MAT 125B
Spring 2011
Problem Set 2 Solutions

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1. We give an example of a function $f : [0,1] \to \mathbb{R}$ such that f is not integrable, but $|f|$ is. We define such a function by
$$f(x) = \begin{cases} 1, & x \in \mathbb{Q} \cap [0,1], \\ -1, & \text{otherwise.} \end{cases}$$
Then f is discontinuous everywhere on $[0,1]$ so it is not Riemann integrable (That is, the same reasoning as Example 1.14 shows f is not integrable.). However, $|f| \equiv 1$, which is integrable.

2. Let $f : [a,b] \to \mathbb{R}$ and suppose that $|f(x)| \leq A < \infty$ for all $x \in [a,b]$.

   (a) We prove that
$$U_\delta(f^2) - L_\delta(f^2) \leq 2A(U_\delta(f) - L_\delta(f)), \quad \forall P_\delta. \tag{1}$$
Without loss of generality, suppose that A is the least such upper bound. Take any partition P_δ and let $a = x_0 < x_1 < \cdots < x_n = b$ be the ordered points in P_δ. In the notation of the notes let $\sup_{x\in[x_{i-1},x_i]} f(x) = M_i$ and $\inf_{x\in[x_{i-1},x_i]} f(x) = m_i$ then
$$\begin{aligned}
U_\delta(f^2) - L_\delta(f^2) &= \sum_{i=1}^{n} \Big(\sup_{u_i\in[x_{i-1},x_i]} (f(u_i))^2 - \inf_{v_i\in[x_{i-1},x_i]} (f(v_i))^2 \Big)(x_i - x_{i-1}) \\
&= \sum_{i=1}^{n} \sup_{u_i\in[x_{i-1},x_i]} \inf_{v_i\in[x_{i-1},x_i]} (f(u_i) + f(v_i))(f(u_i) - f(v_i))(x_i - x_{i-1}) \\
&\leq \sum_{i=1}^{n} \sup_{u_i\in[x_{i-1},x_i]} \inf_{v\in[x_{i-1},x_i]} 2A(f(u_i) - f(v_i))(x_i - x_{i-1}) \\
&= \sum_{i=1}^{n} 2A(M_i - m_i)(x_i - x_{i-1}) \\
&= 2A(U_\delta(f) - L_\delta(f)),
\end{aligned}$$
where we have used the fact that $M_i, m_i \leq A$ for all $i = 1, 2, \ldots, n$ and the fact that $f(u_i) - f(v_i)$ can be no greater than $M_i - m_i$.

   (b) We show that if $f \in \mathcal{R}([a,b])$ then $f^2 \in \mathcal{R}([a,b])$. Take $\epsilon > 0$ arbitrary. Since $f \in \mathcal{R}([a,b])$ choose $\delta' > 0$ such that for any $\delta < \delta'$ we have $U_\delta(f) - L_\delta(f) < \epsilon/(2A)$. Then by (1) we have
$$U_\delta(f^2) - L_\delta(f^2) < 2A \cdot \frac{\epsilon}{2A} = \epsilon.$$

Ergo, $f^2 \in \mathcal{R}([a,b])$.

3. Let $f, g \in \mathcal{R}([a,b])$. By the previous exercise, squares of Riemann integrable functions are Riemann integrable. Since the finite sum of Riemann integrable functions is Riemann integrable, it follows that $f+g, f-g \in \mathcal{R}([a,b])$, $(f+g)^2, -(f-g)^2 \in \mathcal{R}([a,b])$ and so

$$4fg = (f+g)^2 - (f-g)^2 \in \mathcal{R}([a,b]).$$

Since the scalar multiple of an integrable function is integrable, by multiplying the above function by $1/4$, we conclude $fg \in \mathcal{R}([a,b])$.

4. Suppose that $f : [a,b] \to \mathbb{R}$ is continuous, $f(x) \geq 0$ for $x \in [a,b]$, and $f(\xi) > 0$ for some $\xi \in [a,b]$. We show that $\int_a^b f(x)dx > 0$.

Suppose to the contrary that $\int_a^b f(x)dx \leq 0$. Because f is nonnegative, we must have that $\int_a^b f(x)dx = 0$ since no terms in any Riemann sums can be negative. However, $\int_a^b f(x)dx = 0$ is a contradiction as continuity of f and $f(\xi) > 0$ guarantees the existence of an interval on which f is positive—hence an integral of f over this interval must be positive.

To be more explicit, because $f(\xi) > 0$ and if $\xi \in (a,b)$ continuity of f implies there exists an $\epsilon > 0$ sufficiently small such that $f(x) > 0$ for $x \in [\xi - \epsilon, \xi + \epsilon]$; if $\xi = a$ or $\xi = b$, then we consider instead $[a, a+\epsilon]$ or $[b-\epsilon, b]$ respectively (We need only change the considered interval appropriately in the remainder of the argument.). Since $[\xi - \epsilon, \xi + \epsilon]$ is compact, there exists an $m > 0$ such that $f(x) > 0$ for all $x \in [\xi - \epsilon, \xi + \epsilon]$. Then because f is nonnegative it follows that

$$\int_a^b f(x)dx \geq \int_{\xi-\epsilon}^{\xi+\epsilon} f(x)dx \geq 2m\epsilon > 0.$$

5. Define the function $f : [0,1] \to \mathbb{R}$ by

$$f(x) = \begin{cases} \sin(1/x), & x \neq 0, \\ 0, & x = 0. \end{cases}$$

We prove $f \in \mathcal{R}([0,1])$. Take $\epsilon > 0$ arbitrary. Write $E_1 = [0, \epsilon/3]$ and $E_2 = [\epsilon/3, 1]$, so $[0,1] = E_1 \cup E_2$. Since f is continuous on E_2 choose $\delta_1 > 0$ such that $U_{\delta,E_2}(f) - L_{\delta,E_2}(f) < \epsilon/3$ for any partition P_δ of E_2 with $\delta < \delta_1$. Since $\max_{x \in E_1} \sin(1/x) = 1$ and $\min_{x \in E_1} \sin(1/x) = -1$, we have $U_{\epsilon/3,E_1}(f) - L_{\epsilon/3,E_1}(f) \leq 2\epsilon/3$. Therefore, it follows that for any $\delta_2 < \min\{\epsilon/3, \delta_1\}$ we have

$$U_{\delta_2}(f) - L_{\delta_2}(f) \leq (U_{\epsilon/3,E_1}(f) - L_{\epsilon/3,E_1}(f)) + (U_{\delta,E_2}(f) - L_{\delta,E_2}(f)) < \epsilon.$$