MAT 125B
Spring 2011
Problem Set 1 Solutions

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1. Exercise 1. We want to compute $\int_0^1 xdx$ from the definition of the Riemann integral. First, we define a sequence of partitions $\{P_{1/n}\}_{n=1}^{\infty}$ as follows: let $P_{1/n}$ contain the $n+1$ points $x_i = i/n$, where $i = 0, 1, 2, \dots, n$, so

$$0 = x_0 < x_1 < \cdots < x_n = 1,$$

and $P_{1/n}$ has the property that

$$||P_{1/n}|| = \max_{1 \leq i \leq n} \{|x_i - x_{i-1}|\} = \frac{1}{n}.$$

Next, we form our upper and lower Riemann sums associate to P_n :

$$L_n = \sum_{i=0}^{n-1} \inf_{z_i \in [x_i, x_{i+1}]} f(z_i)(x_{i+1} - x_i) = \sum_{i=0}^{n-1} f(x_i)(x_{i+1} - x_i) = \sum_{i=0}^{n-1} \frac{i}{n^2} = \frac{n(n-1)}{2n^2}$$
$$U_n = \sum_{i=0}^{n-1} \sup_{z_i \in [x_i, x_{i+1}]} f(z_i)(x_{i+1} - x_i) = \sum_{i=0}^{n-1} f(x_{i+1})(x_{i+1} - x_i) = \sum_{i=1}^{n} \frac{i}{n^2} = \frac{n(n+1)}{2n^2},$$

where we used the fact that $f(x) = x$ is an increasing function to easily find the infimums and the supremums. Taking the limit gives us the result:

$$\int_0^1 xdx = \lim_{n\to\infty} L_n = \lim_{n\to\infty} U_n = 1/2.$$

2. Exercise 2. Let $f : [0,1] \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} 1, & x = 1, 1/2, 1/3, 1/4, \dots \\ 0, & \text{otherwise.} \end{cases}$$

We prove $\int_0^1 f(x)dx = 0$. By the first proposition of §3, it suffices to show that for any $\epsilon > 0$ there exists a step function $g(x)$ such that $0 \leq f(x) \leq g(x)$ and such that $\int_0^1 g(x)dx < \epsilon$.

Fix $\epsilon > 0$. Then there exists an $N \in \mathbb{N}$ such that for all $n \geq N$ we have $1/n + (n-1)/2^n < \epsilon$. Let $E_j = [1/j - 1/2^N, 1/j + 1/2^N] \cap [0,1]$ for $j = 1, 2, \dots, N-1$. Define the step function

$$g(x) = \begin{cases} 1, & x \in [0, 1/N] \cup E_1 \cup \cdots \cup E_{N-1}, \\ 0, & \text{otherwise.} \end{cases}$$

Then $0 \leq f(x) \leq g(x)$ and

$$\int_0^1 f_N(x)dx < \frac{1}{N} + \frac{N-1}{2^N} < \epsilon.$$

Taking $\epsilon \to 0$ allows us to conclude the result.

3. Exercise 5. We must prove that a continuous real-valued function on a closed interval $[a,b] \subset \mathbb{R}$ is integrable, using only uniform continuity and Lemma 1 from §3. By Lemma 1, we only need to show that for any $\epsilon > 0$ there exists a $\delta > 0$ such that $|S^1_{\delta_1} - S^2_{\delta_2}| < \epsilon$ whenever $S^1_{\delta_1}$ and $S^2_{\delta_2}$ are Riemann sums of f corresponding to partitions of $[a,b]$ of width less than δ. That is, $\delta_1, \delta_2 < \delta$.

   Fix $\epsilon > 0$. Since a continuous function on a compact interval is uniformly continuous we choose $\delta > 0$ sufficiently small such that $|f(x) - f(y)| < \epsilon/(b-a)$ when $|x-y| < \delta$, $x, y \in [a,b]$. Let $P^1_{\delta_1}$ and $P^2_{\delta_2}$ be the partitions of $[a,b]$ such that $\delta_1, \delta_2 < \delta$ and let $S^1_{\delta_1}$ and $S^2_{\delta_2}$ be Riemann sums associated with $P^1_{\delta_1}$ and $P^2_{\delta_2}$ respectively. Write $S^1_{\delta_1} = \sum_{x_i \in P^1_{\delta_1}} f(x'_i)\Delta x_i$ and $S^2_{\delta_2} = \sum_{y_j \in P^2_{\delta_2}} f(y'_j)\Delta y_j$ as usual. Recall that $||P^1_{\delta_1} \cup P^2_{\delta_2}|| \leq \min\{\delta_1, \delta_2\} < \delta$.

   We can define the function $g : [a,b] \to \mathbb{R}$ by $g(z'_r) = f(x'_i) - f(y'_j)$ whenever $z'_r \in [x_i, x_{i+1})$ and $z_r \in [y_j, y_{j+1})$ for $i = 0, 1, \ldots, m-1$ and $j = 0, 1, \ldots, n-1$ or when $z_r \in [x_{m-1}, x_m]$ and $z_r \in [y_{n-1}, y_n]$ and where x'_i and y'_j are defined according to S_1 and S_2 respectively. By construction, we see that $|x'_i - y'_j| < \delta$, which, by uniform continuity, shows that $|g(z_r)| = |f(x'_i) - f(x'_j)| < \epsilon/(b-a)$.

   Hence, it follows that

$$\begin{aligned}
|S^1_{\delta_1} - S^2_{\delta_2}| &= \left| \sum_{x_i \in P^1_{\delta_1}} f(x'_i)\Delta x_i - \sum_{y_j \in P^2_{\delta_2}} f(y'_j)\Delta y_j \right|, \\
&\leq \sum_{z_r \in P_1 \cup P_2} |g(z'_r)||\Delta z_r| \\
&< \frac{\epsilon}{b-a} \sum_{r \in P^1_{\delta_1} \cup P^2_{\delta_2}} \Delta z_r \\
&= \frac{\epsilon}{b-a} \cdot (b-a) \\
&= \epsilon.
\end{aligned}$$

4. Exercise 21.

   (a) We want to find the limit

$$\lim_{n\to\infty} \frac{1^k + 2^k + \cdots + n^k}{n^{k+1}}, \tag{1}$$

   where $k > 0$. To find this limit we want to approximate the sum $S_n = \sum_{m=1}^n m^k$ by an integral, which will be easier to evaluate. Notice that S_n is an upper Riemann sum and a lower Riemann sum from the integrals

$$\begin{aligned}
U_n &= \int_0^n x^k dx \\
L_n &= \int_1^{n+1} x^k dx.
\end{aligned}$$

Notice that $L_n \leq S_n \leq U_n$ (Draw a picture of the graph x^k, evenly partition the intervals $[0, n]$ and $[1, n+1]$ into subintervals of unit length, and draw the corresponding upper and lower Riemann sums of the graph on these intervals.). So

$$U_n = \int_0^n x^k dx = \left[\frac{x^{k+1}}{k+1}\right]_0^n = \frac{n^{k+1}}{k+1}$$
$$L_n = \int_1^{n+1} x^k dx = \left[\frac{x^{k+1}}{k+1}\right]_1^{n+1} = \frac{(n+1)^{k+1} - 1}{k+1}.$$

Then because $L_n \leq S_n \leq U_n$ we have by taking limits

$$\frac{1}{k+1} = \lim_{n\to\infty} \frac{(n+1)^{k+1} - 1}{(k+1)n^{k+1}} \leq \lim_{n\to\infty} \frac{1^k + 2^k + \cdots + n^k}{n^{k+1}} \leq \lim_{n\to\infty} \frac{n^{k+1}}{(k+1)n^{k+1}} = \frac{1}{k+1}.$$

We conclude that the considered limit is therefore $1/(k+1)$.

(b) We must determine the limit

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n} = \lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{n+k}.$$

Again, we will bound the sum S_n from above and below by integrals of the function $1/(x+n)$. Observe that

$$\begin{aligned}
\lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{n+k} &= \lim_{n\to\infty} \sum_{k=1}^{n} \int_k^{k+1} \frac{1}{n+k} dx \\
&\geq \lim_{n\to\infty} \sum_{k=1}^{n} \int_k^{k+1} \frac{1}{x+n} dx \\
&= \lim_{n\to\infty} \sum_{k=1}^{n} \log(k+1+n) - \log(k+n) \\
&= \lim_{n\to\infty} \log\left(\frac{2n+1}{n+1}\right) \\
&= \log 2.
\end{aligned}$$

Similarly,

$$\begin{aligned}
\lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{n+k} &\leq \lim_{n\to\infty} \sum_{k=1}^{n} \int_k^{k+1} \frac{1}{x+n-1} dx \\
&= \lim_{n\to\infty} \sum_{k=1}^{n} \log(k+n) - \log(k+n-1) \\
&= \lim_{n\to\infty} \log(2n) - \log n \\
&= \log 2.
\end{aligned}$$

We conclude by the squeeze theorem that in the limit as $n \to \infty$ we have $S_n \to \log 2$.

5. Exercise 22. We want to show that

$$\gamma_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - \log n$$

decreases as n increases and that the limit converges to a value between 0 and 1. Recall that

$$\log(n+1) = \int_1^{n+1} \frac{1}{x} dx.$$

Notice that

$$\begin{aligned}
\log(n+1) &= \int_1^{n+1} \frac{1}{x} dx \\
&= \sum_{k=1}^{n} \int_k^{k+1} \frac{1}{x} dx \\
&< \sum_{k=1}^{n} \frac{1}{k} \\
&< 1 + \sum_{k=1}^{n-1} \int_k^{k+1} \frac{1}{x} dx \\
&= 1 + \int_1^{n} \frac{1}{x} dx \\
&= 1 + \log n.
\end{aligned} \tag{2}$$

See the Figure 1, to see an illustration of why the inequalities in (2) hold.

Subtracting $\log n$ from the terms in the above inequality shows

$$0 < \log(n+1) - \log n \leq \sum_{k=1}^{n} \frac{1}{k} - \log n \leq 1. \tag{3}$$

Thus, the sequence $\{\gamma_n\}_n$ is bounded from below by 0 and from above by 1.

Furthermore, we also see that

$$\frac{1}{n+1} < \int_n^{n+1} \frac{1}{x} dx = \log(n+1) - \log n.$$

It follows that

$$\gamma_{n+1} - \gamma_n = \frac{1}{n+1} - \log(n+1) + \log n < 0.$$

Hence, the sequence $\{\gamma_n\}_n$ is decreasing. Since the sequence $\{\gamma_n\}_n$ is decreasing an bounded from below by 0 the sequence converges. That is, if $\gamma = \lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{k} - \log n$, then $\gamma \in [0, 1)$.

It remains to show that $\gamma \neq 0$. Consider the sequence $\{\gamma'_n\}_n$ with

$$\gamma'_n = \sum_{k=1}^{n} \frac{1}{k} - \log(n+1).$$

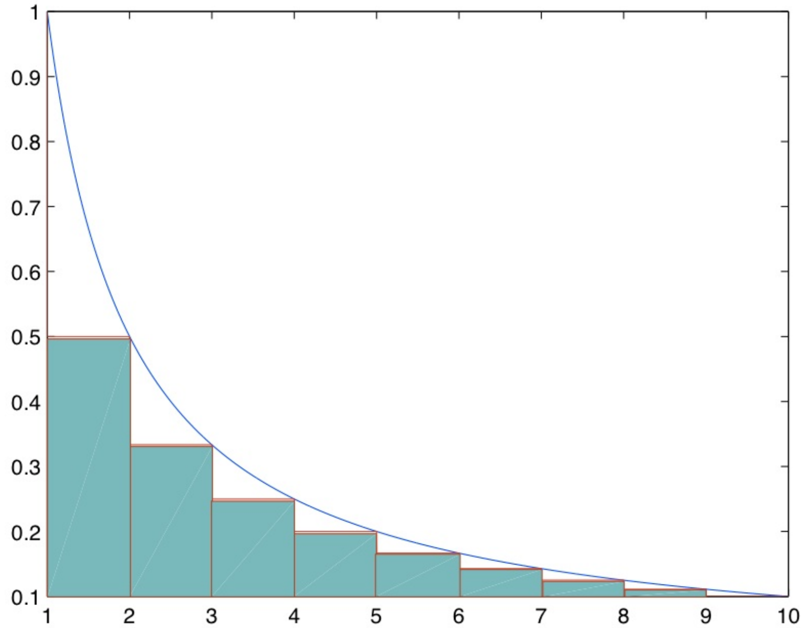

Figure 1: Bounding the sum $\sum_{k=1}^{10} k^{-1}$ by the integral $\int_1^{10} x^{-1} dx$.

From (3) we deduce that $\{\gamma'_n\}_n$ is bounded from below by 0 and from above by 1. Moreover, notice that

$$\frac{1}{n+1} > \int_{n+1}^{n+2} \frac{1}{x} dx = \log(n+2) - \log(n+1),$$

so

$$\gamma'_{n+1} - \gamma'_n = \frac{1}{n+1} - \log(n+2) + \log(n+1) > 0.$$

So the sequence $\{\gamma'_n\}_n$ is increasing. Since $\{\gamma'_n\}_n$ is increasing and bounded from above by 1, the sequence converges to a number $\gamma' \in (0,1]$ with $\gamma' = \lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{k} - \log(n+1)$. Finally, observe that

$$\gamma - \gamma' = \lim_{n\to\infty} \gamma_n - \gamma'_n = \lim_{n\to\infty} \log\left(\frac{n+1}{n}\right) = 0.$$

Thus, the limits of the two sequences are identical. Therefore, $\gamma \in (0,1)$, as advertised.