MAT 125B
Spring 2011
Problem Set 3 Solutions

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1. Problem 1.6. We evaluate the integral

$$\int_0^1 x \arctan x dx.$$

   (a) Integration by parts shows

$$\begin{aligned}\int_0^1 x \arctan x dx &= \left[\frac{x^2}{2}\arctan x\right]_0^1 - \frac{1}{2}\int_0^1 \frac{x^2}{1+x^2}dx \\ &= \frac{\pi}{8} - \frac{1}{2}\int_0^1 \frac{1+x^2}{1+x^2}dx + \frac{1}{2}\int_0^1 \frac{1}{1+x^2}dx \\ &= \frac{\pi}{8} - \frac{1}{2} + \frac{1}{2}\left[\arctan x\right]_0^1 \\ &= \frac{\pi}{4} - \frac{1}{2}.\end{aligned}$$

   (b) Similarly,

$$\int_0^1 x \arctan x dx = \left[\frac{1+x^2}{2}\arctan x\right]_0^1 - \frac{1}{2}\int_0^1 \frac{1+x^2}{1+x^2}dx = \frac{\pi}{4} - \frac{1}{2}.$$

2. Problem 1.7. Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of bounded Riemann integrable functions on $[a, b]$. Suppose $f_n \to f$ uniformly. We prove that f is Riemann integrable on $[a, b]$.

   First, we prove that f is bounded. For any $\epsilon > 0$ choose a positive integer N sufficiently large so that for all $n \geq N$ we have $\sup_{x\in[a,b]} |f(x) - f_n(x)| < \epsilon$. It follows that

$$\sup_{x\in[a,b]} |f(x)| \leq \sup_{x\in[a,b]} |f_N(x)| + \epsilon < \infty.$$

   Next, let $I_n = \int_a^b f_n(x)dx$. Consider the sequence of real numbers $\{I_n\}_n$. We claim that the sequence is Cauchy. Take $\epsilon > 0$ arbitrary. Let N be a positive integer such that for all $m, n \geq N$ we have $\sup_{x\in[a,b]} |f_m(x) - f_n(x)| < \epsilon/(b - a)$. Then

$$|I_m - I_n| \leq \int_a^b |f_m(x) - f_n(x)|dx < \epsilon.$$

Thus, the sequence is Cauchy and converges to some $I = \lim_{n\to\infty} I_n$.

Finally, we use the previous results to prove that f is integrable with integral I. By uniform convergence of f_n to f choose N sufficiently large so that $\sup_{[a,b]} |f(x) - f_n(x)| < \epsilon/(3(b-a))$ for all $n \geq N$. Furthermore, let $\delta > 0$ be sufficiently small so that for all $0 < \delta' < \delta$ we have $|S_{\delta'}(f) - I_n| < \epsilon/3$. The triangle inequality implies

$$|S_{\delta'}(f) - I| \leq |S_{\delta'}(f) - S_{\delta'}(f_n)| + |S_{\delta'}(f_n) - I_n| + |I_n - I| < \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon.$$

3. Problem 1.8. We prove the six propositions in Example 1.48. The first of these is completed in the text and shows by the comparison test that $\int_1^\infty x^p dx$ converges for $p < -1$ and diverges otherwise. We begin by proving the second one, that $\int_0^1 x^p dx$ converges for $p > -1$ and diverges otherwise. To demonstrate the convergence observe that for any $\epsilon > 0$ the integral is

$$\lim_{b\to 0^+} \int_b^1 \frac{1}{x^{1-\epsilon}} dx = \lim_{b\to 0^+} \frac{1}{\epsilon}[x^\epsilon]_b^1 = \frac{1}{\epsilon}.$$

Next, observe that

$$\lim_{b\to 0^+} \int_b^1 \frac{1}{x} dx = \lim_{b\to 0^+} -\log b = \infty.$$

Since $x^p > x^{-1}$ for $p < -1$ and $x \in (0,1)$ it follows by the comparison test that $\int_0^1 x^p dx$ diverges for $p \leq -1$.

Now suppose that the third part, that $\int_1^\infty e^{-x} x^p dx$ converges for all p. If we take $p < -1$, then

$$\int_1^\infty e^{-x} x^p dx \leq \int_1^\infty x^p dx < \infty.$$

So the integral converges by the comparison test. Now for any $p \in \mathbb{R}$ there exists a minimal nonnegative integer n such that $p - n < -1$. To prove that the integral converges for all possible p we prove the result by induction on n. Notice that we have already verified the initial case when $n = 0$. So suppose that the integral $\int_1^\infty e^{-x} x^p dx$ converges for all $p \in \mathbb{R}$ with the property that $p - (n-1) < -1$. We must show that $\int_1^\infty e^{-x} x^p dx$ converges if p has the property that $p - n < -1$. Integration by parts implies that

$$\int_1^\infty e^{-x} x^p dx = [-e^{-x} x^p]_1^\infty + \int_1^\infty p e^{-x} x^{p-1} dx = e^{-1} + \int_1^\infty p e^{-x} x^{p-1} dx$$

The last integral is found to be finite by the induction hypothesis because $p-1$ has the property that $p - 1 - (n-1) = p - n < -1$. Therefore, we conclude that the integral converges for all $p \in \mathbb{R}$.

Next, we prove 4 : that $\int_0^a e^{1/x} x^p dx$ with $a > 0$ diverges for all p. For simplicity, we prove the result when $a = 1$. Taking the change of variables $u = 1/x$ shows that

$$\int_0^1 e^{1/x} x^p dx = \int_1^\infty e^u u^{-p-2} du.$$

By L'Hôpital's rule we see that $\lim_{u\to\infty} e^u u^{-p-2} = \infty$, for any given p so the above integral diverges for any p.

To prove 5. for $a > 0$ observe that the change of variables $u = \log x$ gives

$$\begin{aligned}\int_0^a \log x dx &= \int_{-\infty}^{\log a} ue^u du \\ &= \int_{-\log a}^{\infty} ye^{-y} dy, \quad y = -u, \\ &= \int_{-\log a}^{0} ye^{-y} dy + \int_0^1 ye^{-y} dy + \int_1^{\infty} ye^{-y} dy.\end{aligned}$$

The first two integrals are clearly finite and the last integral exists by part 3 of this exercise.

Finally, we prove 6. Consider the function $f(x) = x - \log x$. Then $f'(x) = 1 - x^{-1} > 0$ for $x \in (1, \infty)$. So f is increasing on $(1, \infty)$. Because $f(1) = 1$ we know that $x - \log x > 0$ for all $x \in (1, \infty)$, whence $\frac{1}{\log x} > \frac{1}{x}$. Hence,

$$\int_1^{\infty} \frac{1}{\log x} dx \geq \int_1^{\infty} \frac{1}{x} dx = \infty.$$

Thus, the integral $\int_1^{\infty} \frac{1}{\log x} dx$ diverges.

4. Problem 1.9. Calculus and Theorem 1.46 imply

$$\int_0^{\infty} \frac{1}{(1+x)^2} dx = \lim_{b \to +\infty} \left[-\frac{1}{(1+x)} \right]_0^b = 1.$$

5. Problem 1.10. The integral $\int_0^{\infty} x^p dx$ is not convergent for any p. Write

$$\int_0^{\infty} x^p dx = \int_0^1 x^p dx + \int_1^{\infty} x^p dx.$$

From Problem 1.8 the integral $\int_1^{\infty} x^p dx$ converges whenever $p < -1$. On the other hand, $\int_0^1 x^p dx$ converges for $p > -1$. Since p cannot satisfy both $p < -1$ and $p > -1$ the considered integral cannot converge for any $p \in \mathbb{R}$.