MAT 125B
Spring 2011
Problem Set 5 Solutions

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1. Problem 2.11. We determine the unit normal to the surface $x^2 - y^2 + xyz = 1$ in $\mathbb{R}^3$ at $(1, 0, 1)$. Recall that the unit normal to the surface is given by the normalized gradient of the function $f(x, y, z) = x^2 - y^2 + xyz - 1$. Then $\text{grad} f(x, y, z) = (2x - yz, -2y + xz, xy)$. So $\text{grad} f(1, 0, 1) = (2, 1, 0)$. Normalizing yields the unit normal $N(1, 0, 1) = 5^{-1/2}(2, 1, 0)$.

2. Problem 2.12. Recall that the direction in which the function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by $f(x, y) = e^{x^2 y}$ is increasing the fastest is precisely the gradient of f, which is $\text{grad} f(x, y) = (2xye^{x^2 y}, x^2 e^{x^2 y})$.

3. Problem 2.13. We prove L'Hôpital's rule. Suppose $f, g : \mathbb{R} \to \mathbb{R}$ are differentiable at $x_0 \in \mathbb{R}$, that $g'(x_0) \neq 0$, and that $f(x_0) = g(x_0) = 0$. We must show that

$$\lim_{x \to x_0} \frac{f(x)}{g(x)} = \frac{f'(x_0)}{g'(x_0)}. \tag{1}$$

   From the suppositions, the Cauchy-Schwarz inequality, and the definition of the derivative we immediately have that

$$\lim_{x \to x_0} \frac{f(x)}{g(x)} = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{g(x) - g(x_0)} = \lim_{x \to x_0} \left(\frac{f(x) - f(x_0)}{x - x_0} \right) \left(\frac{x - x_0}{g(x) - g(x_0)} \right) = \frac{f'(x_0)}{g'(x_0)}.$$

4. Problem 2.14. By L'Hôpital's rule, we have that

$$\lim_{x \to 0} \frac{\sin x}{x} = \lim_{x \to 0} \frac{\cos x}{1} = 1$$
$$\lim_{x \to 0} \frac{e^x - 1}{x} = \lim_{x \to 0} \frac{e^x}{1} = 1.$$

   We can apply L'Hôpital's rule in the above expressions since both limits are indeterminate of the form $0/0$.

5. Problem 2.15. Let $f : A \subset \mathbb{R}^n \to \mathbb{R}$ be differentiable on a convex set A. Suppose that $||\,\text{grad} f(x)|| \leq M$ for $x \in A$. We prove that

$$|f(y) - f(x)| \leq M||y - x||, \quad \forall x, y \in A. \tag{2}$$

Since A is convex, the mean value theorem implies that there exists a point $z = tx + (1-t)y$ with $t \in [0,1]$ such that

$$f(x) - f(y) = Df(z) \cdot (y - x).$$

Then by the Cauchy-Schwarz inequality and the boundedness of the gradient we have

$$|f(y) - f(x)| = |Df(z) \cdot (y - x)| \leq ||\,\mathrm{grad} f(x)|| ||y - x|| \leq M||y - x||.$$

Next, we give a counterexample to the assertion when A is not presumed to be convex.

Let $A_1 = (0, 1/2)$, $A_2 = (1/2, 1)$, and $A = A_1 \cup A_2$. Then A is not convex because $1/2 \notin A$.

Define the map $f : A \subset \mathbb{R} \to \mathbb{R}$ by $f(x) = \mathbf{1}_{A_2}(x)$. Then f is differentiable since it is trivially differentiable on A_1 and A_2 with zero derivative. That is, $||\,\mathrm{grad} f(x)|| = |f'(x)| = 0$.

Now fix any $M \geq 0$. Let $\epsilon \in (0, 1)$ be sufficiently small so that the points $x = 1/2 - \epsilon/M$ and $y = 1/2 + \epsilon/M$ lie in A_1 and A_2, respectively. Then

$$|f(x) - f(y)| = 1 > M|x - y| = \epsilon,$$

so (2) does not hold for our function f.