Research Summary

Patterns are ubiquitous in biological and physical processes, with spiral waves being one of the most universal. Spirals are observed in oscillatory chemical reactions and rotating waves of electrical activity in cardiac tissue. These systems support regular spiral waves of a constant shape, and bifurcations to complex and unstable patterns can have serious consequences. For example, stable spiral waves in cardiac tissue are abnormal and problematic, but wave break up is associated with the onset of life threatening ventricular fibrillation (VF). This clinical significance makes it crucial to study factors contributing to both the initiation and termination phases of spiral waves.

My research primarily focuses on using techniques from dynamical systems and nonlinear waves to mathematically study the existence and stability of spiral waves in order to discover hidden mechanisms driving their structure and transformations. Specifically, I consider spirals formed in reaction-diffusion systems and research topics include:

1. Spiral wave stability and mechanisms for break up.
2. Conditions for arrhythmia initiation and initial formation of spiral waves.

Additionally, I developed an agent-based model to probe patterns in a different biological application: seasonal migrations of blue whales. We sought to answer:

3. How environmental conditions and prey independently influence blue whale migrations.

Summaries of these projects, including future research directions are described in the following sections.

Spiral Wave Stability and Breakup

Spiral waves $U_s = U_s(r, \psi)$ are stationary solutions of the reaction-diffusion system

$$\partial_t U = D\Delta_{r,\psi} U + \omega \partial_{\psi} U + F(U), \quad U \in \mathbb{R}^n, \quad D \in \mathbb{R}^{n \times n}$$

in a polar frame rotating with frequency $\omega$. Stability is analyzed by examining the spectrum of the operator $L_s$ created by linearizing (1) about the spiral wave solution

$$L_s V = D\Delta_{r,\psi} V + \omega \partial_{\psi} V + F'(U_s)V$$

and considering the resulting eigenvalue problem $L_s V = \lambda V$ for eigenvalues $\lambda \in \mathbb{C}$. On bounded domains, the spectrum of $L_s$ is a union of three disjoint sets associated with instabilities of the far-field, core, or boundary conditions of the spiral. The contributions from each set are important factors in understanding how a spiral destabilizes, but in practice these contributions are difficult to disentangle. Therefore, we introduce a novel methodology in [3] to determine which of these sets unstable point eigenvalues belong to and thus which regions generate the observed instabilities.

**Alternans and line defects:** Higher onset rates of VF have been clinically linked to alternans, a beat-to-beat oscillation in the cardiac rhythm that manifests in modulating spiral band width. Further, spirals in chemical oscillations form stationary line defects, with wave amplitudes out of phase across the defect lines. Both instabilities result in patterns with twice the period of the original structure (see Figure) and arise through bifurcations of unstable point eigenvalues. We apply the method in [3] and find that despite similarities in alternans and line defects, the two instabilities are created by different mechanisms. Line defects are a boundary instability, meaning the shape and type of boundary condition directly impacts the emergent structure. In contrast, alternans arise from a core instability, implying
that the domain shape and boundary conditions are insignificant factors in their formation. This result has direct implications for cardiac dynamics, as conclusions for the development of alternans on bounded disks can be extended to the irregular and complex geometry of the heart.

Spirals tips are known to pin to heterogeneities in cardiac tissue, with defect size effecting spiral stability. To date, there are no comprehensive mathematical studies that quantify the stability as a function of defect size or determine how alternans formation is impacted by spiral pinning. In future work, I will extend my current spectral analysis to spirals on annuli to answer these questions.

**Spectra of systems with a rank-deficient diffusion matrix:** Much of spectral theory for reaction-diffusion systems has been developed under the assumption of all species diffusing. Cardiac and neuronal tissues are commonly investigated with ion channel models, systems which can have one or more diffusiveless components, yet the impact of a rank-deficient diffusion matrix $D$ on the spectra of spiral waves remains undetermined. We are using two-component reaction-diffusion systems to study spectral changes between full rank and rank-deficient diffusion matrices and finding abrupt differences. In the limit of sending one diffusion coefficient $\delta$ to zero, we observe the emergence of finite limit points in the essential spectrum and shifting locations of the absolute spectrum; unexpected changes which occur sharply at $\delta = 0$. Our results prove the finite limit points are dictated by the non-diffusing species and furthermore predict locations for the essential and absolute spectrum [4]. These facts have significant consequences for determining the stability of patterns formed in ion channel models, and future work will be focused on the point spectrum under the same limit.

**Spiral Wave Initiation**

Wave reflections caused by traveling pulses interacting with heterogeneities and rapid pacing of cardiac tissue experimentally have been observed to initiate spiral waves and cardiac arrhythmias. These initiation mechanisms remain mathematically not well understood.

**Reflection of pulses:** Properties of reflected waves depend on features of the heterogeneity, but the ability to reflect appears to be intimately linked to the existence of one-dimensional spiral waves; a pattern with a core that sheds alternating forward and backward propagating pulses. Mathematically, we view the 1D spiral as a time-periodic antisymmetric source defect (see Figure) and are using a combination of direct numerical simulation and continuation methods to probe their existence in an idealized model of cardiac tissue. The goal is to determine how the pattern’s existence, and thus the likelihood of unwanted reflections, depends on system parameters, which could suggest preventive treatments for the onset of arrhythmia. Our preliminary results support and extend a previous hypothesis that the structure emerges through a global heteroclinic bifurcation and identifies several key parameters for further study [2].

**Geometry dependent arrhythmias:** Increasingly, antiarrhythmic drugs are tested on small clusters of cells despite differences in the behavior of single cells and spatially extended tissues. To showcase the variations, in a collaboration with the Cohen Lab we designed a detailed ion channel model for synthetic cells which can generate comparable chaotic electrical spiking as cardiac cells and support traveling waves [5]. The model output reproduces the initiation of complex arrhythmic rhythms and dynamical transitions observed in the synthetic experiments and captures how the tissue geometry impacts the behavior of propagating waves under a fast stimulus. Regardless of stimulus frequency, there is a one-to-one relationship between stimuli and voltage spikes in a single cell, but in spatially extended arrays of cells the arrhythmic behavior emerges for fast pacing frequencies and differs in locations near and far from the stimulus. These results have important consequences for the experimental development of antiarrhythmic drug therapies and urge caution when interpreting results of single cell experiments.
Migratory Patterns of Blue Whales

Blue whales are a highly migratory species, spending winters near Baja California and summers in the north Pacific. Sea surface temperature (SST) and prey density have been shown to be significant predictors of migratory habitat, but their individual influences on migrations are difficult to disentangle. The small whale population makes continuous density models unfeasible, thus in a collaboration with researchers at NOAA, we developed an agent-based model to investigate movement of blue whales in the California Current System [1].

Our model is formulated as a state switching model with each state exhibiting characteristic movements associated with transit and forage behaviors. States are probabilistically selected based on SST and krill density, with data from a coupled regional ocean and dynamic prey model. Using both input factors, the model reproduces realistic foraging behavior and timing of latitudinal migrations; demonstrating how individual decision-making leads to emergent migratory behavior at the population level. Ensembles driven with only one input reveal that prey influences fine scale (seasonal and interannual) behaviors, whereas unfavorable SST acts to limit foraging opportunities.

Understanding the movement drivers is important for insight into how species will respond to changing environmental conditions. In future work, I will use climate change projections as input data to predict the impacts on the migratory behavior.

References


