MAT 125B
Spring 2011
Midterm Solutions

Kevin Schenthal

May 8, 2011

1. We prove that $f(x) = e^{-x}x^p$ is integrable on $[0, \infty)$ for any $p \geq 0$ and compute $\int_0^\infty e^{-x}xdx$.

   Write $p = n + k$ for n a nonnegative integer and $k \in (0, 1)$. Observe that by L'Hôpital's rule that

$$\lim_{x\to\infty} \frac{x^p}{e^x} = \lim_{x\to\infty} \frac{px^{p-1}}{e^x} = \cdots = \lim_{x\to\infty} \frac{n! \cdot x^k}{e^x} \lim_{x\to\infty} \frac{n! \cdot k}{x^{1-k}e^x} = 0,$$

   since the limit of each quotient is an indeterminate of the form ∞/∞. Since this limit exists for any $p \geq 0$, there exists a constant $M = M(p)$ such that

$$\frac{x^{p+2}}{e^x} \leq 1, \quad \forall x \geq M.$$

   Or multiplying by x^{-2} we equivalently have

$$\frac{x^p}{e^x} \leq \frac{1}{x^2}, \quad \forall x \geq M.$$

   Then linearity of the integral implies that

$$\begin{aligned}\int_0^\infty e^{-x}x^p dx &= \int_0^M e^{-x}x^p dx + \int_M^\infty e^{-x}x^p dx \\ &\leq \int_0^M e^{-x}x^p dx + \int_M^\infty \frac{1}{x^2} dx.\end{aligned}$$

   The first integral on the right hand side of our inequality converges since $e^{-x}x^p$ is a continuous function on a compact interval. The second integral clearly converges as well. Hence, by the comparison test, we deduce that the integral $\int_1^\infty e^{-x}x^p dx$ converges, proving the result.

   Next, we do the computation using integration by parts:

$$\int_0^\infty e^{-x}xdx = [-e^{-x}x]_0^\infty + \int_0^\infty e^{-x}dx = [-e^{-x}]_0^\infty = 1.$$

2. For $\theta \in [0, 2\pi)$, and $\phi \in [0, \pi]$ spherical coordinates are given by

$$x = \cos\theta \sin\phi, \quad y = \sin\theta \sin\phi, \quad z = \cos\phi.$$

We express $\partial/\partial\theta$ and $\partial/\partial\phi$ as a linear combination of $\partial/\partial x$, $\partial/\partial y$, and $\partial/\partial z$.

Observe that

$$\begin{aligned} \tfrac{\partial x}{\partial \theta} &= -\sin\theta\sin\phi, & \tfrac{\partial x}{\partial \phi} &= \cos\theta\cos\phi \\ \tfrac{\partial y}{\partial \theta} &= \cos\theta\sin\phi, & \tfrac{\partial y}{\partial \phi} &= \sin\theta\cos\phi \\ \tfrac{\partial z}{\partial \theta} &= 0, & \tfrac{\partial z}{\partial \phi} &= -\sin\phi. \end{aligned}$$

The chain rule implies that

$$\begin{aligned} \frac{\partial}{\partial\theta} &= \frac{\partial x}{\partial\theta}\frac{\partial}{\partial x} + \frac{\partial y}{\partial\theta}\frac{\partial}{\partial y} + \frac{\partial z}{\partial\theta}\frac{\partial}{\partial z} \\ &= -\sin\theta\sin\phi\frac{\partial}{\partial x} + \cos\theta\sin\phi\frac{\partial}{\partial y} \\ \frac{\partial}{\partial\phi} &= \frac{\partial x}{\partial\phi}\frac{\partial}{\partial x} + \frac{\partial y}{\partial\phi}\frac{\partial}{\partial y} + \frac{\partial z}{\partial\phi}\frac{\partial}{\partial z} \\ &= \cos\theta\cos\phi\frac{\partial}{\partial x} + \sin\theta\cos\phi\frac{\partial}{\partial y} - \sin\phi\frac{\partial}{\partial z}. \end{aligned}$$

3. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be given by

$$f(x,y) = \begin{cases} (x^2+y^2)\sin\left(\frac{1}{\sqrt{x^2+y^2}}\right), & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0). \end{cases}$$

We prove that f is differentiable at $(x,y) = (0,0)$, but the partial derivatives $\partial f/\partial y$ and $\partial f/\partial y$ are discontinuous at $(x,y) = (0,0)$.

Recall that f is differentiable at (x_0,y_0) whenever there exists a linear map $A : \mathbb{R}^2 \to \mathbb{R}$ such that

$$\lim_{(x,y)\to(x_0,y_0)} \frac{|f(x,y) - f(x_0,y_0) - A\cdot((x,y)-(x_0,y_0))|}{|(x,y)-(x_0,y_0)|} = 0. \tag{1}$$

In our case, let A denote the zero map. Then (1) becomes

$$\begin{aligned} \lim_{(x,y)\to(0,0)} \frac{|f(x,y)|}{\sqrt{x^2+y^2}} &= \lim_{(x,y)\to(0,0)} \sqrt{x^2+y^2}\left|\sin\left(\frac{1}{\sqrt{x^2+y^2}}\right)\right| \\ &\leq \lim_{(x,y)\to(0,0)} \sqrt{x^2+y^2} \\ &= 0. \end{aligned}$$

Hence, by the positive-definiteness of the norm, we deduce that the inequality is in fact an equality, whence the considered limit exists and is 0. We have therefore demonstrated that f is differentiable at $(x,y) = (0,0)$.

However, observe that for $(x,y) \neq 0$ we have

$$\begin{aligned} \frac{\partial f}{\partial x}(x,y) &= 2x\sin\left(\frac{1}{\sqrt{x^2+y^2}}\right) - (x^2+y^2)\cos\left(\frac{1}{\sqrt{x^2+y^2}}\right)\cdot\frac{x}{(x^2+y^2)^{3/2}} \\ &= 2x\sin\left(\frac{1}{\sqrt{x^2+y^2}}\right) - \frac{x}{\sqrt{x^2+y^2}}\cos\left(\frac{1}{\sqrt{x^2+y^2}}\right). \end{aligned}$$

In particular,

$$\frac{\partial f}{\partial x}(x,0) = 2x\sin\left(\frac{1}{|x|}\right) - \frac{x}{|x|}\cos\left(\frac{1}{|x|}\right).$$

The first of the two terms on the right hand side of the equation tends to 0 as $x \to 0$ by the squeeze theorem. But $\lim_{x\to 0}(-x/|x|)\cos(|x|^{-1})$ does not exist. Therefore, $\partial f/\partial x$ has an oscillating discontinuity at $x = 0$. By symmetry, we deduce the same result for $\partial f/\partial y$.