MAT 125B
Spring 2011
Problem Set 7 Solutions

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1. Problem 3.2. Let $f(x) = x + 2x^2 \sin(1/x)$ for $x \neq 0$ and $f(0) = 0$. We prove that f is not locally integrable near $x = 0$. Observe that for $x \neq 0$ the derivative of f exists and

$$f'(x) = 1 + 4x\sin(1/x) - 2\cos(1/x).$$

Furthermore, notice that

$$f'(0) = \lim_{h\to 0} \frac{f(h) - f(0)}{h} = \lim_{h\to 0} \frac{h + 2h^2\sin(1/h)}{h} = 1.$$

So $f'(0) = 1 \neq 0$. Hence, although f is differentiable, the $\lim_{x\to 0} f'(x)$ does not exist so $f \notin C^1(\mathbb{R})$. As a result, the inverse function theorem does not apply.

Next, to show that f is not invertible near $x = 0$, consider the points $1/((2n+1)\pi) < 1/((2n+1/2)\pi) < 1/(2n\pi)$ with n a positive integer. Then $f(1/((2n+1)\pi)) < f(1/((2n+1/2)\pi))$ and $f(1/((2n+1)\pi)) < f(1/(2n\pi))$. However, a computation shows that

$$f\left(\frac{1}{(2n+1/2)\pi}\right) - f\left(\frac{1}{2n\pi}\right) = \frac{(16-2\pi^2)n - \pi^2/2}{(4n)(2n+1/2)^2\pi^2},$$

which is greater than 0 for $n \geq 2$. Thus, $f(1/((2n+1)\pi)) < f(1/(2n\pi)) < f(1/((2n+1/2)\pi))$. Ergo, the intermediate value theorem implies there exists a point $c_n \in (1/((2n+1)\pi), 1/((2n+1/2)\pi))$ such that $f(c_n) = f(1/(2n\pi))$.

Therefore, for any open neighborhood U containing 0, there exists an integer n such that $[1/((2n+1)\pi), 1/(2n\pi)] \subset U$. Since U contains both c_n and $1/(2n\pi)$, we deduce that f is not invertible on U. Consequently, f is not invertible on any open neighborhood of 0. We conclude the desired result.

2. Problem 3.3. Let $L : \mathbb{R}^n \to \mathbb{R}^n$ be a linear isomorphism and let $f(x) = L(x) + g(x)$, where $g : \mathbb{R}^n \to \mathbb{R}^n$ with the property that $||g(x)|| \leq M||x||^2$ and $f \in C^1(\mathbb{R}^n, \mathbb{R}^n$. We prove that f is locally invertible near $x = 0$.

By Problem 2.4 we know that g is differentiable at $x = 0$ and $Dg(0)$ is the zero map. Furthermore, $L(x)$ is differentiable with derivative $DL(0) = L$. Since L is a linear isomorphism, $\det(L) \neq 0$. Hence, by linearity of the derivative and multilinearity of the determinant $\det(Df(0)) = \det(L) \neq 0$. Thus, the inverse function theorem guarantees there exists an open neighborhood A containing the origin such that f is invertible on A.

3. Problem 3.4. We show that the system

$$\begin{aligned} u(x,y,z) &= x + xyz \\ v(x,y,z) &= y + xy \\ w(x,y,z) &= z + 2x + 3z^2 \end{aligned}$$

can be solved for (x,y,z) in terms of (u,v,w) near $(x,y,z) = (0,0,0)$. Let $f : \mathbb{R}^3 \to \mathbb{R}^3$ be defined by $f(x,y,z) = (u,v,w)$. Then the Jacobian of f at $(0,0,0)$ is

$$Df(0,0,0) = \left.\begin{pmatrix} 1+yz & xz & xy \\ y & 1+x & 0 \\ 2 & 0 & 1+6z \end{pmatrix}\right|_{x,y,z=0} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix}.$$

Since $\det(Df(0,0,0)) \neq 0$ the inverse function theorem guarantees that in some open neighborhood of $(0,0,0)$ we have a differentiable inverse of f, which means that we can solve for x, y, z in terms of u, v, w.

4. Problem 3.6. Consider the equation

$$\frac{dy}{dt} = \sqrt{y}. \tag{1}$$

Then the zero function $y \equiv 0$ trivially satisfies the equation. If $y(t_0) > 0$ for some $t_0 \in \mathbb{R}$ then there exists an $\epsilon > 0$ sufficiently small such that $y(t) > 0$ for all $t \in U := (t_0 - \epsilon, t_0 + \epsilon)$. Then for all $t \in U$ we have

$$\frac{dy}{\sqrt{y(t)}} = dt.$$

Integrating the above equality shows that $2\sqrt{y(t)} = t + C$. Since $y(0) = 0$ it follows that $C = 0$, whence $y(t) = t^2/2$ in U. Hence, we conclude that the only admissible nontrivial solution of (1) with this property in U is

$$y(t) = \begin{cases} 0, & t \leq 0, \\ \frac{t^2}{2}, & t > 0. \end{cases}$$

The fundamental theorem of ordinary differential equations is not contradicted since the function $\sqrt{y}$ is not Lipschitz. That is, the slope of the curve $c(y) = \sqrt{y}$ is unbounded as $y \to 0^+$ as evidenced by the fact that for $y > 0$ we have $c'(y) = \frac{1}{2\sqrt{y}}$.

5. Problem 3.8. Let B be an $n \times n$ matrix and consider the linear system

$$\frac{dy}{dt} = B \cdot y(t), \quad y(t) \in \mathbb{R}^n. \tag{2}$$

We show that a solution is given by

$$y(t) = e^{tB} y(0), \qquad \text{where } e^B := \sum_{n=0}^{\infty} \frac{B^n}{n!}$$

Observe that

$$\frac{d}{dt} e^{tB} y(0) = \frac{d}{dt} \left(\sum_{n=0}^{\infty} \frac{t^n B^n}{n!} \right) y(0) = B \left(\sum_{n=0}^{\infty} \frac{t^n B^n}{n!} \right) y(0) = B \cdot e^{tB} y(0).$$

Hence, we have found a solution that holds for all $t \geq 0$.

We prove the result directly from the fundamental theorem of ODEs.

Let $y_0 = y(0)$ and define the sequence $\{y_n(t)\}_{n=1}^{\infty}$ of vectors in $\mathbb{R}^n$ by

$$y_n(t) = y(0) + \int_0^t F(y_{n-1}(s), s)ds = y(0) + B \cdot \int_0^t y_{n-1}(s)ds,$$

where the integral is defined componentwise on $y_{n-1}(s)$. Then $\lim_{n\to\infty} y_n(t) = e^{tB}y(0)$. To prove that the sequence $\{y_n(t)\}$ indeed converges to the solution of (2), by the proof of the fundamental theorem of ODEs, it suffices to determine the domain of $t \in \mathbb{R}$ for which the sequence $\{y_n\}_n$ converges.

Let $F(y(t), t) = B \cdot y(t)$ with $y_0 = y(0)$, $r > 0$ and $a > 0$ so $F : \bar{B}(y_0, r) \times [-a, a] \to \mathbb{R}^n$. Notice that B represents a bounded linear map $B : \mathbb{R}^n \to \mathbb{R}^n$ that is continuous. That is, there exists a constant K such that for all $y \in \mathbb{R}^n$ we have $||By|| \leq K||y||$. In particular,

$$||F(y_1, t) - F(y_2, t)|| = ||B(y_1 - y_2)|| \leq K||y_1 - y_2||, \quad \forall y_1, y_2 \in \bar{B}(y_0, r), t \in [-a, a].$$

Furthermore, since B and y are continuous, we know that

$$\sup_{y \in \bar{B}(y_0, r), t \in [-a, a]} ||F(y, t)|| = M < \infty,$$

since $\bar{B}(y_0, r)$ and $[-a, a]$ are compact. Thus, if we choose $b < \min\{a, r/M, 1/K\}$ then we have met all the hypotheses of the fundamental theorem of ODEs. Observe that such a b does not depend on a or r. So for $t \in [-1/K, 1/K]$ we have the desired convergence of the sequence $\{y_n(t)\}_n$.

By applying the fundamental theorem of ODEs again, this time at $t = 1/K$ with initial condition $y(1/K)$, we see again that $\{y_n(t)\}_n$ converges to the solution of (2) for $t \in [0, 2/K]$. Repeated applications of the fundamental theorem guarantees that $e^{tB}y(0)$ is a solution of (2) for all $t \geq 0$.