MAT 125B
Spring 2011
Problem Set 6 Solutions

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1. Problem 2.16. Let $f : (-1, 1) \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} x^2 \sin(1/x), & x \in (-1, 0) \cup (0, 1), \\ 0 & x = 0. \end{cases}$$

To apply Taylor's theorem we must determine the nonnegative integer k for which $f \in C^k(-1, 1)$.

Observe by the squeeze theorem that $\lim_{x \to 0} x^2 \sin(1/x) = 0$. So $k \geq 0$.

Next, notice that for $x = 0$, we have

$$f'(x) = 2x \sin(1/x) - \cos(1/x).$$

Since the limit $\lim_{x \to 0} f'(x)$ does not exist, we conclude that $f \in C^0(-1, 1)$.

Thus, Taylor's theorem for f about the point $x = 0$ asserts that

$$f(h) = 0 + R_0(x, h)$$

with

$$\lim_{h \to 0} \frac{R_0(x, h)}{|h|} = 0.$$

2. Problem 2.17. We determine the Taylor series of $f(x) = \log(1 - x)$ for $x \in (-1, 1)$ and equals $f(x)$ for each such x.

Observe that for $x < 1$ we have by the power rule and the chain rule that

$$f^{(k)}(x) = -\frac{(k-1)!}{(1-x)^k}, \quad k \in \mathbb{N}.$$

Since $\log(1 - x) \in C^\infty(-1, 1)$ Taylor's theorem implies that the Taylor series around $x_0 = 0$ is given by

$$f(x) = -\sum_{k=1}^{\infty} \frac{x^k}{k}.$$

It remains to prove the series converges uniformly on any closed subinterval of $(-1,1)$. Let $[a,b] \subset (-1,1)$ be arbitrary. Take $0 < M = \max\{|a|,|b|\} < 1$. Then for $x_0 \in [a,b]$ we have

$$\left|\frac{f^{(k)}(0)}{k!}(x-0)^k\right| = \left|\frac{x^k}{k}\right| \leq \frac{M}{k} \to 0 \quad \text{as } k \to \infty.$$

Thus, by Example 2.51, we conclude that the series converges uniformly on any closed subinterval of $(-1,1)$.

3. Problem 2.18. We verify that if the conditions in Example 2.51. are met then we can differentiate the Taylor series term by term to obtain $f'(x)$. Let $f(x) = T(x)$ for all $x \in [x_0-1, x_0+1]$ where $T(x)$ is the Taylor series centered at x_0 given by

$$T(x) = \sum_{k=0}^{\infty} \frac{1}{k!} f^{(k)}(x_0)(x-x_0)^k.$$

Since $|f^{(k)}(x)| \leq M$ for all k and all $x \in [x_0-1, x_0+1]$ we see that

$$\sum_{k=0}^{\infty} \left|\frac{1}{k!} f^{(k)}(x_0)\right| |x-x_0|^k \leq \sum_{k=0}^{\infty} \frac{M}{k!} < \infty.$$

Thus, the series is absolutely summable and converges uniformly on $[x_0-1, x_0+1]$ to $f(x)$ by the Weierstrass M-test. Hence, we may interchange the order of differentiation and summation (Rosenlicht, p.150), i.e.,

$$f'(x) = \frac{d}{dx}\sum_{k=0}^{\infty} \frac{1}{k!} f^{(k)}(x_0)(x-x_0)^k = \sum_{k=0}^{\infty} \frac{d}{dx}\frac{1}{k!} f^{(k)}(x_0)(x-x_0)^k = \sum_{k=1}^{\infty} \frac{1}{(k-1)!} f^{(k)}(x_0)(x-x_0)^{k-1}.$$

This proves the result.

4. Problem 2.19. We investigate the nature of the critical point $(0,0)$ of

$$f(x,y) = x^2 + 2xy + y^2 + 6 = (x+y)^2 + 6$$

Observe that

$$\frac{\partial f}{\partial x}(x,y) = 2x + 2y$$
$$\frac{\partial f}{\partial y}(x,y) = 2x + 2y$$

So $(0,0)$ is indeed a critical point of f and

$$\frac{\partial^2 f}{\partial x \partial y}(x,y) = \frac{\partial^2 f}{\partial y \partial x}(x,y) = \frac{\partial^2 f}{\partial x^2}(x,y) = \frac{\partial^2 f}{\partial y^2}(x,y) = 2.$$

Thus,

$$D^2 f(0,0) = \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix}.$$

Since $2 > 0$ anddet$(D^2f(0,0)) = 0$ we deduce that $D^2f(0,0)$ is positive-semidefinite. Hence, $(0,0)$ is a local minimum, as can be seen by determining the principal directions and principal curvatures. Observe that the eigenvalues of $D^2f(0,0)$ are found by solving

$$\begin{vmatrix} \lambda - 2 & -2 \\ -2 & \lambda - 2 \end{vmatrix} = \lambda(\lambda - 4) = 0.$$

So $\lambda = 0, 4$. For each λ, we solve the system of linear equations

$$\begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \lambda \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}.$$

The unnormalized eigenvectors of $\lambda = 0$ and $\lambda = 4$ are determined to be $v = (v_1, v_2) = (-1, 1)$ and $v' = (1, 1)$ respectively. Thus, in the direction v at $(0,0)$, our function f is neither increasing nor decreasing. While in the direction v' the function is increasing at its fastest possible rate. Therefore, we conclude that the local minima are attained along the span of v'.

5. Problem 2.20. We determine the nature of the critical point $(0,0)$ of the function

$$f(x,y) = x^3 + 2xy^2 - y^4 + x^2 + 3xy + y^2 + 10.$$

Notice that

$$\begin{aligned} \frac{\partial f}{\partial x}(x,y) &= 3x^2 + 2y^2 + 2x + 3y \\ \frac{\partial f}{\partial y}(x,y) &= 4xy - 4y^3 + 3x + 2y. \end{aligned}$$

So $(0,0)$ is in fact a critical point. Furthermore,

$$\begin{aligned} \frac{\partial^2 f}{\partial x^2}(x,y) &= 6x + 2 \\ \frac{\partial^2 f}{\partial y^2}(x,y) &= 4x + 12y^2 + 2 \\ \frac{\partial^2 f}{\partial x \partial y}(x,y) = \frac{\partial^2 f}{\partial y \partial x}(x,y) &= 4y + 3. \end{aligned}$$

Thus, with respect to the standard basis we have

$$D^2f(0,0) = \begin{pmatrix} 2 & 3 \\ 3 & 2 \end{pmatrix}.$$

Thus, $\det(D^2f(0,0)) = -5$. Since $2 > 0$ and $\det(D^2f(0,0)) < 0$ we know that $D^2f(0,0)$ is neither positive-definite nor negative-definite. So $(0,0)$ is a saddle point. We can see this result geometrically by determining the principal directions and principal curvatures for $D^2f(0,0)$. The principal curvatures are found by finding the roots of

$$\begin{vmatrix} \lambda - 2 & -3 \\ -3 & \lambda - 2 \end{vmatrix} = (\lambda - 5)(\lambda + 1).$$

So $\lambda = 5, -1$. For each λ, we solve the system of linear equations

$$\begin{pmatrix} 2 & 3 \\ 3 & 2 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \lambda \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}.$$

The unnormalized eigenvectors of $\lambda = 5$ and $\lambda = -1$ are determined to be $v = (v_1, v_2) = (1, 1)$ and $v' = (-1, 1)$ respectively. We deduce that the graph of f is increasing at $(0, 0)$ in the direction $(1, 1)$ while f is decreasing at $(0, 0)$ in the direction $(-1, 1)$. Thus, we conclude that the point $(0, 0)$ is a saddle point.