MAT 125B
Spring 2011
Problem Set 4 Solutions

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1. Problem 2.3. Let $A = \{(x_1, x_2) \in \mathbb{R}^2 | 0 \leq x_1 \leq 1, x_2 = 0\}$. Then A is not open. We show that f is differentiable at $x_0 \in A$ then the linear transformation $Df(x_0)$ is not uniquely determined.

   Fix such an $x = (x_1, x_2) \in A$. Consider the two linear transformations $L_1, L_2 : \mathbb{R}^2 \to \mathbb{R}$ defined by $L_1(w_1, w_2) = 0$ for all $w = (w_1, w_2) \in \mathbb{R}^2$ and $L_2(w_1, w_2) \cdot (u_1, u_2) = u_2$. So $L_1 \neq L_2$. Then

$$\lim_{w \to x} \frac{||f(w) - f(x) - L_1(w) \cdot (w - x)||}{||w - x||} = \lim_{w \to x} \frac{||0||}{||w - x||} = 0$$

   and

$$\begin{aligned} \lim_{w \to x} \frac{||f(w) - f(x) - L_2(w) \cdot (w - x)||}{||w - x||} &= \lim_{w \to x} \frac{|w_2|}{\sqrt{(w_1 - x_1)^2 + w_2^2}} \\ &= 0, \end{aligned}$$

   where we obtain the last equality by recalling that the limit operator $\lim_{w \to x}$ is taken over all $w \in A$, so $w_2 = 0$. Thus, we see that L_1 and L_2, while they are distinct linear transformations from $\mathbb{R}^2 \to \mathbb{R}$, are both derivatives of f in A.

2. Problem 2.4. Suppose $f : \mathbb{R}^n \to \mathbb{R}^m$ and that there exists a constant M such that for all $x \in \mathbb{R}^n$ we have $||f(x)|| \leq M||x||^2$. We verify that f is differentiable at $x_0 = 0$ and $Df(0) = 0$.

   Notice that since $||f(x)|| \leq M||x||^2$ that $f(0) = 0$. Let A denote the zero map. From the definition of the derivative we see that

$$\begin{aligned} \lim_{x \to x_0} \frac{||f(x) - f(x_0) - A(x) \cdot (x - x_0)||}{||x - x_0||} &= \lim_{x \to 0} \frac{||f(x)||}{||x||} \\ &\leq \lim_{x \to 0} \frac{M||x||^2}{||x||} \\ &= 0. \end{aligned}$$

   Since the limit exists and is zero, we conclude that the zero map satisfies the definition of the derivative. By the uniqueness of the derivative, we conclude the advertised result.

3. Problem 2.7. The Lipschitz condition does not guarantee differentiability. Consider the function $f(x) = |x|$ defined on $\mathbb{R}$. Then $||x| - |y|| \leq |x - y|$ by the reverse triangle inequality, but f is not differentiable at $x = 0$.

4. Problem 2.9. Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$f(x, y) = \begin{cases} \frac{(xy)^2}{\sqrt{x^2+y^2}}, & (x, y) \neq (0, 0), \\ 0, & (x, y) = (0, 0). \end{cases}$$

We show that f is differentiable. Observe that we may rewrite the function in polar coordinates (r, θ), with $r \geq 0$ and $\theta \in [0, 2\pi)$, by

$$f(r, \theta) = \begin{cases} \frac{r^3 \sin^2 2\theta}{4}, & (r, \theta) \neq (0, 0), \\ 0, & (r, \theta) = (0, 0). \end{cases}$$

Then for $(r, \theta) \neq (0, 0)$ we have

$$\begin{aligned} \frac{\partial f}{\partial r}(r, \theta) &= \frac{3r^2 \sin^2 2\theta}{4} \\ \frac{\partial f}{\partial \theta}(r, \theta) &= \frac{r^3 \sin 2\theta}{4}. \end{aligned}$$

Taking the limit as $(r, \theta) \to (0, 0)$ in the above partial derivatives shows that $\partial f/\partial r(0, 0) = \partial f/\partial \theta(0, 0) = 0$. Thus, we conclude that the partial derivatives are everywhere well-defined and are continuous. Hence, f is differentiable.

5. Problem 2.10. Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$f(x, y) = \begin{cases} \frac{xy}{\sqrt{x^2+y^2}}, & (x, y) \neq (0, 0), \\ 0, & (x, y) = (0, 0). \end{cases}$$

We claim that f is not differentiable at $(0, 0)$. Observe that

$$\frac{\partial f}{\partial x}(x, y) = \frac{y^3}{(x^2 + y^2)^{3/2}}, \quad (x, y) \neq (0, 0).$$

Then

$$\lim_{(0,y)\to(0,0)} \frac{\partial f}{\partial x}(0, y) = 1 \neq 0 = \lim_{(x,0)\to(0,0)} \frac{\partial f}{\partial x}(x, 0).$$

So the partial derivative $\partial f/\partial x$ cannot be continuous at $(0, 0)$. Therefore, f cannot be differentiable at $(0, 0)$.