Section 14.3: Partial Derivatives

Definition: Partial Derivative with respect to $x$, $y$

The partial derivative of $f(x, y)$ with respect to $x$ at the point $(x_0, y_0)$ is

$$f_x = \frac{\partial f}{\partial x} \bigg|_{(x_0, y_0)} = \frac{d}{dx} f(x, y_0) \bigg|_{(x_0, y_0)} = \lim_{h \to 0} \frac{f(x_0 + h, y_0) - f(x_0, y_0)}{h}$$

The partial derivative of $f(x, y)$ with respect to $y$ at the point $(x_0, y_0)$ is

$$f_y = \frac{\partial f}{\partial y} \bigg|_{(x_0, y_0)} = \frac{d}{dy} f(x_0, x) \bigg|_{(x_0, y_0)} = \lim_{h \to 0} \frac{f(x_0, y_0 + h) - f(x_0, y_0)}{h}$$

Example: Taking Partial Derivatives

$$f(x, y) = y \sin(xy)$$

To find $f_x$ (the partial derivative of $f$ with respect to $x$), treat $y$ as a constant.

$$f_x = y \cos(xy)y = y^2 \cos(xy)$$

To find $f_y$ (the partial derivative of $f$ with respect to $y$), treat $x$ as a constant.

$$f_y = (1) \sin(xy) + y \cos(xy)x = \sin(xy) + xy \cos(xy)$$

Example: Implicit Differentiation

Find $\frac{\partial z}{\partial x}$ if the equation

$$yz - \ln z = x + y$$

defines $z$ as a function of two independent variables $x$ and $y$ and the partial derivative exists.

$$\frac{d}{dx} (yz - \ln z) = \frac{d}{dx} (x + y)$$
\[ y \frac{\partial z}{\partial x} - \frac{1}{z} \frac{\partial z}{\partial x} = 1 + 0 \]
\[ \left( y - \frac{1}{z} \right) \frac{\partial z}{\partial x} = 1 \]
\[ \frac{\partial z}{\partial x} = \frac{x}{yz - 1} \]

**Definition: Second-Order Partial Derivatives**

When we differentiate a function \( f(x, y) \) twice, we produce its **second-order derivatives**. Notation:

\[ \frac{\partial^2 f}{\partial x^2} \text{ or } f_{xx}, \quad \frac{\partial^2 f}{\partial y^2} \text{ or } f_{yy}, \quad \frac{\partial^2 f}{\partial x \partial y} \text{ or } f_{yx}, \quad \text{and} \quad \frac{\partial^2 f}{\partial y \partial x} = f_{xy} \]

The defining equations are

\[ \frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x} \left( \frac{\partial f}{\partial x} \right), \quad \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x} \left( \frac{\partial f}{\partial y} \right) \]

Note that \( f_{yx} = \frac{\partial^2 f}{\partial x \partial y} \) means you **first** take the derivative with respect to \( y \), and **then** take the derivative with respect to \( x \).

In general, \( f_{xy} \neq f_{yx} \).

**Theorem 2: Mixed Derivative Theorem**

If \( f(x, y) \) and its partial derivatives \( f_x, f_y, f_{xy}, \) and \( f_{yx} \) are defined throughout an open region containing a point \((a, b)\) and are all continuous at \((a, b)\), then

\[ f_{xy}(a, b) = f_{yx}(a, b) \]

**Example: Partial Derivatives of Higher Order**

Find \( f_{xyz} \) if \( f(x, y, z) = 1 - 2xy^2z + x^2y \)

\[ f(x, y, z) = 1 - 2xy^2z + x^2y \]
\[ f_y = -4xyz + x^2 \]
\[ f_{yx} = -4yz + 2x \]
\[ f_{yxy} = -4z \]
\[ f_{xyz} = -4 \]
Definition: Differentiability

A function \( z = f(x,y) \) is differentiable at \((x_0, y_0)\) if \( f_x(x_0, y_0) \) and \( f_y(x_0, y_0) \) exists and \( \Delta z \) satisfies an equation of the form

\[
\Delta z = f_x(x_0, y_0)\Delta x + f_y(x_0, y_0)\Delta y + \epsilon_1 \Delta x + \epsilon_2 \Delta y
\]

in which each of \( \epsilon_1, \epsilon_2 \to 0 \) as both \( \Delta x, \Delta y \to 0 \). We call \( f \) differentiable if it is differentiable at every point in its domain, and say that its graph is a smooth surface.

Theorem 3: The Increment Theorem for Functions of Two Variables

Suppose that the first partial derivatives of \( f(x,y) \) are defined throughout an open region \( R \) containing the point \((x_0, y_0)\) and that \( f_x \) and \( f_y \) are continuous at \((x_0, y_0)\). Then the change

\[
\Delta z = f(x_0 + \Delta x, y_0 + \Delta y) - f(x_0, y_0)
\]

in the value of \( f \) that results from moving from \((x_0, y_0)\) to another point \((x_0 + \Delta x, y_0 + \Delta y)\) in \( R \) satisfies an equation of the form

\[
\Delta z = f_x(x_0, y_0)\Delta x + f_y(x_0, y_0)\Delta y + \epsilon_1 \Delta x + \epsilon_2 \Delta y
\]

in which each of \( \epsilon_1, \epsilon_2 \to 0 \) as both \( \Delta x, \Delta y \to 0 \).

Corollary of Theorem 3

If the partial derivatives \( f_x \) and \( f_y \) of a function \( f(xy) \) are continuous throughout an open region \( R \), then \( f \) is differentiable at every point of \( R \).

Theorem 4 - Differentiability Implies Continuity

If a function \( f(x,y) \) is differentiable at \((x_0, y_0)\), then \( f \) is continuous at \((x_0, y_0)\).

Section 14.4: The Chain Rule

Theorem 5: Chain Rule for Functions of One Independent Variable and Two Intermediate Variables

If \( w = f(x,y) \) is differentiable and if \( x = x(t) \), \( y = y(t) \) are differentiable functions of \( t \), then the composite \( w = f(x(t), y(t)) \) is a differentiable function of \( t \) and

\[
\frac{dw}{dt} = f_x(x(t), y(t)) \cdot x'(t) + f_y(x(t), y(t)) \cdot y'(t)
\]

or

\[
\frac{dw}{dt} = \frac{\partial f}{\partial x} \frac{dx}{dt} + \frac{\partial f}{\partial y} \frac{dy}{dt}
\]
Theorem 6: Chain Rule for Functions of One Independent Variable and Three Intermediate Variables

If \( w = f(x, y, z) \) is differentiable and \( x, y, \) and \( z \) are differentiable functions of \( t \), then \( w \) is a differentiable function of \( t \) and

\[
\frac{dw}{dt} = \frac{\partial w}{\partial x} \frac{dx}{dt} + \frac{\partial w}{\partial y} \frac{dy}{dt} + \frac{\partial w}{\partial z} \frac{dz}{dt}
\]

Example

Find \( \frac{dw}{dt} \) if

\[w = xy + z, \quad x = \cos t, \quad y = \sin t, \quad z = t\]

Use Theorem 6:

\[
\frac{dw}{dt} = (y)(- \sin t) + (x)(\cos t) + (1)(1) = - \sin^2 t + \cos^2 t + 1 = \cos 2t + 1
\]

What is the derivative at \( t = 0 \)?

\[
\left( \frac{dw}{dt} \right)_{t=0} = 1 + \cos 0 = 2
\]

Theorem 7: Chain Rule for Two Independent Variables and Three Intermediate Variables

Suppose that \( w = f(x, y, z) \), \( x = g(r, s) \), \( y = h(r, s) \), and \( z = k(r, s) \). If all four functions are differentiable, then \( w \) has partial derivatives with respect to \( r \) and \( s \) given by the formulas

\[
\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x} \frac{\partial x}{\partial r} + \frac{\partial w}{\partial y} \frac{\partial y}{\partial r} + \frac{\partial w}{\partial z} \frac{\partial z}{\partial r}
\]

\[
\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial w}{\partial y} \frac{\partial y}{\partial s} + \frac{\partial w}{\partial z} \frac{\partial z}{\partial s}
\]

Theorem 8: A Formula for Implicit Differentiation

Suppose that \( F(x, y) \) is differentiable and that the equation \( F(x, y) = 0 \) defines \( y \) as a differentiable function of \( x \). Then at any point where \( F_y \neq 0 \),

\[
\frac{dy}{dx} = -\frac{F_x}{F_y}
\]
Example

Find \( \frac{dy}{dx} \) if \( y^2 - x^2 - \sin xy = 0 \).

\[
\frac{dy}{dx} = \frac{F_x}{F_y} = \frac{-2x - y \cos xy}{2y - x \cos xy} = \frac{2x + y \cos xy}{2y - x \cos xy}
\]

Expansion of Theorem 8 to Three Variables

Suppose \( F(x, y, z) = 0 \) and \( z = f(x, y) \). Assuming \( F \) and \( f \) are differentiable functions, then

\[
\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} \quad \text{and} \quad \frac{\partial z}{\partial y} = -\frac{F_y}{F_z}
\]

Expansion of Chain Rule to Functions of \( n \) variables

In general, suppose \( z = f(x_1, x_2, \ldots, x_n) \) is a differential function of the intermediate variables \( x_1, x_2, \ldots, x_n \) where \( n \) is a positive integer (\( n \in \mathbb{N} \)). Also suppose each \( x_i \) is a differentiable function of the independent variables \( t_1, t_2, \ldots, t_m \), with \( m \in \mathbb{N} \). In equation form,

\[
x_1 = g_1(t_1, t_2, \ldots, t_m)
\]

\[
x_2 = g_2(t_1, t_2, \ldots, t_m)
\]

\[
\vdots
\]

\[
x_n = g_n(t_1, t_2, \ldots, t_m)
\]

Then \( w \) is a differential function of each of the independent variables \( t_1, t_2, \ldots, t_m \), and the partial derivatives of \( w \) with respect to each \( t_i \) are

\[
\frac{\partial w}{\partial t_1} = \frac{\partial w}{\partial x_1} \frac{\partial x_1}{\partial t_1} + \frac{\partial w}{\partial x_2} \frac{\partial x_2}{\partial t_1} + \cdots + \frac{\partial w}{\partial x_n} \frac{\partial x_n}{\partial t_1} = \sum_{i=1}^{n} \frac{\partial w}{\partial x_i} \frac{\partial x_i}{\partial t_1}
\]

\[
\frac{\partial w}{\partial t_2} = \frac{\partial w}{\partial x_1} \frac{\partial x_1}{\partial t_2} + \frac{\partial w}{\partial x_2} \frac{\partial x_2}{\partial t_2} + \cdots + \frac{\partial w}{\partial x_n} \frac{\partial x_n}{\partial t_2} = \sum_{i=1}^{n} \frac{\partial w}{\partial x_i} \frac{\partial x_i}{\partial t_2}
\]

\[
\vdots
\]

\[
\frac{\partial w}{\partial t_m} = \frac{\partial w}{\partial x_1} \frac{\partial x_1}{\partial t_m} + \frac{\partial w}{\partial x_2} \frac{\partial x_2}{\partial t_m} + \cdots + \frac{\partial w}{\partial x_n} \frac{\partial x_n}{\partial t_m} = \sum_{i=1}^{n} \frac{\partial w}{\partial x_i} \frac{\partial x_i}{\partial t_m}
\]

More compactly,

\[
\frac{\partial w}{\partial t_j} = \sum_{i=1}^{n} \frac{\partial w}{\partial x_i} \frac{\partial x_i}{\partial t_j} \quad \text{for } j = 1, 2, \ldots, m
\]
Directional Derivatives and Gradient Vectors

Definition: Directional Derivative

The **derivative of** \( f \) **at** \( P_0(x_0, y_0) \) **in the direction of** the **unit vector** \( u = u_1i + u_2j \) **is the number**

\[
\left( \frac{df}{ds} \right)_{u,P_0} = \lim_{s \to 0} \frac{f(x_0 + su_1, y_0 + su_2) - f(x_0, y_0)}{s}
\]

provided the limit exists. The directional derivative is also denoted

\[
\left( \frac{df}{ds} \right)_{u,P_0} = (Du)f_{P_0}
\]

and is read “The derivative of \( f \) at \( P_0 \) in the direction of \( u \).”

Definition: Gradient Vector

The **gradient vector** (gradient) of \( f(x, y) \) **at** a **point** \( P \) **is the vector**

\[
\nabla f = \frac{\partial f}{\partial x}i + \frac{\partial f}{\partial y}j
\]

Theorem 9: The Directional Derivative is a Dot Product

If \( f(x, y) \) **is differentiable in** an **open region containing** \( P_0(x_0, y_0) \), then

\[
(Du)f_{P_0} = (\nabla f)_{P_0} \cdot u
\]

In words, the derivative of \( f \) **at** \( P_0 \) **in the direction of** \( u \) **is the dot product of** the **gradient** \( \nabla f \) **at** \( P_0 \) **and** \( u \). **In brief,**

\[
(Du)f = \nabla f \cdot u
\]

Example

Find the derivative of \( f(x, y) = xe^y + \cos xy \) **at** the **point** \( (2, 0) \) **in the direction of** \( v = 3i - 4j \).

First, find the unit direction vector

\[
u = \frac{v}{\|v\|} = \frac{v}{5} = \frac{3}{5}i - \frac{4}{5}j
\]

Then we need to find the partial derivatives of \( f \) **at** \( (2, 0) \) because together, they make up the gradient, \( \nabla f \).

\[
f_x = e^y - y \sin xy
\]

\[
f_x(2, 0) = e^0 - 0 \sin(2 \cdot 0) = 1 - 0 = 1
\]

\[
f_y = xe^y - x \sin xy
\]

\[
f_y(2, 0) = 2e^0 - 2 \sin(2 \cdot 0) = 2 - 0 = 2
\]
Plug these values into the definition of gradient.

\[ \nabla f|_{(2,0)} = f_x(2,0)i + f_y(2,0)j \]
\[ = i + 2j \]

Then the directional derivative of \( f \) at \((2,0)\) in the direction of \( u \) is

\[ (D_uf)_{(2,0)} = \nabla f|_{(2,0)} \cdot u \]
\[ = \left( \frac{3}{5}i - \frac{4}{5}j \right) \cdot (i + 2j) \]
\[ = \frac{3}{5} - 2 \cdot \frac{4}{5} \]
\[ = -1 \]

**Properties of the Directional Derivative**

1. The function \( f \) increases most rapidly when \( \cos \theta = 1 \), i.e. when \( \theta = 0 \), i.e. when \( u \) is the direction of \( \nabla f \). That is, at each point \( P \) in its domain, \( f \) increases most rapidly in the direction of the gradient vector \( \nabla f \) at \( P \). The derivative in this direction is

\[ D_uf = \| \nabla f \| \cos 0 = \| \nabla f \| \]

2. Similarly, \( f \) decreases most rapidly in the direction of \(-\nabla f\). The derivative in this direction is

\[ D_uf = \| \nabla f \| \cos \pi = -\| \nabla f \| \]

3. Any direction \( u \) orthogonal to a gradient \( \nabla f \neq 0 \) is a direction of zero change in \( f \) because \( \theta = \frac{\pi}{2} \) and

\[ D_uf = \| \nabla f \| \cos \frac{\pi}{2} = \| \nabla f \| \cdot 0 = 0 \]

**Example**

Let \( f(x, y) = \frac{x^2}{2} + \frac{y^2}{2} \), and consider the point \((1,1)\).

The function increases most rapidly in the direction of \( \nabla f \).

\[ (\nabla f) = xi + yj \implies (\nabla f)_{(1,1)} = i + j \]

The unit vector of \((\nabla f)_{(1,1)}\) is

\[ u = \frac{i + j}{\sqrt{2}} = \frac{1}{\sqrt{2}}i + \frac{1}{\sqrt{2}}j \]

The function decreases most rapidly in the direction \(-\nabla f)_{(1,1)}\)

\[ -\frac{1}{\sqrt{2}}i - \frac{1}{\sqrt{2}}j \]

The directions of zero change at \((1,1)\) are the directions orthogonal to \( \nabla f \):

\[ n = \frac{1}{\sqrt{2}}i + \frac{1}{\sqrt{2}}j \] and \[ -n = \frac{1}{\sqrt{2}}i - \frac{1}{\sqrt{2}}j \]
Important Concept

At every point \((x_0, y_0)\) in the domain of a differentiable function \(f(x, y)\), the gradient of \(f\) is normal to the level curve through \((x_0, y_0)\).

Tangent Line to a Level Curve

\[
f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) = 0
\]

Notice this is the same as point-slope form from elementary algebra.

\[
y - y_0 = m(x - x_0)
\]

where

\[
m = -\frac{f_x}{f_y} = \frac{dy}{dx}
\]

by Theorem 8.

Algebra Rules for Gradients

1. **Sum Rule:** \(\nabla (f + g) = \nabla f + \nabla g\)
2. **Difference Rule:** \(\nabla (f - g) = \nabla f - \nabla g\)
3. **Constant Multiple Rule:** \(\nabla (kf) = k\nabla f\)
4. **Product Rule:** \(\nabla (fg) = f\nabla g + g\nabla f\)
5. **Quotient Rule:** \(\nabla \left(\frac{f}{g}\right) = \frac{g\nabla f - f\nabla g}{g^2}\)

Gradients of Functions of \(n\) variables

For a differential function \(f(x_1, x_2, \ldots, x_n)\) and a unit vector \(u = \langle u_1, u_2, \ldots, u_n \rangle\) in space, we have

\[
\nabla f = \left\langle \frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \ldots, \frac{\partial f}{\partial x_n} \right\rangle
\]

and

\[
D_u f = \nabla f \cdot u = \frac{\partial f}{\partial x_1} u_1 + \frac{\partial f}{\partial x_2} u_2 + \ldots + \frac{\partial f}{\partial x_n} u_n = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} u_i
\]

The Derivative Along a Path

Let \(r(t) = x(t)i + y(t)j + z(t)k\) be a smooth path \(C\) and \(w = f(r(t))\) a scalar function along \(C\). Then

\[
\frac{dw}{dt} = \frac{\partial w}{\partial x} \frac{dx}{dt} + \frac{\partial w}{\partial y} \frac{dy}{dt} + \frac{\partial w}{\partial z} \frac{dz}{dt}
\]

or in vector notation,

\[
\frac{d}{dt} f(u(t)) = \nabla f(r(t)) \cdot r'(t)
\]