Abstract

We extend several notions and results from the classical Patterson-Sullivan theory to the setting of Anosov subgroups of higher rank semisimple Lie groups, working primarily with invariant Finsler metrics on associated symmetric spaces. In particular, we prove the equality between the Hausdorff dimensions of flag limit sets, computed with respect to a suitable Gromov (pre-)metric on the flag manifold, and the Finsler critical exponents of Anosov subgroups.

Contents

1 Geometric preliminaries . . . . . . . . . . . . . . . . . . . . . . . . . 5
1.1 Symmetric spaces . . . . . . . . . . . . . . . . . . . . . . . . . 5
1.2 Boundary at infinity . . . . . . . . . . . . . . . . . . . . . . . . 6
1.3 $\Delta$-valued distances and generalized triangle inequality . . . . . . . . . . . . 7
1.4 Parallel sets, cones, and diamonds . . . . . . . . . . . . . . . . . 7
1.5 Morse embeddings . . . . . . . . . . . . . . . . . . . . . . . . . 8
1.6 Discrete subgroups of $G$ and their limit sets . . . . . . . . . . . . 8
1.7 Illustrating examples . . . . . . . . . . . . . . . . . . . . . . . . 10
2 Critical exponent . . . . . . . . . . . . . . . . . . . . . . . . . . . 11
3 Conformal densities . . . . . . . . . . . . . . . . . . . . . . . . . . 14
4 Hyperbolicity of Morse image . . . . . . . . . . . . . . . . . . . . . 18
5 Gromov distance at infinity . . . . . . . . . . . . . . . . . . . . . 21
6 Shadow lemma . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 25
7 Dimension of a conformal density . . . . . . . . . . . . . . . . . . 29
8 Uniqueness of conformal density . . . . . . . . . . . . . . . . . . . . 31
9 Hausdorff density . . . . . . . . . . . . . . . . . . . . . . . . . . . 36
10 Examples . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 39
  10.1 Product of hyperbolic spaces . . . . . . . . . . . . . . . . . . . . 39
  10.2 Projective Anosov representations . . . . . . . . . . . . . . . . 40
Consider a discrete group $\Gamma$ of isometries of the $n$-dimensional hyperbolic space $\mathbb{H}^n$. The critical exponent $\delta$ is a fundamental numerical invariant associated with $\Gamma$ which measures the asymptotic growth rates of $\Gamma$-orbits in $\mathbb{H}^n$. The relation between the Hausdorff dimension of the limit set $\Lambda(\Gamma)$ of $\Gamma$ and its critical exponent is now a classical result. In an influential paper [Sul79], Sullivan proved the following theorem extending pioneering work by Patterson ([Pat76]) on Fuchsian groups:

**Theorem** ([Sul79, Thm. 8]). Let $\Gamma$ be a convex-cocompact subgroup of the isometry group of $\mathbb{H}^n$. Then the critical exponent $\delta$ of $\Gamma$ equals to the Hausdorff dimension of $\Lambda(\Gamma)$.

Later Sullivan generalized this theorem for geometrically finite Kleinian groups ([Sul84]). An important ingredient of Sullivan’s proof of this theorem is the existence of a finite, non-null Borel measure on $\Lambda(\Gamma)$ that changes conformally under the $\Gamma$-action. The construction of such measure goes back to Patterson’s original idea in [Pat76]. Measures of this type (resp. a class of “well-behaved” measures) are commonly referred as Patterson-Sullivan measures (resp. densities). We refer to Nicholls’ book ([Nic89]) for a self-contained exposition on these results.

Since its introduction, the theory of Patterson and Sullivan has attracted a lot of attention. Further developments have been made by various people who analyzed more general classes of discrete groups and their limit sets. We list some of these developments here. Corlette ([Cor90]) and Corlette-Iozzi ([CI99]) proved the above theorem for geometrically finite groups of isometries of rank-one symmetric spaces, and Bishop-Jones ([BJ97]) extended these results to arbitrary discrete isometry groups of rank-one symmetric spaces. Yue ([Yue96]) and Ledrappier ([Led95]) studied the case of Hadamard spaces of negative curvature. Burger ([Bur93]), Albuquerque ([Alb99]) and Quint ([Qui02b], [Qui02a]) considered the case of Zariski-dense discrete subgroups in the isometry groups of higher-rank symmetric spaces, while Link ([Lin10]) studied the case of products of rank one symmetric spaces.

In the more abstract setting of Gromov hyperbolic spaces, much of Sullivan’s work in [Sul79] was generalized by Coornaert ([Coo93]) to the class of quasiconvex-cocompact groups. See also work of Paulin ([Pau97]) on actions of subgroups of Gromov hyperbolic groups. More recent developments by Das-Simmons-Urbański in [DSU17] achieved generalizations of the Patterson-Sullivan theory (e.g., a generalization of Bishop-Jones’ theorem) in the case of “infinite-dimensional” Gromov hyperbolic spaces.

The goal of this paper is to study the Patterson-Sullivan theory for Anosov subgroups. Anosov subgroups, first introduced by Labourie ([Lab06]) and then further developed by Guichard-Wienhard ([GW12]) and Kapovich-Leeb-Porti ([KLP14, KLP17, KLP18]), extend the class of convex-cocompact subgroups of rank-one semisimple Lie groups to higher rank. In this paper, we mainly work with Kapovich-Leeb-Porti’s characterizations of Anosov subgroups. We briefly review some of these characterizations (and related background) in Section 1.
Let $G$ be a noncompact real semisimple Lie group, $X = G/K$ be the associated symmetric space and $\Gamma$ be a $\tau_{\text{mod}}$-Anosov subgroup of $G$. We will be assuming several conditions on $G$ and $X$; they are labeled as “assumption” in Section 1. We consider two types of $G$-invariant (pseudo-)metrics on $X$, namely, one is the Riemannian metric of the symmetric space $X$ and the other one is Finslerian. The critical exponents of $\Gamma$ with respect to these two metrics, denoted by $\delta_R$ and $\delta_F$, respectively, are defined in the usual fashion, i.e., as the exponents of convergence of associated Poincaré series (see Section 2). Using the classical construction of Patterson, we define a $\Gamma$-invariant conformal density on the flag limit set of $\Gamma$ (see Section 3).

Throughout this paper, the Finsler metric is given more emphasis than its Riemannian counterpart. For example, the construction of the above mentioned Patterson-Sullivan density is carried out in terms of the Finsler metric. The main reason for this choice is that Finsler metrics reflect the asymptotic geometry of $\Gamma$ better than the Riemannian metric.

We should note that many of the results in this paper are often proven for more general classes of discrete subgroups of $G$ with the hope that the results may be useful, for instance, in the study of relatively Anosov subgroups.\(^1\) Regarding Anosov subgroups, the main results of this paper are summarized below.

Let $\sigma_{\text{mod}}$ be a maximal simplex in the Tits building of $X$, $\iota : \sigma_{\text{mod}} \to \sigma_{\text{mod}}$ be the opposition involution, $\tau_{\text{mod}}$ be an $\iota$-invariant face of $\sigma_{\text{mod}}$, $P$ be the maximal parabolic subgroup of $G$ that stabilizes $\tau_{\text{mod}}$, and $\text{Flag}(\tau_{\text{mod}}) = G/P$ be the partial flag manifold associated to the face $\tau_{\text{mod}}$ (see Subsection 1.2).

**Main theorem.** Let $\Gamma$ be a nonelementary $\tau_{\text{mod}}$-Anosov subgroup of $G$ and $\delta_F$ be the Finsler critical exponent for the action of $\Gamma$ on the symmetric space $X = G/K$. Then the Patterson-Sullivan density $\mu$ (constructed with respect to the Finsler metric on $X$) on the flag limit set $\Lambda_{\tau_{\text{mod}}} (\Gamma) \subset \text{Flag}(\tau_{\text{mod}})$ is the unique (up to a constant factor) $\Gamma$-invariant conformal density. Moreover,

(i) The density $\mu$ is non-atomic and its dimension equals to $\delta_F$.

(ii) The support of $\mu$ is $\Lambda_{\tau_{\text{mod}}} (\Gamma)$ and the action $\Gamma \acts \Lambda_{\tau_{\text{mod}}} (\Gamma)$ is ergodic with respect to $\mu$.

(iii) The critical exponent $\delta_F$ (as well as the Riemannian critical exponent $\delta_R$) is positive and finite.

(iv) The Poincaré series of $\Gamma$ diverges at the critical exponent $\delta_F$. In other words, $\Gamma$ has (Finsler) divergence type.

(v) The $\delta_F$-dimensional Hausdorff measure on $\Lambda_{\tau_{\text{mod}}} (\Gamma)$ with respect to a Gromov (pre-)metric\(^2\) is a member of a $\Gamma$-invariant conformal density (called the Hausdorff density). In particular, the Hausdorff dimension of $\Lambda_{\tau_{\text{mod}}} (\Gamma)$ is $\delta_F$.

The uniqueness of conformal density is proven in Corollary 8.4. The main ingredients in the proof are (1) a generalization of Sullivan’s shadow lemma proven in Theorem 6.1, and (2) an ergodicity argument (see Theorem 8.1) also due to Sullivan. The proof of part (i) of the theorem

---

\(^1\)Relatively Anosov subgroups are recent extension by Kapovich-Leeb ([KL18b]) of the class of geometrically finite groups into the higher rank.

\(^2\)See Section 5.
is given in Corollaries 6.2 and 7.5. The second half of part (ii) follows from Theorem 8.3 while the first half follows from the facts that the support of $\mu$ is a closed $\Gamma$-invariant subset of $\Lambda_{\tau_{\text{mod}}}$. The action $\Gamma \curvearrowleft \Lambda_{\tau_{\text{mod}}}$ is minimal. The part (iii) is proven in Propositions 2.3 and 3.1. See also the remarks following these propositions where $\delta\tau$ is analyzed. The part (iv) follows from Corollary 6.5. The Hausdorff density in part (v) is studied in Section 9 (cf. Theorem 9.3). The background Gromov (pre-)metric is introduced in Section 5 where we also prove that the action $\Gamma \curvearrowleft \Lambda_{\tau_{\text{mod}}}$ with respect to this metric is conformal (see Corollary 5.6).

For certain classes of Anosov subgroups, Patterson-Sullivan theory was used by Sambarino in [Sam14, Sam15] to solve certain counting problems, while in [BCLS15] Bridgeman-Canary-Labourie-Sambarino used related thermodynamic formalism to construct pressure metrics on spaces of Hitchin representations. Moreover, Glorieux-Monclair ([GM16]) studied the Patterson-Sullivan theory in the case of convex-cocompact subgroups of the isometry group of $\mathbb{H}^{p,q}$ equipped with the pseudo-Riemannian metric.

While working on this article, we came to know about two recent developments by Pozzetti-Sambarino-Wienhard ([PSW19]) and Glorieux-Monclair-Tholozan ([GMT19]) which are related to our work. In these articles, the authors proved that the Hausdorff dimension of the limit set of a projective Anosov subgroup $\Gamma$ in the real projective space is bounded above by a certain critical exponent, called the “simple root critical exponent” in the second article. In the second article, the authors also obtain upper and lower bounds for the Hausdorff dimension of the flag limit set\(^3\) of $\Gamma$ while mentioning that they also “hoped to get” a lower bound for the limit set in the projective space. We obtain a lower bound for this limit set which turns out to be same as in the case of the flag limit set (see Theorem 10.1).

After this work was completed, Andrés Sambarino informed us that Ledrappier’s methods from [Led95] (in conjunction with results of [BCLS15, sect. 3.2]) can be used to obtain some of the results of our paper; we refer the reader to [Sam14, Sam15] for similar applications of Ledrappier’s work.

**Acknowledgement:** This project is a part of the first author’s dissertation work. The second author was partly supported by the NSF grant DMS-16-04241. We are grateful to Olivier Glorieux for pointing out a mistake in the first example of Section 10 and to Andrés Sambarino for telling us about Ledrappier’s work and related results.

**Notations**

Here we list some commonly used notations.

- $\mathcal{B}(Y)$: Class of Borel subsets of a topological space $Y$
- $B(x,r)$: (Closed) ball of radius $r$ centered at $x$
- $d_F, d_R$: Finsler and Riemannian metrics, respectively, on $X$ (see Sec. 2)

\(^3\)In [GMT19] the flag limit set is called the “symmetric limit set”. See [GMT19, Thm. 1.1] for details.
• \(\overline{xy}, \overline{xy}\): Finsler\(^4\) and Riemannian geodesic segments, respectively, connecting \(x, y \in X\) (see Sec. 2)
• \(\delta_F, \delta_R\): Finsler and Riemannian critical exponents, respectively, of \(\Gamma\) (see Sec. 2)
• \(d^{x,\epsilon}_G\): Gromov premetric (see Def. 5.2)
• \(d^\text{hor}_t\): Horospherical distance (see (3.1))

1. Geometric preliminaries

In this section, we briefly present some background material needed for the paper.

1.1. Symmetric spaces: A symmetric space \(X\) is a Riemannian manifold that has an inversion symmetry or point-reflection with respect to each point \(x \in X\): This is an isometric involution \(s_x : X \rightarrow X\) fixing \(x\) and sending each tangent vector at \(x\) to its negative. In this paper we only consider symmetric spaces which are simply-connected and have noncompact type. The later means that \(X\) has no flat deRham factor and the sectional curvature of \(X\) is non-positive. In particular, \(X\) is a Hadamard manifold and, hence, is diffeomorphic to a euclidean space. We refer to Eberlein’s book [Ebe96] for a detailed discussion of symmetric spaces.

Assumption 1. The symmetric spaces \(X\) is simply-connected and of noncompact type.

A symmetric space \(X\) can be written as \(G/K\) where \(G\) is a semisimple Lie group whose Lie algebra does not have compact and abelian factors, and \(K\) is a maximal compact subgroup of \(G\). Moreover, this group \(G\) can be chosen to have finite center and be commensurable with the isometry group \(\text{Isom}(X)\) of \(X\). For example, one can choose \(G\) to be the identity component of \(\text{Isom}(X)\).

Assumption 2. The semisimple Lie group \(G\) has finite center and is commensurable with the isometry group \(\text{Isom}(X)\) of the symmetric space \(X\).

Each point \(x \in X\) determines a canonical decomposition of the Lie algebra \(\mathfrak{g}\) of \(G\) called the Cartan decomposition,

\[\mathfrak{g} = \mathfrak{t} + \mathfrak{p}\]

where \(\mathfrak{t}\) is tangent to the stabilizer of \(x\) which is a conjugate of \(K\). The dimension of a maximal abelian subalgebra \(\mathfrak{a} \subset \mathfrak{p}\) is called the rank of \(X\). The exponential map \(\exp_x : \mathfrak{g} \rightarrow X\) identifies \(\mathfrak{a}\) with a maximal flat \(F \subset X\) through \(x\) and, hence, the rank of \(X\) can also be defined as the dimension of a maximal totally geodesic flat subspace in \(X\). A chosen maximal flat \(F_{\text{mod}} \subset X\) is called the model flat which we isometrically identify with \(\mathbb{R}^k\) where \(k = \text{rank}(X)\). The image in \(\text{Isom}(F)\) of the \(G\)-stabilizer of \(F_{\text{mod}}\) is isomorphic to \(\mathbb{R}^k \rtimes W\), where the first factor acts on \(F_{\text{mod}} \cong \mathbb{R}^k\) by translations while the second factor \(W\), called the Weyl group, is finite, fixes the origin, and is generated by hyperplane reflections. The closures of the connected components of the complement of the reflecting hyperplanes (for hyperplane reflections in \(W\) in \(F_{\text{mod}}\) are called chambers. A chosen chamber is called the model Weyl chamber; we denote it by \(\Delta\).

\(^4\)Note that Finsler geodesic segments connecting two points in \(X\) are usually non-unique.
1.2. Boundary at infinity: For a symmetric space \( X \), there are multiple notions of (partial) boundary at infinity. The space of equivalence classes of asymptotic rays is called the visual boundary of \( X \) and denoted \( \partial_\infty X \). The visual boundary is naturally identified with the unit tangent sphere \( T^1_x X \) at any point \( x \in X \). The topology it gets from this identification is called the visual topology. Attaching the visual boundary to \( X \) provides a compactification of \( X \).

Another (strictly finer) topology on \( \partial_\infty X \) is given by the \( G \)-invariant Tits angle metric:

\[
\angle_{\text{Tits}}(\zeta, \eta) = \sup_{x \in X} \angle_x(\zeta, \eta)
\]

where \( \angle_x(\zeta, \eta) \) denotes the angle between the rays emanating from \( x \) and asymptotic to \( \zeta \) and \( \eta \). The boundary \( \partial_\infty X \) with this topology is denoted by \( \partial_{\text{Tits}} X \) and called the Tits boundary.

The Tits boundary \( \partial_{\text{Tits}} X \) carries a canonical \( G \)-invariant structure of a spherical simplicial complex called the Tits building of \( X \). This can be understood as follows: Consider the ideal boundary \( \partial_\infty F_{\text{mod}} \) of \( F_{\text{mod}} \) where \( k = \text{rank}(X) \). This is identified with the unit sphere \( a^1 \) of \( a \) and thus, we have an action of the Weyl group \( W \) on \( \partial_{\text{Tits}} F_{\text{mod}} \). The pair \( (\partial_{\text{Tits}} F_{\text{mod}}, W) \) is a spherical Coxeter complex which generates a spherical simplicial complex structure on entire \( \partial_{\text{Tits}} X \) by the \( G \)-action.

**Assumption 3.** We assume that the Tits building is thick, i.e., every simplex of codimension one is a face of three maximal simplices.

We denote the intersection of \( \Delta \) with the unit sphere in \( F_{\text{mod}} \) centered at the origin by \( \sigma_{\text{mod}} \). This is a fundamental domain for the action \( W \curvearrowright \partial_{\text{Tits}} F_{\text{mod}} \) where \( \partial_{\text{Tits}} F_{\text{mod}} \) is identified with the unit sphere in \( F_{\text{mod}} \) centered at the origin. We call \( \sigma_{\text{mod}} \) the model chamber. Any other chamber (i.e., a top-dimensional simplex) in the Tits building is naturally identified with \( \sigma_{\text{mod}} \) via a \( G \)-equivariant map, called the type map,

\[
\theta : \partial_{\text{Tits}} X \rightarrow \sigma_{\text{mod}}
\]

We reserve the notation \( \tau_{\text{mod}} \) for the faces of \( \sigma_{\text{mod}} \). An ideal point \( \zeta \in \partial_{\text{Tits}} X \) (resp. a simplex \( \tau \subset \partial_{\text{Tits}} X \)) is called of type \( \bar{\theta} \in \sigma_{\text{mod}} \) (resp. of type \( \tau_{\text{mod}} \subset \sigma_{\text{mod}} \)) if \( \theta(\zeta) = \bar{\theta} \) (resp. \( \theta(\tau) = \tau_{\text{mod}} \)). For \( \bar{\theta} \in \tau_{\text{mod}} \) and a simplex \( \tau \) of type \( \tau_{\text{mod}} \), we use the notation \( \bar{\theta}(\tau) \) to denote the unique point in \( \tau \) of type \( \bar{\theta} \). The opposition involution \( \iota \) is an automorphism of \( \sigma_{\text{mod}} \) which is defined as the negative of the longest element in the Weyl group.

Two simplices \( \tau_1, \tau_2 \) in the Tits building are called antipodal if there exists a point-reflection \( s_x \) swapping these two. Their types are related by \( \theta(\tau_1) = \iota \theta(\tau_2) \). In particular, when \( \tau_1 \) has an \( \iota \)-invariant type \( \tau_{\text{mod}} \), then any antipodal simplex \( \tau_2 \) also has type \( \tau_{\text{mod}} \). In this paper, we only consider types that are \( \iota \)-invariant.

We now describe an important class of partial boundaries of \( X \) which are central to our study. Consider the action of \( G \) on the Tits building. The stabilizer of a face \( \tau_{\text{mod}} \) of \( \sigma_{\text{mod}} \) is a parabolic subgroup \( P_{\tau_{\text{mod}}} \) of \( G \) and we identify the quotient \( G/P_{\tau_{\text{mod}}} \) with the set of all simplices in the Tits building of type \( \tau_{\text{mod}} \). This quotient \( G/P_{\tau_{\text{mod}}} \) is a smooth compact manifold, called the partial flag manifold of type \( \tau_{\text{mod}} \) and is denoted \( \text{Flag}(\tau_{\text{mod}}) \). The partial compactification of \( X \) by attaching \( \text{Flag}(\tau_{\text{mod}}) \) is denoted

\[
\tilde{X}^{\tau_{\text{mod}}} = X \cup \text{Flag}(\tau_{\text{mod}})
\]
which is topologized via the topology of flag convergence (see Subsection 1.6). In the special case when \( \tau_{\text{mod}} = \sigma_{\text{mod}} \), the associated parabolic subgroup \( P_{\sigma_{\text{mod}}} \) is minimal and \( \text{Flag}(\sigma_{\text{mod}}) = G/P_{\sigma_{\text{mod}}} \) is the full flag manifold, also called the Furstenberg boundary of \( X \).

A subset \( A \subset \text{Flag}(\tau_{\text{mod}}) \) is called antipodal if any two distinct simplices in \( A \) are antipodal.

1.3. \( \Delta \)-valued distances and generalized triangle inequality: There is a canonical map \( d_\Delta : X \times X \to \Delta \) which is defined as follows: For a pair of points \((x, y)\) in \( X \), there is an element \( g \in G \) which maps \( x \) to the origin in \( \Delta \) and \( y \) to a point \( v \in \Delta \). We define \( d_\Delta(x, y) = v \). Note that the norm \( \|d_\Delta(x, y)\| \) (induced by the euclidean inner product on \( F_{\text{mod}} \cong \mathbb{R}^k \)) equals \( d_R(x, y) \) where \( d_R \) denotes the distance function induced by the Riemannian metric on \( X \).

For a pair \((x, y)\) \( \in X \times X \), the value \( d_\Delta(x, y) \) is called the \( \Delta \)-valued distance between \( x \) and \( y \). This is a complete \( G \)-congruence invariant for oriented line segments in \( X \). The \( \Delta \)-valued distances satisfy generalized triangle inequalities (see [KLM09]). In the paper we will need the following triangle inequality. For \( x, y, z \in X \),

\[
\|d_\Delta(x, y) - d_\Delta(x, z)\| \leq d_R(y, z). \tag{1.1}
\]

1.4. Parallel sets, cones, and diamonds: For a detailed discussion on this subsection, we refer to [KLP14, Subsec. 2.4], [KLP17, Subsec. 2.5].

Let \( \tau_+ \) be a pair of antipodal simplices in the Tits building of \( X \). The parallel set \( P(\tau_+, \tau_-) \) is the union of all maximal flats in \( X \) whose ideal boundary contains \( \tau_+ \cup \tau_- \) as a subset. This is a totally geodesic submanifold of \( X \).

For a simplex \( \tau \), the star \( \text{st}(\tau) \) of \( \tau \) is the union of all chambers in the Tits building containing \( \tau \). The open star \( \text{ost}(\tau) \) of \( \tau \) is the union of all the open simplices whose closures contains \( \tau \). For a face \( \tau_{\text{mod}} \) of \( \sigma_{\text{mod}} \) (viewed as a complex), define the open star \( \text{ost}(\tau_{\text{mod}}) \) similarly. The boundary \( \partial \text{st}(\tau_{\text{mod}}) \) is the complement of \( \text{ost}(\tau_{\text{mod}}) \) in \( \sigma_{\text{mod}} \).

Let \( \tau_{\text{mod}} \) be an \( \iota \)-invariant face of \( \sigma_{\text{mod}} \). An ideal point \( \xi \in \partial_\infty X \) is called \( \tau_{\text{mod}} \)-regular if its type is contained in \( \text{ost}(\tau_{\text{mod}}) \). Moreover, given an \( \iota \)-invariant compact subset \( \Theta \subset \text{ost}(\tau_{\text{mod}}) \), an ideal point \( \xi \in \partial_\infty X \) is called \( \Theta \)-regular if its type is contained in \( \Theta \). A nondegenerate geodesic segment (or line or ray) in \( X \) is called \( \tau_{\text{mod}} \)-regular (resp. \( \Theta \)-regular) if the ideal endpoints of its line extension are \( \tau_{\text{mod}} \)-regular (resp. \( \Theta \)-regular).

For a simplex \( \tau \) in the Tits building and a point \( x \in X \), the \( \tau_{\text{mod}} \)-cone \( V(x, \text{st}(\tau)) \) with apex \( x \) is the union of all rays emanating from \( x \) asymptotic to a point \( \xi \in \text{st}(\tau) \). For a \( \tau_{\text{mod}} \)-regular geodesic segment \( \overline{xy} \subset X \), the \( \tau_{\text{mod}} \)-diamond \( \Diamond_{\tau_{\text{mod}}} \) \( x, y \) is the intersection of the opposite cones \( V(x, \text{st}(\tau_+)) \) and \( V(y, \text{st}(\tau_-)) \) containing it. The points \( x \) and \( y \) are called the endpoints of \( \Diamond_{\tau_{\text{mod}}} \) \( x, y \). The cones and parallel sets can be interpreted as limits of diamonds where, respectively, one or both endpoints diverges to infinity. All of these are convex subsets of \( X \) (see [KLP14, Prop. 2.14], [KLP17, Prop. 2.10]). In particular, the cones are nested: For every \( y \in V(x, \text{st}(\tau)) \), \( V(y, \text{st}(\tau)) \subset V(x, \text{st}(\tau)) \).

Let \( \Theta \) be an \( \iota \)-invariant compact subset \( \text{ost}(\tau_{\text{mod}}) \). In a similar way as above, the \( \Theta \)-cone \( V(x, \text{ost}_{\Theta}(\tau)) \) with apex \( x \) is the union of all rays emanating from \( x \) asymptotic to a point \( \xi \in \text{st}(\tau) \) of type \( \Theta \). Note that \( V(x, \text{ost}_{\Theta}(\tau)) \) is strictly contained inside \( V(x, \text{st}(\tau)) \).
1.5. Morse embeddings: The Morse property in higher rank was introduced by Kapovich-Leeb-Porti in [KLP14].

Recall that a quasigeodesic in $X$ is a quasimetric embedding $\phi : I \rightarrow X$ of an interval $I \subset \mathbb{R}$. We say that $\phi$ is $\tau_{\text{mod}}$-regular quasigeodesic if for all sufficiently separated points $t_1, t_2 \in I$, the segment $\phi([t_1, t_2])$ is $\tau_{\text{mod}}$-regular. We say that $\phi$ is a $\tau_{\text{mod}}$-Morse quasigeodesic if it is $\tau_{\text{mod}}$-regular and for all sufficiently separated points $t_1, t_2 \in I$, the image $\phi([t_1, t_2])$ is uniformly close to $\phi_{\tau_{\text{mod}}}((\phi(t_1), \phi(t_2)))$.

Let $Z$ be a geodesic Gromov-hyperbolic metric space (cf. Definition 4.2). A quasimetric map $\phi : Z \rightarrow X$ is called a $\tau_{\text{mod}}$-Morse embedding if the image of every geodesic is a $\tau_{\text{mod}}$-Morse quasigeodesic with uniformly controlled coarse-geometric quantifiers: There exists a constant $D > 0$ and an $\iota$-invariant compact subset $\Theta \subset \text{ost}(\tau_{\text{mod}})$ such that if $z_1, z_2$ is a geodesic segment in $Z$ of length $\geq D$, then $\phi(z_1)\phi(z_2)$ is a $\Theta$-regular geodesic in $X$ and the image $\phi([z_1, z_2])$ is $D$-close to $\phi_{\tau_{\text{mod}}}((\phi(z_1), \phi(z_2)))$.

A discrete finitely generated subgroup (equipped with a word metric) $\Gamma < G$ is called $\tau_{\text{mod}}$-Morse if it is hyperbolic and, for an(y) $x \in X$, the orbit map $\Gamma \rightarrow \Gamma x$ is a $\tau_{\text{mod}}$-Morse embedding.

1.6. Discrete subgroups of $G$ and their limit sets: We consider discrete subgroups with various levels of regularity and their flag limit sets. Most of these notions first appear in the work of Benoist ([Ben97]); our discussion follows [KLP14] and [KLP17].

We first recall the notion of regular sequences in $X$. Let $\tau_{\text{mod}}$ be an $\iota$-invariant face of $\sigma_{\text{mod}}$. Let $V(0, \text{dist}(\tau_{\text{mod}}))$ denote the union of all rays in $\Delta$ emanating from 0 asymptotic to points $\xi \in \text{dist}(\tau_{\text{mod}})$. A sequence $(x_n)$ in $X$ diverging to infinity is $\tau_{\text{mod}}$-regular if for all $x \in X$, the sequence $(d_{\Delta}(x, x_n))_{n \in \mathbb{N}}$ in $\Delta$ diverges away from $V(0, \text{dist}(\tau_{\text{mod}}))$. Furthermore, a $\tau_{\text{mod}}$-regular sequence $(x_n)$ is called uniformly $\tau_{\text{mod}}$-regular if the sequence $(d_{\Delta}(x, x_n))_{n \in \mathbb{N}}$ in $\Delta$ diverges away from $V(0, \text{dist}(\tau_{\text{mod}}))$ at a linear rate,

$$\lim_{n \rightarrow \infty} \frac{d(d_{\Delta}(x, x_n), V(0, \text{dist}(\tau_{\text{mod}})))}{d(0, d_{\Delta}(x, x_n))} > 0,$$

where $d$ denotes the euclidean distance on $\Delta$. Accordingly, a sequence $(g_n)$ in $G$ is $\tau_{\text{mod}}$-regular (resp. uniformly $\tau_{\text{mod}}$-regular) if for some (equivalently, every) $x \in X$, the sequence $(g_n(x))$ is $\tau_{\text{mod}}$-regular (resp. uniformly $\tau_{\text{mod}}$-regular).

For $x \in X$ and $A \subset X$, define the shadow of $A$ on $\text{Flag}(\tau_{\text{mod}})$ from $x$ as

$$S(x : A) = \{ \tau \in \text{Flag}(\tau_{\text{mod}}) \mid A \cap V(x, \text{st}(\tau)) \neq \emptyset \}. \quad (1.2)$$

Let $(g_n)$ be a $\tau_{\text{mod}}$-sequence on $G$. A sequence $(\tau_n)$ on $\text{Flag}(\tau_{\text{mod}})$ is called a shadow sequence of $(g_n)$ if there exists $x \in X$ such that, for every $n \in \mathbb{N}$, $\tau_n = S(x : \{ g_n \})$. A $\tau_{\text{mod}}$-regular sequence $(g_n)$ is said to be $\tau_{\text{mod}}$-flag-convergent to $\tau \in \text{Flag}(\tau_{\text{mod}})$ if any shadow sequence $(\tau_n)$ of $(g_n)$ converges to $\tau$.

The notion of flag-convergence leads to the definition of flag limit sets of discrete subgroups $\Gamma < G$. The $\tau_{\text{mod}}$-flag limit set of $\Gamma$ denoted by $\Lambda_{\tau_{\text{mod}}}^\Gamma$ is the subset of $\text{Flag}(\tau_{\text{mod}})$ which consists of all limit simplices of $\tau_{\text{mod}}$-flag-convergent sequences on $\Gamma$. The flag limit set $\Lambda_{\tau_{\text{mod}}}^\Gamma$ is $\Gamma$-invariant.
More generally, one defines $\tau_{\text{mod}}$-flag-limit sets of a subset $Z \subset X$ as the accumulation subset of $Z$ in $\text{Flag}(\tau_{\text{mod}})$ with respect to the topology of flag-convergence.

Next, we review definitions of several classes of discrete subgroups of $G$.

(R) A discrete subgroup $\Gamma < G$ is $\tau_{\text{mod}}$-regular if for all $x \in X$ and all sequences of distinct elements $(\gamma_n)$ in $\Gamma$, the sequence $(\gamma_n x)$ is $\tau_{\text{mod}}$-regular. For $\tau_{\text{mod}}$-regular subgroups $\Gamma$, the flag limit set $\Lambda_{\tau_{\text{mod}}} (\Gamma)$ provides a compactification of the orbit $\Gamma x \subset X$, i.e., $\Gamma x \sqcup \Lambda_{\tau_{\text{mod}}} (\Gamma)$ is compact.

(RA) A $\tau_{\text{mod}}$-regular subgroup $\Gamma$ is $\tau_{\text{mod}}$-RA (regular antipodal) if its limit set $\Lambda_{\tau_{\text{mod}}} (\Gamma)$ is antipodal, i.e., every two distinct elements of $\Lambda_{\tau_{\text{mod}}} (\Gamma)$ are antipodal to each other. For $\tau_{\text{mod}}$-RA subgroups $\Gamma$, the action $\Gamma \curvearrowright \Lambda_{\tau_{\text{mod}}} (\Gamma)$ is a convergence action (see [KLP14, Prop. 5.38]). A $\tau_{\text{mod}}$-RA subgroup $\Gamma$ is called nonelementary if $\Lambda_{\tau_{\text{mod}}} (\Gamma)$ consists of at least three (hence infinitely many) points; otherwise $\Gamma$ is called elementary. If $\Gamma$ is nonelementary then the action $\Gamma \curvearrowright \Lambda_{\tau_{\text{mod}}} (\Gamma)$ is minimal, i.e., every orbit of $\Gamma$ is dense, and $\Lambda_{\tau_{\text{mod}}} (\Gamma)$ is perfect.\footnote{This follows from a general result for convergence actions by Gehring-Martin ([GM87]) and Tukia ([Tuk94]). See also [KLP14, Subsec. 3.2] or [KLP17, Subsec. 3.3].}

(RC) For a $\tau_{\text{mod}}$-regular subgroup $\Gamma$, a limit simplex $\tau \in \Lambda_{\tau_{\text{mod}}} (\Gamma)$ is a conical limit point if there exists $x \in X$, $c > 0$ and a sequence $(\gamma_n)$ of pairwise distinct isometries on $\Gamma$ such that

$$d_R(\gamma_n x, V(x, st(\tau))) \leq c$$

where $d_R$ denotes the Riemannian distance on $X$. The set of all conical limit simplices is denoted by $\Lambda_{\tau_{\text{mod}}}^\text{con} (\Gamma)$. A subgroup $\Gamma < G$ is called $\tau_{\text{mod}}$-RC if $\Lambda_{\tau_{\text{mod}}} (\Gamma) = \Lambda_{\tau_{\text{mod}}}^\text{con} (\Gamma)$.

(RCA) A subgroup $\Gamma$ is $\tau_{\text{mod}}$-RCA if it is both $\tau_{\text{mod}}$-RA and $\tau_{\text{mod}}$-RC.

(U) A finitely generated subgroup $\Gamma < G$ (equipped with the word metric) is said to be undistorted if one (equivalently, every) orbit map $\Gamma \rightarrow \Gamma x \subset X$ is a quasiisometric embedding.

(UR) A discrete subgroup $\Gamma < G$ is uniformly $\tau_{\text{mod}}$-regular if for all $x \in X$ and all sequences of distinct elements $(\gamma_n)$ in $\Gamma$, the sequence $(\gamma_n x)$ is uniformly $\tau_{\text{mod}}$-regular.

(URU) A subgroup $\Gamma < G$ is said to be $\tau_{\text{mod}}$-URU if it is both $\tau_{\text{mod}}$-uniformly regular and undistorted.

In [KLP17, Equiv. Thm. 1.1] and [KLP18], the properties Morse, RCA and URU are proven to be equivalent to the Anosov property defined by Labourie [Lab06] and Guichard-Wienhard [GW12].

**Theorem 1.1** ([KLP17, Equiv. Thm. 1.1]). The following classes of nonelementary discrete subgroups of $G$ are equal:

(i) $\tau_{\text{mod}}$-RCA,
1.7. Illustrating examples: In this paper, we consider the following two classes of examples.

Example 1.2 (Product of rank-one symmetric spaces). Let $X$ be a product of $k$ rank-one symmetric spaces,

$$X = X_1 \times \cdots \times X_k.$$  

The rank of $X$ is $k$. Let $G$ be a semisimple Lie group commensurable with the isometry group of $X$. (For example, we may take $G = \text{Isom}(X_1) \times \cdots \times \text{Isom}(X_k)$.) The Assumption 3 amounts to the requirement that $G$ preserves the factors of the direct product decomposition of $X$.

The model maximal flat $F_{\text{mod}}$ can be viewed as the product of some chosen geodesic lines (coordinate axes), one for each deRham factor. The Weyl group $W$ is generated by reflections along the coordinate hyperplanes and the longest element in it is the reflection about the origin. The model Weyl chamber $\Delta$ can be realized as the nonnegative orthant. The opposition involution $\iota$ acts on it trivially.

Recall that the Tits boundary of a product of two symmetric spaces is the simplicial join of their individual Tits buildings and, for rank-one symmetric spaces, the Tits boundary is discrete. These two facts imply that the $(p-1)$-simplices in the Tits building of $X$ for $1 \leq p \leq k$ can be parametrized by $p$-tuples $(\xi_{r_1}, \ldots, \xi_{r_p}) \in \partial_\infty X_{r_1} \times \cdots \times \partial_\infty X_{r_p}$, $1 \leq r_1 < \cdots < r_p \leq k$,

$$(\xi_{r_1}, \ldots, \xi_{r_p}) \leftrightarrow \tau = \text{span}\{\xi_{r_1}, \ldots, \xi_{r_k}\}.$$  

We say that such a simplex $\tau$ has type $\tau_{\text{mod}} = (r_1, \ldots, r_p)$. The incidence structure can be understood as follows: Two simplices have a common $q$-face if and only if they have $q$ equal coordinates.

The star $\text{st}(\tau)$ of $\tau = (\xi_{r_1}, \ldots, \xi_{r_p})$ is the minimal subcomplex of the Tits building containing all chambers $(\xi_1, \ldots, \xi_p)$ satisfying $\xi_{r_i} = \xi_{r_i}$, for all $i \in \{1, \ldots, p\}$.

Since the opposition involution $\iota$ fixes each chamber point-wise, every face $\tau_{\text{mod}}$ of $\sigma_{\text{mod}}$ and every type is $\iota$-invariant. Every two chambers (resp. faces of the same type) in $\partial_{\text{Tits}} X$ are antipodal to each other unless they have a common face (resp. sub-face).

Example 1.3 ($X = \text{SL}(k+1, \mathbb{R})/\text{SO}(k+1, \mathbb{R})$). We take $G = \text{SL}(k+1, \mathbb{R})$, $K = \text{SL}(k+1, \mathbb{R})$; the symmetric space $X = G/K$ is identified with the set of all positive definite, symmetric matrices in $\text{SL}(k+1, \mathbb{R})$. In this case $\text{rank}(X) = k$ and $X$ is irreducible. The standard choice of a model flat $F_{\text{mod}}$ is the subset of all diagonal matrices $a = \text{diag}(a_1, \ldots, a_{k+1}) \in \text{SL}(k+1, \mathbb{R})$ with positive diagonal entries. We identify the model flat with $a$ via the logarithm map

$$\log : a = \text{diag}(a_1, \ldots, a_{k+1}) \mapsto (\log a_1, \ldots, \log a_{k+1})$$

where $a$ is viewed as the hyperplane in $\mathbb{R}^{k+1}$ consisting of all points with zero sum of coordinates.
The Cartan projection\footnote{Or the $\Delta$-valued distance in the sense that $d_{\Delta}(x, gx) = \rho(g)$.} $\rho : \text{SL}(k+1, \mathbb{R}) \to a_+$ can be written as $g \mapsto \log a$ where $a$ is associated to $g$ via the singular value decomposition $g = uav$, $u, v \in \text{SO}(k+1, \mathbb{R})$. The logarithm of $i$-th singular value of $g$ will be denoted by $\sigma_i(g)$. The opposition involution $\iota$ sends $(\sigma_1, \ldots, \sigma_{k+1}) \in a_+$ to $(-\sigma_{k+1}, \ldots, -\sigma_1)$.

The Tits building of $X$ can be identified with the incidence geometry of flags in $\mathbb{R}^{k+1}$. The Furstenberg boundary consists of full flags

$$V_1 \subset \cdots \subset V_{k+1} = \mathbb{R}^{k+1}, \quad \dim(V_i) = i.$$  

The partial flags are

$$V : V_{r_1} \subset \cdots \subset V_{r_p} \subset V_{r_{p+1}} = \mathbb{R}^{k+1}, \quad \dim(V_{r_i}) = r_i,$$  

$1 \leq r_1 < \cdots < r_p < r_{p+1} = k + 1$, which are elements of Flag($\tau_{\text{mod}}$) where $\tau_{\text{mod}} = (r_1, \ldots, r_p)$. The opposition involution sends $\tau_{\text{mod}}$ to $\iota\tau_{\text{mod}} = (k + 1 - r_p, \ldots, k + 1 - r_1)$. It follows that $\tau_{\text{mod}}$ is $\iota$-invariant if and only if $r_i + r_{p+1-i} = k + 1$, for each $i = 1, \ldots, p$. The partial flag manifold Flag($\tau_{\text{mod}}$) consisting of all partial flags $V$ of type $\tau_{\text{mod}} = (r_1, \ldots, r_p)$ naturally embeds into the product of Grassmanians $\text{Gr}_{r_1}(\mathbb{R}^{k+1}) \times \cdots \times \text{Gr}_{r_p}(\mathbb{R}^{k+1})$.

Suppose that $\tau_{\text{mod}} = (r_1, \ldots, r_p)$ is $\iota$-invariant. Then a pair $V^\pm \in \text{Flag}(\tau_{\text{mod}})$ is antipodal if and only if $V_{r_i}^+ + V_{r_{p+1-i}}^- = \mathbb{R}^{k+1}$ for each $i = 1, \ldots, p$.

2. Critical exponent

On a symmetric space $X = G/K$, we consider two natural (pseudo-)metrics. Let $d_R(\cdot, \cdot)$ denote the distance function on $X$ of the (fixed) $G$-invariant Riemannian metric on $X$. Furthermore, for a fixed $\iota$-invariant face $\tau_{\text{mod}}$ of $\sigma_{\text{mod}}$ and a fixed $\iota$-invariant type $\bar{\theta}$ in the interior of $\tau_{\text{mod}}$, we let $d_F$ denote the polyhedral Finsler (pseudo-)metric on $X$:

$$d_F(x, y) = \langle d_{\Delta}(x, y)|\bar{\theta} \rangle \quad (2.1)$$

(cf. [KL18a, Subsec. 5.1]). The inner product above is the euclidean inner product on $F_{\text{mod}}$ coming from the Riemannian metric on $X$. These two metrics are related by the inequality

$$d_F(x, y) \leq d_R(x, y). \quad (2.2)$$

Since the Finsler metric $d_F$ inherently depends on the choice of $\tau_{\text{mod}}$ and $\bar{\theta}$, from now on we fix $\bar{\theta}$ and use the notation $d_F$ to denote the corresponding Finsler metric.

The metric space $(X, d_F)$ is a complete Riemannian manifold and, in particular, it is geodesic, i.e., any two points in $X$ can be connected by a geodesic segment. The (pseudo-)metric space $(X, d_F)$ is also a geodesic space. The geodesics in $(X, d_F)$ are called Finsler geodesics. All the Riemannian
geodesics are also Finsler, however, there are other Finsler geodesics when rank($X$) $\geq 2$. The precise description of all Finsler geodesics is given in [KL18a, Subsec. 5.1.3]. We merely use this description as a definition of Finsler geodesics.

**Definition 2.1** (Finsler geodesics). Let $I \subset \mathbb{R}$. A path $\ell : I \to X$ is called a **Finsler geodesic** if there exists a pair of antipodal flags $\tau_\pm \in \text{Flag}(\tau_{\text{mod}})$ such that $\ell(I) \subset P(\tau_+, \tau_-)$ and

$$\ell(t_2) \in V(\ell(t_1), \text{st}(\tau_+)), \quad \forall t_1 \leq t_2.$$  

Moreover, given an $\iota$-invariant compact subset $\Theta \subset \text{ost}(\tau_{\text{mod}})$, a Finsler geodesic $\ell : I \to X$ is called a **$\Theta$-Finsler geodesic** if, in addition to the above, it satisfies the following stronger condition:

$$\ell(t_2) \in V(\ell(t_1), \text{ost}_\Theta(\tau_+)), \quad \forall t_1 \leq t_2.$$  

**Remark.** Finsler geodesics give alternative description of diamonds, namely, the $\tau_{\text{mod}}$-diamond $\diamondsuit_{\tau_{\text{mod}}}(x, y)$ is the union of all Finsler geodesics connecting the endpoints $x$ and $y$. See [KL18a, Subsec. 5.1.3].

**Notation.** In this paper, we use the notation $\overline{xy}$ to denote the Riemannian geodesic segment connecting a pair of points $x, y \in X$. To denote a Finsler geodesic segment connecting $x$ and $y$, we use the notation $\overline{xy}$.

Below we let $\ast$ be either R or F. Let $\Gamma < G$ be a subgroup, and $x, x_0 \in X$. Define the **orbital counting function** $N_\ast(r, x, x_0) : [0, \infty) \to [0, \infty],

N_\ast(r) = N_\ast(r, x, x_0) = \text{card}\{y \in \Gamma \mid d_\ast(x, yx_0) < r\}.$

Using $N_\ast(r)$, following [Alb99] and [Qui02b], we define the **critical exponent** $\delta_\ast$ of $\Gamma$ by

$$\delta_\ast = \limsup_{r \to \infty} \frac{\log N_\ast(r)}{r} \in [0, \infty]. \quad (2.3)$$

The critical exponents $\delta_F$ and $\delta_R$ will be called the **Finsler critical exponent** and **Riemannian critical exponent**, respectively.

**Remark.** The discussion in [Alb99] and [Qui02b] is mostly limited to the case when $\tilde{\Theta}$ is regular, i.e., belongs to the interior of $\sigma_{\text{mod}}$.

We note that the critical exponent is independent of the chosen points $x$ and $x_0$. This can be proved as follows: Consider the Poincaré series

$$g_s^\ast(x, x_0) = \sum_{y \in \Gamma} \exp(-s d_\ast(x, yx_0)). \quad (2.4)$$

It is a standard fact that $g_s^\ast(x, x_0)$ converges if $s > \delta_\ast(x, x_0)$ and diverges if $s < \delta_\ast(x, x_0)$ where $\delta_\ast(x, x_0)$ denotes the right side of (2.3). Using the triangle inequality, we obtain

$$\exp(-s d_\ast(x, x_0)) g_s^\ast(x_0, x_0) \leq g_s^\ast(x, x_0) \leq \exp(s d_\ast(x, x_0)) g_s^\ast(x_0, x_0).$$

Hence, convergence or divergence of $g_s^\ast(x, x_0)$ is independent of the choice of $x$ and so is $\delta_\ast(x, x_0)$. For a similar reason, it is also independent of the choice of $x_0$. 

12
Definition 2.2. A discrete subgroup $\Gamma$ of $G$ is of (Finsler) convergence type if the Poincaré series $g_F^\Gamma(x, x_0)$ converges at the critical exponent $\delta_F$. Otherwise, we say that $\Gamma$ has (Finsler) divergence type.

Since the action $\Gamma \curvearrowright X$ is properly discontinuous, $\delta_R$ is bounded above by the volume entropy of $X$ which is finite.\footnote{Finiteness of the volume entropy of a symmetric space follows, for instance, from the fact that $X$ has curvature bounded below combined with the Bishop-Günter volume comparison theorem, see e.g. [BC01, Sec. 11.10, Cor. 4].} For the Finsler critical exponent, (2.2) implies the following lower bound,

$$\delta_R \leq \delta_F.$$ (2.5)

Finiteness of $\delta_F$ is more subtle because, in general, $d_F$ is only a pseudo-metric and therefore, the orbital counting function $N_F$ may take infinity as a value. However, if the angular radius of the model Weyl chamber $\sigma_{\text{mod}}$ with respect to $\bar{\theta}$ is $< \pi/2$, then $d_F$ is a metric equivalent to $d_R$ and, consequently, $\delta_F$ is finite in this case. In particular, when $G$ is simple, then diameter of $\sigma_{\text{mod}}$ is $< \pi/2$ and therefore, $\delta_F$ is finite.

The following finiteness result holds in the general pseudo-metric case.

Proposition 2.3. For a uniformly $\tau_{\text{mod}}$-regular subgroup $\Gamma < G$, the Finsler critical exponent $\delta_F$ is finite.

Proof. When $\Gamma$ is uniformly $\tau_{\text{mod}}$-regular, the Riemannian and Finsler (pseudo-)metrics restricted to an orbit $\Gamma x$ are coarsely equivalent: There exist $L \geq 1, A \geq 0$ such that, for all $x_1, x_2 \in \Gamma x$,

$$L^{-1}d_R(x_1, x_2) - A \leq d_F(x_1, x_2) \leq d_R(x_1, x_2).$$ (2.6)

The right side of this inequality comes from (2.2). From this we get $\delta_R \leq \delta_F \leq L \delta_R$. Since $\delta_R$ is finite, $\delta_F$ is also finite. \hfill \Box

Remark. 1. It is clear from the proof of Proposition 2.3 that when $\Gamma$ is uniformly $\tau_{\text{mod}}$-regular, then $\delta_F$ is positive if and only if $\delta_R$ is positive.

2. As Anosov subgroups are uniformly regular (see Theorem 1.1), the above proposition applies to the class of Anosov subgroups.

Before closing this section, we compute Finsler metrics in two examples.

Example 2.4 (Product of rank-one symmetric spaces). We continue with the discussion from Example 1.2. The Finsler metric can be described as follows. Let $\tau_{\text{mod}} = (r_1, \ldots, r_p)$ be a face of the model chamber, let $\bar{\theta} = (1/\sqrt{p}, \ldots, 1/\sqrt{p})$ be its barycenter, and let $d_F$ be the corresponding metric on $X$. Given $x = (x_1, \ldots, x_k), y = (y_1, \ldots, y_k) \in X$, the $\Delta$-valued distance is

$$d_\Delta(x, y) = (d_{X_1}(x_1, y_1), \ldots, d_{X_k}(x_k, y_k))$$

where $d_{X_i}$ denotes the Riemannian distance function on $X_i$. Then

$$d_F(x, y) = \frac{1}{\sqrt{p}} \sum_{j=1}^p d_{X_j}(x_{r_j}, y_{r_j}).$$ (2.7)
**Example 2.5** ($X = \text{SL}(k+1,\mathbb{R})/\text{SO}(k+1,\mathbb{R})$). We continue with the discussion from Example 1.3. The Riemannian metric on $X$ is given by the restriction of the Killing form $B$ of $\mathfrak{g} = \text{sl}(k+1,\mathbb{R})$ to $\mathfrak{p}$,

$$B(P, Q) = 2(k+1) \text{tr}(PQ^T), \quad P, Q \in \mathfrak{g}. \quad (2.8)$$

Note that the inner product $B$ on $a$ (which we identify with $F_{\text{mod}}$) can be written as

$$\langle (\sigma_1, \ldots, \sigma_{k+1})|(\sigma'_1, \ldots, \sigma'_{k+1}) \rangle = 2(k+1) \sum_{i=1}^{k+1} \sigma_i \sigma'_i. \quad (2.9)$$

Let $\tau_{\text{mod}} = (r_1, \ldots, r_p)$ be an $\iota$-invariant face of the model chamber $\sigma_{\text{mod}}$ and let $\Delta_{\tau_{\text{mod}}}$ be the corresponding face of the model euclidean Weyl chamber $\Delta$.

$$\Delta_{\tau_{\text{mod}}} = \{ \sigma \in a_+ \mid \sigma = (\sigma_1, \ldots, \sigma_{r_1-1}, \sigma_i, \sigma_i, \sigma_{r_1}, \ldots, \sigma_{r_1-1}, \sigma_{p+1}, \ldots, \sigma_{p+1}) \}. \quad (2.10)$$

For notational convenience we denote $\sigma$ in the above expression simply by the $(p+1)$-vector $(\sigma_1, \ldots, \sigma_{p+1})$ (by identifying the repeated entries). With this convention, the opposition involution acts by

$$\iota(\sigma_1, \ldots, \sigma_{p+1}) = (-\sigma_{p+1}, \ldots, -\sigma_1).$$

We identify $\tau_{\text{mod}}$ with the unit sphere (w.r.t. the metric in (2.9)) in $\Delta_{\tau_{\text{mod}}}$ centered at the origin, i.e., $\tau_{\text{mod}}$ consists of all elements $(\sigma_1, \ldots, \sigma_{p+1}) \in \Delta_{\tau_{\text{mod}}}$ satisfying $2(k+1) \sum_{i=1}^{p+1} (r_i - r_{i-1})\sigma_i^2 = 1$. An element $\tilde{\theta} = (\sigma_1, \ldots, \sigma_{p+1}) \in \tau_{\text{mod}}$ lies in the interior of $\tau_{\text{mod}}$ if and only if $\sigma_1 > \cdots > \sigma_{p+1}$. Moreover, $\tilde{\theta}$ is $\iota$-invariant if and only if $\sigma_i + \sigma_{p+2-i} = 0$ for all $i = 1, \ldots, p + 1$.

The Finsler metric corresponding to $\tilde{\theta}$ can be calculated explicitly in terms of the above formulas. In the special case when $\tau_{\text{mod}} = (1, k)$ and $\tilde{\theta} = (1/2\sqrt{k} + 1, 0, -1/2\sqrt{k} + 1)$, for all $g \in \text{SL}(k+1,\mathbb{R})$ and all $x \in X$, we have

$$d_T(x, g x) = \sqrt{k+1} (\sigma_1(g) - \sigma_{k+1}(g)). \quad (2.10)$$

### 3. Conformal densities

Recall that Busemann functions define the notion of “distance from infinity”. For $\tau \in \text{Flag}(\tau_{\text{mod}})$, let $b_{\tau} : X \to \mathbb{R}$ denote the Busemann function based at the ideal point $\tilde{\theta}(\tau) \in \partial_{\infty} X$ normalized at $x_0$ (i.e., $b_{\tau}(x_0)$ is set to be zero). Using Busemann functions, one defines the horospherical signed distance functions as

$$d^\text{hor}_T(x, y) = b_{\tau}(x) - b_{\tau}(y). \quad (3.1)$$

(Note that these functions can take negative values. However, their absolute values $|d^\text{hor}_T(x, y)|$ satisfy the triangle inequality and, hence, are pseudo-metrics on $X$.) These functions are related by Finsler distance functions by

$$d^\text{hor}_T(x, y) = \lim_{n \to \infty} (d_F(x, z_n) - d_F(y, z_n)). \quad (3.2)$$
whenever \((z_n)\) is a sequence in \(X\) flag-converging to \(\tau\), cf. [KL18a, Prop. 5.43].

We define conformal densities on \(\text{Flag}(\tau_{\text{mod}})\) using these horospherical distance functions.

For a topological space \(S\), we let \(M_+(S)\) denote the set of positive, totally finite, regular Borel measures on \(S\). Recall that a group \(G\) of self-homeomorphisms of \(S\) acts on \(M_+(S)\) by pull-back: For every \(B \in \mathcal{B}(S), h \in H,\)

\[
\mu \mapsto h^* \mu, \quad h^* \mu(B) = \mu(h^{-1}(B)).
\]

Let \(\Gamma < G\) be a discrete subgroup and let \(A \subset X\) be a nonempty \(\Gamma\)-invariant subset. By a \(\Gamma\)-invariant conformal \(A\)-density \(\mu\) of dimension \(\beta \geq 0\) (or “conformal \(A\)-density” in short) on \(\text{Flag}(\tau_{\text{mod}})\), we mean a continuous \(\Gamma\)-equivariant map

\[
\mu : A \rightarrow M^+(\text{Flag}(\tau_{\text{mod}})), \quad a \mapsto \mu_a,
\]

satisfying the following properties:

(i) For each \(a \in A\), \(\text{supp}(\mu_a) \subset \Lambda_{\tau_{\text{mod}}} (\Gamma)\).

(ii) (Invariance) \(\mu\) is \(\Gamma\)-invariant, i.e., \(\gamma^* \mu_a = \mu_{\gamma a}\) for each \(\gamma \in \Gamma\) and each \(a \in A\).

(iii) (Conformality) For every pair \(a, b \in A\), \(\mu_a \ll \mu_b\), i.e., \(\mu_a\) is absolutely continuous with respect to \(\mu_b\), and the Radon Nikodym derivative \(d\mu_a / d\mu_b\) can be expressed as

\[
\frac{d\mu_a}{d\mu_b}(\tau) = \exp\left(-\beta d^\text{hor}_{\tau}(a, b)\right), \quad \forall \tau \in \text{Flag}(\tau_{\text{mod}}). \quad (3.3)
\]

**Remark.** Though we define conformal densities for any discrete subgroup of \(G\), for the purpose we restrict our discussion only to \(\tau_{\text{mod}}\)-regular subgroups.

A conformal \(X\)-density \(\mu\) is simply called a **conformal density**. Note that conformal \(X\)-densities and conformal \(A\)-densities are in a one-to-one correspondence:

\[
\{\text{conformal \(X\)-densities}\} \leftrightarrow \{\text{conformal \(A\)-densities}\}. \quad (3.4)
\]

From an \(X\)-density, define an \(A\)-density by restricting the family. On the other hand, given an \(A\)-density \(\mu\), extend it to an \(X\)-density \(\{\mu_x\}_{x \in X}\) by

\[
\frac{d\mu_x}{d\mu_a}(B) = \int_B \exp\left(-\beta d^\text{hor}_{\tau}(x, a)\right) d\mu_a(\tau), \quad B \in \mathcal{B}(\text{Flag}(\tau_{\text{mod}}))
\]

where \(\mu_a\) is a density in the family \(\mu\). Note that this extension is unique because \(\mu_x\) and \(\mu_a\) are absolutely continuous with respect to each other. To check \(\Gamma\)-invariance, note that

\[
\gamma^* \mu_x(B) = \int_{\gamma^{-1}B} \frac{d\mu_x}{d\mu_a}(\tau) d\mu_a(\tau) = \int_B \exp\left(-\beta d^\text{hor}_{\gamma^{-1}\tau}(x, a)\right) d\mu_a(\gamma^{-1} \tau) = \int_B \exp\left(-\beta d^\text{hor}_{\tau}(\gamma x, \gamma a)\right) d\mu_{\gamma a}(\tau) = \int_B \frac{d\mu_{\gamma x}}{d\mu_{\gamma a}}(\tau) d\mu_{\gamma a}(\tau) = \mu_{\gamma x}(B),
\]

15
for every $B \in \mathfrak{B}(\text{Flag}(\tau_{\text{mod}}))$. The other two defining properties are also satisfied.

Next we construct a conformal density using the Patterson-Sullivan construction. This definition is standard and already appeared in the work of Albuquerque and Quint, although only in the setting of Zariski dense subgroups $\Gamma < G$ and regular vectors $\tilde{\theta}$; we present it here for the sake of completeness. We let $\Gamma < G$ be a $\tau_{\text{mod}}$-regular subgroup and let $Z$ denote the $\Gamma$-orbit of a point $x_0 \in X$. The union

$$\tilde{Z} = Z \cup \Lambda_{\tau_{\text{mod}}} (\Gamma) \subset \tilde{X}_{\tau_{\text{mod}}},$$

equipped with the topology of flag-convergence, is a compactification of $Z$.

For $s > \delta_F$ we define a family of positive measures $\mu_s = \{\mu_{x,s}\}_{x \in X}$ on $\tilde{Z}$ by

$$\mu_{x,s} = \frac{1}{g_s^F(x_0, x_0)} \sum_{\gamma \in \Gamma} \exp (-s d_F(x, \gamma x_0)) D(\gamma x_0), \quad (3.5)$$

where $D(\gamma x_0)$ denotes the Dirac point mass of weight one at $\gamma x_0$. Note that $\mu_{x,s}$ is a probability measure when $x \in Z$. Also, note that $\Lambda_{\tau_{\text{mod}}} (\Gamma)$ is a null set for these measures. For $\gamma \in \Gamma$ a straightforward computation shows that

$$\gamma^* \mu_{x,s} = \mu_{x,s}. \quad (3.6)$$

Moreover, it is easy to see that the measures in the family $\mu_s$ are absolutely continuous with respect to each other. Using (3.5) we compute the Radon-Nikodym derivatives $d\mu_{x,s} / d\mu_{x_0,s}$,

$$\psi_s(z) = \frac{d\mu_{x,s}}{d\mu_{x_0,s}}(z), \quad (3.7)$$

where $s \geq 0$,

$$\psi_s(z) := \exp (-s (d_F(z, x) - d_F(z, x_0))).$$

The formula for $\psi_s$ above only makes sense when $z \in Z$. Since $\Lambda_{\tau_{\text{mod}}} (\Gamma)$ is a null set, we extend $\psi_s$ continuously to $\Lambda_{\tau_{\text{mod}}} (\Gamma)$ by setting

$$\psi_s(\tau) = \exp (-s d^\text{hor}_R(x, x_0)).$$

The continuity of this function can be verified using properties of Finsler distances (e.g., see [KL18a, Sec. 5.1.2] and (3.2)).

Next we prove that $\psi_s \to \psi_{\delta_F}$ uniformly as $s \to \delta_F$. For $S \geq s, s' > \delta_F$ and $z \in Z$,

$$|\psi_{s'}(z) - \psi_s(z)| = |\exp (-s' (d_F(z, x) - d_F(z, x_0))) - \exp (-s (d_F(z, x) - d_F(z, x_0)))|$$

$$= \exp (-s (d_F(z, x) - d_F(z, x_0))) \exp ((s' - s) (d_F(z, x) - d_F(z, x_0))) - 1|$$

$$\leq \exp (S d_R(x, x_0)) \exp ((s' - s) (d_F(z, x) - d_F(z, x_0))) - 1|.$$

Switching $s$ and $s'$ in the above, we also get

$$|\psi_{s'}(z) - \psi_s(z)| \leq \exp (S d_R(x, x_0)) \exp ((s' - s) (d_F(z, x) - d_F(z, x_0))) - 1|.$$
Combining the above two inequalities, we get

$$\abs{\psi_s'(z) - \psi_s(z)} \leq \exp(Sd_R(x, x_0)) (\exp(|s' - s| \cdot d_F(z, x) - d_F(z, x_0)) - 1) \leq \exp(Sd_R(x, x_0)) (\exp(|s' - s|d_R(x, x_0)) - 1).$$

Since $Z$ is dense in $ar{Z}$, the above yields

$$\|\psi_s - \psi_s'\|_\infty \leq \exp(Sd_R(x, x_0)) (\exp(|s' - s|d_R(x, x_0)) - 1).$$

Therefore, $\psi_s \to \psi_{\delta_F}$ uniformly as $s \to \delta_F$.

Now we construct a conformal density as a limit of the family of densities $\{\mu_s\}_{s > \delta_F}$. We first assume that $\Gamma$ has divergence type. Then, as $s$ decreases to $\delta_F$, the family $\mu_s = \{\mu_{x,s}\}_{x \in X}$ weakly accumulates to a density $\mu$ supported on some subset of $\Lambda_{\text{mod}}(\Gamma)$.

By (3.6) we have the $\Gamma$-invariance of $\mu$, namely, for $\gamma \in \Gamma$,

$$\gamma^* \mu_x = \mu_{\gamma x}. \tag{3.8}$$

Any such limit density is called a Patterson-Sullivan density.

Since $\mu_x$ is obtained as a weak limit of the measures $\mu_{x,s}$ and the derivatives $\psi_s = d\mu_{x,s} / d\mu_{x_0,s}$ converge uniformly to $\psi_{\delta_F}$, it follows that the Radon-Nikodym derivative $d\mu_x / d\mu_{x_0}$ exists and equals to the limit

$$\lim_{s \to \delta_F} \frac{d\mu_{x,s}}{d\mu_{x_0,s}} = \psi_{\delta_F},$$

or more explicitly,

$$\frac{d\mu_x}{d\mu_{x_0}}(\tau) = \exp\left(-\delta_F h_{\tau}(x, x_0)\right). \tag{3.9}$$

Note that in general weak limits are not unique. In Corollary 8.4 we will prove that for Anosov subgroups $\Gamma$ we get a unique density in this limiting process.

When $\Gamma$ has convergence type, we change weights of the Dirac masses by a small amount (as in [Nic89, Sec. 3.1]) in the definition (3.5) to force the Poincaré series to diverge. Define

$$\mu_{x,s} = \frac{1}{g_s^F(x_0, x_0)} \sum_{\gamma \in \Gamma} \exp\left(-sd_F(x, \gamma x_0)\right) h(d_F(x, \gamma x_0)) D(\gamma x_0)$$

where $h : \mathbb{R}_+ \to \mathbb{R}_+$ is a subexponential function such that the following modified Poincaré series

$$\tilde{g}_s^F(x, x_0) = \sum_{\gamma \in \Gamma} \exp\left(-sd_F(x, \gamma x_0)\right) h(d_F(\gamma x, x_0))$$

diverges at the critical exponent $s = \delta_F$. In this case also, limit density $\mu$ has the properties (3.8) and (3.9).

The existence of a conformal density implies that the Finsler critical exponent of $\Gamma$ is positive.

---

\[\text{Footnote: This will be the case for Anosov subgroups. See Corollary 6.5.}\]
Proposition 3.1. Suppose that $\Gamma$ is a nonelementary $\tau_{\text{mod}}$-regular antipodal subgroup. Then, the critical exponent $\delta_{\mathcal{F}}$ is positive.

**Proof.** Suppose to the contrary that $\delta_{\mathcal{F}} = 0$. Let $\mu$ be a Patterson-Sullivan density constructed above. It follows from the $\Gamma$-invariance and conformality that for all $\gamma \in \Gamma$,

$$\mu_x(\gamma A) = \mu_{\gamma^{-1}x}(A) = \mu_x(A), \quad \forall A \in \mathcal{B}(\Lambda_{\tau_{\text{mod}}}(\Gamma)). \quad (3.10)$$

Note that this implies that $\mu$ is atom-free. For if $\tau \in \Lambda_{\tau_{\text{mod}}}(\Gamma)$ were an atom, then, by the minimality of the action $\Gamma \curvearrowright \Lambda_{\tau_{\text{mod}}}(\Gamma)$ and (3.10), $\Lambda_{\tau_{\text{mod}}}(\Gamma)$ would have infinite $\mu_x$-mass.

Let $(\gamma_n)$ be a sequence on $\Gamma$ such that $\gamma_n^{-1} \rightarrow \tau_{\pm} \in \Lambda_{\tau_{\text{mod}}}(\Gamma)$. Let $(U_n), U_n \subset \text{Flag}(\tau_{\text{mod}})$, be a contraction sequence$^9$ for $(\gamma_n)$. By the definition,

(i) $(U_n)$ exhausts $\text{Flag}(\tau_{\text{mod}})$ in the sense that every compact set antipodal to $\tau_-$ is contained in $U_n$ for all sufficiently large $n$.

(ii) The sequence $\gamma_n U_n$ Hausdorff-converges to $\tau_+$.

Let $A \subset \Lambda_{\tau_{\text{mod}}}(\Gamma) - \{\tau_+\}$ be a compact set of positive mass (this exists because $\tau_+$ has zero mass). Therefore, by property (1), there exists $n_0 \in \mathbb{N}$ such that $\mu_x(U_n) \geq \mu_x(A) > 0$, for all $n \geq n_0$, and together with property (2) above, we get

$$\mu_x(\tau_+) \geq \lim_{n \to \infty} \mu_x(\gamma_n U_n) \geq \mu_x(A) > 0$$

Hence $\tau_+$ is an atom which gives a contradiction. $\square$

Remark. As a corollary to the above proposition, the Riemannian critical exponent $\delta_{\mathcal{R}}$ of a nonelementary uniformly $\tau_{\text{mod}}$-regular antipodal subgroup is also positive. See the remark after Proposition 2.3.

4. Hyperbolicity of Morse image

In this section we prove that the image of a Morse map is Gromov-hyperbolic with respect to the Finsler pseudo-metric $d_\mathcal{F}$. As a corollary, we prove that each orbit of an Anosov subgroup is also Gromov-hyperbolic with respect to the Finsler metric.

We first recall two notions of hyperbolicity.

**Definition 4.1** (Rips hyperbolic). Let $(Z, d)$ be a geodesic metric space. Then, $(Z, d)$ is called $\delta(\geq 0)$-hyperbolic in the sense of Rips (or Rips hyperbolic) if every geodesic triangle $\triangle$ is $\delta$-thin, i.e., each side of $\triangle$ lies in the $\delta$-neighborhood of the union of the other two sides.

$^9$See [KLP14, Def. 5.9, Prop. 5.14] or [KLP17, Defn. 4.1, Prop. 4.13].
**Definition 4.2** (Gromov hyperbolic). Let \((Z, d)\) be a geodesic metric space. For any three points \(z, z_1, z_2 \in Z\), the *Gromov product* is defined as

\[
\langle z_1 | z_2 \rangle_z = \frac{1}{2} [d(z, z_1) + d(z, z_2) - d(z_1, z_2)].
\]

Then \((Z, d)\) is called \(\delta(\geq 0)\)-hyperbolic in the sense of Gromov (or *Gromov hyperbolic*) if the Gromov product satisfies the following *ultrametric inequality*: For all \(z, z_1, z_2, z_3 \in Z\),

\[
\langle z_1 | z_2 \rangle_z \geq \min \{\langle z_1 | z_3 \rangle_z, \langle z_2 | z_3 \rangle_z\} - \delta.
\]

It should be noted that Gromov’s definition applies to all metric spaces whereas Rips’ definition works only for geodesic metric spaces. Moreover, Gromov hyperbolicity is not quasiisometric invariant whereas Rips hyperbolicity is (as a consequence of Morse lemma, cf. [DK18, Cor. 11.43])). For geodesic metric spaces, these two notions of hyperbolicity are equivalent (e.g., see [DK18, Lemma 11.27]).

Let \((Z', d')\) be Rips hyperbolic and \(f : (Z', d') \rightarrow (X, d_R)\) be a \(\tau_{\text{mod}}\)-Morse map. We denote the image \(f(Z')\) by \(Z\). Recall that the Finsler metric is coarsely equivalent to the Riemannian metric on \(Z\). Therefore, since \(f\) is a quasiisometric embedding with respect to \(d_R\), it is also a quasiisometric embedding with respect to \(d_f\). Moreover, the image of a geodesic (of length bounded below by a constant) in \(Z\) stays within a uniformly bounded Riemannian distance, say \(\lambda_0 \geq 0\), from a \(\tau_{\text{mod}}\)-regular Finsler geodesic connecting the images of the endpoints. This is a consequence of the Morse property ([KLP18, Thm. 1.1]), see also [KL18a, Prop. 12.2]. A consequence of this is that \(Z\) is \(\lambda_0\)-quasiconvex in \(X\) with respect to the Finsler metric (or *Finsler quasiconvex*).

For \(\lambda \geq \lambda_0\), let \(Y = Y_\lambda\) be the Riemannian \(\lambda\)-neighborhood of \(Z\) in \(X\). From the discussion above, it is clear that any two points \(z_1, z_2 \in Z\) (with \(d_R(z_1, z_2)\) sufficiently large) can be connected by a Finsler geodesic \(\hat{z}_1 \hat{z}_2\) in \(Y\).

**Proposition 4.3.** Let \(c\) and \(c'\) be two Finsler geodesics in \(Y\) connecting two points \(z_1, z_2\). Then they are uniformly Hausdorff close. Here the Hausdorff distance is induced by either of Riemannian or Finsler metric.

**Proof.** Since Riemannian and Finsler metrics are comparable on \(Y\), it is enough to prove the proposition for the Riemannian metric.

Let \(\hat{c}\) and \(\hat{c}'\) be the respective nearest point projections of \(c\) and \(c'\) to \(Z\). Applying the coarse inverse of \(f\), \(\hat{c}\) and \(\hat{c}'\) map to uniform quasigeodesics \(\hat{c}\) and \(\hat{c}'\), respectively, in \(Z'\). Since \(Z'\) is Rips hyperbolic, \(\hat{c}\) and \(\hat{c}'\) are uniformly close. Applying \(f\) to \(\hat{c}\) and \(\hat{c}'\), we see that \(c\) and \(c'\) are uniformly close. Hence \(c\) and \(c'\) are also uniformly close. \(\square\)

Next we observe that geodesic triangles in \((Y, d_f)\) with vertices on \(Z\) are uniformly thin.

**Proposition 4.4.** There exists \(\delta \geq 0\) such that every Finsler geodesic triangle \(\Delta = \Delta(z_1, z_2, z_3)\) in \(Y\) is \(\delta\)-thin both in Riemannian and Finsler sense.

---

\(^{10}\)This is also true for any finite Riemannian tubular neighborhood of \(Z\).
Proof. Since $Z'$ is Rips hyperbolic, geodesic triangles in $Z'$ are $\delta'$-thin, for some $\delta' \geq 0$. We map $\triangle$ to a uniformly quasigeodesic triangle $\triangle' \subset Z'$ via the coarse inverse map $Y \to Z'$ of the map $f$. Since $Z'$ is Rips-hyperbolic, the Morse quasigeodesic triangle $\triangle'$ is uniformly thin. Therefore, $\triangle$ is also uniformly thin as well. □

Imitating the proof of [DK18, Lem. 11.27], we prove that $(Z, d_F)$ is Gromov-hyperbolic.

**Theorem 4.5** (Hyperbolicity of Morse maps). Let $Z \subset X$ be the image of a $\tau_{\text{mod}}$-Morse map $f : (Z', d') \to (X, d_R)$. Then $(Z, d_F)$ is Gromov-hyperbolic.

Proof. Let $\delta$ be as in Proposition 4.4. Then the following holds.

**Lemma 4.6.** Let $z, z_1, z_2 \in Z$, and let $\overline{z_1z_2}$ be any Finsler geodesic in $Y$ connecting $z_1$ and $z_2$. Then,

\[ \langle z_1|z_2 \rangle_z \leq d_F(z, \overline{z_1z_2}) \leq \langle z_1|z_2 \rangle_z + 2\delta. \]

Proof. The proof is exactly same as [DK18, Lem. 11.22]. Note that the proof uses $\delta$-thinness of a triangle with vertices $z, z_1, z_2$. □

Let $z, z_1, z_2, z_3$ be any four points in $Z$, and let $\triangle$ be a Finsler geodesic triangle in $Y$ with the vertices $z_1, z_2, z_3$. Let $m$ be a point on the side $\overline{z_1z_2}$ nearest to $z$. By Proposition 4.4, since $\triangle$ is $\delta$-thin, $d_F(m, \overline{z_2z_3} \cup \overline{z_1z_3}) \leq \delta$. Without loss of generality, assume that there is a point $n$ on $z_2, z_3$ which is $\delta$-close to $m$. Then, using the above lemma, we get

\[ \langle z_2|z_3 \rangle_z \leq d_F(z, \overline{z_2z_3}) \leq d_F(z, \overline{z_1z_2}) + \delta, \]

and

\[ d_F(z, \overline{z_1z_2}) \leq \langle z_1|z_2 \rangle_z + 2\delta. \]

The theorem follows from this. □

Quasiisometry of hyperbolic metric spaces extends to a homeomorphism of their Gromov boundaries. At the same time, it is proven in [KLP18] that each $\tau_{\text{mod}}$-Morse map

\[ f : Z' \to Z = f(Z') \subset X \]

extends continuously (with respect to the topology of flag-convergence) to a homeomorphism

\[ \partial_{\infty} f : \partial_{\infty} Z' \to \Lambda \subset \text{Flag}(\tau_{\text{mod}}). \]

Thus, we obtain

**Corollary 4.7.** The Gromov boundary $\partial_{\infty} Z$ of $(Z, d_F)$ is naturally identified with the flag-limit set $\Lambda \subset \text{Flag}(\tau_{\text{mod}})$ of $Z$: A sequence $(z_n)$ in $Z$ converges to a point in $\partial_{\infty} Z$ if and only if $(z_n)$ flag-converges to some $\tau \in \Lambda$.

For a $\tau_{\text{mod}}$-Anosov subgroup $\Gamma$ we know that the orbit map $\Gamma \to \Gamma x_0$ is a $\tau_{\text{mod}}$-Morse embedding (see Subsection 1.6). Then, using Theorem 4.5 we obtain:

**Corollary 4.8** (Hyperbolicity of Anosov orbits). For $x_0 \in X$, let $Z = \Gamma x_0$ where $\Gamma$ is a $\tau_{\text{mod}}$-Anosov subgroup. Then $(Z, d_F)$ is Gromov hyperbolic. The Gromov boundary of $(Z, d_F)$ is naturally identified with the $\tau_{\text{mod}}$-limit set $\Lambda_{\tau_{\text{mod}}} (\Gamma)$. 

20
5. Gromov distance at infinity

The definition of horospherical signed distances given in (3.1) is free of choice of any particular normalization for the Busemann functions. Note that

\[-d_F(x_1, x_2) \leq d_\tau^{\text{hor}}(x_1, x_2) \leq d_F(x_1, x_2).\]

Furthermore, \(d_\tau^{\text{hor}}\) satisfies the cocycle condition: For each triple \(x_1, x_2, x_3 \in X\),

\[d_\tau^{\text{hor}}(x_1, x_2) + d_\tau^{\text{hor}}(x_2, x_3) = d_\tau^{\text{hor}}(x_1, x_3).\] \hspace{1cm} (5.1)

For a pair of antipodal simplices \(\tau_+ \in \text{Flag}(\tau_{\text{mod}})\), the Gromov product with respect to a base point \(x \in X\) is defined as

\[\langle \tau_+ | \tau_- \rangle_x = \frac{1}{2} \left( d_\tau^{\text{hor}}(x, z) + d_\tau^{\text{hor}}(x, z) \right),\] \hspace{1cm} (5.2)

where \(z\) is some point on the parallel set \(P(\tau_+, \tau_-)\) spanned by \(\tau_\pm\).

The following lemma proves that the Gromov products do not depend on the chosen \(z \in P(\tau_+, \tau_-)\).

**Lemma 5.1.** For \(z_1, z_2 \in P(\tau_+, \tau_-)\), one has \(b_{\tau_+}(z_1) + b_{\tau_-}(z_1) = b_{\tau_+}(z_2) + b_{\tau_-}(z_2)\).

**Proof.** Let \(z\) be the midpoint of \(z_1z_2\) and let \(s_z : X \to X\) be the point reflection about \(z\). Assuming that Busemann functions are normalized at \(z\), \(s_z\) transforms \(b_{\tau_+}(z_1) + b_{\tau_-}(z_1)\) into \(b_{\tau_+}(z_2) + b_{\tau_-}(z_2)\). Hence the quantities are equal.

Using the Gromov product, we define a premetric\(^{11}\) on \(\text{Flag}(\tau_{\text{mod}})\).

**Definition 5.2** (Gromov premetric). Given fixed \(x \in X\), \(\epsilon > 0\), define the Gromov premetric \(d^{\text{G,}\epsilon}_x\) on \(\text{Flag}(\tau_{\text{mod}})\) as

\[d^{\text{G,}\epsilon}_x(\tau_1, \tau_2) = \begin{cases} \exp(-\epsilon \langle \tau_1 | \tau_2 \rangle_x), & \text{if } \tau_1, \tau_2 \text{ are antipodal,} \\ 0, & \text{otherwise.} \end{cases}\]

Note that a pair of points \(\tau_\pm \in \text{Flag}(\tau_{\text{mod}})\) is antipodal if and only if \(d^{\text{G,}\epsilon}_x(\tau_+, \tau_-) \neq 0\).

**Lemma 5.3.** \(d^{\text{G,}\epsilon}_x\) is a continuous function.

**Proof.** The claim follows from [Bey17, Lem. 3.8]. □

**Lemma 5.4.** Let \(\gamma \in G\) and \(\Lambda \subset \text{Flag}(\tau_{\text{mod}})\) be a \(\gamma\)-invariant antipodal subset. Then the map \(\gamma : \Lambda \to \Lambda\) is conformal with respect to the premetric \(d^{\text{G,}\epsilon}_x\).

---

\(^{11}\) A *premetric* on \(X\) is a symmetric, continuous function \(d : X \times X \to [0, \infty)\) such that \(d(x, x) = 0\) for all \(x \in X\).
Proof. Given distinct points \( \tau_+ \in \Lambda \),

\[
d^x_G(\gamma \tau_+, \gamma \tau_-) = \exp(-\epsilon(\gamma \tau_+ | \gamma \tau_-),_x)
= \exp\left(-\frac{\epsilon}{2} \left(d^\text{hor}_{\gamma \tau_+}(x, z) + d^\text{hor}_{\gamma \tau_-}(x, z)\right)\right)
= \exp\left(-\frac{\epsilon}{2} \left(d^\text{hor}_{\tau_+}(\gamma^{-1}x, \gamma^{-1}z) + d^\text{hor}_{\tau_-}(\gamma^{-1}x, \gamma^{-1}z)\right)\right)
= \exp\left(-\frac{\epsilon}{2} \left(d^\text{hor}_{\tau_+}(\gamma^{-1}x, x) + d^\text{hor}_{\tau_-}(\gamma^{-1}x, x)\right)\right)\] 

where the last equality follows from the cocycle condition (5.1). Moreover, the continuity of Busemann functions \( b_\tau \) as a function of \( \tau \) implies that

\[
\lim_{\tau_- \to \tau_+} d_{\tau_-}^\text{hor}(\gamma^{-1}x, x) = d_{\tau_+}^\text{hor}(\gamma^{-1}x, x).
\]

Therefore,

\[
\lim_{\tau_- \to \tau_+} \frac{d^x_G(\gamma \tau_+, \gamma \tau_-)}{d^x_G(\tau_+, \tau_-)} = E(\gamma, \tau_+) := \exp\left(-\epsilon d_{\tau_+}^\text{hor}(\gamma^{-1}x, x)\right).
\]

The lemma follows from this. \( \square \)

The premetric \( d^x_G \) is not a metric in general since:

(i) Pairs of distinct non-antipodal points have zero distance.

(ii) The triangle inequality may fail.

However, as we shall see below, \( d^x_G \) defines a metric when restricted to “nice” antipodal subsets \( \Lambda \subset \text{Flag}(\tau_{\text{mod}}) \) for sufficiently small \( \epsilon > 0 \).

**Theorem 5.5.** Let \( Z \subset X \) be the image of a \( \tau_{\text{mod}} \)-Morse map \( f : (Z', d') \to (X, d) \), and \( \Lambda \subset \text{Flag}(\tau_{\text{mod}}) \) be the flag limit set of \( Z \). There exists \( \epsilon_0 > 0 \) such that, for all \( 0 < \epsilon \leq \epsilon_0 \) and all \( x \in Z \), the premetric \( d^x_G \) restricts to a metric on \( \Lambda \). Moreover, the topology induced by \( d^x_G \) on \( \Lambda \) coincides with the subspace topology of \( \Lambda \subset \text{Flag}(\tau_{\text{mod}}) \).

Proof. For the first part of the theorem we only need to check that \( d^x_G \) satisfies the triangle inequality for sufficiently small \( \epsilon > 0 \). The idea of the proof is due to Gromov [Gro87]: We show that the Gromov product defined in (5.2) restricted to \( \Lambda \) satisfies an ultrametric inequality (see (5.7)).

Let \( Y \subset X \) be a Riemannian \( \lambda \)-neighborhood of \( Z \). We assume that \( \lambda \) here is so large such that \( x \in Y \) and the image of any complete geodesic \( l \) in \( Z' \) lies within distance \( \lambda \) from the parallel set spanned by the images of the ideal endpoints of \( l \) under \( \tilde{f} : \partial_\infty Z' \to \text{Flag}(\tau_{\text{mod}}) \). Note that \( \lambda \) satisfying the last condition exists as a consequence of the Morse property.

Observe that \( (Y, d_F) \) is a Gromov \( \delta \)-hyperbolic metric space for some \( \delta \geq 0 \). This follows from the Gromov hyperbolicity of \( (Z, d_F) \) (cf. Theorem 4.5) and the fact that \( Z \) and \( Y \) are (Hausdorff) \( \lambda \)-close to each other.
We recall from Väisälä [Väi05, Sec. 5] that there are multiple ways to define Gromov products on \( \Lambda \) viewed as the Gromov boundary of \((Z, d_F)\) and, hence, of \((Y, d_F)\). For a distinct pair \( \tau_{\pm} \in \Lambda \), define using the Gromov product \( \langle \cdot | \cdot \rangle \) on \((Y, d_F)\) the following two products:

\[
\langle \tau_+ | \tau_- \rangle^{\text{inf}}_x = \inf \left\{ \lim_{i,j \to \infty} \langle y^+_i | y^-_j \rangle_x \mid (y^+_n) \subset Y, y^+_n \to \tau_+ \right\}
\]

and

\[
\langle \tau_+ | \tau_- \rangle^{\text{sup}}_x = \sup \left\{ \limsup_{i,j \to \infty} \langle y^+_i | y^-_j \rangle_x \mid (y^+_n) \subset Y, y^+_n \to \tau_+ \right\}.
\]

Then the difference of the above two quantities is uniformly bounded (see [Väi05, 5.7]), namely, for all distinct pairs \( \tau_{\pm} \in \Lambda \),

\[
0 \leq \langle \tau_+ | \tau_- \rangle^{\text{sup}}_x - \langle \tau_+ | \tau_- \rangle^{\text{inf}}_x \leq 2\delta. \tag{5.4}
\]

Finally, \( \langle \cdot | \cdot \rangle^{\text{inf}}_x \) satisfies the ultrametric inequality (see [Väi05, 5.12]), i.e., for distinct triples \( \tau_1, \tau_2, \tau_3 \in \Lambda \),

\[
\langle \tau_1 | \tau_2 \rangle^{\text{inf}}_x \geq \min \left\{ \langle \tau_1 | \tau_3 \rangle^{\text{inf}}_x, \langle \tau_2 | \tau_3 \rangle^{\text{inf}}_x \right\} - \delta. \tag{5.5}
\]

By (5.4), \( \langle \cdot | \cdot \rangle^{\text{sup}}_x \) also satisfies the ultrametric inequality but with a different constant, \( 5\delta \).

Next we compare Väisälä’s Gromov products with ours (see (5.2)). Let \( \tau_{\pm} \in \Lambda \) be a pair of antipodal points and let \( P = P(\tau_+, \tau_-) \). Note that our assumption on largeness of \( \lambda \) implies that there exist uniformly \( \tau_{\text{mod}} \)-regular sequences \( (y^+_n) \) and \( (y^-_n) \) on \( Y \cap P \) such that \( y^+_n \to \tau_+ \) as \( n \to \infty \). Let \( p \in P(\tau_+, \tau_-) \). Then, the additivity of Finsler distances on \( \tau_{\text{mod}} \)-cones (cf. [KL18a, Lem. 5.10]) yields, for large \( n \), \( \langle y^+_n | y^-_n \rangle_p = 0 \). By definition,

\[
\langle y^+_n | y^-_n \rangle_x = \langle y^+_n, y^-_n \rangle_x + \frac{1}{2} \left[ (d_F(y^+_n, x) - d_F(y^+_n, p)) + (d_F(y^-_n, x) - d_F(y^-_n, p)) \right],
\]

and for large \( n \),

\[
\langle y^+_n | y^-_n \rangle_x = \frac{1}{2} \left[ (d_F(y^+_n, x) - d_F(y^+_n, p)) + (d_F(y^-_n, x) - d_F(y^-_n, p)) \right].
\]

The limit, as \( n \to \infty \), of the right side of this equation equals \( \langle \tau_+ | \tau_- \rangle_x \) (cf. (3.2)). Therefore,

\[
\langle \tau_+ | \tau_- \rangle^{\text{inf}}_x \leq \langle \tau_+, \tau_- \rangle_x \leq \langle \tau_+ | \tau_- \rangle^{\text{sup}}_x. \tag{5.6}
\]

Hence, by (5.4) and (5.5), \( \langle \cdot | \cdot \rangle_x \) satisfies the ultrametric inequality with constant \( 5\delta \), i.e., for distinct points \( \tau_1, \tau_2, \tau_3 \in \Lambda \),

\[
\langle \tau_1 | \tau_2 \rangle_x \geq \min \{ \langle \tau_1 | \tau_3 \rangle_x, \langle \tau_2 | \tau_3 \rangle_x \} - 5\delta. \tag{5.7}
\]

This completes the proof of the first part of the theorem.

For the second part, note that the inequality (5.6) implies that \( d_{G}^{\text{ray}} \) induces the standard topology on \( \Lambda \) as the Gromov boundary of \((Y, d_F)\) (see [Väi05, 5.29]). Since, as we noted earlier, this topology is the same as the subspace topology of the flag-manifold \( \text{Flag}(\tau_{\text{mod}}) \), the second claim of the theorem follows as well. \( \square \)
Corollary 5.6 (Conformal metric on Anosov limit set). Let $\Gamma$ be a $\tau_{\text{mod}}$-Anosov subgroup, $x \in X$. Then there exists $\epsilon_0 > 0$ such that for all $0 < \epsilon \leq \epsilon_0$ and all $z \in \Gamma x$, $d_G^{x,\epsilon}$ is a metric on $\Lambda_{\tau_{\text{mod}}} (\Gamma)$. Moreover, the action $\Gamma \curvearrowright \Lambda_{\tau_{\text{mod}}} (\Gamma)$ is conformal with respect to $d_G^{x,\epsilon}$.

Proof. Since Anosov subgroups satisfy the Morse property, corollary follows from Theorem 5.5 combined with Lemma 5.4. \hfill \Box

Example 5.7 (Product of rank-one symmetric spaces). We continue with Example 2.4. Let $\tau = (\xi_{r_1}, \ldots, \xi_{r_p})$ be a simplex in the Tits building of type $\tau_{\text{mod}} = (r_1, \ldots, r_p)$ and $\bar{\theta} = (1/\sqrt{p}, \ldots, 1/\sqrt{p}) \in \tau_{\text{mod}}$. We compute the horospherical distance, Gromov distance associated with $\tau_{\text{mod}}$ and type $\bar{\theta}$.

Let $x = (x_1, \ldots, x_k)$, $y = (y_1, \ldots, y_k) \in X$. Then

$$d_{\tau}^{\text{hor}} (x, y) = \lim_{t \to \infty} (d_X (\ell (t), x) - t)$$

where $\ell (t)$ is a geodesic ray emanating from $y$ and asymptotic to $\bar{\theta} (\tau)$. A direct computation yields

$$d_{\tau}^{\text{hor}} (x, y) = \frac{1}{\sqrt{p}} \sum_{j=1}^{p} \left( b_{\xi_j} (x_j) - b_{\xi_j} (y_j) \right) = \frac{1}{\sqrt{p}} \sum_{j=1}^{p} d_{\text{hor}}^{\xi_j} (x_j, y_j).$$

Hence the Gromov product can be written as

$$\langle x, y \rangle_x = \frac{1}{\sqrt{p}} \sum_{j=1}^{p} \langle \xi_j^{+}, \xi_j^{-} \rangle_{x_j}, \quad \forall x = (x_1, \ldots, x_p), y = (y_1, \ldots, y_p) \in \text{Flag} (\tau_{\text{mod}})$$

and, finally the Gromov predistance is

$$d_{G}^{x, 1/\sqrt{p}} (\tau, \tau_{-}) = \prod_{j=1}^{p} d_{G}^{x_j, 1/p} (\xi_j^{+}, \xi_j^{-}). \quad (5.8)$$

Example 5.8 ($X = \text{SL}(k+1, \mathbb{R}) / \text{SO}(k+1, \mathbb{R})$). In this case the computations of Busemann functions (see [Hat95]) and Gromov products (see [Bey17]) are explicitly known, and therefore, the Gromov distance can also be computed explicitly. We only give a formula for the Gromov distance in the special case when $\tau_{\text{mod}} = (1, k)$ that corresponds to the partial flags \{line $\subset$ hyperplane\} of $\mathbb{R}^{k+1}$.

We continue with the notations from Example 2.5. The unique $\iota$-invariant type is

$$\bar{\theta} = (1/2\sqrt{k+1}, 0, -1/2\sqrt{k+1}).$$

After equipping $\mathbb{R}^{k+1}$ with the inner product induced by the choice of $x \in X$, the Gromov product (with respect to $x = I_{k+1}$, the identity matrix) can be written as

$$\langle (l_1, h_1) | (l_2, h_2) \rangle_x = -\frac{\sqrt{k+1}}{2} \log (\sin \angle (l_1, h_2) \cdot \sin \angle (l_2, h_1))$$

where $\angle (l, h)$ denotes the angle between the line $l$ and the hyperplane $h$. Thus, the Gromov predistance can be written as

$$d_{G}^{x, 1/\sqrt{k+1}} ((l_1, h_1), (l_2, h_2)) = \sin \angle (l_1, h_2)^{1/2} \cdot \sin \angle (l_2, h_1)^{1/2}. \quad (5.9)$$
6. Shadow lemma

In this section we prove a generalization Sullivan’s shadow lemma in higher rank. The proof we present here is inspired by that of Albuquerque’s ([Alb99, Thm. 3.3]) who treated the case of full flag manifold and Quint ([Qui02b]) who treated general flag-manifolds but only in the case of regular vectors $\bar{\theta}$.

Recall the notion of shadow from (1.2). We mainly consider shadows of closed balls (with respect to the Riemannian metric) of non-zero radii in $X$ from a fixed base point $x \in X$. The topology generated by these shadows is the topology of flag convergence.

The main result in this section is the following.

**Theorem 6.1** (Shadow lemma). Let $\Gamma$ be a nonelementary $\tau_{\text{mod}}$-RA subgroup, $x \in X$, and $\mu$ a $\Gamma$-invariant conformal density of dimension $\beta$. There exists $r_0 > 0$ such that for all $r \geq r_0$ and all $\gamma \in \Gamma$ satisfying $d_R(x, \gamma x_0) > r$,

$$C^{-1} \exp (-\beta d_F(x, \gamma x_0)) \leq \mu_x(S(x : B(\gamma x_0, r))) \leq C \exp (-\beta d_F(x, \gamma x_0)),$$

for some constant $C \geq 1$.

Before presenting the proof, we note two consequences of this theorem.

**Corollary 6.2.** Let $\Gamma$ be a nonelementary uniformly $\tau_{\text{mod}}$-RA subgroup. Then any conformal density $\mu$ does not have conical limit points as atoms.

**Proof.** Any conical limit point $\tau \in \Lambda_{\tau_{\text{mod}}} \Gamma$ lies in infinitely many shadows $S(x, B(\gamma x_0, r))$ for sufficiently large $r > 0$ (depending on $\tau$). If $\tau$ is an atom, then (by Theorem 6.1) the Poincaré series

$$g^F_{\beta}(x, x_0) = \sum_{\gamma \in \Gamma} \exp (-\beta d_F(x, \gamma x_0))$$

(6.1)

diverges for every $\beta \geq 0$. Hence $\delta_F$ must be infinite. But this contradicts Proposition 2.3. □

The second application of shadow lemma will be given for the following class of subgroups.

**Definition 6.3** (Uniform conicality). A $\tau_{\text{mod}}$-RA subgroup is called uniformly conical if for a given pair of points $x, x_0 \in X$, there is a constant $r > 0$ such that for each conical limit point $\tau \in \Lambda_{\tau_{\text{mod}}} \Gamma$, there exists a sequence $(\gamma_k)$ on $\Gamma$ flag-converging to $\tau$ satisfying $d_R(\gamma_k x_0, V(x, \text{st}(\tau))) < r, \forall k \in \mathbb{N}$.

We observe that Anosov subgroups satisfy the uniform conicality condition:

**Proposition 6.4.** Anosov subgroups are uniformly conical.

**Proof.** This follows from the fact that the orbit map $\Gamma \rightarrow \Gamma x_0 \subset X$ is a Morse embedding. Let $\tau \in \Lambda_{\tau_{\text{mod}}} \Gamma$ be any point and $\xi \in \partial_\infty \Gamma$ be the preimage of $\tau$ under the boundary map. Let $(\gamma_k), \gamma_1 = 1_\Gamma$ be a geodesic sequence in $\Gamma$ asymptotic to $\xi$. Then the sequence $(\gamma_k x_0)$ is a Morse quasigeodesic in $X$ that is uniformly close to $V(x, \text{st}(\tau))$ (by definition of a Morse embedding). □
Corollary 6.5. Let $\Gamma$ be a nonelementary uniformly conical $\tau_{\text{mod}}$-RA subgroup and $\mu$ be a conformal density of dimension $\beta$. If the conical limit set $\Lambda_{\tau_{\text{mod}}}^{\text{con}}(\Gamma)$ is non-null, then the Poincaré series $g_{\beta}^F(x, x_0)$ (see (6.1)) diverges.

Proof. Writing the elements of $\Gamma$ in a sequence $(\gamma_n)$, define

$$S_N = \sum_{n \geq N} \exp(-\beta d_F(\gamma_n x_0, x)).$$

Convergence of the series (6.1) asserts that $\lim_{N \to \infty} S_N = 0$. Since $\Gamma$ is uniformly conical, there exists $r > 0$ such that for all $N \in \mathbb{N}$,

$$\Lambda_{\tau_{\text{mod}}}^{\text{con}}(\Gamma) \subset \bigcup_{n \geq N} S(x : B(\gamma_n x_0, r)).$$

Applying Theorem 6.1, we get

$$\mu_\tau(\Lambda_{\tau_{\text{mod}}}^{\text{con}}(\Gamma)) \leq \sum_{n \geq N} \mu_\tau(S(x : B(\gamma_n x_0, r))) \leq \text{const} \cdot S_N$$

and, the bound above approaches to zero as $N \to \infty$. Hence we must have $\mu_\tau(\Lambda_{\tau_{\text{mod}}}^{\text{con}}(\Gamma)) = 0$. \qed

The proof of shadow lemma occupies the rest of the section.

Proof of Theorem 6.1. In this proof, we equip $\text{Flag}(\tau_{\text{mod}})$ with a $G_x$-invariant Riemannian metric. We use the notation $L(\tau)$ to denote the set of all $\tau' \in \text{Flag}(\tau_{\text{mod}})$ which are not antipodal to $\tau$. The complement of $L(\tau)$ in $\text{Flag}(\tau_{\text{mod}})$ is denoted by $C(\tau)$. Note that $L(\tau)$ is closed and hence, compact. Moreover, if $\tau_n \to \tau_0$, then the sequence of sets $(L(\tau_n))$ Hausdorff-converges to $L(\tau_0)$.

Lemma 6.6. For every $\varepsilon > 0$, there exists $\delta > 0$ such that, for every $\tau_0 \in \text{Flag}(\tau_{\text{mod}})$ and every $\tau \in B(\tau_0, \delta)$,

$$N_{\varepsilon/2}(L(\tau)) \subset N_\varepsilon(L(\tau_0)).$$

Proof. We equip the set

$$Y = \{ L(\tau) : \tau \in \text{Flag}(\tau_{\text{mod}}) \}$$

with the Hausdorff distance $d_H$. Then, as we noted above, the function $f : \text{Flag}(\tau_{\text{mod}}) \to Y$, $\tau \mapsto L(\tau)$, is continuous and, hence, uniformly continuous. Therefore, for every $\varepsilon > 0$, there exists $\delta > 0$ such that $d(\tau, \tau_0) < \delta$ implies $d_H(L(\tau), L(\tau_0)) < \varepsilon/2$, which then implies $L(\tau) \subset N_{\varepsilon/2}(L(\tau_0))$. The lemma follows from this. \qed

Let $m = \mu_\tau(\Lambda_{\tau_{\text{mod}}}^{\text{con}}(\Gamma))$ denote the total mass of $\mu_\tau$, and $l = \sup\{ \mu_\tau(\tau) \mid \tau \in \Lambda_{\tau_{\text{mod}}}^{\text{con}}(\Gamma) \}$. Since $\mu_\tau$ is a regular measure and $\Lambda_{\tau_{\text{mod}}}^{\text{con}}(\Gamma)$ is compact, $l$ is realized, i.e., if $\mu_\tau$ has an atomic part, then it has a largest atom. Moreover, since $\Gamma$ is nonelementary, $\text{supp}(\mu_\tau)$ is not singleton. In fact, if $\tau$ is an atom, then the every point in the orbit $\Gamma \tau$ (which has infinite number of points) is an atom. In particular, $l < m$.  

26
Lemma 6.7. Given \( 1 < q < \infty \), there exists an \( \varepsilon_0 > 0 \) such that for all \( \tau \in \Lambda_{\text{mod}}(\Gamma) \) and all \( B \in \mathcal{B}(\text{Flag}(\tau_{\text{mod}})) \) contained in \( N_{\varepsilon_0}(L(\tau)) \), \( \mu_x(B) \leq q \).

Proof. If this were false, then we would get a sequence \( (B_n) \) of Borel sets, a sequence \( (\varepsilon_n) \) positive numbers converging to zero, and a sequence \( (\tau_n) \) on \( \Lambda_{\text{mod}}(\Gamma) \) converging to a point \( \tau_0 \) such that for every \( n \in \mathbb{N} \),

\[
B_n \subset N_{\varepsilon_n}(L(\tau_n)), \quad \mu_x(B_n) > q.
\]

To get a contradiction, we will show that \( \mu_x(\tau_0) \geq q \). Let \( U \) be an open neighborhood of \( L(\tau_0) \). As \( L(\tau_0) \) is compact, there exists \( \varepsilon > 0 \) such that \( N_{\varepsilon}(L(\tau_0)) \subset U \). Let \( \delta > 0 \) be a number that corresponds to this \( \varepsilon \) as in Lemma 6.6. Choose \( n \) so large such that \( \tau_n \in B(\tau_0, \delta) \) and \( \varepsilon_n \leq \varepsilon / 2 \). By Lemma 6.6, we get \( N_{\varepsilon_n}(L(\tau_n)) \subset N_{\varepsilon}(L(\tau_0)) \) and, consequently, \( B_n \subset U \). This shows that every open set \( U \) containing \( L(\tau_0) \) has mass \( \mu_x(U) > q \). Therefore, \( \mu_x(L(\tau_0)) = \mu_x(\tau_0) \geq q \). \( \Box \)

Lemma 6.8. Given \( \varepsilon > 0 \) there exists \( r_1 > 0 \) such that for all \( r \geq r_1 \), the complement of \( S(x : B(x_0, r)) \) in \( \text{Flag}(\tau_{\text{mod}}) \) is contained in \( N_{\varepsilon}(L(\tau)) \), for some \( \tau \in S(x_0 : \{x\}) \).

Proof. For \( r > 0 \) and \( \tau_0 \in \text{Flag}(\tau_{\text{mod}}) \), \( \tau' \in C(\tau_0) \), consider

\[
U(\tau_0, x_0, r) = \{ \tau' \in \text{Flag}(\tau_{\text{mod}}) \mid P(\tau_0, \tau') \cap B(x_0, r) \neq \emptyset \}.
\]

This is an analogue of shadows (1.2) as viewed from the infinity. It is easy to verify that

\[
\bigcup_{r \geq 0} U(\tau_0, x_0, r) = C(\tau_0).
\]

Moreover, for \( g \in G \), these shadows from infinity transform as \( gU(\tau_0, x_0, r) = U(\tau_0, x_0, r) \).

If \( k \in K = G_{x_0} \), the stabilizer of \( x_0 \), then \( kU(\tau_0, x_0, r) = U(k\tau_0, x_0, r) \). Since \( K \) is compact, there exists \( M \geq 1 \) such that the action \( k \curvearrowright \text{Flag}(\tau_{\text{mod}}) \) is \( M \)-Lipschitz for all \( k \in K \). Let \( r_1 > 0 \) be such that \( U(\tau_0, x_0, r_1 / 2)^c \subset N_{\varepsilon/M}(L(\tau_0)) \). Here and below, for \( A \subset \text{Flag}(\tau_{\text{mod}}) \), \( A^c = \text{Flag}(\tau_{\text{mod}}) - A \).

Then, for any \( \tau \in \text{Flag}(\tau_{\text{mod}}) \),

\[
U(\tau, x_0, r / 2)^c \subset N_{\varepsilon}(L(\tau)), \quad \forall r \geq r_1. \quad (6.2)
\]

For \( x \in X \), let \( \tau \in \text{Flag}(\tau_{\text{mod}}) \) be a simplex such that \( x \in V(x_0, \text{st}(\tau)) \). Then there exists a parameterized geodesic ray \( x_t \) starting from \( x_0 \), passing through \( x \) and asymptotic to some \( \xi \in \text{st}(\tau) \).

Claim. For all \( r > 0 \), \( S(x : B(x_0, 2r)) \supset U(\tau, x_0, r) \).

Proof of claim. Pick \( \tau' \in U(\tau, x_0, r) \) and let \( \bar{x}_0 \in P(\tau, \tau') \) denote the nearest point projection of \( x_0 \). In addition to the ray \( x_t \), we define another parameterized geodesic ray \( \bar{x}_t \), starting at \( \bar{x}_0 \) and asymptotic to \( \xi \). Due to the convexity of the Riemannian distance function on \( X \), the distance \( d_R(x_t, \bar{x}_t) \) monotonically decreases with \( t \). Moreover, the cones \( V(\bar{x}_t, \text{st}(\tau')) \) are nested as \( t \) decreases. Then,

\[
d_R(x_0, V(x_t, \text{st}(\tau'))) \leq d_R(x_0, V(\bar{x}_t, \text{st}(\tau'))) + d_R(x_t, \bar{x}_t) \\
\leq d_R(x_0, V(\bar{x}_0, \text{st}(\tau'))) + r \leq d_R(x_0, \bar{x}_0) + r \leq 2r.
\]

Therefore, \( \tau' \in S(x : B(x_0, 2r)) \). \( \Box \)
Using (6.2) it follows from the above claim that whenever \( r \geq r_1 \), the complement of the shadow \( S(x : B(x_0, r)) \) is contained in \( N_\varepsilon(L(\tau)) \) for some \( \tau \) satisfying \( x \in V(x_0, \text{st}(\tau)) \). \( \square \)

**Lemma 6.9.** For all \( r > 0 \) and all \( \tau \in S(x : B(x_0, r)) \),

\[
|d_F(x, x_0) - d^\text{hor}_\tau(x, x_0)| \leq 2r.
\]

**Proof.** We recall that the Finsler distance can alternatively be defined as

\[
d_F(y, z) = \max_{\tau \in \text{Flag}(\tau_{\text{mod}})} d^\text{hor}_\tau(y, z),
\]

where the maximum above occurs at any point in \( S(y : \{z\}) \) (see [KL18a, Sec. 5.1.2]). Fix some \( \tau_0 \in S(x, \{x_0\}) \). Then for any \( \tau_1 \in S(x : B(x_0, r)) \),

\[
|d_F(x, x_0) - d^\text{hor}_{\tau_1}(x, x_0)| = |b_{\tau_0}(x_0) - b_{\tau_1}(x_0)| \\
= |b_{\tau_0}(x_0) - b_{\tau_0}(k^{-1}x_0)| \\
\leq d_R(x_0, k^{-1}x_0) = d_R(kx_0, x_0),
\]

where \( k \in K \), stabilizer of \( x \), is some isometry satisfying \( \tau_1 = k\tau_0 \). In the above we chose the normalizations of the Busemann functions at \( x \).

Let \( y \in V(x, \text{st}(\tau)) \cap B(x_0, r) \). Then \( y \in V(x, \sigma) \) for some chamber \( \sigma \) in \( \text{st}(\tau) \). We identify \( V(x, \sigma) \) with the model Weyl chamber \( \Delta \). Let \( k_1 \in K \) such that \( k_1x \in V(x, \sigma) \). Then \( k_1x_0 = d_\Delta(x, x_0) \) via the identification above. Moreover, since the map

\[
X \rightarrow \Delta, \quad z \mapsto d_\Delta(x, z)
\]

is 1-Lipschitz (by the triangle inequality for \( \Delta \)-distances (1.1)) and \( d_\Delta(x, y) = y \), we obtain,

\[
d_R(y, k_1x_0) \leq d(y, x_0) < r
\]

and, in particular, \( d(x_0, k_1x_0) < 2r \). \( \square \)

Using the above lemmata, we now complete the proof of Theorem 6.1. We first fix some auxiliary quantities. Let \( q \in (l, m) \) and \( \varepsilon_0 \) be corresponding constant as given in Lemma 6.7. Let \( \delta \) be a constant given by Lemma 6.6 which corresponds to \( \varepsilon = \varepsilon_0 \). By \( \Lambda \) we denote the \( \delta \)-neighborhood of \( \Lambda_{\tau_{\text{mod}}} \) and let

\[
V = \bigcup_{\tau \in \Lambda} V(x, \text{st}(\tau)) \subset X.
\]

Since \( \Gamma \) is discrete, the elements of \( \Gamma \) which send \( x_0 \) outside \( V \) form a finite set \( \Phi \). Let

\[
\varepsilon_0 = \max\{r_1, d_R(x, \gamma x_0) \mid \gamma \in \Phi\}
\]

where \( r_1 \) is a constant that corresponds to \( \varepsilon_0/2 \) as in Lemma 6.8.
For every $\gamma \in \Gamma$ satisfying $d_R(x, \gamma x_0) > r \geq r_0$, we assign an element $\tau_\gamma \in S(x : \{\gamma x_0\}) \cap \Lambda$ (the intersection is nonempty by above). Using Lemma 6.6, for every such $\tau_\gamma$ there exists $\tau_0 \in \Lambda_{\tau \text{mod}}$ so that

$$N_{x_0/2}(L(\tau_\gamma)) \subset N_{x_0}(L(\tau_0)).$$

By Lemmata 6.7 and 6.8, $\mu_x(S(\gamma^{-1} x : B(x_0, r))) \geq m - q$ and by properties of conformal measures,

$$\mu_x(S(x : B(x_0, r))) = \mu_{\gamma^{-1} x}(S(\gamma^{-1} x : B(x_0, r))) = \int_{S(\gamma^{-1} x : B(x_0, r))} \exp \left( -\beta d_{\tau}^{\text{hor}}(\gamma^{-1} x, x) \right) d\mu_x$$

$$\approx \exp \left( -\beta d_F(x, \gamma x_0) \right)$$

where in the last step we have additionally used Lemma 6.9. This completes the proof. \qed

7. Dimension of a conformal density

In this section, we establish a lower bound for the dimension of a conformal density. For Anosov subgroups, we prove that the dimension equals the Finsler critical exponent (see Corollary 7.5).

**Theorem 7.1.** Suppose that $\Gamma$ is a nonelementary $\tau_{\text{mod}}$-RA subgroup. Let $\mu$ be a $\Gamma$-invariant conformal density of dimension $\beta$. Then $\beta$ has the following lower bound:

$$\beta \geq \delta_F - \delta_F^c. \quad (7.1)$$

The proof of this theorem is given at the end of this section. The number $\delta_F^c$ above quantifies the maximal exponential growth rate of the orbit $\Gamma x_0$ in a conical direction. The precise definition is given below.

**Definition 7.2 (Critical exponent in conical directions).** Suppose that $\Gamma$ is a $\tau_{\text{mod}}$-regular subgroup. For $\tau \in \Lambda_{\tau_{\text{mod}}} \setminus \{\gamma \in \Gamma \mid d_F(x, \gamma x_0) < r, d_R(\gamma x_0, V(x, \text{st}(\tau))) < c\}$ and

$$N_F^c(r, c, x, x_0, \tau) = \text{card}\{\gamma \in \Gamma \mid d_F(x, \gamma x_0) < r, d_R(\gamma x_0, V(x, \text{st}(\tau))) < c\}$$

and

$$\delta_F^c(\Gamma) = \sup_{\tau \in \Lambda_{\tau_{\text{mod}}} \setminus \{\gamma \in \Gamma \mid d_F(x, \gamma x_0) < r, d_R(\gamma x_0, V(x, \text{st}(\tau))) < c\}} \left( \lim_{c \to \infty} \sup_{r \to \infty} \frac{\log N_F^c(r, c, x, x_0, \tau)}{r} \right).$$

Note that it is sufficient to take the supremum in the definition of $\delta_F^c(\Gamma)$ over the conical limit set $\Lambda_{\tau_{\text{mod}}}^{\text{con}}(\Gamma)$. For rank-one symmetric spaces, and, more generally, for $\sigma_{\text{mod}}$-regular subgroups, this number is zero. Below we see that for $\tau_{\text{mod}}$-Anosov subgroups also, $\delta_F^c(\Gamma) = 0$. It should be noted that, however, for general discrete subgroups, $\delta_F^c$ could be $\infty$.

**Proposition 7.3.** Suppose that $\Gamma$ is a nonelementary $\tau_{\text{mod}}$-Anosov subgroup. Then the function $N(r) = N_F^c(r, c, x, x_0, \tau)$ grows linearly with $r$. In particular, $\delta_F^c(\Gamma) = 0$. 

29
Proof. Without loss of generality, we can assume that \( x = x_0 \).

**Lemma 7.4.** Fix \( c > 0 \). For any \( \tau \in \Lambda_{\tau_{\text{mod}}}(\Gamma) \), the set

\[
\{ \gamma x_0 \mid \gamma \in \Gamma, \ d_R(\gamma x_0, V(x, st(\tau))) < c \}
\]

is within a uniformly bounded distance from a uniform \( \tau_{\text{mod}} \)-Morse quasiray \( \alpha \) emanating from \( x_0 \) and asymptotic to \( \tau \).

**Proof.** Let \( \tau \in \Lambda_{\tau_{\text{mod}}}(\Gamma) \) be arbitrary. Denote the preimage of \( \tau \) in \( \partial_{\infty} \Gamma \) under the boundary homeomorphism \( \partial_{\infty} \Gamma \rightarrow \Lambda_{\tau_{\text{mod}}}(\Gamma) \) by \( \zeta \). Since \( \Gamma \) is discrete, we can arrange the elements of \( \{ \gamma \in \Gamma \mid d_R(\gamma x_0, V(x, st(\tau))) < c \} \) in a sequence \( (\gamma_n) \). The sequence \( x_n = \gamma_n x_0 \) converges conically to \( \tau \). Let \( \alpha : \mathbb{Z}_{\geq 0} \rightarrow X \) be the image (under the orbit map \( \Gamma \rightarrow \Gamma x \)) of a parametrized geodesic ray \( \mathbb{Z}_{\geq 0} \rightarrow \Gamma \) starting at \( 1 \Gamma \) and asymptotic to \( \zeta \). Then \( \alpha \) is a uniform \( \tau_{\text{mod}} \)-Morse quasiray starting at \( x_0 \) and asymptotic to \( \tau \). Hence \( \alpha \) is uniformly close to \( V(x_0, st(\tau)) \). Since both sequence \( (x_n) \) and \( (\alpha(n)) \) are uniformly close to \( V(x_0, st(\tau)) \), it is enough to understand the simpler case when \( \alpha(n), x_n \in V(x_0, st(\tau)) \), for all \( n \in \mathbb{N} \).

We claim that that the sequence \( (x_n) \) is uniformly close to \( \alpha \). Otherwise, after extraction, \( (x_n) \) would diverge away from \( \alpha \). Since \( \alpha \) is a Morse quasiray, \( \alpha \) eventually enters each cone \( V(x_n, st(\tau)) \), but further and further away from the tip \( x_n \) as \( n \) grows. Since the separation between two successive points on \( \alpha \) (being a quasigeodesic) is uniformly bounded, we could find arbitrarily large \( m \)’s such that \( \alpha(m) \) is uniformly close to the boundary of a cone \( V(x_n, st(\tau)) \) and is arbitrarily far away from its tip \( x_n \). But this would contradict the \( \tau_{\text{mod}} \)-regularity of the group \( \Gamma \). \( \Box \)

We continue with the notations from the proof of the lemma. Since any \( \tau_{\text{mod}} \)-Anosov subgroup \( \Gamma < G \) is uniformly \( \tau_{\text{mod}} \)-regular (cf. Theorem 1.1), we may work with the Riemannian metric in place of the Finsler metric. Moreover, we may assume that the sequence \( (x_n) \) is sufficiently spaced. Let \( \bar{x}_n \) denote the nearest-point projection of \( x_n \) to the image of \( \alpha \). The above lemma implies that \( d(x_n, \bar{x}_n) \) is uniformly bounded. Since \( x_n \)’s are sufficiently spaced, \( \bar{x}_n \)’s are also sufficiently spaced which guarantees that \( d_R(\bar{x}_n, x_0) \geq \text{const} \cdot n \), for all large \( n \), which in turn implies that \( d_R(x_n, x_0) \geq \text{const} \cdot n \). The proposition follows from this. \( \Box \)

As a corollary of the above results, we obtain that any \( \Gamma \)-invariant conformal density must have dimension \( \delta_{\text{F}} \) when \( \Gamma \) is \( \tau_{\text{mod}} \)-Anosov. The Patterson-Sullivan densities constructed in Section 3 also had this dimension.

**Corollary 7.5.** Suppose that \( \Gamma \) is a nonelementary \( \tau_{\text{mod}} \)-Anosov subgroup. Let \( \mu \) be a \( \Gamma \)-invariant conformal density of dimension \( \beta \). Then \( \beta = \delta_{\text{F}} \).

**Proof.** By Corollary 6.5 we know that the Poincaré series \( g_{\beta}^F(x, x_0) \) diverges and, consequently, \( \beta \leq \delta_{\text{F}} \). The reverse inequality is obtained in combination of Theorem 7.1 and Proposition 7.3. \( \Box \)

To close this section, we prove Theorem 7.1.

---

\footnote{Note that the number \( \delta_{\text{F}}(\Gamma) \) does not depend on \( x \) and \( x_0 \) as we have seen in the case of \( \delta_{\text{F}} \) in Sec. 2.}
Proof of Theorem 7.1. We fix some \( r \geq r_0 \) where \( r_0 \) is given by Theorem 6.1. Assume that the stabilizer of \( x_0 \) in \( \Gamma \) is trivial in which case the function \( N(R) = N_F(R, x, x_0) \) counts the number of orbit points (in \( \Gamma x_0 \)) within the Finsler \( r \)-ball centered at \( x \). The general case follows immediately.

We place a Riemannian ball of radius \( r \) at each point in the orbit. In this proof, we reserve the word \textit{ball} to specify these balls. Let

\[
c = \min_{1 \neq \gamma \in \Gamma} \{ d_R(x_0, \gamma x_0) \}.
\]

There exists a number \( N \in \mathbb{N} \) that depends only on \( r, c, \) and \( X \) such that any ball intersect at most \( N \) other balls (including itself). Note that the shadows in \( \text{Flag}(\tau_{\text{mod}}) \) from \( x \) of two distinct \( \tau_{\text{mod}} \)-cone with tip at \( x \). Also note that, at large distances from \( x \), the balls do not intersect the boundaries of the \( \tau_{\text{mod}} \)-cones because of the \( \tau_{\text{mod}} \)-regularity of the orbit.

Let \( n_R \) denote the maximal number of balls in \( B_F(x, R) \) that intersect a particular \( \tau_{\text{mod}} \)-cone \( V(x, \text{st}(\tau)) \). It follows from the definition of \( \delta_F^c(\Gamma) \) that

\[
\limsup_{R \to \infty} \frac{\log n_R}{R} \leq \delta_F^c(\Gamma). \tag{7.2}
\]

On the other hand, for each \( \tau \in \Lambda_{\tau_{\text{mod}}}(\Gamma) \), the maximal number of balls in \( B_F(x, R) \) whose shadows intersect \( \tau \) is \( n_R \). Therefore,

\[
\frac{N_F(R, x, x_0)}{N \cdot n_R} s(R) \leq m = \text{total mass of } \mu_x, \tag{7.3}
\]

where \( s(R) \) is any lower bound for the measures of the shadows of balls in \( B_F(x, R) \). We note that the shadow lemma (Theorem 6.1) produces such a positive lower bound\(^{13}\), namely, we may take \( s(R) = \text{const} \cdot e^{-\beta R} \). Then (7.3) yields

\[
N_F(R, x, x_0) \leq \frac{mN \cdot n_R e^{\beta R}}{\text{const}}.
\]

Together with (7.2), the above results in (7.1).

\[\square\]

8. Uniqueness of conformal density

Recall that an action of a group \( H \) on a measure space \((S, \sigma)\) is said to be \textit{ergodic} if each \( H \)-invariant measurable set \( B \subset S \) is either null or co-null. In [Sul79], Sullivan proved that for a discrete group \( \Gamma \) of Möbius transformations of the Poincaré ball \( \mathbb{B}^3 \), a \( \Gamma \)-invariant conformal density \( \mu \) of non-zero dimension is unique (here and henceforth, by “unique” we mean unique up-to a constant factor) in the class of all conformal densities of same dimension if and only if the action \( \Gamma \) on the limit set \( \Lambda(\Gamma) \) is ergodic with respect to any \( \mu_x \in \mu \). See also [Nic89, Thm. 4.2.1]. Generalizing this statement in our setting, we obtain the following result. The proof is essentially same of Sullivan’s theorem, hence we omit the details.

\(^{13}\)We may need to disregard a finite number of balls from the picture.
Theorem 8.1. Suppose that $\Gamma$ is a nonelementary $\tau$-mod-RA subgroup. A $\Gamma$-invariant conformal density $\mu$ of dimension $\beta > 0$ is unique in the class of all $\Gamma$-invariant conformal densities of dimension $\beta$ if and only if the action $\Gamma \acts \Lambda_{\tau_{\text{mod}}} (\Gamma)$ is ergodic with respect to any $\mu_x \in \mu$.

It is then natural to ask

Question 8.2. For which $\tau$-mod-regular subgroups $\Gamma$, the action $\Gamma \acts \Lambda_{\tau_{\text{mod}}} (\Gamma)$ is ergodic with respect to a conformal measure?

In this section we prove that the Anosov property is a sufficient condition:

Theorem 8.3 (Anosov implies ergodic). Suppose that $\Gamma$ is a nonelementary $\tau$-mod-Anosov subgroup and $\mu$ be a $\Gamma$-invariant conformal density. Then the action $\Gamma \acts \Lambda_{\tau_{\text{mod}}} (\Gamma)$ is ergodic with respect to any $\mu_x \in \mu$.

As a corollary, we obtain that when $\Gamma$ is $\tau$-mod-Anosov, then, up to a constant factor, there is exactly one $\Gamma$-invariant conformal density, namely, the Patterson-Sullivan density.

Corollary 8.4 (Existence and uniqueness of conformal density). Suppose that $\Gamma$ is a nonelementary $\tau$-mod-Anosov subgroup. Then, up to a constant factor, there exists a unique $\Gamma$-invariant conformal density $\mu$, namely, the Patterson-Sullivan density.

Proof. First of all, by Proposition 3.1, any such density must have a positive dimension. Secondly, by Corollary 7.5 this dimension equals to the critical exponent $\delta_F$. Then the uniqueness follows from the combination of Theorems 8.1 and 8.3. \qed

Now we return to the proof of Theorem 8.3.

Proof of Theorem 8.3. Let $\mu$ be a $\Gamma$-invariant conformal density. Note that the dimension $\beta$ of $\mu$ must be positive (by Proposition 3.1 and Corollary 7.5).

Let $B$ be a $\Gamma$-invariant Borel subset of $\Lambda_{\tau_{\text{mod}}} (\Gamma)$. We need to prove that if $B$ is not a null set, then it is co-null. From now on, we assume that $B$ is not a null set, i.e., $\mu_x (B) > 0$.

We need the following lemmata.

Lemma 8.5. There exists $r_1 > 0$ such that for every $r \geq r_1$ and every $\gamma \in \Gamma$, the shadow $S(x, B(\gamma x_0, r))$ intersects $\Lambda_{\tau_{\text{mod}}} (\Gamma)$.

Proof. The proof simply follows from the Morse property of the Anosov subgroup $\Gamma$. \qed

We assume that the $r_1$ in the lemma also satisfies the “uniform conicality” property for $\Gamma$ (cf. Proposition 6.4).

Lemma 8.6. Let $r \geq \max \{r_0, r_1\}$ where $r_0$ is as in Theorem 6.1. For $\mu_x$-a.e. $\tau \in B$ and every sequence $(\gamma_n)$ on $\Gamma$, $\gamma_n \to \tau$, satisfying $\tau \in S_n := S(x : B(\gamma_n x_0, r))$, we have

$$\lim_{n \to \infty} \frac{\mu_x (S_n \cap B)}{\mu_x (S_n)} = 1. \quad (8.1)$$
Sublemma 8.7. Lebesgue density theorem (cf. [Fed69, Subsec. 2.9.11, 2.9.12]) if we knew that
Lemmata 6.7 and 6.8. Therefore, by (8.2),
Proof of Lemma 8.6.
The proof follows [Rob03, Subsec. 1E] (see also [Lin06, Sec. 3]).

Flag
\(\mu\)

for all large

\(\tau\)

Let \(\tau \in B\) be a density point, i.e., \(\tau\) satisfies (8.1). Such point exist by Lemma 8.6 because \(B\) has positive mass. Note that, \(\Gamma\)-invariance of \(B\) and \(\mu\) implies that

\[
\frac{\mu_x(S(y_n^{-1}x : B(x_0,r)) \cap B)}{\mu_x(S(y_n^{-1}x : B(x_0,r)))} = \frac{\mu_{\gamma_n,x}(S_n \cap B)}{\mu_{\gamma_n,x}(S_n)} = 1 - \frac{\mu_{\gamma_n,x}(S_n - B)}{\mu_{\gamma_n,x}(S_n)} = 1 - \frac{\int_{S_n-B} \exp (-\beta d_{\tau}^{\text{hor}}(\gamma_n x, x)) \, d\mu_x}{\int_{S_n} \exp (-\beta d_{\tau}^{\text{hor}}(\gamma_n x, x)) \, d\mu_x} \geq 1 - \text{const} \cdot \frac{\mu_x(S_n - B)}{\mu_x(S_n)},
\]

where the inequality follows by Lemma 6.9. Together with (8.1), we get

\[
\lim_{n \to \infty} \frac{\mu_x(S(y_n^{-1}x : B(x_0,r)) \cap B)}{\mu_x(S(y_n^{-1}x : B(x_0,r)))} = 1. \tag{8.2}
\]

Note that by Corollary 6.2, \(\mu\) is atom-free. Therefore, for every \(\varepsilon > 0\) there exists \(r > r_1\) such that

\[
\mu_x(S(y_n^{-1}x : B(x_0,r))) \geq m - \varepsilon,
\]

for all large \(n\), where \(m\) denotes the total mass of \(\mu_x\). The above follows from the combination of Lemmata 6.7 and 6.8. Therefore, by (8.2),

\[
\mu_x(B) \geq \lim_{n \to \infty} \mu_x(S(y_n^{-1}x : B(x_0,r)) \cap B) \geq m - \varepsilon,
\]

which holds for every \(\varepsilon > 0\). Hence \(\mu_x(B) = m\). This completes the proof of the theorem. \(\square\)

Now we prove Lemma 8.6. The lemma would have followed from a generalization of the Lebesgue density theorem (cf. [Fed69, Subsec. 2.9.11, 2.9.12]) if we knew that \(\mu_x\) is, e.g., a doubling measure. Since this property is unclear, we adopt a more direct approach. The idea of the proof follows [Rob03, Subsec. 1E] (see also [Lin06, Sec. 3]).

Proof of Lemma 8.6. The proof requires a version of the Lebesgue differentiation theorem.

Sublemma 8.7. For every bounded measurable function \(\Phi : \text{Flag}(\tau_{\mod}) \to \mathbb{R}_{\geq 0}\),

\[
\Phi(\tau) = \lim_{n \to \infty} \frac{1}{\mu_x(S(x : B(\gamma_n x_0,r)))} \int_{S(x : B(\gamma_n x_0,r))} \Phi \, d\mu_x.
\]

for \(\mu_x\)-a.e. \(\tau \in \Lambda_{\tau_{\mod}}\) and all \(\gamma_n \in \Gamma\) satisfying \(\tau \in S(x : B(\gamma_n x_0,r))\).

Proof. For every bounded measurable function \(\Psi : \text{Flag}(\tau_{\mod}) \to \mathbb{R}_{\geq 0}\), define a function \(\Psi^*\) on \(\text{Flag}(\tau_{\mod})\) which is zero outside \(\Lambda_{\tau_{\mod}}(\Gamma)\) and on \(\Lambda_{\tau_{\mod}}(\Gamma)\) it is defined by

\[
\Psi^*(\tau) = \limsup_{N \to \infty} \frac{1}{\mu_x(S(x : B(\gamma x_0,r)))} \int_{S(x : B(\gamma x_0,r))} \Psi \, d\mu_x, \tag{8.3}
\]

33
Here and in the following the limit superior is taken over all \( \gamma \in \Gamma \) that satisfy \( d_R(x, \gamma x_0) \geq N \) and \( \tau \in S(x : B(\gamma x_0, r)) \).

Let \( \Phi_k \) be a sequence of continuous functions converging to \( \Phi \) \( \mu_x \)-almost surely such that
\[
\int_{\text{Flag}(\tau_{\mod})} |\Phi_k - \Phi| d\mu_x < \frac{1}{k}, \quad \forall k \in \mathbb{N}.
\]
Then for every \( \tau \in \text{Flag}(\tau_{\mod}) \) and \( \gamma \in \Gamma \), we have
\[
\limsup_{N \to \infty} \frac{1}{\mu_x(S(x : B(\gamma x_0, r)))} \int_{S(x : B(\gamma x_0, r))} \Phi d\mu_x - \Phi(\tau) \leq |\Phi - \Phi_k|^*(\tau) + |\Phi_k(\tau) - \Phi(\tau)|
\]
\[
+ \limsup_{N \to \infty} \frac{1}{\mu_x(S(x : B(\gamma x_0, r)))} \int_{S(x : B(\gamma x_0, r))} \Phi_k d\mu_x - \Phi_k(\tau).
\] (8.4)

Since \( \Phi_n \) are continuous, the last quantity in the right side of the above vanishes. Moreover, the limit of \( |\Phi_k(\tau) - \Phi(\tau)| \) as \( k \to \infty \) vanishes at \( \mu_x \)-a.e. \( \tau \in \text{Flag}(\tau_{\mod}) \). Therefore, we only need to control the first term of the right side of (8.4): We show that, for all bounded nonnegative measurable functions \( \Psi \) on \( \text{Flag}(\tau_{\mod}) \) and all \( \varepsilon > 0 \),
\[
\mu_x (\{\Psi^* > \varepsilon\}) \leq \frac{\text{const}}{\varepsilon} \int_{\text{Flag}(\tau_{\mod})} \Psi d\mu_x
\] (8.5)
where the constant does not depend on \( \varepsilon \) or \( \Psi \). The sublemma follows from this as follows: Setting \( \Psi = |\Phi - \Phi_k| \) and taking limit as \( k \to \infty \) in (8.5), we see that \( |\Phi - \Phi_k|^* \mu_x \)-a.s. to zero. Hence left-hand side of (8.4) also converges to zero for \( \mu_x \)-a.e. \( \tau \in \Lambda_{\tau_{\mod}} \).

Now we verify (8.5). Let \( \varepsilon > 0 \) be arbitrary. For \( d \geq 0 \), let \( \Gamma_d \) be the set of all elements \( \gamma \in \Gamma \) such that \( d_F(x, \gamma x_0) \geq d \) and
\[
\int_{S(x : B(\gamma x_0, r))} \Psi d\mu_x \geq \frac{\varepsilon}{2} \mu_x(S(x : B(\gamma x_0, r))).
\] (8.6)

**Claim 1.** The union of all shadows \( S(x : B(\gamma x_0, r)) \) over \( \gamma \in \Gamma_d \) covers \( \{\Psi^* > \varepsilon\} \).

**Proof of claim.** The proof is straightforward. \( \square \)

We recursively construct a sequence of subsets, \( (\Gamma_{d,N}) \), of \( \Gamma_d \) in the following way: Let \( \Gamma_{d,1} = \{\gamma \in \Gamma_d \mid 0 \leq d_F(x, \gamma x_0) < 1\} \), and, for \( N \geq 2 \), define
\[
\Gamma_{d,N} = \left\{ \gamma \in \Gamma_d \mid \begin{array}{c}
N - 1 \leq d_F(x, \gamma x_0) < N \\
S(x : B(\gamma x_0, r)) = 0, \forall \phi \in \Gamma_{d,1} \cup \cdots \cup \Gamma_{d,N-1} \end{array} \right\}.
\]
Set \( \Gamma_d^+ = \bigcup_{N \geq 1} \Gamma_{d,N} \).

**Claim 2.** There exists a constant \( R \geq r \) such that, for every \( d \geq 0 \),
\[
\{\Psi^* > \varepsilon\} \subset \bigcup_{\phi \in \Gamma_d^+} S(x : B(\phi x_0, R)).
\]
Proof of claim. It is enough to prove the claim for very large $d$. In fact, we assume that $d$ is so large such that $x(\gamma x_0)$ is uniformly $\tau_{\text{mod}}$-regular for all $\gamma \in \Gamma_d$.

Let $\tau \in \{\Psi^* \succ \varepsilon\}$ be arbitrary. Then there exists $\gamma \in \Gamma_d$ such that $\tau \in S(x : B(\gamma x_0, r))$. Assume that $\gamma \not\in \Gamma_d^\ast$. By construction of $\Gamma_d^\ast$, there exists $\phi \in \Gamma_d^\ast$ such that $S(x : B(\gamma x_0, r)) \cap S(x : B(\phi x_0, r)) \neq \emptyset$ and $d_F(x, \phi x_0) < d_F(x, \gamma x_0)$.

By Lemma 7.4, both $\gamma x_0$ and $\phi x_0$ stay uniformly close to a $\tau_{\text{mod}}$-uniform Morse quasigeodesic $\alpha$ with one endpoint at $x$. Since $d_F(x, \phi x_0) < d_F(x, \gamma x_0)$, we may assume that the other endpoint of $\alpha$ is uniformly close to $\gamma x_0$. It follows that $\phi x_0$ is uniformly close to the diamond $\diamondsuit_{\Theta}(x, \gamma x_0)$, since $\alpha$ is, for some $\Theta \subset \tau_{\text{mod}}$. By Lemma 7.4, both $\gamma x_0$ and $\phi x_0$ lie in an annulus $\{\tau' \in X \mid N - 1 \leq d_F(x', x) < N\}$ in the following way: Let $\tau \in S(x : B(\gamma x_0, r)) \cap S(x : B(\phi x_0, r))$. Let $z \in V(x, \text{st}(\tau))$ be a point uniformly close to $\gamma x_0$. By $\delta$-hyperbolicity,

$$\langle \gamma x_0 | \phi x_0 \rangle_x + \delta \geq \min \{\langle \gamma x_0 | z \rangle_x, \langle \phi x_0 | z \rangle_x\}. \quad (8.8)$$

Expanding the left side, we get

$$\langle \gamma x_0 | \phi x_0 \rangle_x + \delta = \frac{1}{2} (d_F(\gamma x_0, x) + d_F(\phi x_0, x) - d_F(\gamma x_0, \phi x_0)) + \delta \leq \left( d_F(\phi x_0, x) - \frac{1}{2} d_F(\gamma x_0, \phi x_0) \right) + \delta + \frac{1}{2}, \quad (8.9)$$

and expanding the right side, we get

$$\min \left\{\frac{1}{2} (d_F(\gamma x_0, x) + d_F(x, z) - d_F(\gamma x_0, z))\right\} = \min \left\{\frac{1}{2} (d_F(\phi x_0, z) + d_F(x, z) - d_F(\phi x_0, z))\right\}. \quad (8.8)$$

Taking $z \to \tau$ in the right side of the last one and using (3.2), we get

$$\min \left\{\frac{1}{2} \left( d_F(\gamma x_0, x) + d_\tau^{\text{hor}}(x, \gamma x_0) \right), \frac{1}{2} \left( d_F(\phi x_0, x) + d_\tau^{\text{hor}}(x, \phi x_0) \right) \right\}. \quad (8.9)$$

which, by Lemma 6.9, is at least

$$\min \{d_F(\gamma x_0, x), d_F(\phi x_0, x)\} - r \geq d_F(\gamma x_0, x) - r - 1.$$
Combining this with (8.8) and (8.9), we get
\[ d_F(\gamma x_0, \phi x_0) \leq 2r + 2\delta + 3. \]
\[ \square \]

In particular, for each \( \tau \in \mu_x (\{ \Psi^* > \varepsilon \}), \# \{ \phi \in \Gamma_d^* \mid \tau \in S(x : B(\phi x_0, r)) \} \) is uniformly bounded, say, by \( D > 0 \). Therefore,
\[ \sum_{\phi \in \Gamma^*_R} \mu_x (S(x : B(\phi x_0, r))) \leq D \mu_x \left( \bigcup_{\phi \in \Gamma^*_R} S(x : B(\phi x_0, r)) \right). \] (8.10)

We would like to use the shadow lemma (Theorem 6.1). To this end, we have
\[ \mu_x (\{ \Psi^* > \varepsilon \}) \leq \sum_{\phi \in \Gamma^*_R} \mu_x (S(x : B(\phi x_0, R))) \leq C' \sum_{\phi \in \Gamma^*_R} \exp (-\beta d_F(x, \phi x_0)) \] (8.11)
where the first inequality is given by (8.7) and the last inequality is given by the shadow lemma with \( r_0 \leq r = R \). Note that the necessary condition \( d_F(x, \phi x_0) \geq R \) which we needed to apply the shadow lemma in the above follows from the definition of \( \Gamma^*_R \). Moreover, applying shadow lemma again with \( r_0 \leq r = r \), we get another constant \( C > 0 \) such that
\[ C^{-1} \sum_{\phi \in \Gamma^*_R} \exp (-\beta d_F(x, \phi x_0)) \leq \sum_{\phi \in \Gamma^*_R} \mu_x (S(x : B(\phi x_0, r))). \] (8.12)

Combined with (8.10), the inequalities in (8.11) and (8.12) give
\[ \mu_x (\{ \Psi^* > \varepsilon \}) \leq D C' C \mu_x \left( \bigcup_{\phi \in \Gamma^*_R} S(x : B(\phi x_0, r)) \right). \]

Finally, the above and (8.6) yield
\[ \mu_x (\{ \Psi^* > \varepsilon \}) \leq \frac{2DC'C}{\varepsilon} \int_{\text{Flag}(\tau \mod \Lambda)} \Psi d\mu_x. \]
This proves (8.5).
\[ \square \]

The proof of the lemma follows from the sublemma by taking \( \Phi \) in the sublemma to be the indicator function for \( B \).
\[ \square \]

**9. Hausdorff density**

In this section, we restrict our attention to Anosov subgroups. Usually, one defines Hausdorff measures and Hausdorff dimension for metric spaces. In the appendix to this paper we verify that the theory goes through for premetrics as well. The reader who prefers to work with metrics can assume that \( \varepsilon > 0 \) is chosen so that \( d_G^{x,\varepsilon} \) defines a metric on \( \Lambda_{\tau \mod \Gamma} \) (cf. Corollary 5.6).
For \( \beta \geq 0 \) we let \( \mathcal{H}_x^\beta \) denote the \( \beta \)-dimensional Hausdorff measure on \((\Lambda_{\text{mod}}(\Gamma), d_G^{x,e})\) (defined with respect to the premetric \( d_G^{x,e} \) as in the appendix). The Hausdorff dimension of a Borel subset \( B \subset \Lambda_{\text{mod}}(\Gamma) \) is then defined as

\[
\text{Hd}(B) = \inf \{ \beta \mid \mathcal{H}_x^\beta(B) = 0 \} = \sup \{ \beta \mid \mathcal{H}_x^\beta(B) = \infty \}.
\]

Note that if for some \( \beta \geq 0 \), \( \mathcal{H}_x^\beta(B) \in (0, \infty) \), then \( \text{Hd}(B) = \beta \).

**Proposition 9.1.** Suppose that for some \( \beta \geq 0 \)

\[
\mathcal{H}_x^\beta(\Lambda_{\text{mod}}(\Gamma)) \in (0, \infty).
\]  

(9.1)

Let \( Z = \Gamma.x \). Then \( \mathcal{H}_x^\beta = \{ \mathcal{H}_x^\beta \}_{z \in Z} \) is a \( \beta \varepsilon \)-dimensional \( \Gamma \)-invariant conformal \( Z \)-density.

**Proof.** Let \( y, z \in Z \). Define a function \( f : \Lambda_{\text{mod}}(\Gamma) \times \Lambda_{\text{mod}}(\Gamma) \to \mathbb{R}_{\geq 0} \) by

\[
f(\tau_1, \tau_2) = \begin{cases} d_G^{x,e}(\tau_1, \tau_2)/d_G^{x,e}(\tau_1, \tau_2), & \tau_1 \neq \tau_2, \\ \exp(-\varepsilon d_\tau^{\text{hor}}(y, z)), & \tau_1 = \tau_2 = \tau. \end{cases}
\]

By a calculation similar to the proof of Proposition 5.4, we obtain

\[
\lim_{\tau_1, \tau_2 \to \tau} \frac{d_G^{x,e}(\tau_1, \tau_2)}{d_G^{x,e}(\tau_1, \tau_2)} = \exp\left(-\varepsilon d_\tau^{\text{hor}}(y, z)\right)
\]

which shows that \( f \) is continuous. For \( \tau \in \Lambda_{\text{mod}}(\Gamma) \) and small \( \eta > 0 \), let \( U_\eta \) be a neighborhood of \( \tau \) in \( \Lambda_{\text{mod}}(\Gamma) \) such that \( \forall \tau_1, \tau_2 \in U_\eta \),

\[
d_G^{x,e}(\tau_1, \tau_2) \leq \left\{ \exp\left(-\varepsilon d_\tau^{\text{hor}}(y, z)\right) + \eta \right\} d_G^{x,e}(\tau_1, \tau_2).
\]

Hence the identity map \( \text{id} : (\Lambda_{\text{mod}}(\Gamma), d_G^{x,e}) \to (\Lambda_{\text{mod}}(\Gamma), d_G^{x,e}) \) on \( U_\eta \) is \( L_\varepsilon \)-Lipschitz, where \( L_\varepsilon = \exp\left(-\varepsilon d_\tau^{\text{hor}}(y, z)\right) + \eta \). In particular, the map \( \text{id} \) is locally Lipschitz. Therefore, for any \( B \in \mathcal{B}(U) \),

\[
\mathcal{H}_y^\beta(B) \leq L_\varepsilon^{\beta} \mathcal{H}_z^\beta(B).
\]

This also shows that \( \mathcal{H}_y^\beta \ll \mathcal{H}_z^\beta \). Taking limit as \( \eta \to 0 \), we obtain

\[
\frac{d \mathcal{H}_y^\beta}{d \mathcal{H}_z^\beta}(\tau) \leq \exp\left(-\beta \varepsilon d_\tau^{\text{hor}}(y, z)\right)
\]

and switching the role of \( y \) and \( z \) in the above we also obtain the reverse inequality. Hence

\[
\frac{d \mathcal{H}_y^\beta}{d \mathcal{H}_z^\beta}(\tau) = \exp\left(-\beta \varepsilon d_\tau^{\text{hor}}(y, z)\right)
\]

which proves conformality. Suppose that \( y = \gamma z \) for some \( \gamma \in \Gamma \). Then for any \( B \in \mathcal{B}(\Lambda_{\text{mod}}(\Gamma)) \),

\[
\mathcal{H}^\beta_{\gamma z}(B) = \int_B \exp\left(-\beta \varepsilon d_\tau^{\text{hor}}(y z, z)\right) d \mathcal{H}_z^\beta = \int_B d \left( \gamma^* \mathcal{H}_z^\beta \right) = \gamma^* \mathcal{H}_z^\beta(B)
\]

and \( \Gamma \)-invariance also follows. Therefore, \( \mathcal{H}^\beta \) is a conformal \( Z \)-density of dimension \( \beta \varepsilon \). \( \square \)
Remark. 1. Note that if such a family \( \{ \mathcal{H}_z^\beta \mid z \in Z \} \) exists, then it may be extended to a full conformal density via the correspondence in (3.4).

2. By the uniqueness of conformal density (Theorem 8.4), the number \( \beta \) in Proposition 9.1 equals to \( \delta G/\varepsilon \).

3. In the following we shall see that, indeed, the \( \delta G/\varepsilon \)-dimensional Hausdorff measure \( \mathcal{H}_x^{\delta G/\varepsilon} \) is finite and non-null (i.e., it satisfies (9.1)).

Next we show that if \( \beta = \delta G/\varepsilon \), then the \( \beta \)-dimensional Hausdorff measure \( \mathcal{H}_x^\beta \) satisfies (9.1). Let us first discuss the simpler case, namely, when the Finsler pseudo-metric \( d_F \) is a metric. There is an abundance of examples when this occurs, e.g., in the case when \( X = G/K \) is an irreducible symmetric space.

Let \((Y, d)\) be a proper, geodesic, Gromov hyperbolic metric space and \( \Gamma \) be a nonelementary discrete group of isometries acting properly discontinuously on \( \partial_\infty Y \). Let \( \Lambda \) be the limit set of \( \Gamma \) in \( \partial_\infty Y \). Further, assume that \( \Gamma \) is quasiconvex-cocompact, i.e., the quasiconvex hull \( \text{QCH}(\Lambda) \) is nonempty and the quotient \( \text{QCH}(\Lambda)/\Gamma \) is compact. In [Coo93], Coornaert proved the following result.

Theorem 9.2 ([Coo93, Cor. 7.6]). Suppose that the critical exponent \( \delta \) of \( \Gamma \) is finite. Then the \( \delta \)-dimensional Hausdorff measure on \( \Lambda \) with respect to a Gromov metric \( d_G \) is finite and non-null.

To apply this theorem to our case, we need an appropriate setup. In Section 4, we proved that the orbit \( Z = \Gamma x \) is a Gromov hyperbolic space with respect to the Finsler metric (cf. Corollary 4.8) and it is also proper. But \( Z \) fails to be geodesic. This problem can be remedied by taking a uniform neighborhood \( Y \) of \( Z \) in \( X \) such that \( Z \) is quasiconvex in \( Y \), and then putting the intrinsic path-metric \( d \) on \( Y \) induced by \( d_F \) (this requires positivity of \( d_F \)), and finally by completing \( Y \) in this metric. Then \((Y, d)\) is proper, geodesic and Gromov hyperbolic. Moreover, \((Y, d)\) and the isometrically embeded \((Z, d_F)\) are Hausdorff-close and, in particular, \((Y, d)\) is quasimetric to \((Z, d_f)\) by a \((1, A)\)-quasiisometry. This implies that there is a bi-Lipschitz homeomorphism from \( \partial_\infty Y \) (equipped with the metric \( d_G^\varepsilon \) defined by \( d_G^\varepsilon(\xi_1, \xi_2) = d_G(\xi_1, \xi_2)^\varepsilon \) where \( d_G \) is a Gromov metric on \( \partial_\infty Y \)) to \((\Lambda_{\text{mod}}(\Gamma), d_G^\varepsilon)\). Note that the action \( \Gamma \sim (Y, d) \) satisfies all the properties needed to apply Theorem 9.2. Therefore, by this theorem the \( \delta G/\varepsilon \)-dimensional Hausdorff measure on \( \partial_\infty Y \) (and, consequently, also on \( \Lambda_{\text{mod}}(\Gamma) \)) is finite and non-null.

In the general case where the positivity of \( d_F \) is unknown, the above argument still works after some modifications. Let us go back to our construction in the above paragraph. Let \( Y \) be a uniform Riemannian neighborhood of \( Z \) in which \( Z \) is Finsler quasiconvex. Define a new \( \Gamma \)-invariant metric \( d_f \) on \( Y \) by
\[
d_f(y, z) = \max \{ d_F(y, z), \varepsilon d_G(y, z) \}, \quad \forall y, z \in Y
\]
where \( \varepsilon > 0 \) is some number that is strictly lesser than \( L^{-1} \) given in (2.6). Note that for \( y, z \in Z \), if \( d_F(y, z) \) is sufficiently large, then \( d_f(y, z) = d_F(y, z) \). Moreover, for a given \( \varepsilon \)-invariant compact subset \( \Theta \subset \text{ost}(\tau_{\text{mod}}) \) and a possibly smaller \( \varepsilon \) (depending on the choice of \( \Theta \)), any \( \Theta \)-Finsler
geodesic (see Definition 2.1) connecting these two points remains a geodesic in this new metric. In other words, $Z$ remains quasiconvex in $Y$ with respect to $\bar{d}_F$.

Observe that the identity embedding $(Z, d_F) \to (Y, \bar{d}_F)$ is a $(1, A)$-quasiisometric embedding for some large enough $A$ and the image is Hausdorff-close to $Y$. Therefore, in this case also we get a natural identification of the Gromov boundaries of $(Z, d_F)$ and $(Y, \bar{d}_F)$. Next, considering intrinsic metrics, we complete $Y$ as before to get a proper, geodesic, Gromov hyperbolic space $(Y, d)$. The rest of the argument works as before.

Using Proposition 9.1, we obtain the following result.

**Theorem 9.3.** Suppose that $\Gamma$ is a nonelementary $\tau_{\text{mod}}$-Anosov subgroup. If $\beta = \delta_F/\epsilon$, then the $\beta$-dimensional Hausdorff density $H^\beta = \{H^\beta_z\} \subset \Gamma$ is a $\Gamma$-invariant conformal density of dimension $\delta_F$. In particular, the Hausdorff dimension with respect to the metric $d_G$ satisfies

$$Hd(\Lambda_{\tau_{\text{mod}}}(\Gamma)) = \delta_F/\epsilon.$$

Moreover, $H^\beta$ equals to a non-zero multiple of the Patterson-Sullivan density.

We have mostly completed the proof of this theorem. The remaining “moreover” part follows from the uniqueness of $\Gamma$-invariant conformal density (Theorem 8.4).

**Corollary 9.4.** With respect to the Gromov premetric $d_G^\infty := d_G^{\infty,1}$ the Hausdorff dimension satisfies

$$Hd(\Lambda_{\tau_{\text{mod}}}(\Gamma)) = \delta_F.$$

### 10. Examples

**10.1. Product of hyperbolic spaces:** Let $\Gamma_1$, $\Gamma_2$ be isomorphic discrete cocompact subgroups of $\text{PSL}(2, \mathbb{R})$ where the isomorphism is given by $\phi : \Gamma_1 \to \Gamma_2$. We let $f : S^1 \to S^1$ be the equivariant homeomorphism of ideal boundaries of hyperbolic planes determined by $\phi$.

The discrete subgroup

$$\Gamma = \{(\gamma_1, \phi \gamma_1) \mid \gamma_1 \in \Gamma_1\} < G = \text{PSL}(2, \mathbb{R}) \times \text{PSL}(2, \mathbb{R})$$

acts on $X = \mathbb{H}^2 \times \mathbb{H}^2$ as a $\sigma_{\text{mod}}$-Anosov subgroup. (This follows, for instance, from the fact that $\Gamma$ is an URU subgroup of $G$.) The $\sigma_{\text{mod}}$-limit set of $\Gamma$ (in the full flag-manifold $S^1 \times S^1$) equals the graph of the map $f$.

We denote $d_1$ (resp. $d_2$) the distance functions of the constant $-1$ curvature Riemannian metrics on the first (resp. second) factor of the product $\mathbb{H}^2 \times \mathbb{H}^2$.

Unlike in section 5, we work with the Finsler metric on $\mathbb{H}^2 \times \mathbb{H}^2$ given by

$$d_F((x_1, x_2), (y_1, y_2)) = \frac{d_1(x_1, y_1) + d_2(x_2, y_2)}{2}.$$

(We multiply the distance function (2.7), for $p = 2$, by a factor $1/\sqrt{2}$ in order to avoid cumbersome radical constants.)
By the formula of the Gromov predistance (5.8), for \( \epsilon = 1 \) and \( x = (x_1, x_2) \), \( d_G^{x,1}(\tau_+, \tau_-) \) is bi-Lipschitz equivalent to the product
\[
\sqrt{\alpha_1 \alpha_2},
\]
where \( \tau_\pm = (\xi_1^\pm, \xi_2^\pm) \) and \( \alpha_i \) is the angle between \( \xi_i^+, \xi_i^- \) as measured from \( x_i, i = 1, 2 \).

By [BS93, Thm. 2 & 3] we note that the Finsler critical exponent \( \delta_F \) of \( \Gamma \) is at most 1. This can also be obtained by comparing the Hausdorff dimensions as follows. Note that by the formula of the Gromov predistance, the identity map
\[
(S^1 \times S^1, \rho) \to (\text{Flag}(\sigma_{\text{mod}}), d_G^{x,1})
\]
is Lipschitz, where \( \rho \) is a Riemannian distance function on \( S^1 \times S^1 = \partial_\infty \mathbb{H}^2 \times \partial_\infty \mathbb{H}^2 \). Moreover, the limit set of \( \Gamma \) in \( S^1 \times S^1 \) is the graph of a BV function, hence, is a rectifiable curve, and, thus, has Hausdorff dimension 1 with respect to \( \rho \). Consequently, with respect to \( d_G^{x,1}, \text{Hd}(\Lambda_{\sigma_{\text{mod}}}(\Gamma)) \leq 1 \).

By Theorem 9.3, \( \delta_F \leq 1 \) as well. Moreover, by [BS93, Thm. 2], \( \delta_F = 1 \) if and only if \( \phi \) is induced by an isometry of \( \partial_\infty \mathbb{H}^2 \), equivalently, \( f \) is a Möbius transformation.

We further note that one can use [Bur93] as an alternative argument for both inequality and the equality case.

10.2. Projective Anosov representations: Recall that a representation \( \Gamma \to \text{SL}(k + 1, \mathbb{R}) \), \( k \geq 2 \), is called projective Anosov if it is \( \tau_{\text{mod}} \)-Anosov for \( \tau_{\text{mod}} = (1, k) \) (see Examples 1.3, 2.5, and 5.8 for notations). The Finsler critical exponent associated to the \( \iota \)-invariant type
\[
\tilde{\theta} = (1/2\sqrt{k + 1}, 0, -1/2\sqrt{k + 1})
\]
will be denoted by \( \delta_F \).

Let \( \Gamma \to \text{SL}(k + 1, \mathbb{R}) \) be a projective Anosov representation. In [GMT19], the authors defined the following two critical exponents of \( \Gamma \), namely, the Hilbert critical exponent
\[
\delta_{1,k+1} = \limsup_{r \to \infty} \frac{\log \text{card}\{ \gamma \in \Gamma \mid \sigma_1(\gamma) - \sigma_{k+1}(\gamma) < r \}}{r}
\]
and the simple root critical exponent
\[
\delta_{1,2} = \limsup_{r \to \infty} \frac{\log \text{card}\{ \gamma \in \Gamma \mid \sigma_1(\gamma) - \sigma_2(\gamma) < r \}}{r}.
\]
A direct computation yields
\[
\sqrt{k + 1} \delta_F = \delta_{1,k+1} \leq \delta_{1,2}/2,
\]
where the left equality follows from the formula for the Finsler metric given by (2.10) and the right inequality follows from \( 2(\sigma_1 - \sigma_2) \leq \sigma_1 - \sigma_{k+1} \). Also note that (by (5.9)) for a pair of partial flags \( (l_1, h_1), (l_2, h_2) \in \text{Flag}(\tau_{\text{mod}}), \)
\[
d_G^{x,1/\sqrt{k+1}}((l_1, h_1), (l_2, h_2)) \leq \sin \angle(l_1, l_2)
\]
where the right side equals the distance (with respect to the constant curvature Riemannian metric on $\mathbb{R}P^k$ determined by $x \in X$) between the lines $l_1, l_2$ in $\mathbb{R}P^k$. This together with Theorem 9.3 implies that

$$\delta_{1,k+1} = \delta_F \sqrt{k + 1} = \text{Hd}(A_{\text{mod}}(\Gamma)) \leq \text{Hd}_R(\xi^1(\partial_\infty \Gamma)) \tag{10.1}$$

where $\xi^1 : \partial_\infty \Gamma \to \mathbb{R}P^k$ is the $\Gamma$-equivariant embedding of $\partial_\infty \Gamma$ into $\mathbb{R}P^k$ and $\text{Hd}_R$ denotes the Hausdorff dimension with respect to the Riemannian metric. Together with a recently obtained upper-bound for $\text{Hd}_R(\xi^1(\partial_\infty \Gamma))$ (see [PSW19, Prop. 4.1] or [GMT19, Thm. 4.1]), we obtain the following result.

**Theorem 10.1.** Let $\Gamma \to \text{SL}(k + 1, \mathbb{R})$ be a projective Anosov representation. Then

$$\delta_{1,k+1} \leq \text{Hd}_R(\xi^1(\partial_\infty \Gamma)) \leq \delta_{1,2}.$$ 

Also compare [GMT19, Cor. 1.2] where the authors obtain identical bounds for the Hausdorff dimension of the flag limit set equipped with a certain Gromov metric.

**Appendix: Hausdorff measures on premetric spaces**  

Let $X$ be a metrizable topological space. Recall that an outer measure is a function $\mu : \mathcal{P}(X) \to [0, \infty]$ that satisfies

(i) $\mu(\emptyset) = 0$,

(ii) for all $A, B \in \mathcal{P}(X)$ with $A \subset B$, $\mu(A) \leq \mu(B)$, and

(iii) for all countable collection $(A_k | k \in \mathbb{N})$ of subsets of $X$,

$$\mu \left( \bigcup_{k \in \mathbb{N}} A_k \right) \leq \sum_{k \in \mathbb{N}} \mu(A_k).$$

A set $A \subset X$ is called $\mu$-measurable if for every $E \in \mathcal{P}(X)$, $\mu(A) = \mu(A \cap E) + \mu(A \cap E^c)$. By Carathéodory’s theorem (cf. [Fol99, Thm. 1.11]), $\mu$-measurable sets form a $\sigma$-algebra to which $\mu$ restricts as a complete measure.

Assume now that $X$ is compact. The outer measure $\mu$ is called good if additionally,

(iv) for all $A, B \subset X$ with $\bar{A} \cap \bar{B} = \emptyset$, $\mu(A \cup B) = \mu(A) + \mu(B)$.

The next lemma asserts that, for outer measures $\mu$ on compact metrizable spaces, the $\sigma$-algebra of Borel sets is a subalgebra of the $\sigma$-algebra of $\mu$-measurable sets.

**Lemma A.1.** Let $X$ be a compact metrizable space. If $\mu$ is a good outer measure on $X$, then every Borel set $B \in \mathcal{B}(X)$ is measurable.

\footnote{Composition of the $\Gamma$-equivariant boundary embedding $\partial_\infty \Gamma \to \text{Flag}(\tau_{\text{mod}})$ and the projection map $\text{Flag}(\tau_{\text{mod}}) \to \mathbb{R}P^k = \text{Gr}_1(\mathbb{R}^{k+1})$.}
Proof. Let $d$ be a metric on $X$. Then the condition (iv) above implies that

(iv') for all $A, B \subset X$ with $d(A, B) > 0$, $\mu(A \cup B) = \mu(A) + \mu(B)$.

Therefore, $\mu$ is a metric outer measure on $(X, d)$. By [Fol99, Prop. 11.16], Borel subsets of $X$ are measurable. □

**Definition A.2** (Premetric space). Let $X$ be a topological space. A symmetric continuous function $d : X \times X \to [0, \infty]$ is called a premetric on $X$. A pair $(X, d)$ consisting of a metrizable topological space $X$ and a premetric $d$ on $X$ is called a premetric space.

In what follows, we consider only positive premetrics, i.e.,

$$d(x, y) > 0 \iff x \neq y, \forall x, y \in X$$

Let $(X, d)$ be a compact positive premetric space. Then $d$ satisfies the following *separation property*:

$$d(A, B) > 0 \iff \overline{A} \cap \overline{B} = \emptyset, \forall A, B \subset X.$$  \hfill (A.1)

Let $\varepsilon > 0$, $\beta > 0$. For every $A \subset X$, define

$$\mathcal{H}^\beta_\varepsilon(A) = \inf_{\mathcal{U}} \left\{ \sum_{k \in \mathbb{N}} \text{diam}_d(U_k)^\beta \left| \mathcal{U} = \{U_k \mid k \in \mathbb{N}\} \text{ covers } A, \text{mesh}(\mathcal{U}) \leq \varepsilon \right\}.$$  

In the above, mesh$(\mathcal{U})$ is the supremum of the $d$-diameters of the members of $\mathcal{U}$. Then

$$\mathcal{H}^\beta_\varepsilon : \mathcal{P}(X) \to [0, \infty]$$

is an outer measure on $X$ (cf. [Fol99, Prop. 1.10]). Define the $\beta$-dimensional Hausdorff measure $\mathcal{H}^\beta$ by

$$\mathcal{H}^\beta(A) = \lim_{\varepsilon \to 0} \mathcal{H}^\beta_\varepsilon(A).$$

**Theorem A.3.** The Hausdorff measure $\mathcal{H}^\beta$ is a good outer measure.

**Proof.** We need to check the properties (i)-(iv) above. Since, for all $\varepsilon > 0$, $\mathcal{H}^\beta_\varepsilon$ is an outer measure, taking limit $\varepsilon \to 0$, properties (i)-(iii) are easily verified. Therefore, we only need to check that $\mathcal{H}^\beta$ satisfies property (iv).

Let $A, B \subset X$ such that $\overline{A} \cap \overline{B} = \emptyset$. By (A.1), $d(A, B) = d_0 > 0$. Let $\varepsilon < d_0$ be a positive number and $\mathcal{U}$ be a countable open cover of $A \cup B$ with mesh$(\mathcal{U}) \leq \varepsilon$. If such open cover does not exist, then $\mathcal{H}^\beta_\varepsilon(A \cup B)$ (and hence, $\mathcal{H}^\beta(A \cup B)$) is infinity. Otherwise, $\mathcal{U}$ can be written as a disjoint union $\mathcal{U}_A \sqcup \mathcal{U}_B$ where $\mathcal{U}_A$ consists of all open sets in $\mathcal{U}$ that intersect $A$ and $\mathcal{U}_B$ consists of the rest. Clearly, $\mathcal{U}_A$ and $\mathcal{U}_B$ are open covers of $A$ and $B$, respectively. Therefore,

$$\sum_{E \in \mathcal{U}} \text{diam}_d(E)^\beta = \sum_{E \in \mathcal{U}_A} \text{diam}_d(E)^\beta + \sum_{E \in \mathcal{U}_B} \text{diam}_d(E)^\beta \geq \mathcal{H}^\beta_\varepsilon(A) + \mathcal{H}^\beta_\varepsilon(B).$$
Since the above holds for any cover $\mathcal{U}$ with mesh $\leq \varepsilon$, we have

$$\mathcal{H}_\varepsilon^\beta(A \cup B) \geq \mathcal{H}_\varepsilon^\beta(A) + \mathcal{H}_\varepsilon^\beta(B).$$

Taking limit $\varepsilon \to 0$, we get $\mathcal{H}^\beta(A \cup B) \geq \mathcal{H}^\beta(A) + \mathcal{H}^\beta(B)$. The reverse inequality follows from property (iii). Therefore, $\mathcal{H}^\beta(A \cup B) = \mathcal{H}^\beta(A) + \mathcal{H}^\beta(B)$. This completes the proof. □

By Lemma A.1 and the above theorem, we obtain the following result.

**Corollary A.4.** Every Borel subset of $X$ is $\mathcal{H}^\beta$-measurable.

The Hausdorff dimension of a Borel subset $B \subset (X, d)$ is then defined as

$$\text{Hd}(B) = \inf \{ \beta \mid \mathcal{H}^\beta(B) = 0 \} = \sup \{ \beta \mid \mathcal{H}^\beta(B) = \infty \}.$$  

**References**


Subhadip Dey  
Department of Mathematics  
University of California, Davis  
One Shields Avenue, Davis, CA 95616  
E-mail address: sdey@math.ucdavis.edu

Michael Kapovich  
Department of Mathematics  
University of California, Davis  
One Shields Avenue, Davis, CA 95616  
E-mail address: kapovich@math.ucdavis.edu