to solve Poisson’s equation with the Neumann boundary condition:

$$
\Delta u = K \quad \text{in } \Omega; \quad \partial_\nu u = \partial_\nu f \quad \text{on } \partial \Omega,
$$

where the constant source term $K$ ($=\text{the integration of } \partial_\nu f \text{ along } \partial \Omega \text{ normalized by the area of } \Omega$) is necessary for solvability of the Neumann problem.

Then, the Fourier cosine coefficients of the residual decay as $O(||k||^{-4})$ for $\nu \in C^2(\overline{\Omega})$ because $\partial_\nu \nu = 0 \text{ on } \partial \Omega$. 

Green’s second identity claims that for any \( u, v \in C^1(\overline{\Omega}) \),

\[
\int_{\Omega} (u \Delta v - v \Delta u) \, dx = \int_{\partial \Omega} (u \partial_{\nu} v - v \partial_{\nu} u) \, d\sigma(x),
\]

where \( d\sigma(x) \) is a surface (or boundary) measure. Setting \( v = 1 \) with the Neumann boundary condition, we have

\[
\int_{\Omega} \Delta u \, dx = \int_{\partial \Omega} \partial_{\nu} u \, d\sigma(x) = \int_{\partial \Omega} \partial_{\nu} f \, d\sigma(x).
\]

This is a necessary condition that \( u \) must satisfy. Now, the source term of Poisson’s equation is \( K := \frac{1}{|\Omega|} \int_{\partial \Omega} \partial_{\nu} f \, d\sigma(x) \), where \( |\Omega| \) is the volume of the block \( \Omega \).
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Want to achieve the PHLCT representation of $f = u + v$ entirely in the DCT domain, $F = U + V$.

Let $f(x, y) \in C(\Omega)$, and $f_{i,j}$ be a sample $f(x_i, y_j)$ with $x_i = (i + 0.5)/N$, $y_j = (j + 0.5)/N$, $i, j = 0, 1, \ldots, N - 1$.

Let $F \in \mathbb{R}^{N \times N}$ be a DCT coefficient matrix of $\{f_{i,j}\}$:

$$F_{k_1, k_2} := \lambda_{k_2} \sqrt{\frac{2}{N}} \sum_{j=0}^{N-1} \left( \lambda_{k_1} \sqrt{\frac{2}{N}} \sum_{i=0}^{N-1} f(x_i, y_j) \cos \pi k_1 x_i \right) \cos \pi k_2 y_j$$

where $\lambda_0 = 1/\sqrt{2}$, $\lambda_k = 1$ for all $k \geq 1$.

Now let’s compute the DCT coefficient matrix $U$ of the polyharmonic component $u$ using $F$. 
Assume for the moment that the discretized Neumann boundary data at each edge of $\overline{\Omega} = [0, 1]^2$ are available:

$$g_i^{(1)} := -f_y(x_i, 0), \quad g_i^{(2)} := f_y(x_i, 1), \quad g_j^{(3)} := -f_x(0, y_j), \quad g_j^{(4)} := f_x(1, y_j).$$
Let \( \{ G_k^{(l)} \} \) be the 1D-DCT coefficients of \( \{ g_i^{(l)} \} \).

Then, we have a solution to Poisson’s equation as (see Yamatani-Saito for details):

\[
\begin{align*}
    u(x, y) &= \sqrt{\frac{2}{N}} \sum_{k=0}^{N-1} \lambda_k \left\{ \left( G_k^{(1)} \psi_k(y - 1) + G_k^{(2)} \psi_k(y) \right) \cos \pi kx ight. \\
    &\quad + \left( G_k^{(3)} \psi_k(x - 1) + G_k^{(4)} \psi_k(x) \right) \cos \pi ky \left. \right\} + c,
\end{align*}
\]

where \( c \) is a constant to be determined and

\[
\psi_k(t) := \begin{cases} 
    t^2/2 & \text{if } k = 0; \\
    (\cosh \pi kt)/(\pi k \sinh \pi k) & \text{otherwise}.
\end{cases}
\]
Applying 2D DCT to \( u \) above, we obtain \( U = (U_{k_1,k_2}) \) as

\[
U_{k_1,k_2} = G_{k_1}^{(1)} \eta_{k_1,k_2} + G_{k_1}^{(2)} \eta_{k_1,k_2}^* + G_{k_2}^{(3)} \eta_{k_2,k_1} + G_{k_2}^{(4)} \eta_{k_2,k_1}^* ,
\]

where \( \eta_{k_1,k_2}, \eta_{k_1,k_2}^* \) are independent from the input image:

\[
\eta_{k,m} := \lambda_m \sqrt{\frac{2}{N}} \sum_{i=0}^{N-1} \psi_k(x_i - 1) \cos \pi m x_i ,
\]

\[
\eta_{k,m}^* := \lambda_m \sqrt{\frac{2}{N}} \sum_{i=0}^{N-1} \psi_k(x_i) \cos \pi m x_i ,
\]

Can set the DC component \( U_{0,0} \equiv 0 \) because the solution to the Poisson-Neumann problem is unique modulo an additive constant. In fact this is achieved by \( c = -\frac{4N^2-1}{24N^2.5} \left( G_0^{(1)} + G_0^{(2)} + G_0^{(3)} + G_0^{(4)} \right) \). This will become important in our algorithms.
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In practice, we need to estimate the Neumann boundary data \( \{g^{(\ell)}_i\} \) from the image samples of the current and adjacent blocks. Let 
\[ f_{i,j}^{(s,t)} = f(x_i + s, y_j + t) \] 
and \( \Omega^{(s,t)} \) be:

Let \( \mathcal{I}_5 := \{(0, -1), (-1, 0), (0, 0), (1, 0), (0, 1)\} \) be the indices of the current and adjacent blocks.
Approximation of the Neumann Boundary Data . . .

- Approximate \{g_i^{(\ell)}\} using column & row averages:

\[ g_i^{(1)} \approx X_i^{(-1)} - X_i^{(0)}; \quad g_i^{(2)} \approx X_i^{(1)} - X_i^{(0)}; \quad g_j^{(3)} \approx Y_j^{(-1)} - Y_j^{(0)}; \quad g_j^{(4)} \approx Y_j^{(1)} - Y_j^{(0)} \]

\[ X_i^{(t)} := \frac{1}{N} \sum_{j=0}^{N-1} f_{i,j}^{(0,t)}, \quad Y_j^{(s)} := \frac{1}{N} \sum_{i=0}^{N-1} f_{i,j}^{(s,0)}, \]

- Then, \{G_k^{(\ell)}\} can be expressed using the first row & column of \( F^{(s,t)} \).

Consequently, for \((k_1, k_2) \neq (0, 0)\), we have

\[ U_{k_1,k_2} = \frac{1}{\sqrt{N}} \left\{ \left( F_{k_1,0}^{(0,-1)} - F_{k_1,0} \right) \eta_{k_1,k_2} + \left( F_{k_1,0}^{(0,1)} - F_{k_1,0} \right) \eta_{k_1,k_2}^* 
\right. \\
+ \left. \left( F_{0,k_2}^{(-1,0)} - F_{0,k_2} \right) \eta_{k_2,k_1} + \left( F_{0,k_2}^{(1,0)} - F_{0,k_2} \right) \eta_{k_2,k_1}^* \right\}. \quad (1) \]
Approximation of the Neumann Boundary Data...
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Modifying PHLCT for Practice

Approximating Eq. (1) only using the DC components $F_{0,0}$ and $F_{0,0}^{(s,t)}$ allows us to simplify our algorithms: $U_{k_1,k_2} =$

\[
\begin{align*}
&\left\{ \begin{array}{ll}
0 & \text{ if } k_1 = k_2 = 0; \\
\frac{1}{\sqrt{N}} \left\{ \left( F_{0,0}^{(-1,0)} - F_{0,0} \right) \eta_{0,k_1} + \left( F_{0,0}^{(1,0)} - F_{0,0} \right) \eta_{0,k_1}^* \right\} & \text{ if } k_1 \neq 0 = k_2; \\
\frac{1}{\sqrt{N}} \left\{ \left( F_{0,0}^{(0,-1)} - F_{0,0} \right) \eta_{0,k_2} + \left( F_{0,0}^{(0,1)} - F_{0,0} \right) \eta_{0,k_2}^* \right\} & \text{ if } k_1 = 0 \neq k_2; \\
U_{k_1,k_2} & \text{ as Eq. (1)} \\
\end{array} \right. \\
\end{align*}
\]

Now set $V_{k_1,k_2} = F_{k_1,k_2} - U_{k_1,k_2}, \ \forall k_1, k_2$. Note $V_{0,0} = F_{0,0}!!$

Note also that if we know $V$ of the current and adjacent blocks, we can reconstruct $F$. **No need to store $U$!** See next page.

Strictly speaking, this new version of $u$ does not satisfy Poisson’s equation, but satisfies the Neumann condition.
Inverse PHLCT

1. Assuming $V^{(s,t)}$, $(s, t) \in \mathcal{I}_5$, are available, recover the first column and row of $U$ using the DC components, $F_{0,0}^{(s,t)} = V_{0,0}^{(s,t)}$, $(s, t) \in \mathcal{I}_5$ via (2);

2. Recover the first column and row of $F^{(s,t)}$, $(s, t) \in \mathcal{I}_5$ by summing those of $U$ and $V$ (see (2));

3. Recover other entries of $U$ via (1) and the results of Step 2;

4. Set $F = U + V$;

5. Apply Inverse 2D DCT to $F$ to recover $f$. 
Inverse PHLCT: Step 0

(a) $F$

(b) $U$

(c) $V$
Inverse PHLCT: Step 1

(a) $F$

(b) $U$

(c) $V$
Inverse PHLCT: Step 2

(a) $F$

(b) $U$

(c) $V$
Inverse PHLCT: Step 3

(a) $F$

(b) $U$

(c) $V$
Inverse PHLCT: Step 4

(a) $F$

(b) $U$

(c) $V$
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FPHLCT adds simple procedures in both the encoder and the decoder parts of the JPEG Baseline method.

In the encoder part, the only difference from JPEG is to: 1) compute \( U \) from \( F \); and 2) compute the residual \( V = F - U \) and store the quantized version \( V^Q \) instead of \( F^Q \).

In the decoder part, the only difference from JPEG is to: 1) compute \( U^Q \), the estimate of \( U \) from \( V^Q \); and 2) compute \( U^Q + V^Q \) as an improved estimate of \( F \) over \( F^Q \).

Because \( V \) decays faster than \( F \), the decompressed image quality gets better than JPEG if it is compressed at the same bit rate.
Partial Mode PHLCT (PPHLCT)

- Only the decoder part of the JPEG Baseline method is modified: PPHLCT accepts the JPEG-compressed files.
- The JPEG encoder kills small DCT coefficients $F_{k'}$, i.e., $F_{k'}^Q = 0$.
- PPHLCT replaces those $F_{k'}^Q$ by $U_{k'}^Q$ if $U_{k'}^Q$ are also small.
- This is possible because $U^Q$ can be computed solely from the first column & row of $F^Q$ and those of the adjacent blocks $F^{(s,t)Q}$; see Eqs.(1), (2).
- Our reasoning to do this is $F_k \approx U_k$ for large $k$ because $V_k$ decays quickly.
- We also add some quadratic polynomial to reduce the blocking artifacts further. This can be done also in the DCT domain. (See Yamatani-Saito for the details.)
Numerical Experiments

Two test images.

(a) Barbara

(b) Gabor
Numerical Experiments . . .

(a) JPEG, 23.61 dB  
(b) FPHLCT, 24.19 dB  
(c) PPHLCT, 23.97 dB

Compressed at 0.15 bits/pixel. Numerical values indicate the Peak Signal-to-Noise Ratio (PSNR).
Numerical Experiments . . .

(a) JPEG, 25.67 dB
(b) FPHLCT, 26.05 dB
(c) PPHLCT, 25.73 dB

Compressed at 0.30 bits/pixel. Numerical values indicate the Peak Signal-to-Noise Ratio (PSNR).
Comparison of PSNR gain by various methods for the Barbara image over the JPEG Baseline method.
Numerical Experiments . . .

(a) JPEG, 31.41 dB
(b) FPHLCT, 39.21 dB
(c) PPHLCT, 35.69 dB

Compressed at 0.15 bits/pixel. Numerical values indicate the Peak Signal-to-Noise Ratio (PSNR).
Numerical Experiments . . .

(a) JPEG, 38.12 dB  
(b) FPHLCT, 47.02 dB  
(c) PPHLCT, 40.89 dB

Compressed at 0.30 bits/pixel. Numerical values indicate the Peak Signal-to-Noise Ratio (PSNR).
Comparison of PSNR gain by various methods for the Gabor image over the JPEG Baseline method.
More extensive numerical experiments (see Yamatani-Saito) indicate that FPHLCT reduces the bit rates about 15% over JPEG whereas PPHLCT does about 7% to achieve the same PSNR in the relatively low bit rate range.

If one can afford to use the higher bit rates, then our methods naturally approach to the performance of JPEG.

PPHLCT is particularly useful because it accepts the files compressed by the JPEG standard.

On the other hand, FPHLCT is better than PPHLCT if one can afford to modify the encoder part of the JPEG standard.

Additional computational cost of both methods over JPEG is small: linearly proportional to the number of pixels of an input image.

Should be useful for zooming, interpolation, feature extraction.
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Thank you very much for your attention!