MATH 22A: LINEAR ALGEBRA

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LINEAR ALGEBRA IS AWESOME!

A BEAUTIFUL part of MATHEMATICS, it is super USEFUL too!

- **Networks and Graphs**
  1. Electrical/Mechanical/Transportation networks
  2. World Wide Web Searching

- **Data Analysis**
  1. Least Squares and Interpolation
  2. Vector Recognition and Machine Learning

- **Information Processing**
  1. Error-correcting codes
  2. Data compression and Noise removal

- **Computer Graphics and Animation**
  1. Transformations of image, deformations
  2. Perspective views
MOTIVATION:

Abstractly, Linear Algebra studies **Hyperplanes** and **Vectors** in high-dimensional spaces.
A network or graph consists of nodes (or vertices) and edges (or connections).

Examples

- **street network** (nodes are crossings, edges represent streets)
- The **facebook graph** (nodes are people, edges represent friends)
- The **wikipedia graph** (nodes are concepts, edges represent relations via links)

Graphs and networks are great tools in mathematical research, electrical engineering, computer programming and networking, business administration, sociology, economics, marketing,
A graph is **directed** if its edges are directed (that means they have a specific direction). Oriented edges are called **arcs** or **arrows**.

A graph with numeric values on its edges or vertices is called a **network**. E.g., electrical networks

1. **Ohm's Law**: The voltage drop across a resistor is the product of the current and the resistance:
   \[ V = R \]

2. **Kirchhoff's first Law**: The sum of the currents flowing into a node is equal to the sum of the current flowing out.

3. **Kirchhoff's second Law**: The sum of the voltage drops around a closed loop is equal to the total voltage in the loop.
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Example In the network below

Applying Kirchhoff's first Law to either of the nodes B or C, we find

\[ I_1 = I_2 + I_3 \]

or

\[ I_1 - I_2 - I_3 = 0. \]

Applying Kirchhoff's second Law to the loops BDCB and BCAB, we obtain

\[ -10I_3 + 10I_2 = 10 \]

and

\[ 20I_1 + 10I_2 = 5. \]

This gives a linear system of equations:

\[ I_1 - I_2 - I_3 = 0, \]

\[ -10I_3 + 10I_2 = 10, \]

\[ 20I_1 + 10I_2 = 5. \]

whose solution gives the electric current in each channel.
Example In the network below

Applying Kirchhoff’s first Law to either of the nodes B or C, we find $I_1 = I_2 + I_3$ or $I_1 - I_2 - I_3 = 0$. 

Applying Kirchhoff’s second Law to the loops BDCB and BCAB, we obtain 

\[-10I_3 + 10I_2 = 10\] 
\[20I_1 + 10I_2 = 5\]

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\[I_1 - I_2 - I_3 = 0\]
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Applying Kirchhoff’s second Law to the loops \( BDCB \) and \( BCAB \), we obtain \(-10I_3 + 10I_2 = 10\) and \(20I_1 + 10I_2 = 5\).
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\[
\begin{align*}
I_1 - I_2 - I_3 &= 0, \\
-10I_3 + 10I_2 &= 10, \\
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\end{align*}
\]

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GOAL 1

SOLVE SYSTEMS OF LINEAR EQUATIONS FAST

Key words: Linear combination, Gaussian Elimination, Matrices, Matrix operations, Inverses
One of the important applications we will discuss is the method of regression or curve fitting. The process finds equations of approximating curves to a set of raw data. We desire to have a curve with minimal deviation or error from all data points.
Suppose that the data points are \((x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)\) where \(x\) is the independent variable and \(y\) is the dependent variable. The fitting curve \(f(x)\) has a deviation error for each point of \(d_i = y_i - f(x_i)\). We are looking for the curve \(f(x)\) that minimizes

\[
d_1^2 + d_2^2 + d_3^2 + \cdots + d_n^2
\]

The approximation by a polynomial \(f(x)\) will depend on the degree of the polynomial but as we will see this is done via a system of linear conditions. This is related to minimizing quadratic constraints over the points inside a linear space.

We will learn how and why this method works so well. (invented by Legendre and Gauss back in the 1800’s)
GOAL 2

HOW DO LEAST-SQUARES APPROXIMATIONS WORK??

Key words: Vector spaces and subspaces, Linear independence, Orthogonality, Projections, Gram-Schmidt Process
Suppose a taxi company serves Davis, Sacramento, and Woodland. Records of the company indicate:

- 10% of the customers taking a taxi in Davis go to Sacramento and 30% go to Woodland.
- 30% of customers taking a taxi in Sacramento go to Davis and 30% go to Woodland.
- 40% of customers taking a taxi in Woodland go to Davis and 30% go to Sacramento.

The company wants to know where will the taxis end up on average? We can study this using a matrix.

The entries $t_{ij}$ of the transition matrix represent the transitional probabilities that the system will switch from city $j$ to city $i$ (note the reversal of indices). For example:

$$T = \begin{bmatrix}
0.6 & 0.3 & 0.4 \\
0.1 & 0.4 & 0.3 \\
0.3 & 0.3 & 0.3
\end{bmatrix}$$
We start the month with initial proportions of taxis at each place \((u_1^0, u_2^0, u_3^0)\).

The equation \(u^{(k+1)} = Tu^{(k)}\) represents a chain of probability vectors.

If we repeat and repeat (taxis move and move), eventually, no matter how the taxis were initially distributed, 47% of taxis will be in Davis, 23% in Sacramento and 30% in Woodland.

This can be read from the eigenvectors and eigenvalues of \(T\).

So linear algebra can be used to predict dynamic behavior.
To compute eigenvalues and eigenvectors one way is to use determinants!

A fascinating application is to count the number of **spanning trees** of a graph: A spanning tree is a subgraph that is connected but has no cycles. Important for communications and planning.

How many different spanning trees are there? This is given by a determinant of a matrix!
GOAL 3

EIGENVALUES AND EIGENVECTORS

Key words: Determinants, Characteristic polynomials, Eigenspaces, Diagonalization