1 125B: Final Exam Solutions

Problem 1. (25 pts) Prove that the function

$$f(x,y) = \begin{cases} \frac{\sin(x)}{y}\,, & y \neq 0 \\ 0\,, & y = 0 \end{cases}$$

is not differentiable at $(0,0)$.

Proof. This problem was very similar to Problem 6 on the list of practice problems, and the proof is the same. Namely, f is not continuous at the origin $(0,0)$, and hence it is not differentiable at $(0,0)$. To see that f is not continuous at the origin, we note that $f(0,0) = 0$, and hence the limit of $f(x,y)$ as $(x,y) \to (0,0)$ should also vanish. Along the line $x = y$, however, L'Hospital's rule shows that

$$\lim_{x\to 0} \frac{\sin(x)}{x} = \lim_{x\to 0} \cos(x) = 1 \neq 0\,.$$

One may also consider the limit of $f(x,y)$ along the path $y = \sin x$, which also produces a limiting value of 1. Yet another alternative is to use the fact that if f is differentiable then all the partial derivatives *exist*, which is equivalent to the fact that if partial derivatives do *not exist*, meaning that the limit of the difference quotient does not exist, then f is not differentiable. Note, however, that it is not correct to argue that if the partial derivatives are not continuous, then f is not differentiable!! □

Problem 2. (25 pts) Using the <u>definition</u> of the derivative of a function $f : \mathbb{R}^2 \to \mathbb{R}^2$, prove that

$$Df(x,y) = \begin{bmatrix} 1 & -1 \\ 2x & 2y \end{bmatrix}$$

is the derivative of $f(x,y) = (x - y, x^2 + y^2)$ at a point (x,y).

Proof. This problem is very similar to Problem 7 on the practice problems. The definition of the derivative states that $Df(x,y)$ is the derivative of f at (x,y) if

$$\lim_{(h_1,h_2)\to(0,0)} \frac{||f(x+h_1, y+h_2) - f(x,y) - Df(x,y)\cdot(h_1,h_2)||}{||(h_1,h_2)||} = 0\,.$$

Hence, we must show that the linear map $Df(x,y) = \begin{bmatrix} 1 & -1 \\ 2x & 2y \end{bmatrix}$ satisfies this definition.

We substitute this matrix into the definition:

$$\lim_{(h_1,h_2)\to(0,0)} \frac{||f(x+h_1,y+h_2)-f(x,y)-Df(x,y)\cdot(h_1,h_2)||}{||(h_1,h_2)||}$$
$$= \lim_{(h_1,h_2)\to(0,0)} \frac{||(x+h_1-y-h_2,(x+h_1)^2+(y+h_2)^2)-(x-y,x^2+y^2)-(h_1-h_2,2xh_1+2yh_2)||}{||(h_1,h_2)||}$$
$$= \lim_{(h_1,h_2)\to(0,0)} \frac{||(0,h_1^2+h_2^2)||}{||(h_1,h_2)||}$$
$$= \lim_{(h_1,h_2)\to(0,0)} \sqrt{h_1^2+h_2^2} = 0\,.$$

□

Problem 3. (25 pts) Let $f:\mathbb{R}^2\to\mathbb{R}^2$ be given by

$$f(x_1,x_2) = (x_1x_2,\ x_1^2+x_2)$$

and let $\vec{x}_0$ denote the 2-vector (x_{01},x_{02}). Find $\vec{x}_0\in\mathbb{R}^2$ such that $f(\vec{x}_0)=\vec{y}_0\in\mathbb{R}^2$ with $\vec{y}_0=(0,-1)$. Prove that $f^{-1}:W\to U$ exists and is differentiable in some nonempty open set W containing $\vec{y}_0$ and some nonempty open set U containing $\vec{x}_0$, and compute $D(f^{-1})(\vec{y}_0)$.

Proof. This problem is similar to Problem 4 on the practice list.

We first find the point $\vec{x}_0\in\mathbb{R}^2$ such that $f(\vec{x}_0)=\vec{y}_0=(0,-1)$. We have two equations and two unknowns:

$$x_1x_2 = 0 \text{ and } x_1^2+x_2=-1\,.$$

Thus, either $x_1=0$ or $x_2=0$. If $x_2=0$, then $x_1=\pm\sqrt{-1}$ which we cannot allow, and if $x_1=0$, then $x_2=-1$. Hence,

$$\vec{x}_0=(0,-1)\,.$$

Given $f(x_1,x_2)=(f_1(x_1,x_2),f_2(x_1,x_2)=(x_1x_2,\ x_1^2+x_2)$, we see that both f_1 and f_2 are polynomials and hence smooth or C^∞ functions. In particular $f\in C^1(\mathbb{R}^2;\mathbb{R}^2)$. As f is continuously differentiable on all of $\mathbb{R}^2$, we compute the derivative:

$$Df(x_1,x_2)=\begin{bmatrix} x_2 & x_1\\ 2x_1 & 1\end{bmatrix},$$

from which it follows that

$$\det Df(x_{01},x_{02}) = \det\begin{bmatrix} -1 & 0\\ 0 & 1\end{bmatrix} = -1\neq 0\,.$$

Thus, by the inverse function theorem, there exists neighborhoods U of $\vec{x}_0$ and W of $\vec{y}_0$, respectively, such that $f : U \to W$ has a C^1 inverse $f^{-1} : W \to U$. Furthermore, the inverse function theorem gives us a formula for computing the derivative of f^{-1} on W: for all $y \in W$,

$$D(f^{-1})(y) = [Df(f^{-1})(y)]^{-1}\,.$$

Since we have found $\vec{x}_0 = f^{-1}(\vec{y}_0)$, we see that

$$D(f^{-1})(\vec{y}_0) = [Df(\vec{x}_0)]^{-1} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}.$$

$\square$

Problem 4. (25 pts) Which of the following are true and which are false (you do <u>not</u> have to give a proof) for a function $f : \mathbb{R}^n \to \mathbb{R}^m$:

(a) If f is differentiable then f is continuous. (TRUE)

(b) If f is not differentiable at some $x \in \mathbb{R}^n$, then some partial derivatives $\frac{\partial f^i}{\partial x_j}$ do not exist. (FALSE)

(c) If for some i, j, the partial derivative $\frac{\partial f^i}{\partial x_j}$ exists but is not continuous, then f is not differentiable. (FALSE)

(d) If $f : \mathbb{R} \to \mathbb{R}$ is a C^∞ function, then in a small enough open set U containing $x_0 \in \mathbb{R}$,

$$f(x) = f(x_0) + \sum_{l=1}^{\infty} \frac{d^l f}{dx^l}(x_0)(x - x_0)^l \text{ for all } x \in U\,.$$

(FALSE)

(e) If $A \subset \mathbb{R}^n$ is open, $f : A \to \mathbb{R}$ is C^2 and $x_0 \in A$ is a critical point of f such that the Hessian $D^2 f(x_0)$ is negative definite, then f has a local maximum at x_0. (TRUE)