Please answer all 10 problems marked with an asterisk (e.g., ∗1.) plus 5 additional problems of your choice (if you turn in more than 5 additional problems only the first 5 will be graded). Please show all of your working.

The Small Angle Approximation for \( \sin x \)

Recall that \( \lim_{x \to 0} \frac{\sin x}{x} = 1 \), which is best interpreted as saying that \( \sin x \) is very well approximated by \( x \) when \( x \) is very close to 0. (You can try plotting the two functions, \( \sin x \) and \( x \), in Desmos and zooming in on the origin to convince yourself of this. You can also see that the limit appears to be one from looking at the graph, as shown.)

For good measure, we include here a geometric proof of this important limit (such a proof can also be found in Section 2.4 in the textbook). We will use the Greek letter \( \theta \), rather than \( x \), which we think of as an angle in radians. The idea will be to establish that

\[
\cos \theta \leq \frac{\sin \theta}{\theta} \leq 1 \quad \text{for} \quad -\frac{\pi}{2} < \theta < \frac{\pi}{2}, \quad \theta \neq 0
\]

and then apply the Sandwich/Squeeze Theorem to conclude that \( \lim_{x \to 0} \frac{\sin x}{x} = 1 \).

The inequalities in (1) can be established by some simple area computations. Recall that the area of a sector of angle \( \theta \) in the unit circle is \( \frac{1}{2} \theta \) (to check this, note that the sector of angle \( 2\pi \) is the...
entire circle and that $\frac{1}{2}2\pi$ gives the correct area of $\pi$ for the full unit circle). By computing the areas in Figure 3 we obtain that for $0 < \theta < \frac{\pi}{2}$,

$$\frac{1}{2} \sin \theta \leq \frac{1}{2} \theta \leq \frac{1}{2} \tan \theta. \quad (2)$$

![Figure 3. Area of some triangles (half base times height) and a sector.](image)

The first inequality in (2) gives us that $\frac{\sin \theta}{\theta} \leq 1$ (dividing both sides by $\theta$) and the second gives that $\theta \leq \frac{\sin \theta}{\cos \theta}$ which gives that $\cos \theta \leq \frac{\sin \theta}{\theta}$ (under the assumption that $0 < \theta < \frac{\pi}{2}$). This proves (1) for $0 < \theta < \frac{\pi}{2}$, but the functions in (1) are all even (i.e. they are symmetric under reflection through the $y$-axis) so (1) also holds for $-\frac{\pi}{2} < \theta < 0$. Hence, by the Sandwich/Squeeze Theorem,

$$\lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1. \quad (3)$$

Q.E.D.

A consequence of this is that $\lim_{\theta \to 0} \frac{1 - \cos \theta}{\theta} = 0$. Intuitively speaking this says that $1 - \cos \theta$ is very very tiny compared to $\theta$ when $\theta$ is close to zero (and more and more so as $\theta$ gets closer and closer to zero). You may want to plot $1 - \cos \theta$ and $\theta$ in Desmos to see this, but a better way to see it is to note that for very small angles $\theta$ the function $1 - \cos \theta$ is very well approximated by $\frac{1}{2}\theta^2$ (check this by plotting $1 - \cos x$ and $\frac{1}{2}x^2$ in Desmos) which can be used to justify writing

$$\lim_{\theta \to 0} \frac{1 - \cos \theta}{\theta} = \lim_{\theta \to 0} \frac{\frac{1}{2}\theta^2}{\theta} = \lim_{\theta \to 0} \frac{1}{2}\theta = 0. \quad (4)$$

Since we haven’t fully justified this calculation, let us now give a proof that this limit is zero using the fact that $\lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1$:

$$\lim_{\theta \to 0} \frac{1 - \cos \theta}{\theta} = \lim_{\theta \to 0} \frac{1 - \cos \theta}{\theta} \frac{1 + \cos \theta}{1 + \cos \theta} = \lim_{\theta \to 0} \frac{1 - \cos^2 \theta}{\theta(1 + \cos \theta)} = \lim_{\theta \to 0} \frac{\sin^2 \theta}{\theta(1 + \cos \theta)} = \lim_{\theta \to 0} \frac{\sin \theta}{\theta} \lim_{\theta \to 0} \frac{\sin \theta}{1 + \cos \theta} = 1 \cdot \frac{0}{2} = 0. \quad (5)$$

2
Problems

*1. Imitate the intuitive calculation in (4) to show that \( \lim_{\theta \to 0} \frac{1 - \cos \theta}{\theta^2} = \frac{1}{2} \), then imitate the rigorous calculation in (5) to reach the same conclusion.

*2. Compute the following limits

(a) \( \lim_{\theta \to 0} \frac{\sin 5\theta}{\sin 3\theta} \)

(b) \( \lim_{x \to 0} \frac{\tan 7x}{\tan 2x} \)

(c) \( \lim_{h \to 0} \frac{\cos 3h - 1}{\cos 2h - 1} \)

3. Compute the following limits

(a) \( \lim_{x \to 2^+} \frac{|x - 2|}{x - 2} \)

(b) \( \lim_{x \to 1} \cos \left( \frac{\sqrt{x} - 1}{x - 1} \right) \)

(c) \( \lim_{x \to \pi} \sin \left( \frac{x}{2} + \sin x \right) \)

(d) \( \lim_{x \to -1} 2^{\frac{x^2 - x - 2}{x+1}} \)

4. Relativistic length contraction. In relativity theory, the length of an object, say a rocket, appears to an observer to depend on the speed at which the object is travelling with respect to the observer. If the observer measures the rocket’s length as \( L_0 \) at rest, then at speed \( v \) the length will appear to be

\[
L = L_0 \sqrt{1 - \frac{v^2}{c^2}}
\]

where \( c \) is the speed of light in a vacuum (about \( 3 \times 10^8 \) m/sec). What happens to \( L \) as \( v \) increases? Find \( \lim_{v \to c^-} L \). Why was the left hand limit needed?

5. The Michaelis-Menten equation. When an enzyme is combined with a substrate of concentration \( s \) (in millimolars), the reaction rate (in micromolars/min) is

\[
R(s) = \frac{As}{K + s} \quad (A, K \text{ constants}).
\]

(a) Show, by computing \( \lim_{s \to \infty} R(s) \), that \( A \) is the limiting reaction rate as the concentration \( s \) approaches \( \infty \).

(b) Show that the reaction rate \( R(s) \) attains one-half of the limiting value \( A \) when \( s = K \).

(c) For a certain reaction, \( K = 1.25 \text{ mM} \) and \( A = 0.1 \). For which concentration \( s \) is \( R(s) \) equal to 75% of its limiting value?

6. Measuring cup. The interior of a typical 1-L measuring cup is a right circular cylinder of radius 6 cm. The volume of water we put in we put in the cup is is therefore a function of the level \( h \) to which the cup is filled, the formula being \( V = \pi 6^2 h = 36\pi h \). How closely must we measure \( h \) to measure out 1 L of water (1000 cm\(^3\)) with an error of no more than 1% (10 cm\(^3\))?
7. At which points are the functions listed below continuous?

(a) \( y = |x - 1| + \sin x \)
(b) \( y = \frac{x \tan x}{x^2 + 1} \)
(c) \( y = \sqrt[3]{3x - 1} \)
(d) \( y = (2 - x)^{1/5} \)
(e) \( f(x) = \begin{cases} \frac{x^3 - 8}{x^2 - 4}, & x \neq \pm 2 \\ 3, & x = 2 \\ 4, & x = -2 \end{cases} \)

8. **Removable discontinuity.** Give an example of a function \( f(x) \) that is continuous for all values of \( x \) except \( x = 2 \), where it has a removable discontinuity. Explain how you know that \( f \) is discontinuous at \( x = 2 \), and how you know the discontinuity is removable.

9. **Nonremovable discontinuity.** Give an example of a function \( g(x) \) that is continuous for all values of \( x \) except \( x = -1 \), where it has a nonremovable discontinuity. Explain how you know that \( g \) is discontinuous at \( x = -1 \) and why the discontinuity is not removable.

10. **Dirichlet’s Function:** A function discontinuous at every point. Use the fact that between any two distinct real numbers there exist (infinitely many) rational numbers and (infinitely many) irrational numbers to show that the function

\[
f(x) = \begin{cases} 
0, & \text{if } x \text{ is rational} \\
1, & \text{if } x \text{ is irrational}
\end{cases}
\]

is discontinuous at every point. To gain intuition about this function, think about how you would draw it’s graph. The only reasonable way that I know to draw the graph of this function is as two horizontal lines, one at height 0 and one at height 1 (but the function still passes the vertical line test because the horizontal lines have invisible holes in them in just the right places!). The function values are jumping up and down between 0 and 1 so fast that the function looks like it is taking both values at the same time; much in the same way that the rotor blades on a helicopter appear to be in more than one place at the same time when they are spinning rapidly.

**Just for fun:** In calculus (in particular in 21B) the notion of “area under the graph” of a function \( y = f(x) \) is fundamental. How would one define/compute the area under the graph of Dirichlet’s function, say between \( x = 0 \) and \( x = 1 \)? Should this area be 0, 1, \( \frac{1}{2} \), or none of these? (This problem confounded mathematicians for some time. Dirichlet’s function may or may not be relevant to real life, but it lead to some important developments in advanced calculus that are.)

11. **A function continuous at only one point.** Let

\[
f(x) = \begin{cases} 
|x|, & \text{if } x \text{ is rational} \\
0, & \text{if } x \text{ is irrational}
\end{cases}
\]
(a) Show that $f$ is continuous at $x = 0$. Hint: You can use the Sandwich/Squeeze Theorem for this one.

(b) Explain why the function is not continuous at any other point.

12. The **sign preserving property of continuous functions.** Explain why it is true that if a continuous function $f(x)$ is positive somewhere (say $f(c) > 0$) then it is also positive nearby (i.e. there is an interval $(c - \delta, c + \delta)$ about $c$ on which $f$ is positive).

*13. Use the **Intermediate Value Theorem** to prove that the equation $\cos x = x$ has a solution (there is only one solution). Then use Desmos, WolframAlpha or some other calculator to find the solution (to four decimal places).

14. **Just for fun:** Is there any reason to believe that there is always a pair of antipodal (diametrically opposite) points on Earth’s equator where the temperatures are the same? Explain.

*15. Compute the following limits

(a) \( \lim_{x \to \infty} \frac{9x^4 + x}{2x^4 + 5x^2 - x + 6} \)

(b) \( \lim_{x \to -\infty} \frac{5x^7 - 2x^4 + 8}{3 + x - 4x^3} \)

(c) \( \lim_{x \to -\infty} \frac{x + 1}{x^2 - x - 2} \)

(d) \( \lim_{x \to \infty} \left( \sqrt{x + 9} - \sqrt{x + 4} \right) \)

**Hint for (d):** Try multiplying and dividing by the “conjugate” expression.

16. Use limits to determine the equations for all vertical asymptotes of the following functions.

(a) \( y = \frac{x^2 + 4}{x - 3} \)

(b) \( y = \frac{x^2 - x - 2}{x^2 - 2x + 1} \)

(c) \( y = \frac{x^2 + x - 6}{x^2 + 2x - 8} \)

17. Use limits to determine the equations for all horizontal asymptotes of the following functions.

(a) \( y = \frac{1 - x^2}{x^2 + 1} \)

(b) \( y = \frac{\sqrt{x + 4}}{\sqrt{x + 4}} \)

(c) \( y = \frac{\sqrt{x^4 + 4}}{x} \)

(d) \( y = \frac{\sqrt{x^2 + 9}}{9x^2 + 1} \)
*18. For each of the following functions, find the equation for the tangent line to the curve at the given point.

(a) \( y = x^2 + 3x - 4 \) at \((1, 0)\)

(b) \( y = 2\sqrt{x} \) at \((4, 4)\)

(c) \( y = x^3 \) at \((-2, -8)\)

(d) \( y = \frac{1}{x^3} \) at \((-2, -\frac{1}{8})\)

19. Compute the derivative of the following functions.

(a) \( f(x) = x^5 + 4x^3 + 2x^2 + 9 \)

(b) \( f(x) = x^e - x^{-\pi} \)

(c) \( f(x) = \sqrt{x}\sin x + 7, \ x > 0 \)

(d) \( f(x) = \frac{4x}{x^2 + 1} \)

20. Rates of change.

(a) **Object dropped from a tower.** An object is dropped from the top of a 100m-high tower. Its height above ground after \( t \) sec is \( 100 - 4.9t^2 \) m. How fast is the object falling 2 sec after it is dropped?

(b) **Speed of a rocket.** At \( t \) seconds after liftoff, the height of a rocket is \( 3t^2 \) ft. How fast is the rocket climbing 10 sec after liftoff?

(c) **Circle's changing area.** What is the rate of change of the area \( (A = \pi r^2) \) of a circle with respect to radius when the radius is \( r = 3 \)? How does this relate to the circumference of the circle?

(d) **Ball's changing volume.** What is the rate of change of the volume of a ball \( (V = \frac{4}{3}\pi r^3) \) with respect to the radius when the radius is \( r = 2 \)? How does this relate to the surface area of the ball?

*21. Sketching functions and derivatives.** Answer each of these problems by giving a sketch of an example function and a sketch of its derivative (where defined) having the required properties. No formulae are required.

(a) A function \( f(x) \) that is continuous on \( \mathbb{R} \) and differentiable for \( x \neq 2 \), but for which \( f'(x) \) has a jump discontinuity at \( x = 2 \).

(b) A differentiable function \( g(x) \) such that \( g'(1) = 0, \ g'(x) > 0 \) for \( x < 1 \), and \( g'(x) < 0 \) for \( x > 1 \).

(c) A differentiable function \( h : [0, 4] \to \mathbb{R} \) such that \( h(x) \) is increasing as \( x \) goes from 0 to 1, decreasing as \( x \) goes from 1 to 2, constant as \( x \) goes from 2 to 3, and decreasing again as \( x \) goes from 3 to 4.

(d) A differentiable function \( k : \mathbb{R} \to \mathbb{R} \) that is increasing, but for which \( \lim_{x \to \infty} k'(x) = 0 \).
22. **Temperature.** The given graph shows the outside temperature $T$ in °F, between 6am and 7pm.

(a) Estimate the rate of temperature change at the times (i) 7am, (ii) 9am, (iii) 2pm, and (iv) 4pm.
(b) At what time does the temperature increase most rapidly? Decrease most rapidly? What is the rate for each of those times?
(c) Sketch a graph of the derivative $\frac{dT}{dt}$ of temperature against time $t$.

23. **House prices.** Average single-family house prices $P$ (in thousands of dollars) in Sacramento, California, are shown in the accompanying figure from 2006 through 2015.

(a) During what years did house prices decrease? increase?
(b) Estimate home prices at the end of (i) 2007, (ii) 2012, and (iii) 2015.
(c) Estimate the rate of change of house prices at the beginning of (i) 2007, (ii) 2010, and (iii) 2014.
(d) During what year did home prices drop most rapidly and what is an estimate of this rate?
(e) During what year did home prices rise most rapidly and what is an estimate of this rate?
(f) Sketch a graph of the derivative $\frac{dP}{dt}$ of home price versus time $t$. 

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24. **Lunar projectile motion.** A rock thrown vertically upward from the surface of the moon at a velocity of 24 m/sec (about 86 km/h or 54 mph) reaches a height of \( s = 24t - 0.8t^2 \) in \( t \) sec.

   (a) Find the rock’s velocity and acceleration at time \( t \) (the latter is the acceleration due to gravity on the moon).
   
   (b) How long does the rock take to reach its highest point?
   
   (c) How high does the rock go?

25. **Gallileo’s free-fall formula.** Galileo developed a formula for a body’s velocity during free fall by rolling balls from rest down increasingly steep inclined planks and looking for a limiting formula that would predict the balls behavior when the plank was vertical and the ball fell freely; see part (a) of the accompanying figure. He found that, for any angle of the plank, the balls velocity \( t \) sec into motion was a constant multiple of \( t \). That is, the velocity was given by the formula \( v = kt \). The value of the constant \( k \) depended on the inclination of the plank.

   In modern notation–part (b) of the figure–with distance in meters and time in seconds, what Galileo determined by experiment was that, for any given angle \( \theta \), the ball’s velocity \( t \) sec into the roll was \( v = 9.8(\sin \theta)t \) m/sec.

26. The accompanying figure shows the velocity \( v = f(t) \) of a particle moving (back and forwards) along a straight line.

   (a) What is the equation for the ball’s velocity during free fall?
   
   (b) Based on your work in (a), what constant acceleration does a freely falling body experience near the surface of the earth?

   26. The accompanying figure shows the velocity \( v = f(t) \) of a particle moving (back and forwards) along a straight line.

   (a) When does the particle move forward? Move backward? Speed up? Slow down?
(b) When is the particle's acceleration positive? Negative? Zero?
(c) When does the particle move at its greatest speed?
(e) When does the particle stand still for more than an instant?

27. **Launching a rocket.** When a model rocket is launched, the propellant burns for a few seconds, accelerating the rocket upward. After burnout, the rocket coasts upward for a while and then begins to fall. A small explosive charge pops out a parachute shortly after the rocket starts down. The parachute slows the rocket to keep it from breaking when it lands.

The figure here shows velocity data from the flight of the model rocket.

Use the data to answer the following.

(a) How fast was the rocket climbing when the engine stopped?
(b) For how many seconds did the engine burn?
(c) When did the rocket reach its highest point?
(d) When did the parachute pop out? How fast was the rocket falling then?
(e) How long did the rocket fall before the parachute opened?
(f) When was the rocket’s acceleration the greatest?
(g) When was the acceleration constant? What was its value then (to the nearest integer)?

*28. **Airplane takeoff.** Suppose that the distance an aircraft along a runway before taking off is given by \( D = \left( \frac{10}{9} \right) t^2 \), where \( D \) is measured in meters from the starting point and \( t \) is measured in seconds from the time the brakes are released. The aircraft will become airborne when its speed reaches 200 km/h. How long will it take to become airborne, and what distance will it travel in that time?
29. **Vehicular stopping distance.** Based on data from the U.S. Bureau of Public Roads, a model for the total stopping distance of a moving car in terms of its speed is

\[ s = 1.1v + 0.054v^2, \]

where \( s \) is measured in ft and \( v \) in mph. The linear term 1.1\( v \) models the distance the car travels during the time the driver perceives a need to stop until the breaks are applied, and the quadratic term 0.054\( v^2 \) models the additional breaking distance once they are applied. Find \( \frac{ds}{dv} \) at \( v = 35 \) and \( v = 70 \) mph, and interpret the meaning of the derivative.

30. The graphs in the accompanying figures show (in two different scenarios) the position \( s \), velocity \( v = ds/dt \), and acceleration \( a = d^2s/dt^2 \) of a body moving along a coordinate line as functions of time \( t \). For each of the two scenarios (a) and (b) determine which graph is which. Give reasons for your answers.

*31. Together with your solutions to these exercises, please turn in to Canvas scanned copies of handwritten notes you have created on any two of the following topics (or subtopics of these):

2.6 Limits involving Infinity; Asymptotes of Graphs
3.1 Tangent Lines and the Derivative at a Point
3.2 The Derivative as a Function
3.3 Differentiation Rules
3.4 The Derivative as a Rate of Change
3.5 Derivatives of Trigonometric Functions

Please submit at least two pages of notes for any topic you pick (so you must submit at least four pages of notes total). You may combine more than just two topics in the notes you submit. If you send in your own annotated lecture slides, then you must instead submit three complete sets of slides, with your own detailed annotations. (If you do not have notes already, then it is a good idea to use this as an opportunity to make your own review notes for later.)