

WELL-POSEDNESS OF THE FREE-BOUNDARY COMPRESSIBLE 3-D EULER EQUATIONS WITH SURFACE TENSION AND THE ZERO SURFACE TENSION LIMIT*

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Abstract. We prove that the three-dimensional compressible Euler equations with surface tension along the moving free-boundary are well-posed; we then establish the limit as surface tension tends to zero. Specifically, we consider isentropic dynamics and consider an equation of state, modeling a liquid, given by Courant and Friedrichs [*Supersonic Flow and Shock Waves*, Appl. Math. Sci. 21, Springer-Verlag, New York, 1976] as $p(\rho) = \alpha\rho^\gamma - \beta$ for constants $\gamma > 1$ and $\alpha, \beta > 0$. The analysis is made difficult by two competing nonlinearities associated with the potential energy: *compression* in the bulk and *surface area dynamics* on the free-boundary. Unlike the analysis of the incompressible Euler equations, wherein boundary regularity controls regularity in the interior, the compressible Euler equation requires the additional analysis of nonlinear wave equations generating sound waves. An existence theory is developed by a specially chosen parabolic regularization together with the vanishing viscosity method. The artificial parabolic term is chosen so as to be asymptotically consistent with the Euler equations in the limit of zero viscosity. Having solutions for the positive surface tension problem, we proceed to obtain a priori estimates which are independent of the surface tension parameter. This requires choosing initial data which satisfy the Taylor sign condition. By passing to the limit of zero surface tension, we prove the well-posedness of the compressible Euler system without surface on the free-boundary and without derivative loss.

Key words. compressible flow, Euler equations, free boundary problems, surface tension

AMS subject classifications. 35L65, 35L70, 35L80, 35Q35, 35R35, 76B03

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1. Introduction.

1.1. The compressible Euler equations in Eulerian variables. The motion of an inviscid *compressible* liquid with a moving free-boundary is modeled by the Euler equations:

$$(1.1a) \quad \partial_t(\rho u) + \operatorname{div}(\rho u \otimes u + p \operatorname{Id}) = 0 \quad \text{in } \Omega(t),$$

$$(1.1b) \quad \partial_t \rho + \operatorname{div}(\rho u) = 0 \quad \text{in } \Omega(t),$$

$$(1.1c) \quad p = \sigma H(t) \quad \text{on } \Gamma(t),$$

$$(1.1d) \quad \mathcal{V}(\Gamma(t)) = u \cdot n(t),$$

$$(1.1e) \quad (u, \rho) = (u_0, \rho_0) \quad \text{on } \Omega(0),$$

$$(1.1f) \quad \Omega(0) = \Omega,$$

where $\Omega(t)$ denotes an open and bounded subset of \mathbb{R}^3 , $\Gamma(t) = \partial\Omega(t)$ is the moving free-boundary, and $t \in [0, T]$ denotes time. We use the notation $\mathcal{V}(\Gamma(t))$ for the normal velocity of boundary $\Gamma(t)$, which is equal to the normal component of the fluid velocity

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$u \cdot n$, where $n(t)$ is the outward-pointing unit normal to $\Gamma(t)$, $u = (u_1, u_2, u_3)$ denotes the velocity field, p denotes the pressure, and ρ denotes the density.

The first two equations are conservation laws for momentum and mass. The boundary condition (1.1c) is often referred to as the Laplace–Young condition, stating that the fluid stress is proportional to the mean curvature $H(t)$ of the moving surface, the proportionality constant defining the surface tension parameter σ . The last two equations provide the initial conditions for the dynamics. The initial data consists of the initial domain Ω , the initial density function ρ_0 , and the initial velocity vector field u_0 .

In order to model the motion of a compressible liquid, we use the equation-of-state given by Courant and Friedrichs [8] as

$$(1.2) \quad p(x, t) = \alpha \rho(x, t)^\gamma - \beta \quad \text{for } \gamma > 1,$$

where $\alpha > 0$ and $\beta > 0$. For convenience, we set $\alpha = 1$.¹

Using the equation of state (1.2), the momentum equations (1.1a) and Laplace–Young boundary condition (1.1c) are equivalently written as

$$(1.3a) \quad \rho[\partial_t u + (u \cdot D)u] + D\rho^\gamma = 0 \quad \text{in } \Omega(t),$$

$$(1.3b) \quad \rho^\gamma = \beta + \sigma H \quad \text{on } \Gamma(t).$$

We assume that the initial density function is strictly positive and that

$$\rho_0 \geq \lambda > 0 \quad \text{in } \bar{\Omega}.$$

In the absence of surface tension, we further require the initial pressure function p_0 to satisfy the *Taylor sign condition* (see, for example, [30] and [26]), given by

$$0 < \nu \leq -\frac{\partial p_0}{\partial N} \quad \text{on } \Gamma,$$

where N denotes the outward unit normal to Γ . This is equivalent to

$$(1.4) \quad 0 < \nu \leq -\frac{\partial \rho_0^\gamma}{\partial n} \quad \text{on } \Gamma.$$

1.2. Prior results on the Euler equations with moving free-boundary.

1.2.1. The incompressible setting. There has been a recent explosion of interest in the analysis of the free-boundary incompressible Euler equations, particularly in irrotational form, that has produced a number of different methodologies for obtaining a priori estimates. The accompanying existence theories have relied mostly on the Nash–Moser iteration to deal with derivative loss in linearized equations when arbitrary domains are considered, or on complex analysis tools for the irrotational problem with infinite depth. We refer the reader to [2, 3, 11, 14, 21, 22, 27, 36, 37, 40] for a partial list of papers on this topic. For the viscous case, see [15, 28, 29, 39].

1.2.2. The compressible setting. The mathematical analysis of moving hypersurfaces in the multidimensional compressible Euler equations began with the existence and stability of the shock-front solution initiated in [24] and was extensively studied by [16, 17, 18, 25]. (See the references in these articles for a thorough review of the literature in this area). More delicate than the noncharacteristic case

¹Using (1.2), liquid water is modeled using the values $\gamma = 7$, $\alpha = 3001$, and $\beta = 3000$.

of the shock-front solution, the characteristic boundary case is encountered in the study of vortex sheets or current vortex sheets. This class of problems has been studied by [4, 6, 7, 33, 35] (and see the references therein); the linearization of the two-dimensional compressible vortex-sheet problem produces derivative loss, similar to that experienced by many authors in the setting of incompressible flow (for both irrotational flows and for flows with vorticity).

The problem of the expansion of a compressible gas with the so-called physical vacuum singularity has been studied in [9, 12, 13, 19, 20, 38] and is degenerate because of the vanishing of the density function on the moving free-boundary.

For the model of a compressible liquid considered in this paper, the Euler equations are uniformly hyperbolic thanks to the equation-of-state (1.2). In the absence of surface tension, an existence theory was given in [23] using Lagrangian coordinates and a Nash–Moser construction, but the estimates had derivative loss. Using the theory of symmetric hyperbolic systems, the paper [34] gave a different proof for the existence of solutions, also with derivative loss. We prove well-posedness for the motion of a compressible liquid with and without surface tension, and with no derivative loss. We also establish the asymptotic limit of zero surface tension.

1.3. Fixing the domain and Lagrangian variables. To transform the system (1.1) into Lagrangian variables, we let $\eta(x, t)$ denote the flow of a fluid particle x at time t . We define $\eta(x, t)$ as the solution of the following differential equation:

$$\partial_t \eta = u \circ \eta \text{ for } t > 0 \text{ and } \eta(x, 0) = x,$$

where \circ denotes composition, so that $[u \circ \eta](x, t) = u(\eta(x, t), t)$. We then introduce the following Lagrangian variables on the reference domain Ω :

$$\begin{aligned} v &= u \circ \eta && (\text{Lagrangian velocity}), \\ f &= \rho \circ \eta && (\text{Lagrangian density}), \\ A &= [D\eta]^{-1} && (\text{inverse of the deformation tensor}), \\ J &= \det D\eta && (\text{Jacobian determinant}), \\ a &= JA && (\text{cofactor matrix of the deformation tensor}). \end{aligned}$$

Using the notation for partial-derivatives and summation defined below in section 2.1.2, the Lagrangian version of (1.1) on the fixed domain Ω is given by

$$(1.5a) \quad f v_t^i + A_i^k f^\gamma_{,k} = 0 \quad \text{in } \Omega \times (0, T],$$

$$(1.5b) \quad f_t + f A_i^j v^i_{,j} = 0 \quad \text{in } \Omega \times (0, T],$$

$$(1.5c) \quad f^\gamma = \beta + \sigma H(\eta) \quad \text{on } \Gamma \times (0, T],$$

$$(1.5d) \quad (\eta, v, f)|_{t=0} = (e, u_0, \rho_0) \quad \text{on } \Omega,$$

where $\Gamma = \partial\Omega$ and $e(x) = x$ denotes the identity map on Ω .

Time-differentiating the Jacobian determinant yields $J_t = a_i^j v^i_{,j}$ (see 2.5 below). Since f satisfies (1.5b), and since $\eta(x, 0) = x$, we see that the Lagrangian density function f satisfies the important identity

$$(1.6) \quad f = \rho_0 J^{-1},$$

so that the initial density function appears only as a parameter in the Lagrangian description of the flow. It is convenient to set $\gamma = 2$ so that $p(\rho) = \rho^2$ or $q(f) = f^2$.

In this case, we write the compressible Euler equations (1.5) as

$$(1.7a) \quad \rho_0 v_t^i + a_i^k (\rho_0^2 J^{-2})_{,k} = 0 \quad \text{in } \Omega \times (0, T],$$

$$(1.7b) \quad \rho_0^2 J^{-2} = \beta + \sigma H(\eta) \quad \text{on } \Gamma \times (0, T],$$

$$(1.7c) \quad (\eta, v)|_{t=0} = (e, u_0) \quad \text{on } \Omega.$$

For $\sigma > 0$, we shall refer to (1.7a–c) as the *surface tension problem*. The function $H(\eta)$ is defined in (2.13).

For reference, since $A_i^k = J^{-1} a_i^k$, we see that the momentum equations (1.7a) are equivalent to

$$(1.8) \quad v_t^i + 2A_i^k (\rho_0 J^{-1})_{,k} = 0 \quad \text{in } \Omega \times (0, T].$$

For general values of $\gamma > 1$, we can replace the density variable with the enthalpy variable and recover the analysis of the case that $\gamma = 2$.

1.4. The higher-order energy functions $E(t)$ and $\mathcal{E}(t)$. While the physical energy $\int_{\Omega} [\rho_0 \frac{1}{2} |v|^2 + \rho_0^2 J^{-1} + \beta J] + \sigma \mathcal{A}(t)$, $\mathcal{A}(t)$ denoting the surface area of $\Gamma(t)$, is a conserved quantity, it is too weak to control the regularity of the evolving free boundary.

We define the higher-order energy functions $E(t)$ and $\mathcal{E}(t)$ to, respectively, correspond with the surface tension problem and the zero surface tension limit. Although neither $E(t)$ or $\mathcal{E}(t)$ is conserved, we will establish that each of $\sup_{t \in [0, T]} E(t)$ and $\sup_{t \in [0, T]} \mathcal{E}(t)$ is bounded on a sufficiently small time-interval of existence $[0, T]$.

1.4.1. The higher-order energy function for $\sigma > 0$. We define the higher-order energy function (or norm) $E(t)$ as

$$(1.9) \quad E(t) = 1 + \sum_{a=0}^5 \|\partial_t^a \eta(t)\|_{5-a}^2 + |v_{ttt} \cdot n(t)|_1^2 + \sum_{a=0}^2 |\bar{\partial}^2 \partial_t^a v \cdot n(t)|_{2.5-a}^2.$$

We let $M_0 \geq 0$ denote a generic constant given by a polynomial function P of $E(0)$:

$$(1.10) \quad M_0 = P(E(0)).$$

1.4.2. The higher-order energy function for $\sigma = 0$. We define the higher-order energy function $\mathcal{E}(t)$ as

$$(1.11) \quad \mathcal{E}(t) = 1 + \sum_{a=0}^7 \|\partial_t^a \eta(t)\|_{4.5-\frac{1}{2}a}^2 + \sum_{a=0}^5 \|\partial_t^a J(t)\|_{4.5-\frac{1}{2}a}^2 + \|\partial_t^6 J(t)\|_1^2.$$

We let $\mathcal{M}_0 \geq 0$ denote a generic constant given by a polynomial function P of $\mathcal{E}(0)$:

$$(1.12) \quad \mathcal{M}_0 = P(\mathcal{E}(0)).$$

Section 2.1 explains the notation for the norms used in (1.9) and (1.11).

1.5. Compatibility conditions for initial data (ρ_0, u_0, Ω) . We assume that ρ_0 and u_0 are given and sufficiently smooth. We require that the initial density ρ_0 satisfy

$$(1.13) \quad \rho_0 \geq 2\lambda > 0 \quad \text{in } \bar{\Omega}.$$

Using the identities $\eta(0) = e$ and $J_t = a_r^s v^r$, we let J_1 be defined by

$$J_1 = \partial_t(J^{-2})(0) = -2 \operatorname{div} u_0.$$

We let v_1 and v_2 be the vectors, respectively, given by

$$v_1^i = -2\partial_i \rho_0 \quad \text{and} \quad v_2^i = -\rho_0^{-1}[\partial_i(\rho_0^2 J_1) + \partial_t a_i^k(0)(\rho_0^2)_{,k}],$$

where $\partial_t a(0)$ is a smooth function of Du_0 . We define, as a function of ρ_0 and u_0 ,

$$J_a = \partial_t^a(J^{-2})|_{t=0} \quad \text{for } a = 0, 1, 2, 3.$$

1.5.1. The case of positive surface tension. For $\sigma > 0$, we assume that Ω is an H^5 -class domain and that ρ_0 and u_0 are in $H^4(\Omega)$. We define, as a function of ρ_0 and u_0 ,

$$H_a = \partial_t^a H(\eta)|_{t=0} \quad \text{for } a = 0, 1, 2, 3.$$

We require that the initial data satisfy the following compatibility conditions:

$$(1.14) \quad \rho_0^2 J_a = \partial_t^a \beta + \sigma H_a \quad \text{on } \Gamma \text{ for } a = 0, 1, 2, 3.$$

For β as in (1.2) and λ as in (1.13), when $a = 0$ in (1.14), we see that H must satisfy

$$\sigma H_0 \geq 4\lambda^2 - \beta.$$

1.5.2. The case of zero surface tension. For $\sigma = 0$, we assume that Ω is an $H^{4.5}$ -class domain and that ρ_0 is in $H^{4.5}(\Omega)$ and u_0 is in $H^4(\Omega)$.

We require that ρ_0 and Ω satisfy the Taylor sign condition:

$$(1.15) \quad 0 < 2\nu \leq -\frac{\partial \rho_0^2}{\partial N} \quad \text{on } \Gamma.$$

We require that the initial data satisfy the following compatibility conditions:

$$(1.16) \quad \rho_0^2 J_a = \partial_t^a \beta \quad \text{on } \Gamma \text{ for } a = 0, \dots, 6.$$

Remark 1. The twice-mean-curvature function $H(\eta)$ is not present in the zero surface tension limit of (1.7). Accordingly, there is no restriction on the curvature of the initial surface Γ .

1.6. Main results. The main results of this paper are the existence and uniqueness of solutions to the compressible Euler equations with and without surface tension.

THEOREM 1.1 (existence and uniqueness for $\sigma > 0$). *Suppose that the initial data (ρ_0, u_0, Ω) verify*

1. $M_0 = P(E(0)) < \infty$,
2. the lower-bound condition (1.13), and
3. the compatibility conditions (1.14).

Then for some $T > 0$, there exists a solution to (1.7) on the time-interval $[0, T]$ such that $\rho(t) \geq \lambda$ in $\bar{\Omega}(t)$, $\sigma H(t) > -\beta$ on $\Gamma(t)$, and

$$\sup_{t \in [0, T]} E(t) \leq 2M_0.$$

Furthermore, the solution is unique if the initial data is such that

$$\sum_{a=0}^6 \|\partial_t^a \eta(0)\|_{6-a}^2 + |v_{tttt} \cdot n(0)|_1^2 + \sum_{a=0}^3 |\bar{\partial}^2 \partial_t^a v \cdot n(0)|_{3.5-a}^2 < \infty.$$

After establishing Theorem 1.1, we prove σ -independent estimates for solutions to the surface tension problem under an additional Rayleigh–Taylor stability condition on the initial data. By passing to the limit as $\sigma \rightarrow 0$, we establish the following.

THEOREM 1.2 (existence and uniqueness for $\sigma = 0$). *Suppose that the initial data (ρ_0, u_0, Ω) verify*

1. $\mathcal{M}_0 = P(\mathcal{E}(0)) < \infty$,
2. the lower-bound condition (1.13),
3. the Taylor sign condition (1.15), and
4. the compatibility conditions (1.16).

Then for some $T > 0$, there exists a solution to (1.7) with $\sigma = 0$ on the time-interval $[0, T]$ such that $\rho(t) \geq \lambda$ in $\bar{\Omega}(t)$, $-\frac{\partial \rho^2(t)}{\partial n(t)} \geq \nu$ on $\Gamma(t)$, and

$$\sup_{t \in [0, T]} \mathcal{E}(t) \leq 2\mathcal{M}_0.$$

Furthermore, the solution is unique if the initial data is such that

$$\sum_{a=0}^9 \|\partial_t^a \eta(0)\|_{5.5-\frac{1}{2}a}^2 + \sum_{a=0}^7 \|\partial_t^a J(0)\|_{5.5-\frac{1}{2}a}^2 + \|\partial_t^8 J(0)\|_1^2 < \infty.$$

Remark 2. The proofs of Theorems 1.1 and 1.2 do not rely on our choice of $\gamma = 2$ and as such are valid for general $\gamma > 1$, since by introducing a new enthalpy-type variable to replace the density function, (1.5), we can reduce the case of general $\gamma > 1$ to the case that $\gamma = 2$.

1.7. Structure of the proofs of Theorems 1.1 and 1.2.

1.7.1. Existence for the surface tension problem (1.7). An existence theory is obtained via the vanishing viscosity method, but with a very special choice of artificial viscosity which does not alter the vorticity equation, given in Lagrangian coordinates as

$$\text{curl}_\eta v_t = 0,$$

where the Lagrangian curl operator curl_η is defined below in section 2.1.3. By defining a parabolic approximation of the Euler equations (1.7a) which preserves the vorticity equation, we ensure that the vorticity estimates for the approximate problem are independent of the parabolic approximation parameter κ . The parabolic approximate κ -problem defined in section 3.

The structure of the Euler equation (1.7a) shows that an a priori estimate for v_t provides an estimate for the gradient of J , which then yields an estimate for the gradient of $\text{div } \eta$. As such, yet another constraint on the choice of parabolic regularization operator is that its presence must still permit estimates for $D \text{div } \eta$ whenever estimates for v_t are known. We choose our κ -parabolic operator so that the parabolically regularized momentum equations have the form

$$F + \kappa F_t = G$$

for F representing the Lagrangian gradient of J . Such a structure allows us to obtain κ -independent estimates for F , given estimates for G . We must further add parabolic regularization to the surface tension boundary conditions (1.7b), which also have the same structure, in order to infer the maximal regularity of the free-boundary.

Our strategy is to obtain energy estimates for the highest number of time-derivatives in the problem and then use the structure of the momentum equations to infer estimates for the divergence of the time-differentiated velocity vectors and flow map. In conjunction with curl-estimates and boundary regularity, we boot-strap the full regularity of the velocity, its time-derivatives, and the flow map η .

In section 4, we perform energy estimates in the fourth (highest) time-differentiated approximate κ -problem. From these energy estimates, we are able to close estimates for $v_{tttt}(t)$ in $L^2(\Omega)$ and $v_{ttt}(t)$ in $H^1(\Omega)$. We then infer the optimal regularity of v_{tt} and v_t . This, in turn, enables us to get the optimal regularity for v and η . All estimates are found to be independent of the artificial viscosity parameter κ .

In section 5, we construct smooth solutions to the approximate κ -problem. We utilize (the Lagrangian variables version of) the basic vector identity $-\Delta = \text{curl curl} - D \text{div}$ in order to replace the gradient of the divergence of v with the Laplacian of v , modulo lower-order terms. Additionally, we use the fact that a sufficiently regular vector field ξ satisfies

$$N^j A_r^j A_r^k \xi_{,k} = \sqrt{g} J^{-1} (\text{curl}_\eta \xi) \times n + \sqrt{g} J^{-1} (\text{div}_\eta \xi) n + b(\xi, \eta),$$

where $b(\xi, \eta)$ is a first-order differential boundary operator with respect to ξ and a second-order differential boundary operator with respect to η . In our approximation scheme, the regularity of the flow map η is limited by the regularity of the velocity v ; hence, we employ a horizontal convolution-by-layers operator (first introduced in [11]) in order to view $b(v, \eta)$ as a lower-order term. With the horizontal convolution-by-layers approximation in place, solutions can be found via the existence theory for uniformly parabolic second-order equations. We must then find estimates for such solutions which are indeed independent of the convolution parameter and pass to the limit.

In section 6, we use the κ -independent a priori estimates established in section 4 and the construction of solutions to our approximate κ -problem in section 5 to establish the existence and uniqueness of solutions to the surface tension problem (1.7).

1.7.2. The zero surface tension limit. In section 7, we establish our existence theory for the zero surface tension limit of (1.7) via σ -independent a priori estimates. For initial data satisfying the Taylor sign condition (1.15), we show that independently of $\sigma > 0$,

$$0 < \nu \int_\Gamma |\bar{\partial}^4 \eta \cdot n|^2 < \infty$$

on a sufficiently small time-interval $[0, T]$. This estimate allows us to obtain σ -independent a priori estimates and hence pass to the limit as $\sigma \rightarrow 0$.

2. Preliminaries.

2.1. Notation.

2.1.1. The three-dimensional gradient vector. Throughout this paper the symbol D will be used to denote the three-dimensional gradient vector

$$D = \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3} \right).$$

2.1.2. Notation for partial differentiation and Einstein’s summation convention. The k th partial derivative of F will be denoted by $F_{,k} = \frac{\partial F}{\partial x_k}$. Repeated Latin indices i, j, k , etc., are summed from 1 to 3, and repeated Greek indices α, β, γ , etc., are summed from 1 to 2. For example, $F_{,ii} = \sum_{i=1}^3 \frac{\partial^2 F}{\partial x_i \partial x_i}$, and $F^i_{, \alpha} I^{\alpha\beta} G^i_{, \beta} = \sum_{i=1}^3 \sum_{\alpha=1}^2 \sum_{\beta=1}^2 \frac{\partial F^i}{\partial x_\alpha} I^{\alpha\beta} \frac{\partial G^i}{\partial x_\beta}$.

2.1.3. The divergence and curl operators. We use the notation $\operatorname{div} u$ to denote the divergence of a vector field u on Ω ,

$$\operatorname{div} u = u^1_{,1} + u^2_{,2} + u^3_{,3},$$

and we use the notation $\operatorname{curl} u$ to denote the curl of a vector u on Ω ,

$$\operatorname{curl} u = (u^3_{,2} - u^2_{,3}, u^1_{,3} - u^3_{,1}, u^2_{,1} - u^1_{,2}).$$

We recall that the permutation symbol ε_{ijk} is defined by

$$\varepsilon_{ijk} = \begin{cases} 1, & \text{even permutation of } \{1, 2, 3\}, \\ -1, & \text{odd permutation of } \{1, 2, 3\}, \\ 0 & \text{otherwise.} \end{cases}$$

This allows for the curl of a vector field u to be expressed as $\operatorname{curl} u = \varepsilon_{.jk} u^k_{,j}$. Letting $v = u(\eta)$ for a given flow map η , we use the notation $\operatorname{div}_\eta v$ to denote the Lagrangian divergence of v on Ω ,

$$\operatorname{div}_\eta v = A^s_r v^r_{,s},$$

and we use the notation $\operatorname{curl}_\eta v$ to denote the Lagrangian curl of v on Ω ,

$$\operatorname{curl}_\eta v = \varepsilon_{.jk} A^s_j v^k_{,s}.$$

2.1.4. The scalar- and cross-product of vectors in \mathbb{R}^3 . Let u and v be vectors in \mathbb{R}^3 . The scalar-product of u and v , denoted $u \cdot v$, is defined as

$$(2.1) \quad u \cdot v = u^1 v^1 + u^2 v^2 + u^3 v^3.$$

The cross-product of u and v , denoted $u \times v$, is defined as

$$(2.2) \quad u \times v = \varepsilon_{.jk} u^j v^k.$$

2.1.5. Local coordinates near Γ . We let $\Omega \subset \mathbb{R}^3$ denote an open, bounded subset of class H^s for $s \geq 4$, and we let $\{U_l\}_{l=1}^K$ denote an open covering of $\Gamma = \partial\Omega$, such that for each $l \in \{1, 2, \dots, K\}$ with

$$\begin{aligned} \mathcal{B}_l &= B(0, r_l), \text{ denoting the open ball of radius } r_l \text{ centered at the origin,} \\ \mathcal{B}_l^+ &= \mathcal{B}_l \cap \{x_3 > 0\}, \\ \mathcal{D}_l &= \mathcal{B}_l \cap \{x_3 = 0\}, \end{aligned}$$

there exists an H^s -class chart θ_l satisfying

$$\begin{aligned} \theta_l: \mathcal{B}_l &\rightarrow U_l \text{ is an } H^s \text{ diffeomorphism,} \\ \theta_l(\mathcal{B}_l^+) &= U_l \cap \Omega, \\ \theta_l(\mathcal{D}_l) &= U_l \cap \Gamma. \end{aligned}$$

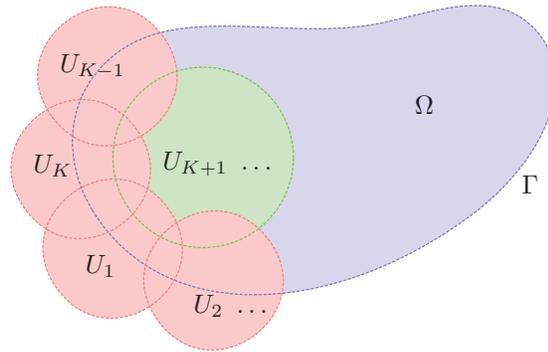


FIG. 2.1. Indexing convention for the open cover $\{U_l\}_{l=1}^L$ of Ω .

For $L > K$, we let $\{U_l\}_{l=K+1}^L$ denote a family of open balls of radius r_l properly contained in Ω such that $\{U_l\}_{l=1}^L$ is an open cover of Ω (see Fig. 2.1). We let

$\{\xi_l\}_{l=1}^L$ denote a C^∞ partition-of-unity subordinate to the open covering of Ω .

2.1.6. Tangential derivatives. On each $U_l \cap \Omega$, for $1 \leq l \leq L$, we let $\bar{\partial}_l$ denote the *local tangential-derivative*. That is, for a differentiable function f on Ω , the α th component of the local tangential-derivative of f is defined in $U_l \cap \Omega$ by

$$\bar{\partial}_{l,\alpha} f = \left(\frac{\partial}{\partial x_\alpha} [f \circ \theta_l] \right) \circ \theta_l^{-1} = \left((Df \circ \theta_l) \frac{\partial \theta_l}{\partial x_\alpha} \right) \circ \theta_l^{-1},$$

where for $K+1 \leq l \leq L$, we have set θ_l to be the identity map e .

We let $\bar{\partial}$ denote the *tangential-derivative* whose α th component is given by

$$\bar{\partial}_\alpha = \sum_{l=1}^L \xi_l \bar{\partial}_{l,\alpha}.$$

We use $\bar{\partial}_\alpha f$ or $f_{,\alpha}$ to denote the components of the tangential-derivative of f .

2.1.7. Geometry of the moving surface $\Gamma(t)$. The vectors $\eta_{,\alpha}$ for $\alpha = 1, 2$, span the tangent space to the moving surface $\Gamma(t) = \eta(\Gamma, t)$ in \mathbb{R}^3 . The *surface metric* g on $\Gamma(t)$ has components

$$g_{\alpha\beta} = \eta_{,\alpha} \cdot \eta_{,\beta}.$$

We let g_0 denote the surface metric of the initial surface Γ . The components of the inverse metric $[g]^{-1}$ are denoted by $g^{\alpha\beta}$. We use \sqrt{g} to denote $\sqrt{\det g}$; we note that $\sqrt{g} = |\eta_{,1} \times \eta_{,2}|$, so that $n = [\eta_{,1} \times \eta_{,2}] / \sqrt{g}$. Equivalently,

$$(2.3) \quad \sqrt{g} n = \eta_{,1} \times \eta_{,2}.$$

The Laplace–Beltrami operator Δ_g is defined on Γ as

$$\Delta_g = \sqrt{g}^{-1} \bar{\partial}_\alpha [\sqrt{g} g^{\alpha\beta} \bar{\partial}_\beta].$$

2.1.8. Sobolev spaces on Ω . For integers $k \geq 0$ and a smooth, open domain Ω of \mathbb{R}^3 , we define the Sobolev space $H^k(\Omega)$ ($H^k(\Omega; \mathbb{R}^3)$) to be the closure of $C^\infty(\overline{\Omega})$ ($C^\infty(\overline{\Omega}; \mathbb{R}^3)$) in the norm

$$\|u\|_k^2 = \sum_{|a| \leq k} \int_{\Omega} |D^a u(x)|^2$$

for a multiindex $a \in \mathbb{Z}_+^3$ with the convention that $|a| = a_1 + a_2 + a_3$. For real numbers $s \geq 0$, the Sobolev spaces $H^s(\Omega)$ and the norms $\|\cdot\|_s$ are defined by interpolation. We will write $H^s(\Omega)$ instead of $H^s(\Omega; \mathbb{R}^3)$ for vector-valued functions. We use $H^s(\Omega)'$ to denote the dual space of $H^s(\Omega)$.

2.1.9. Sobolev spaces on Γ . For functions $u \in H^k(\Gamma)$, $k \geq 0$, we set

$$|u|_k^2 = \sum_{|a| \leq k} \int_{\Gamma} |\bar{\partial}^a u(x)|^2$$

for a multiindex $a \in \mathbb{Z}_+^2$. For real $s \geq 0$, the Hilbert space $H^s(\Gamma)$ and the boundary norm $|\cdot|_s$ is defined by interpolation. The negative-order Sobolev spaces $H^{-s}(\Gamma)$ are defined via duality. That is, for real $s \geq 0$,

$$H^{-s}(\Gamma) = H^s(\Gamma)'$$

Remark 3. Throughout this paper, we suppress the Euclidean measure dx by letting \int_{Ω} represent $\int_{\Omega} dx$. Similarly, the notation \int_{Γ} represents $\int_{\Gamma} dS_0$, where $dS_0 = \sqrt{g_0} dx_1 dx_2$ is the surface measure of the initial surface Γ . Equally, the time integral \int_0^t should be read as $\int_0^t ds$.

Remark 4. We let $|\cdot|_{s, \mathcal{D}_l}$ denote the $H^s(\mathcal{D}_l)$ -norm.

2.2. Differentiation and geometric identities and properties.

2.2.1. An identity for the Jacobian determinant J . With $\dim \Omega = 3$, we have that the Jacobian determinant J is written as

$$(2.4) \quad J = \frac{1}{\dim \Omega} [a_r^s \eta^r_{,s}] ,$$

which follows from the definition of the cofactor matrix a in section 1.3.

2.2.2. Time-differentiating the Jacobian determinant J and the cofactor matrix a . We record the following basic differentiation formulas:

$$(2.5) \quad \partial_t J = a_r^s v^r_{,s} ,$$

$$(2.6) \quad \partial_t a_i^k = J^{-1} [a_r^s a_i^k - a_i^s a_r^k] v^r_{,s} .$$

Using (2.5) and (2.6) and the fact that $a = JA$, we have that

$$(2.7) \quad \partial_t A_i^k = -A_i^s A_r^k v^r_{,s} .$$

We note that the time-differentiation formulas (2.5)–(2.7) at once become formulas for tangential-differentiation by replacing $v^r_{,s}$ with $\bar{\partial} \eta^r_{,s}$ in the right-hand sides of (2.5)–(2.7).

The formulas (2.5) and (2.6) imply the following scaling relations:

$$\begin{aligned} \partial_t J &\sim a Dv, & \partial_t a &\sim a^2 Dv, \\ \partial_t^2 J &\sim a^2 (Dv)^2 + a Dv_t, & \partial_t^2 a &\sim a^3 (Dv)^2 + a^2 Dv_t. \end{aligned}$$

These scaling relations are particularly useful when estimating error-terms.

2.2.3. The Piola identity. Columns of every cofactor matrix are divergence-free. Thus,

$$(2.8) \quad a_i^k{}_{,k} = 0.$$

2.2.4. Relating the normal vectors of Γ and $\Gamma(t)$. With N as the outward unit normal to the reference surface Γ , the outward-normal direction of the moving surface $\Gamma(t)$ is

$$a_i^k N^k = |\eta_{,1} \times \eta_{,2}| n^i.$$

The identity $\sqrt{g} = \sqrt{\det g} = |\eta_{,1} \times \eta_{,2}|$ implies that

$$(2.9) \quad a_i^k N^k = \sqrt{g} n^i.$$

2.2.5. Derivatives of the inverse metric $g^{\alpha\beta}$, Jacobian \sqrt{g} , and unit normal n . A tangential-derivative of the inverse metric $g^{\alpha\beta}$, Jacobian determinant \sqrt{g} , and moving outward normal unit vector n are given by the formulas

$$(2.10a) \quad \bar{\partial} g^{\alpha\beta} = -g^{\alpha\mu} \bar{\partial} g_{\mu\nu} g^{\nu\beta},$$

$$(2.10b) \quad \bar{\partial} \sqrt{g} = \frac{1}{2} \sqrt{g} g^{\mu\nu} \bar{\partial} g_{\mu\nu},$$

$$(2.10c) \quad \bar{\partial} n = -g^{\gamma\delta} [\bar{\partial} \eta_{,\delta} \cdot n] \eta_{,\gamma}.$$

Also,

$$(2.11a) \quad \partial_t g^{\alpha\beta} = -g^{\alpha\mu} \partial_t g_{\mu\nu} g^{\nu\beta},$$

$$(2.11b) \quad \partial_t \sqrt{g} = \frac{1}{2} \sqrt{g} g^{\mu\nu} \partial_t g_{\mu\nu},$$

$$(2.11c) \quad \partial_t n = -g^{\gamma\delta} [v_{,\delta} \cdot n] \eta_{,\gamma}.$$

Remark 5. The right-hand side of (2.10c) and (2.11c) is a vector that is tangent to the embedded surface.

2.2.6. Relating the Laplace–Beltrami operator Δ_g to the unit normal n . With the formulas (2.10a) and (2.10b), we have that the Laplace–Beltrami operator $\Delta_g = \sqrt{g}^{-1} \bar{\partial}_\alpha [\sqrt{g} g^{\alpha\beta} \bar{\partial}_\beta]$ applied to the particle flow η decomposes into normal and tangential components as

$$\sqrt{g} \Delta_g(\eta) = \sqrt{g} \underbrace{(g^{\alpha\beta} \eta_{,\beta\alpha} - \eta_{,\mu\alpha} \cdot \eta_{,\nu} g^{\alpha\mu} g^{\nu\beta} \eta_{,\beta})}_{g^{\alpha\beta} [\eta_{,\alpha\beta} \cdot n] n} + \sqrt{g} \eta_{,\mu\alpha} \cdot \eta_{,\nu} (g^{\alpha\beta} g^{\mu\nu} - g^{\alpha\mu} g^{\nu\beta}) \eta_{,\beta}.$$

We therefore have the identity

$$(2.12) \quad \sqrt{g} \Delta_g(\eta) = \sqrt{g} g^{\alpha\beta} [\eta_{,\alpha\beta} \cdot n] n.$$

For reference, we recall the identity $\Delta_g(\eta) = -H(\eta)n$ or, equivalently,

$$(2.13) \quad H(\eta) = -\Delta_g(\eta) \cdot n.$$

2.3. Two identities for the Euler equations in Lagrangian variables.

2.3.1. The Lagrangian vorticity equation. With the operator curl_η defined in section 2.1.3,

$$(2.14) \quad \text{curl}_\eta v_t = 0 \quad \text{in } \Omega.$$

The identity (2.14) is obtained by taking the Lagrangian curl of the Euler equations (1.8).

2.3.2. A tangential identity for v_t on the boundary Γ . Setting $\sigma = 1$,

$$(2.15) \quad v_t \cdot \eta_{,\alpha} = f^{-1} \bar{\partial}_\alpha (g^{\mu\nu} [\eta_{,\mu\nu} \cdot n]) \quad \text{on } \Gamma.$$

The identity (2.15) is established using the Euler equations (1.8), the Laplace–Young boundary condition (1.7b), and the formula (2.12).

2.4. General inequalities.

2.4.1. Trace estimates. For $s > \frac{1}{2}$ and some constant C independent of $w \in H^s(\Omega)$, the trace theorem [1] states that the trace of w is defined in $H^{s-\frac{1}{2}}(\Gamma)$ with the estimate

$$|w|_{s-\frac{1}{2}} \leq C \|w\|_s.$$

LEMMA 2.1. *Let $w \in L^2(0, T; H^1(\Omega)) \cap L^\infty(0, T; L^2(\Omega))$. Then for $\delta > 0$,*

$$(2.16) \quad \int_0^T |w|_{0.25}^2 \leq \delta \int_0^T \|w\|_1^2 + C_\delta T \sup_{t \in [0, T]} \|w(t)\|_0^2,$$

where the constant C_δ depends on $1/\delta$.

Proof. Using interpolation and Young’s inequality with $\delta > 0$, we have that

$$\int_0^T \|w\|_{0.75}^2 \leq C \int_0^T \|w\|_0^{\frac{1}{2}} \|w\|_1^{\frac{3}{2}} \leq \delta \int_0^T \|w\|_1^2 + C_\delta T \sup_{t \in [0, T]} \|w(t)\|_0^2.$$

The proof is complete thanks to the trace theorem. □

LEMMA 2.2 (normal trace theorem). *Let w be a vector field defined on Ω such that $\bar{\partial}w \in L^2(\Omega)$ and $\text{div } w \in L^2(\Omega)$, and let N denote the outward unit normal vector to Γ . Then the normal trace $\bar{\partial}w \cdot N$ exists in $H^{-0.5}(\Gamma)$ with the estimate*

$$(2.17) \quad |\bar{\partial}w \cdot N|_{-0.5}^2 \leq C \left[\|\bar{\partial}w\|_{L^2(\Omega)}^2 + \|\text{div } w\|_{L^2(\Omega)}^2 \right]$$

for some constant C independent of w .

See [32] for the proof of Lemma 2.2 in the case that $\bar{\partial}w$ is replaced by w . For $\phi \in H^1(\Omega)$, $\int_\Gamma \bar{\partial}w \cdot n \phi dS = \int_\Omega \bar{\partial}w \cdot D\phi dx - \int_\Omega \text{div } w \bar{\partial}\phi dx$ because we can integrate by parts with $\bar{\partial}$ without any boundary contributions. Thus, the identical proof given in [32] proves Lemma 2.2. Similarly, we have the following.

LEMMA 2.3 (tangential trace theorem). *Let w be a vector field defined on Ω such that $\bar{\partial}w \in L^2(\Omega)$ and $\text{curl } w \in L^2(\Omega)$, and let τ_1, τ_2 denote the unit tangent vectors to Γ , so that any vector field u on Γ can be uniquely written as $u^\alpha \tau_\alpha$. Then the tangential trace $\bar{\partial}w \cdot \tau_\alpha$ exists in $H^{-0.5}(\Gamma)$ with the estimate*

$$(2.18) \quad |\bar{\partial}w \cdot \tau_\alpha|_{-0.5}^2 \leq C \left[\|\bar{\partial}w\|_{L^2(\Omega)}^2 + \|\text{curl } w\|_{L^2(\Omega)}^2 \right]$$

for some constant C independent of w .

See [5] for the proof of Lemma 2.3.

2.4.2. An elliptic estimate which is independent of κ . The following lemma is used to establish our κ -independent a priori estimates. The proof is given in section 6 of [10].

LEMMA 2.4. Let $\kappa > 0$, $s \geq 0$, and $g \in L^\infty(0, T; H^s(\Omega))$ be given. Suppose that $f \in H^1(0, T; H^s(\Omega))$ satisfies

$$(2.19) \quad f + \kappa f_t = g \quad \text{in } \Omega \times (0, T).$$

Then,

$$\|f\|_{L^\infty(0, T; H^s(\Omega))} \leq C \max\{\|f(0)\|_s, \|g\|_{L^\infty(0, T; H^s(\Omega))}\}.$$

2.4.3. A technical lemma. The following technical lemma is established in [11].

LEMMA 2.5. There exists a constant C such that

$$\|\bar{\partial}w\|_{H^{0.5}(\Omega)'} \leq C\|w\|_{H^{0.5}(\Omega)} \quad \forall w \in H^{0.5}(\Omega).$$

2.4.4. The Hodge decomposition elliptic estimates. The following Hodge-type elliptic estimate is well known and follows from the identity $-\Delta w = \text{curl curl } w - D \text{ div } w$, together with estimates of divergence-form elliptic operators with Sobolev-class coefficients.

PROPOSITION 2.1. For an H^r -class domain Ω , $r \geq 3$, if $w \in L^2(\Omega; \mathbb{R}^3)$ with $\text{curl } w \in H^{s-1}(\Omega; \mathbb{R}^3)$, $\text{div } w \in H^{s-1}(\Omega)$, and $w \cdot N|_\Gamma \in H^{s-\frac{1}{2}}(\Gamma)$ for $1 \leq s \leq r$, then there exists a constant $C > 0$ depending only on Ω such that

$$\|w\|_s \leq C \left[\|w\|_0 + \|\text{curl } w\|_{s-1} + \|\text{div } w\|_{s-1} + |\bar{\partial}w \cdot N|_{s-\frac{3}{2}} \right],$$

where N denotes the outward unit-normal to Γ .

When the domain is smooth and the boundary norm $|\bar{\partial}w \cdot N|_{s-\frac{3}{2}}$ is replaced with $|w \cdot N|_{s-\frac{1}{2}}$, this is the classical Hodge decomposition, and we refer the reader to section 5.9 of [31] for a review. On the other hand, using the boundary norm $|\bar{\partial}w \cdot N|_{s-\frac{3}{2}}$ is already sufficient to establish the full regularity of the vector field w , whenever the divergence and curl of F are estimated. The stated Sobolev-class regularity of the domain follows immediately from using the identity $-\Delta w = \text{curl curl } w - D \text{ div } w$ and the analysis of second-order elliptic operators with Sobolev-class coefficients.

2.4.5. A polynomial-type inequality. For a constant $M \geq 0$, suppose $f: t \mapsto f(t) \geq 0$ continuously and satisfies

$$(2.20) \quad f(t) \leq M + tP(f(t)), \quad t \geq 0,$$

where P denotes a generic polynomial function. Then for t taken sufficiently small,

$$f(t) \leq 2M.$$

3. A parabolic κ -approximation of the surface tension problem (1.7).

In this section, we define a parabolic approximation of the surface tension problem (1.7), which we term the κ -problem. The κ -problem is defined by adding artificial viscosity terms to the Euler equations and the Laplace–Young boundary condition. The salient feature of the κ -problem is its compatibility with our energy-estimates methodology based on Proposition 2.1 in that (1) the transport structure of the Euler equations is maintained and (2) the momentum equations of the κ -problem are equivalently expressed in the form $f + \kappa f_t = g$ with f equalling the gradient of J . These structural properties of the κ -problem, respectively, yield the κ -independent curl- and divergence-estimates.

3.1. Smoothing the initial data. For the purpose of constructing of solutions to the surface tension problem (1.7), it is convenient to smooth our initial data (ρ_0, u_0, Ω) (which has limited Sobolev regularity). This smoothing process must produce C^∞ -class data and satisfy the conditions (1.13) and (1.14), so standard smoothing via convolution alone does not suffice. We explain our smoothing process in Appendix A; this permits us to use C^∞ -class data for our construction. Later, in section 6.2, we recover the optimal Sobolev-class regularity of the initial data stated in Theorem 1.1.

3.2. The parabolic approximation of the surface tension problem (1.7).

We recall that an equivalent expression of the surface tension problem (1.7) is

$$\begin{aligned} v_t^i + 2A_i^k(\rho_0 J^{-1})_{,k} &= 0 && \text{in } \Omega \times (0, T], \\ \rho_0^2 J^{-2} &= \beta - \sigma g^{\alpha\beta} \eta_{,\alpha\beta} \cdot n && \text{on } \Gamma \times (0, T]. \end{aligned}$$

We have used the identity $H(\eta) = -g^{\alpha\beta} \eta_{,\alpha\beta} \cdot n$ in writing the boundary condition.

The variables in the following problem a priori depend on the parabolic parameter κ . To indicate this dependence, we place the symbol \sim above each of the variables.

DEFINITION 3.1 (the κ -problem). For $\kappa > 0$, we define \tilde{v} as the solution of

$$\begin{aligned} (3.1a) \quad \tilde{v}_t^i + 2\tilde{A}_i^k(\rho_0 \tilde{J}^{-1})_{,k} - \kappa \tilde{A}_i^k(\rho_0 \tilde{J}_t)_{,k} &= 0 && \text{in } \Omega \times (0, T_\kappa], \\ (3.1b) \quad \rho_0^2 \tilde{J}^{-2} - \kappa \rho_0^2 \tilde{J}^{-1} \tilde{J}_t &= \tilde{H}_\kappa && \text{on } \Gamma \times (0, T_\kappa], \\ (3.1c) \quad (\tilde{\eta}, \tilde{v})|_{t=0} &= (e, u_0) && \text{on } \Omega. \end{aligned}$$

The function \tilde{H}_κ appearing in the right-hand side of (3.1b) is defined as

$$(3.2) \quad \tilde{H}_\kappa = \beta(t) - \sigma \tilde{g}^{\alpha\beta} \tilde{\eta}_{,\alpha\beta} \cdot \tilde{n} - \kappa \tilde{g}^{\alpha\beta} \tilde{v}_{,\alpha\beta} \cdot \tilde{n},$$

where the function $\beta(t)$ appearing in the right-hand side of (3.2) is defined as

$$(3.3) \quad \beta(t) = \beta + \kappa \sum_{a=0}^3 \frac{t^a}{a!} \partial_t^a \left[-\rho_0^2 \tilde{J}^{-1} \tilde{J}_t + \tilde{g}^{\alpha\beta} \tilde{v}_{,\alpha\beta} \cdot \tilde{n} \right]_{t=0}.$$

Remark 6. The particular artificial viscosity term $-\kappa \tilde{A}_i^k(\rho_0 \tilde{J}_t)_{,k}$ appearing in (3.1a) preserves the transport structure of the Euler equations; in particular, taking the Lagrangian curl of momentum equations produces the same vorticity evolution as for the original Euler system. In comparison, an artificial viscosity of the form $-\kappa \tilde{A}_r^j[\tilde{A}_r^k \tilde{v}^i]_{,j}$ would not preserve the transport structure of the Euler equations.

Remark 7. The sum over a in (3.3) is used to ensure that the initial data satisfy the compatibility conditions (1.14).

3.3. The constant-in-time vectors v_a for $a = 1, 2, 3, 4$. The vector field $\tilde{v}_t|_{t=0}$ is computed using the momentum equations (3.1a), as follows:

$$\tilde{v}_t|_{t=0} = \left(\kappa \tilde{A}^k(\rho_0 \tilde{J}_t)_{,k} - 2\tilde{A}^k(\rho_0 \tilde{J}^{-1})_{,k} \right) |_{t=0} = D(\kappa \rho_0 \operatorname{div} u_0 - 2\rho_0).$$

Similarly, for all $a \in \mathbb{N}$,

$$\partial_t^a \tilde{v}|_{t=0} = \frac{\partial^{a-1}}{\partial t^{a-1}} \left(\kappa \tilde{A}^k(\rho_0 \tilde{J}_t)_{,k} - 2\tilde{A}^k(\rho_0 \tilde{J}^{-1})_{,k} \right) |_{t=0} \quad \text{on } \Omega.$$

This formula makes it clear that each $\partial_t^a \tilde{v}|_{t=0}$ is a function of space-derivatives of the initial data u_0 and ρ_0 . We define the constant-in-time vectors v_a as

$$(3.4) \quad v_a = \frac{\partial^{a-1}}{\partial t^{a-1}} \left(\kappa \tilde{A}^k(\rho_0 \tilde{J}_t)_{,k} - 2 \tilde{A}^k(\rho_0 \tilde{J}^{-1})_{,k} \right) \Big|_{t=0} \quad \text{for } a = 1, 2, 3, 4.$$

Since $2 \tilde{A}^k(\rho_0 \tilde{J}^{-1})_{,k} = \rho_0^{-1} \tilde{a}_i^k(\rho_0^2 \tilde{J}^{-2})_{,k}$, we have that $v_a \rightarrow v_a$, $a = 1, 2$, as $\kappa \rightarrow 0$, where v_a are defined in section 1.5. We use (3.1b) to compute the following identities: for $a = 0, 1, 2, 3$,

$$(3.5) \quad \partial_t^a [\rho_0^2 \tilde{J}^{-2} - \kappa \rho_0^2 \tilde{J}^{-3} \tilde{J}_t] \Big|_{t=0} = \partial_t^a [\beta(t) - \sigma \tilde{g}^{\alpha\beta} \tilde{\eta}_{,\alpha\beta} \cdot \tilde{n} - \kappa \tilde{g}^{\alpha\beta} \tilde{v}_{,\alpha\beta} \cdot \tilde{n}] \Big|_{t=0}.$$

4. A priori estimates for the κ -problem (3.1). We establish our κ -independent a priori estimates in this section; the precise estimate is stated below in Lemma 4.1. The existence of solutions to the κ -problem (3.1) is established in section 5.

For $\kappa > 0$, we define the following higher-order energy function:

$$(4.1) \quad \begin{aligned} E^\kappa(t) = & 1 + \sum_{a=0}^5 \|\partial_t^a \tilde{\eta}(t)\|_{5-a}^2 + |\tilde{v}_{ttt} \cdot \tilde{n}(t)|_1^2 + \sum_{a=0}^2 |\bar{\partial}^2 \partial_t^a \tilde{v} \cdot \tilde{n}(t)|_{2.5-a}^2 \\ & + \int_0^T |\sqrt{\kappa} \bar{\partial} \tilde{v}_{tttt} \cdot \tilde{n}|_0^2 + \int_0^T \|\sqrt{\kappa} \tilde{v}_{tttt}\|_1^2 \\ & + \sum_{a=0}^3 \|\kappa \partial_t^a \tilde{v}(t)\|_{5-a}^2 + \sum_{a=0}^2 |\kappa \bar{\partial}^2 \partial_t^a \tilde{v}_t \cdot \tilde{n}(t)|_{2.5-a}^2. \end{aligned}$$

Remark 8. The inequality stated in Theorem 5.1 ensures that $E^\kappa(t)$ is continuous in time.

We make the following definition to allow for constants to depend on $1/\delta$.

DEFINITION 4.1 (notational convention for constants depending on $1/\delta > 0$). We let \mathcal{P} denote a generic polynomial with constant and coefficients depending on $1/\delta > 0$.

We define the constant $\mathcal{N}_0 > 0$ by

$$(4.2) \quad \mathcal{N}_0 = \mathcal{P}(\|u_0\|_{100}, \|\rho_0\|_{100}).$$

We let \mathcal{R} denote generic lower-order terms satisfying

$$\int_0^T \mathcal{R} \leq \mathcal{N}_0 + \delta \sup_{t \in [0, T]} E^\kappa(t) + T \mathcal{P} \left(\sup_{t \in [0, T]} E^\kappa(t) \right).$$

The artificially high $H^{100}(\Omega)$ -norm in defining \mathcal{N}_0 is acceptable as the initial data (ρ_0, u_0, Ω) is of C^∞ -class. We shall assume that

$$(4.3) \quad \frac{1}{2} \leq \tilde{J} \leq \frac{3}{2} \quad \forall t \in [0, T] \text{ here and } x \in \Omega.$$

The bounds (4.3) are possible by taking $T > 0$ sufficiently small, since thanks to Theorem 5.1,

$$(4.4) \quad \|\tilde{J} - 1\|_{L^\infty(\Omega)} \leq C \left\| \int_0^t \partial_t \tilde{J} \right\|_2 \leq C\sqrt{T}.$$

THEOREM 4.1 (a priori κ -independent estimates for the approximate κ -problem).
 We let \tilde{v} solve the κ -problem (3.1) on a time-interval $[0, T]$ for some $T = T_\kappa > 0$.
 Then independent of $\kappa > 0$,

$$(4.5) \quad \sup_{t \in [0, T]} E^\kappa(t) \leq \int_0^T \mathcal{R}.$$

The proof of Theorem 4.1 is divided into six steps.

Step 1: The κ -independent curl-estimates. Following the proofs of Lemma 10.1 in [11] and Proposition 3 in [13] and in establishing

LEMMA 4.1 (the κ -independent curl-estimates).

$$\begin{aligned} \sup_{t \in [0, T]} \sum_{a=0}^4 \|\operatorname{curl} \partial_t^a \tilde{\eta}(t)\|_{4-a}^2 + \int_0^T \|\sqrt{\kappa} \operatorname{curl} \tilde{v}_{tttt}\|_0^2 \\ + \sup_{t \in [0, T]} \sum_{a=0}^3 \|\kappa \operatorname{curl} \partial_t^a \tilde{v}(t)\|_{4-a}^2 \leq \int_0^T \mathcal{R}. \end{aligned}$$

Proof. The Lagrangian curl of (3.1a) yields $\operatorname{curl}_{\tilde{\eta}} \tilde{v}_t = 0$. Setting

$$B(\tilde{A}, D\tilde{v}) = \varepsilon_{.ji} \tilde{A}_{tj}^s \tilde{v}^i_{,s},$$

we find that $(\operatorname{curl}_{\tilde{\eta}} \tilde{v})_t = B(\tilde{A}, D\tilde{v})$. By the fundamental theorem of calculus,

$$(4.6) \quad \operatorname{curl}_{\tilde{\eta}} \tilde{v}(t) = \operatorname{curl} u_0 + \int_0^t B(\tilde{A}, D\tilde{v}).$$

Applying the gradient operator D to (4.6) and a second application of the fundamental theorem of calculus, we find that

$$(4.7) \quad \begin{aligned} D \operatorname{curl} \tilde{\eta}(t) &= tD \operatorname{curl} u_0 - \varepsilon_{.ji} D \tilde{\eta}^i_{,s} \int_0^t \tilde{A}_{tj}^s \\ &+ \varepsilon_{.ji} \int_0^t [\tilde{A}_{tj}^s D \tilde{\eta}^i_{,s} - D \tilde{A}_{tj}^s \tilde{v}^i_{,s}] + \int_0^t \int_0^{t'} DB(\tilde{A}, D\tilde{v}), \end{aligned}$$

where we have used the identity $\operatorname{curl}_{\tilde{\eta}} D \tilde{\eta} = \operatorname{curl} D \tilde{\eta} + \varepsilon_{.ji} D \tilde{\eta}^i_{,s} \int_0^t \tilde{A}_{tj}^s$.

The differentiation formula (2.7) equally holds for when the gradient D replaces ∂_t ; hence, the first three terms on the right-hand side of (4.7) are, with respect to the action of D^3 , each bounded by $\int_0^T \mathcal{R}$.

We next analyze the highest-order term created in $\int_0^t \int_0^{t'} D^4 B(\tilde{A}, D\tilde{v})$. With (2.7),

$$DB(\tilde{A}, D\tilde{v}) = -\varepsilon_{.ji} [\tilde{A}_j^q D \tilde{v}^r_{,q} \tilde{A}_r^s \tilde{v}^i_{,s} + \tilde{A}_j^q \tilde{v}^r_{,q} \tilde{A}_r^s D \tilde{v}^i_{,s} + \tilde{v}^r_{,q} \tilde{v}^i_{,s} D(\tilde{A}_j^q \tilde{A}_r^s)],$$

from which it follows that the highest-order term of $D^4 B(\tilde{A}, D\tilde{v})$ is

$$-\varepsilon_{.ji} \int_0^t \int_0^{t'} [\tilde{A}_j^q D^4 \tilde{v}^r_{,q} \tilde{A}_r^s \tilde{v}^i_{,s} + \tilde{A}_j^q \tilde{v}^r_{,q} \tilde{A}_r^s D^4 \tilde{v}^i_{,s}].$$

With a relaxation of the precise structure of the summands in the integrands of the

highest-order terms of $D^4 B(\tilde{A}, D\tilde{v})$, we highlight the derivative count that results from integration by parts in time by writing

$$\int_0^t \int_0^{t'} D^4 B(\tilde{A}, D\tilde{v}) = - \int_0^t \int_0^{t'} D^5 \tilde{\eta} (D\tilde{v} \tilde{A} \tilde{A})_t + \int_0^t D^5 \tilde{\eta} D\tilde{v} \tilde{A} \tilde{A}.$$

With such a temporal-integration-by-parts computation, the action of D^3 in (4.7) yields

$$(4.8) \quad \sup_{t \in [0, T]} \|\text{curl} \tilde{\eta}(t)\|_4^2 \leq \int_0^T \mathcal{R},$$

and by the same arguments, the action of D^3 and κD^4 in (4.6) yield

$$(4.9) \quad \sup_{t \in [0, T]} \|\text{curl} \tilde{v}(t)\|_3^2 + \sup_{t \in [0, T]} \|\kappa \text{curl} \tilde{v}(t)\|_4^2 \leq \int_0^T \mathcal{R}.$$

By returning to the Lagrangian vorticity equation $\text{curl}_{\tilde{\eta}} \tilde{v}_t = 0$, we find that

$$(4.10) \quad \text{curl} \tilde{v}_t = \varepsilon_{j \cdot i} \tilde{v}_{t,s}^i \int_0^t \tilde{A}_{tj}^s,$$

and by considering the action of D^2 and κD^3 in the identity (4.10), we infer that

$$(4.11) \quad \sup_{t \in [0, T]} \|\text{curl} \tilde{v}_t(t)\|_2^2 + \sup_{t \in [0, T]} \|\kappa \text{curl} \tilde{v}_t(t)\|_3^2 \leq \int_0^T \mathcal{R}.$$

By considering the action of $D\partial_t$, ∂_t^2 , $\kappa D^2 \partial_t$, $\kappa D \partial_t^2$, and $\sqrt{\kappa} \partial_t^3$ in (4.10), and using the fundamental-theorem-of-calculus identity $\partial_t^a \tilde{v}_t = \partial_t^a \tilde{v}_t|_{t=0} + \int_0^t \partial_t^{a+1} \tilde{v}_t$ in the lower-order terms, we establish that

$$(4.12) \quad \sup_{t \in [0, T]} \sum_{a=0}^1 \|\text{curl} \partial_t^a \tilde{v}_{tt}(t)\|_{1-a}^2 + \sum_{a=0}^1 \|\kappa \text{curl} \partial_t^a \tilde{v}_{tt}(t)\|_{2-a}^2 + \int_0^T \|\kappa \text{curl} \tilde{v}_{tttt}(t)\|_0^2 \leq \int_0^T \mathcal{R}.$$

The sum of the inequalities (4.8)–(4.12) completes the proof. \square

Step 2: The κ -independent estimates for \tilde{v}_{tttt} and \tilde{v}_{ttt} . We equivalently write the momentum equations (3.1a) and boundary condition (3.1b) of the κ -problem as

$$(4.13a) \quad \rho_0 \tilde{v}_t^i + \tilde{a}_i^k (\rho_0^2 \tilde{J}^{-2})_{,k} - \kappa \rho_0 \tilde{J}^{-1} \tilde{a}_i^k (\rho_0 \tilde{J}_t)_{,k} = 0 \quad \text{in } \Omega \times (0, T],$$

$$(4.13b) \quad \rho_0^2 \tilde{J}^{-2} - \kappa \rho_0^2 \tilde{J}^{-1} \tilde{J}_t = \tilde{H}_\kappa \quad \text{on } \Gamma \times (0, T],$$

where \tilde{H}_κ is given by (3.2). From (3.1a), we have the identity

$$\tilde{v}_t \cdot \tilde{\eta}_\gamma = \bar{\partial}_\gamma [-2\rho_0 \tilde{J}^{-1} + \kappa \rho_0 \tilde{J}_t].$$

Multiplying this identity by $\rho_0 \tilde{J}^{-1}$ yields

$$\rho_0 \tilde{J}^{-1} \tilde{v}_t \cdot \tilde{\eta}_\gamma = \bar{\partial}_\gamma [-\rho_0^2 \tilde{J}^{-2} + \kappa \rho_0^2 \tilde{J}^{-1} \tilde{J}_t] - \kappa (\rho_0 \tilde{J}^{-1})_{,\gamma} \rho_0 \tilde{J}_t.$$

We use the boundary condition (4.13b) to obtain the following tangential identity for v_t :

$$(4.14) \quad \rho_0 \tilde{J}^{-1} \tilde{v}_t \cdot \tilde{\eta}_{,\gamma} = \bar{\partial}_\gamma [\sigma \tilde{g}^{\mu\nu} \tilde{\eta}_{,\mu\nu} \cdot \tilde{n} + \kappa \tilde{g}^{\mu\nu} \tilde{v}_{,\mu\nu} \cdot \tilde{n} - \beta(t)] - \kappa (\rho_0 \tilde{J}^{-1})_{,\gamma} \rho_0 \tilde{J}_t.$$

PROPOSITION 4.1 (energy estimates for the fourth time-differentiated problem).

$$\begin{aligned} \sup_{t \in [0, T]} \|\tilde{v}_{tttt}(t)\|_0^2 + \sup_{t \in [0, T]} \|\partial_t^4 \tilde{J}(t)\|_0^2 + \sup_{t \in [0, T]} |\tilde{v}_{ttt} \cdot \tilde{n}(t)|_1^2 \\ + \int_0^T |\sqrt{\kappa} \partial_t^4 \tilde{v} \cdot \tilde{n}(t)|_1^2 + \int_0^T \|\sqrt{\kappa} \partial_t^5 \tilde{J}\|_0^2 \leq \int_0^T \mathcal{R}. \end{aligned}$$

Proof. Testing four time-derivatives of (4.13a) against $\partial_t^4 \tilde{v}$ in the $L^2(\Omega)$ -inner product, and integrating by parts with respect to ∂_k in the interior integrals $\int_\Omega \tilde{a}_i^k \partial_t^4 (\rho_0^2 \tilde{J}^{-2})_{,k} \partial_t^4 \tilde{v}^i$ and $-\kappa \int_\Omega \tilde{a}_i^k \partial_t^4 [\rho_0 \tilde{J}^{-1} (\rho_0 \tilde{J}_t)_{,k}] \partial_t^4 \tilde{v}^i$, we find that

$$\begin{aligned} \int_\Omega \partial_t^4 [\rho_0 \tilde{v}_t^i] \partial_t^4 \tilde{v}^i - \underbrace{\int_\Omega \partial_t^4 [\rho_0^2 \tilde{J}^{-2}] \tilde{a}_i^k \partial_t^4 \tilde{v}^i}_{\mathcal{I}_1} + \kappa \underbrace{\int_\Omega \partial_t^4 [\rho_0^2 \tilde{J}^{-1} \tilde{J}_t] \tilde{a}_i^k \partial_t^4 \tilde{v}^i}_{\mathcal{I}_2} \\ + \underbrace{\int_\Gamma \partial_t^4 [\rho_0^2 \tilde{J}^{-2} - \kappa \rho_0^2 \tilde{J}^{-1} \tilde{J}_t] \tilde{a}_i^k \partial_t^4 \tilde{v}^i N^k}_{\mathcal{I}} = \mathcal{R}. \end{aligned}$$

It is convenient to rewrite this equation as

$$(4.15) \quad \frac{1}{2} \frac{d}{dt} \int_\Omega \rho_0 |\partial_t^4 \tilde{v}|^2 + \frac{d}{dt} \int_\Omega \rho_0^2 \tilde{J}^{-3} |\partial_t^4 \tilde{J}|^2 + \kappa \int_\Omega \rho_0^2 \tilde{J}^{-1} |\partial_t^5 \tilde{J}|^2 + \mathcal{I} = \mathcal{R},$$

where the identity $\tilde{J}_t = \tilde{a}_r^s \tilde{v}^r_{,s}$ implies that the error created in order to write $\tilde{a}_i^k \partial_t^4 \tilde{v}^i_{,k}$ as $\partial_t^5 \tilde{J}$ in \mathcal{I}_1 and \mathcal{I}_2 is of lower-order and so is absorbed in \mathcal{R} .

Rewriting the boundary integral in (4.15). Using the outward normal identity (2.9) and the boundary condition (4.13b) with $\sigma = 1$, we find that

$$\begin{aligned}
 (4.16) \quad i &= \frac{1}{2} \frac{d}{dt} \int_{\Gamma} \sqrt{\tilde{g}} \tilde{g}^{\alpha\beta} \tilde{v}_{ttt,\alpha} \cdot \tilde{n} \tilde{v}_{ttt,\beta} \cdot \tilde{n} + \int_{\Gamma} \sqrt{\tilde{g}} \tilde{g}^{\alpha\beta} \sqrt{\kappa} \tilde{v}_{ttt,\alpha} \cdot \tilde{n} \sqrt{\kappa} \tilde{v}_{ttt,\beta} \cdot \tilde{n} \\
 &\quad - \underbrace{\int_{\Gamma} \sqrt{\tilde{g}} \tilde{g}^{\alpha\beta} \tilde{v}_{ttt,\alpha} \cdot \partial_t \tilde{n} \tilde{v}_{ttt,\beta} \cdot \tilde{n}}_{\mathfrak{r}_1} - \frac{1}{2} \underbrace{\int_{\Gamma} \partial_t [\sqrt{\tilde{g}} \tilde{g}^{\alpha\beta}] \tilde{v}_{ttt,\alpha} \cdot \tilde{n} \tilde{v}_{ttt,\beta} \cdot \tilde{n}}_{\mathcal{R}} \\
 &\quad + \underbrace{\int_{\Gamma} \tilde{v}_{ttt,\beta} \cdot \tilde{n} \sqrt{\tilde{g}} \tilde{g}^{\alpha\beta} \tilde{v}_{tttt} \cdot \tilde{n}_{,\alpha} + \int_{\Gamma} \tilde{v}_{ttt,\beta} \cdot \tilde{n}_{,\alpha} \sqrt{\tilde{g}} \tilde{g}^{\alpha\beta} \tilde{v}_{tttt} \cdot \tilde{n}}_{\mathfrak{r}_2} \\
 &\quad + \underbrace{\int_{\Gamma} \tilde{v}_{ttt,\beta} \cdot \tilde{n} (\sqrt{\tilde{g}} \tilde{g}^{\alpha\beta})_{,\alpha} \tilde{v}_{tttt} \cdot \tilde{n} - \sum_{l=0}^3 c_l \int_{\Gamma} \sqrt{\tilde{g}} \tilde{g}^{\alpha\beta} \partial_t^l \tilde{\eta}^j_{,\alpha\beta} \partial_t^{4-l} \tilde{n}^j \tilde{v}_{tttt} \cdot \tilde{n}}_{\mathfrak{r}_3} \\
 &\quad - \underbrace{\sum_{l=1}^4 c_l \int_{\Gamma} \partial_t^l \tilde{g}^{\alpha\beta} \partial_t^{4-l} [\tilde{\eta}_{,\alpha\beta} \cdot \tilde{n}] \sqrt{\tilde{g}} \tilde{v}_{tttt} \cdot \tilde{n}}_{\mathfrak{r}_4} \\
 &\quad + \underbrace{\kappa \int_{\Gamma} \tilde{v}_{ttt,\beta}^i (\tilde{n}^i \tilde{g}^{\alpha\beta} \tilde{n}^j)_{,\alpha} \sqrt{\tilde{g}} \tilde{v}_{tttt}^j - \kappa \sum_{l=1}^4 c_l \int_{\Gamma} \partial_t^l (\tilde{g}^{\alpha\beta} \tilde{n}^i) \partial_t^{4-l} \tilde{\eta}_{,\alpha\beta} \sqrt{\tilde{g}} \tilde{v}_{tttt} \cdot \tilde{n}}_{\mathfrak{r}_5}.
 \end{aligned}$$

In our analysis of $\int_0^T \mathfrak{r}_i$, $i = 1, 2, 3$, we adopt the convention of letting

$\tilde{\ell}$ denote a function of $L^\infty(\Gamma)$ -class and \tilde{l} a function of $H^{0.5}(\Gamma)$ -class.

We recycle the symbols j , j_A , j_B , etc., in terming boundary integrals that require explanation.

Analysis of $\int_0^T \mathfrak{r}_1$ in the time-integral of (4.16). The action of $\bar{\partial}_\alpha \partial_t^2$ in the tangential identity (4.14) provides that

$$(4.17) \quad \tilde{v}_{ttt,\alpha} \cdot \tilde{\eta}_{,\gamma} = \rho_0^{-1} \tilde{J} [\bar{\partial}_{\alpha\gamma} (\tilde{g}^{\mu\nu} \tilde{\eta}_{,\mu\nu} \cdot \tilde{n} + \kappa \tilde{g}^{\mu\nu} \tilde{v}_{,\mu\nu} \cdot \tilde{n} - \beta(t))_{tt} - \tilde{l}_{\gamma\alpha}],$$

where the lower-order $\tilde{l}_{\gamma\alpha} \in H^{0.5}(\Gamma)$ is given by

$$\tilde{l}_{\gamma\alpha} = \kappa \bar{\partial}_\alpha [(\rho_0 \tilde{J}^{-1})_{,\gamma} \rho_0 \tilde{J}_t]_{tt} + \tilde{v}_{ttt} \cdot (\tilde{\eta}_{,\gamma} \rho_0 \tilde{J}^{-1})_{,\alpha} + \sum_{a=1}^2 c_a [\partial_t^a \tilde{v} \cdot \partial_t^{3-a} (\tilde{\eta}_{,\gamma} \rho_0 \tilde{J}^{-1})]_{,\alpha}.$$

Setting $\tilde{\ell}_\gamma^{\alpha\beta} = \rho_0^{-1} \tilde{J} \sqrt{\tilde{g}} \tilde{g}^{\alpha\beta} \tilde{g}^{\gamma\delta} \tilde{v}_{,\delta} \cdot \tilde{n}$, we use the tangential identity (4.17), together with the outward normal differentiation formula (2.11c), to find that

$$(4.18) \quad \mathfrak{r}_1 = \underbrace{\int_{\Gamma} \tilde{\ell}_\gamma^{\alpha\beta} \bar{\partial}_{\gamma\alpha} [\tilde{g}^{\mu\nu} \tilde{\eta}_{,\mu\nu} \cdot \tilde{n}]_{tt} \tilde{v}_{ttt,\beta} \cdot \tilde{n}}_{\mathfrak{r}_1'} + \underbrace{\kappa \int_{\Gamma} \tilde{\ell}_\gamma^{\alpha\beta} \bar{\partial}_{\gamma\alpha} [\tilde{g}^{\mu\nu} \tilde{v}_{,\mu\nu} \cdot \tilde{n}]_{tt} \tilde{v}_{ttt,\beta} \cdot \tilde{n}}_j + \mathcal{R}.$$

Concerning τ_1' , we integrate by parts with respect to a time-derivative to write

$$\begin{aligned} & \int_0^T \tau_1' \\ &= - \underbrace{\int_{\Gamma} \bar{\partial}_{\gamma} [\tilde{g}^{\mu\nu} \tilde{\eta}_{,\mu\nu} \cdot \tilde{n}]_{tt} \bar{\partial}_{\alpha} [\tilde{\ell}_{\gamma}^{\alpha\beta} \tilde{v}_{tt,\beta} \cdot \tilde{n}]}_{j_A} \Big|_0^T + \int_0^T \underbrace{\int_{\Gamma} \bar{\partial}_{\gamma} [\tilde{g}^{\mu\nu} \tilde{\eta}_{,\mu\nu} \cdot \tilde{n}]_{tt} \bar{\partial}_{\alpha} [\tilde{\ell}_{\gamma}^{\alpha\beta} \tilde{v}_{tt,\beta} \cdot \tilde{n}_t]}_{j_B} \\ &+ \int_0^T \underbrace{\int_{\Gamma} \bar{\partial}_{\gamma} [\tilde{g}^{\mu\nu} \tilde{\eta}_{,\mu\nu} \cdot \tilde{n}]_{tt} \bar{\partial}_{\alpha} [\partial_t \tilde{\ell}_{\gamma}^{\alpha\beta} \tilde{v}_{tt,\beta} \cdot \tilde{n}]}_{j_C} + \int_0^T \underbrace{\int_{\Gamma} \bar{\partial}_{\gamma} [\tilde{g}^{\mu\nu} \tilde{\eta}_{,\mu\nu} \cdot \tilde{n}]_{ttt} \bar{\partial}_{\alpha} [\tilde{\ell}_{\gamma}^{\alpha\beta} \tilde{v}_{tt,\beta} \cdot \tilde{n}]}_{j_D}, \end{aligned}$$

where we have further integrated by parts with respect to $\bar{\partial}_{\alpha}$ on the right-hand side. Interpolation and two applications of Young's inequality provide that

$$|\tilde{v}_{tt,\beta} \cdot \tilde{n}(T)|_1^2 \leq C |\tilde{v}_{tt} \cdot \tilde{n}(T)|_{1.5} |\bar{\partial} \tilde{v}_{tt} \cdot \tilde{n}(T)|_{1.5} \leq C |\tilde{v}_{tt} \cdot \tilde{n}(T)|_{0.5}^2 + 2\delta |\bar{\partial} \tilde{v}_{tt} \cdot \tilde{n}(T)|_{1.5}^2.$$

Hence,

$$j_A(t) \Big|_0^T \leq \int_0^T \mathcal{R}.$$

We record that via Lemma 2.3,

$$|\tilde{v}_{tt,\alpha\beta} \cdot \tilde{\eta}_{,\sigma} |_{H^{-0.5}(\Gamma)}^2 \leq C [\|\bar{\partial}^2 \tilde{v}_{tt}\|_0^2 + \|\text{curl} \bar{\partial} \tilde{v}_{tt}\|_0^2] \leq C \|\tilde{v}_{tt}\|_2^2.$$

Since $\tilde{n}_t = -\tilde{\eta}_{,\sigma} \tilde{g}^{\sigma\rho} \tilde{v}_{,\rho} \cdot \tilde{n}$ and $\bar{\partial}^2 \tilde{v}_t \cdot \tilde{n}$ is in $H^{1.5}(\Gamma)$, we use an $H^{-0.5}(\Gamma)$ -duality pairing in the highest-order term to write

$$\int_0^T j_B \leq \int_0^T \mathcal{R} + C \int_0^T |\bar{\partial}^2 \tilde{v}_t \cdot \tilde{n}(t)|_{1.5} |\tilde{v}_{tt,\alpha\beta} \cdot \tilde{\eta}_{,\sigma} |_{H^{-0.5}(\Gamma)} \leq \int_0^T \mathcal{R}.$$

We use the Cauchy–Schwarz inequality for the estimate

$$\int_0^T j_C \leq \int_0^T \mathcal{R}.$$

Regarding $\int_0^T j_D$, for the integral $\int_0^T \int_{\Gamma} \bar{\partial}_{\gamma} [\tilde{g}^{\mu\nu} \tilde{\eta}_{,\mu\nu} \cdot \tilde{n}]_{ttt} \bar{\partial}_{\alpha} \tilde{\ell}_{\gamma}^{\alpha\beta} \tilde{v}_{tt,\beta} \cdot \tilde{n}$, we integrate by parts with respect to $\bar{\partial}_{\gamma}$ and estimate using the Cauchy–Schwarz inequality, and for the integral $\int_0^T \int_{\Gamma} \bar{\partial}_{\gamma} [\tilde{g}^{\mu\nu} \tilde{\eta}_{,\mu\nu} \cdot \tilde{n}]_{ttt} \tilde{\ell}_{\gamma}^{\alpha\beta} \tilde{v}_{tt,\beta} \cdot \tilde{n}_{,\alpha}$, we also integrate by parts with respect to $\bar{\partial}_{\gamma}$ and then estimate using an $H^{-0.5}(\Gamma)$ -duality pairing. These methods work equally well in all terms of the spacetime-integral $\int_0^T \int_{\Gamma} \bar{\partial}_{\gamma} [\tilde{g}^{\mu\nu} \tilde{\eta}_{,\mu\nu} \cdot \tilde{n}]_{ttt} \tilde{\ell}_{\gamma}^{\alpha\beta} \tilde{v}_{tt,\alpha\beta} \cdot \tilde{n}$, as well as in $\int_0^T j$, where j is defined in (4.18). Thus, we have established that

$$(4.19) \quad \int_0^T \tau_1 \leq \int_0^T \mathcal{R}.$$

Analysis of $\int_0^T \tau_2$ in the time-integral of (4.16). We have that

$$(4.20) \quad \tau_2 = \underbrace{\int_{\Gamma} \tilde{v}_{ttt,\beta} \cdot \tilde{n} \sqrt{\tilde{g}} \tilde{g}^{\alpha\beta} \tilde{v}_{tttt} \cdot \tilde{n}_{,\alpha}}_{\tau_{2a}} + \underbrace{\int_{\Gamma} \tilde{v}_{ttt,\beta} \cdot \tilde{n}_{,\alpha} \sqrt{\tilde{g}} \tilde{g}^{\alpha\beta} \tilde{v}_{tttt} \cdot \tilde{n}}_{\tau_{2b}}.$$

We write the action of ∂_t^3 in the tangential identity (4.14) as

$$(4.21a) \quad \tilde{v}_{tttt} \cdot \tilde{\eta}_{,\gamma} = \rho_0^{-1} \tilde{J} \left[\bar{\partial}_\gamma (\tilde{g}^{\mu\nu} \tilde{\eta}_{,\mu\nu} \cdot \tilde{n} + \kappa \tilde{g}^{\mu\nu} \tilde{v}_{,\mu\nu} \cdot \tilde{n} - \beta(t))_{ttt} - \tilde{l}_\gamma \right],$$

where the lower-order $\tilde{l}_\gamma \in H^{0.5}(\Gamma)$ is given by

$$(4.21b) \quad \tilde{l}_\gamma = \kappa [(\rho_0 \tilde{J}^{-1})_{,\gamma} \rho_0 \tilde{J}_t]_{ttt} + \sum_{l=1}^3 c_a \partial_t^l \tilde{v} \cdot \partial_t^{4-l} (\tilde{\eta}_{,\gamma} \rho_0 \tilde{J}^{-1}).$$

Letting $\tilde{\ell}_\gamma^\beta = -\rho_0^{-1} \tilde{J} \sqrt{\tilde{g}} \tilde{g}^{\alpha\beta} \tilde{g}^{\gamma\delta} \tilde{\eta}_{,\delta\alpha} \cdot \tilde{n}$, we use (4.21) to write

$$\mathbf{r}_{2a} = - \underbrace{\int_\Gamma \tilde{\ell}_\gamma^\beta [\tilde{g}^{\mu\nu} \tilde{\eta}_{,\mu\nu} \cdot \tilde{n}]_{ttt} \tilde{v}_{ttt,\beta\gamma} \cdot \tilde{n}}_{\mathbf{r}_{2a}' } + \kappa \underbrace{\int_\Gamma \tilde{\ell}_\gamma^\beta \bar{\partial}_\gamma [\tilde{g}^{\mu\nu} \tilde{v}_{ttt,\mu\nu} \cdot \tilde{n}] \tilde{v}_{ttt,\beta} \cdot \tilde{n}}_{\mathbf{r}_{2a}''} + \mathcal{R},$$

where we have used integration by parts with respect to $\bar{\partial}_\gamma$ to determine \mathbf{r}_{2a}' . We integrate by parts with respect to time in order to write

$$\begin{aligned} - \int_0^T \mathbf{r}_{2a}' &= - \int_\Gamma \tilde{\ell}_\gamma^\beta \tilde{v}_{tt,\beta\gamma} \cdot \tilde{n} [\tilde{g}^{\mu\nu} \tilde{\eta}_{,\mu\nu} \cdot \tilde{n}]_{ttt} \Big|_0^T + \int_0^T \int_\Gamma \tilde{\ell}_\gamma^\beta \tilde{v}_{tt,\beta\gamma} \cdot \tilde{n}_t [\tilde{g}^{\mu\nu} \tilde{\eta}_{,\mu\nu} \cdot \tilde{n}]_{ttt} \\ &\quad + \int_0^T \int_\Gamma \partial_t \tilde{\ell}_\gamma^\beta \tilde{v}_{tt,\beta\gamma} \cdot \tilde{n} [\tilde{g}^{\mu\nu} \tilde{\eta}_{,\mu\nu} \cdot \tilde{n}]_{ttt} + \int_0^T \int_\Gamma \tilde{\ell}_\gamma^\beta \tilde{v}_{tt,\beta\gamma} \cdot \tilde{n} [\tilde{g}^{\mu\nu} \tilde{\eta}_{,\mu\nu} \cdot \tilde{n}]_{tttt} \\ &= \int_0^T \underbrace{\int_\Gamma \tilde{\ell}_\gamma^\beta \tilde{v}_{tt,\beta\gamma} \cdot \tilde{n} \tilde{g}^{\mu\nu} \tilde{v}_{ttt,\mu\nu} \cdot \tilde{n}}_j + \int_0^T \mathcal{R}. \end{aligned}$$

Letting $\tilde{\ell} = \rho_0^{-1} \tilde{J} \sqrt{\tilde{g}} \tilde{g}^{\gamma\delta} \tilde{\eta}_{,\gamma\delta} \cdot \tilde{n}$, we utilize the symmetry of $\tilde{\ell}_\gamma^\beta$ to exchange $\bar{\partial}_\alpha$ and $\bar{\partial}_\gamma$ via integration by parts for

$$(4.22) \quad \begin{aligned} \int_0^T j &= \frac{1}{2} \int_\Gamma \tilde{\ell} |\tilde{g}^{\mu\nu} \tilde{v}_{tt,\mu\nu} \cdot \tilde{n}|^2 \Big|_0^T \\ &\quad - \int_0^T \int_\Gamma \tilde{\ell} \tilde{v}_{tt,\alpha\beta} \cdot \tilde{n} \tilde{v}_{tt,\mu\nu} \cdot (n \tilde{g}^{\mu\nu})_t - \frac{1}{2} \int_0^T \int_\Gamma \tilde{\ell}_t |\tilde{g}^{\mu\nu} \tilde{v}_{tt,\mu\nu} \cdot \tilde{n}|^2 = \int_0^T \mathcal{R}. \end{aligned}$$

Next, integrating by parts in time, we find that

$$\begin{aligned} \kappa \int_0^T \mathbf{r}_{2a}'' &= \int_\Gamma \tilde{\ell}_\gamma^\beta \bar{\partial}_\gamma [\tilde{g}^{\mu\nu} \tilde{v}_{tt,\mu\nu} \cdot \tilde{n}] \kappa (\tilde{v}_{ttt,\beta} \cdot \tilde{n}) \Big|_0^T \\ &\quad + \int_0^T \int_\Gamma [\tilde{g}^{\mu\nu} \tilde{v}_{tt,\mu\nu} \cdot \partial_t \tilde{n}] \bar{\partial}_\gamma [\tilde{\ell}_\gamma^\beta \kappa (\tilde{v}_{ttt,\beta} \cdot \tilde{n})] \\ &\quad - \int_0^T \int_\Gamma \tilde{\ell}_\gamma^\beta \bar{\partial}_\gamma [\tilde{g}^{\mu\nu} \kappa (\tilde{v}_{tt,\mu\nu} \cdot \tilde{n})] \tilde{v}_{ttt,\beta} \cdot \partial_t \tilde{n} \\ &\quad - \int_0^T \underbrace{\int_\Gamma \tilde{\ell}_\gamma^\beta \bar{\partial}_\gamma [\tilde{g}^{\mu\nu} \sqrt{\kappa} \tilde{v}_{tt,\mu\nu} \cdot \tilde{n}] \sqrt{\kappa} \tilde{v}_{tttt,\beta} \cdot \tilde{n}}_j = \int_0^T \mathcal{R} + \int_0^T j. \end{aligned}$$

Interpolation and Young’s inequality provide for the estimate

$$\begin{aligned} \int_0^T j &\leq \delta \int_0^T |\sqrt{\kappa} \bar{\partial} \tilde{v}_{tttt} \cdot \tilde{n}|_0^2 + C_\delta \int_0^T |\sqrt{\kappa} \bar{\partial} \tilde{v}_{tt} \cdot \tilde{n}|_2^2 \\ &\leq \delta \int_0^T |\sqrt{\kappa} \bar{\partial} \tilde{v}_{tttt} \cdot \tilde{n}|_0^2 + C_\delta T |\bar{\partial} \tilde{v}_{tt} \cdot \tilde{n}|_{1.5} |\kappa \bar{\partial} \tilde{v}_{tt} \cdot \tilde{n}|_{2.5} \leq \int_0^T \mathcal{R}. \end{aligned}$$

We have thus established that

$$\int_0^T \mathfrak{r}_{2a} = \int_0^T \mathcal{R}.$$

The analysis of $\int_0^T \mathfrak{r}_{2b}$ is similar. We set $\tilde{\ell}^{\gamma\beta} = \sqrt{\tilde{g}} \tilde{g}^{\alpha\beta} \tilde{g}^{\gamma\delta} \tilde{\eta}_{,\delta\alpha} \cdot \tilde{n}$ and write

$$\begin{aligned} \int_0^T \mathfrak{r}_{2b} &= \int_\Gamma \tilde{v}_{ttt,\beta} \cdot \tilde{n}_{,\alpha} \sqrt{\tilde{g}} \tilde{g}^{\alpha\beta} \tilde{v}_{ttt} \cdot \tilde{n} \Big|_0^T - \int_0^T \int_\Gamma \tilde{v}_{ttt,\beta}^j [\tilde{n}^j_{,\alpha} \sqrt{\tilde{g}} \tilde{g}^{\alpha\beta} \tilde{n}^i]_t \tilde{v}_{ttt}^i \\ &\quad + \underbrace{\int_0^T \int_\Gamma \tilde{\ell}^{\gamma\beta} \tilde{v}_{ttt,\beta} \cdot \tilde{\eta}_{,\gamma} \tilde{v}_{ttt} \cdot \tilde{n}}_{\mathfrak{r}_{2b'}} = \int_0^T \mathcal{R} + \int_0^T \mathfrak{r}_{2b'}. \end{aligned}$$

Regarding $\int_0^T \mathfrak{r}_{2b'}$, we use the identity

$$(4.23) \quad \tilde{v}_{ttt,\beta} \cdot \tilde{\eta}_{,\gamma} = \rho_0^{-1} \tilde{J} [\bar{\partial}_\beta \bar{\partial}_\gamma (\tilde{g}^{\mu\nu} \tilde{\eta}_{,\mu\nu} \cdot \tilde{n} + \kappa \tilde{g}^{\mu\nu} \tilde{v}_{,\mu\nu} \cdot \tilde{n} - \beta(t))]_{ttt} - \tilde{l}'_{\beta\gamma},$$

where $\tilde{l}'_{\beta\gamma} = \tilde{v}_{ttt} \cdot \tilde{\eta}_{,\gamma\beta} \rho_0 \tilde{J}^{-1} + \tilde{v}_{ttt} \cdot \tilde{\eta}_{,\gamma} (\rho_0 \tilde{J}^{-1})_{,\beta} + \bar{\partial}_\beta \tilde{l}_\gamma$ with \tilde{l}_γ given by (4.21b). We integrate by parts with respect to both $\bar{\partial}_\beta$ and $\bar{\partial}_\gamma$ in $\int_0^T \int_\Gamma \rho_0^{-1} \tilde{J} \bar{\partial}_{\beta\gamma} [\tilde{g}^{\mu\nu} \tilde{\eta}_{,\mu\nu} \cdot \tilde{n}]_{ttt} \tilde{\ell}^{\gamma\beta} \tilde{v}_{ttt} \cdot \tilde{n}$, where the highest-order term produced by $\bar{\partial}_{\beta\gamma}$ -integration by parts is (4.22). To estimate the integral where $\bar{\partial}_\beta \tilde{l}_\gamma$ appears, we have the choice of integration by parts with respect to $\bar{\partial}_\beta$ or an $H^{-0.5}(\Gamma)$ -duality pairing. In the integral where $\tilde{v}_{ttt} \cdot \tilde{\eta}_{,\gamma} (\rho_0 \tilde{J}^{-1})_{,\beta}$ appears, we use the identity (4.21a). Hence, we conclude that

$$\int_0^T \mathfrak{r}_{2b'} = \int_0^T \mathcal{R} - \underbrace{\int_0^T \int_\Gamma \tilde{\ell}^{\gamma\beta} \bar{\partial}_\beta \tilde{l}_\gamma \tilde{v}_{ttt} \cdot \tilde{n}}_{\mathcal{R}}.$$

Thus, we have established that

$$(4.24) \quad \int_0^T \mathfrak{r}_2 \leq \int_0^T \mathcal{R}.$$

Analysis of $\int_0^T \mathfrak{r}_3$ in the time-integral of (4.16). We have that

$$\mathfrak{r}_3 = \underbrace{\int_\Gamma \tilde{v}_{ttt,\beta} \cdot \tilde{n} (\sqrt{\tilde{g}} \tilde{g}^{\alpha\beta})_{,\alpha} \tilde{v}_{tttt} \cdot \tilde{n}}_{\mathfrak{r}_{3a}} - \sum_{l=0}^3 c_l \underbrace{\int_\Gamma \sqrt{\tilde{g}} \tilde{g}^{\alpha\beta} \partial_t^l \tilde{\eta}^j_{,\alpha\beta} \partial_t^{4-l} \tilde{n}^j \tilde{v}_{tttt} \cdot \tilde{n}}_{\mathfrak{r}_{3b,l}}.$$

We will first establish that $\int_0^T \mathfrak{r}_{3a}$ cancels with a term arising in $\int_0^T \mathfrak{r}_{3b,0}$. Recalling that $\tilde{n}_t = -\tilde{g}^{\gamma\delta} [\tilde{v}_{,\delta} \cdot \tilde{n}]_{\tilde{\eta}_{,\gamma}}$, the highest-order term of \tilde{n}_{tttt} is $-\tilde{g}^{\gamma\delta} [\tilde{v}_{ttt,\delta} \cdot \tilde{n}]_{\tilde{\eta}_{,\gamma}}$. It follows

that

$$(4.25) \quad - \int_0^T \mathfrak{r}_{3b,0} = \int_0^T \int_{\Gamma} \underbrace{\sqrt{\tilde{g}} \tilde{g}^{\alpha\beta} \tilde{g}^{\gamma\delta} \tilde{\eta}_{,\alpha\beta} \cdot \tilde{\eta}_{,\gamma} \tilde{v}_{ttt,\delta} \cdot \tilde{n} \tilde{v}_{tttt} \cdot \tilde{n}}_j + \int_0^T \mathcal{R},$$

where the estimation of the lower-order terms of $\int_0^T \mathfrak{r}_{3b,0}$ requires integration by parts with respect to a time-derivative of \tilde{v}_{tttt} . Using the differentiation formulas (2.10a) and (2.10b), we write

$$j = - \underbrace{\int_{\Gamma} \tilde{v}_{ttt,\beta} \cdot \tilde{n} (\sqrt{\tilde{g}} \tilde{g}^{\alpha\beta})_{,\alpha} \tilde{v}_{tttt} \cdot \tilde{n}}_{\mathfrak{r}_{3a}}.$$

Thus,

$$\int_0^T \mathfrak{r}_{3a} - \int_0^T \mathfrak{r}_{3b,0} = \int_0^T \mathcal{R}.$$

The terms $\mathfrak{r}_{3b,1}$ and $\mathfrak{r}_{3b,2}$ are analyzed by integrating by parts with respect to a time-derivative of \tilde{v}_{tttt} , followed by elementary estimates. Thus,

$$\int_0^T \mathfrak{r}_{3b,1} + \int_0^T \mathfrak{r}_{3b,2} = \int_0^T \mathcal{R}.$$

We next examine $\mathfrak{r}_{3b,3} = \int_{\Gamma} \sqrt{\tilde{g}} \tilde{g}^{\alpha\beta} \tilde{v}_{tt,\alpha\beta} \cdot \tilde{n}_t \tilde{v}_{tttt} \cdot \tilde{n}$. After integration by parts with respect to a time-derivative of \tilde{v}_{tttt} , we find that

$$\mathfrak{r}_{3b,3} = - \int_{\Gamma} \sqrt{\tilde{g}} \tilde{g}^{\alpha\beta} \tilde{v}_{ttt,\alpha\beta} \cdot \tilde{n}_t \tilde{v}_{ttt} \cdot \tilde{n} + \mathcal{R} = \underbrace{\int_{\Gamma} \sqrt{\tilde{g}} \tilde{g}^{\alpha\beta} \tilde{v}_{ttt,\beta} \cdot \tilde{n}_t \tilde{v}_{ttt,\alpha} \cdot \tilde{n}}_{\mathfrak{r}_{3b,3'}} + \mathcal{R}.$$

Letting $\tilde{\ell}_{\alpha\beta}^{\gamma} = \sqrt{\tilde{g}} \tilde{g}^{\alpha\beta} \tilde{g}^{\gamma\delta} \tilde{v}_{,\delta} \cdot \tilde{n}$, we have that

$$\begin{aligned} \int_0^T \mathfrak{r}_{3b,3'} &= \int_{\Gamma} \tilde{\ell}_{\alpha\beta}^{\gamma} \tilde{v}_{ttt,\beta} \cdot \tilde{\eta}_{,\gamma} \tilde{v}_{tt,\alpha} \cdot \tilde{n} \Big|_0^T - \int_0^T \int_{\Gamma} \tilde{v}_{ttt,\beta} \cdot \tilde{\eta}_{,\gamma} \tilde{v}_{tt,\alpha} \cdot (\tilde{n}^i \tilde{\eta}^j_{,\beta} \tilde{\ell}_{\alpha\beta}^{\gamma})_t \\ &\quad - \underbrace{\int_0^T \int_{\Gamma} \tilde{\ell}_{\alpha\beta}^{\gamma} \tilde{v}_{tttt,\beta} \cdot \tilde{\eta}_{,\gamma} \tilde{v}_{tt,\alpha} \cdot \tilde{n}}_j = - \int_0^T j + \int_0^T \mathcal{R}. \end{aligned}$$

By using the identity (4.23) in $\int_0^T j$, we note that the term corresponding with $\tilde{\ell}_{\gamma}$ has an elementary estimate after integration by parts with respect to $\bar{\partial}_{\beta}$. The remaining terms are similarly analyzed, except for $\int_0^T \int_{\Gamma} \tilde{\ell}_{\alpha\beta}^{\gamma} \rho_0^{-1} \tilde{J} \bar{\partial}_{\beta\gamma} [\tilde{g}^{\mu\nu} \tilde{v}_{tt,\mu\nu} \cdot \tilde{n}] \tilde{v}_{tt,\alpha} \cdot \tilde{n}$, in which we integrate by parts with respect to both $\bar{\partial}_{\beta}$ and $\bar{\partial}_{\gamma}$. For a certain highest-order term created by $\bar{\partial}_{\beta\gamma}$ -integration by parts, we form an exact derivative: setting $\tilde{\ell}_{\gamma} = \rho_0^{-1} \tilde{J} \sqrt{\tilde{g}} \tilde{g}^{\gamma\delta} \tilde{v}_{,\delta} \cdot \tilde{n}$,

$$\int_0^T \int_{\Gamma} \tilde{\ell}_{\gamma} [\tilde{g}^{\alpha\beta} \tilde{v}_{tt,\alpha\beta} \cdot \tilde{n}]_{,\gamma} \tilde{g}^{\mu\nu} \tilde{v}_{tt,\mu\nu} \cdot \tilde{n} = - \frac{1}{2} \int_0^T \int_{\Gamma} \bar{\partial}_{\gamma} \tilde{\ell}_{\gamma} |\tilde{g}^{\mu\nu} \tilde{v}_{tt,\mu\nu} \cdot \tilde{n}|^2.$$

This establishes that $\int_0^T j = \int_0^T \mathcal{R}$. We thus conclude that

$$\int_0^T \mathbf{r}_{3b,3} = \int_0^T \mathcal{R}.$$

Hence,

$$(4.26) \quad \int_0^T \mathbf{r}_3 \leq \int_0^T \mathcal{R}.$$

Analysis of $\int_0^T \mathbf{r}_4$ in the time-integral of (4.16). We integrate by parts with respect to a time-derivative of \tilde{v}_{ttt} in $\int_0^T \mathbf{r}_4$ and, if need be, spatially integrate by parts. For example, letting $\mathbf{r}_4 = \sum_{l=1}^4 \mathbf{r}_{4,l}$, we find that after integration by parts with respect to time,

$$\begin{aligned} \int_0^T \mathbf{r}_{4,1} &= \int_0^T \int_{\Gamma} (\sqrt{\tilde{g}} \tilde{g}^{\alpha\beta})_t (\tilde{\eta}_{,\alpha\beta} \cdot \tilde{n})_{ttt} \tilde{v}_{ttt} \cdot \tilde{n} + \int_0^T \mathcal{R} \\ &= - \int_0^T \int_{\Gamma} \tilde{v}_{ttt,\alpha} \cdot \bar{\partial}_{\beta} [\tilde{n} (\sqrt{\tilde{g}} \tilde{g}^{\alpha\beta})_t \tilde{v}_{ttt} \cdot \tilde{n}] + \int_0^T \mathcal{R} = \int_0^T \mathcal{R}. \end{aligned}$$

Similarly, integration by parts with respect to time provides for the expression

$$\int_0^T \mathbf{r}_{4,2} + \int_0^T \mathbf{r}_{4,3} = \int_0^T \mathcal{R}.$$

Finally, using the differentiation formulas (2.10a) and (2.10b),

$$\int_0^T \mathbf{r}_{4,4} = \int_0^T \int_{\Gamma} \sqrt{\tilde{g}} (2\tilde{g}^{\alpha\mu} \tilde{g}^{\nu\beta} - \tilde{g}^{\alpha\beta} \tilde{g}^{\mu\nu}) \tilde{v}_{ttt,\mu} \cdot \tilde{\eta}_{,\nu} \tilde{\eta}_{,\alpha\beta} \cdot \tilde{n} \tilde{v}_{ttt} \cdot \tilde{n} + \int_0^T \mathcal{R} = \int_0^T \mathcal{R},$$

where the second equality follows from our above analysis of $\int_0^T \mathbf{r}_{2b}$.

Hence,

$$(4.27) \quad \int_0^T \mathbf{r}_4 \leq \int_0^T \mathcal{R}.$$

Analysis of $\int_0^T \mathbf{r}_5$ in the time-integral of (4.16). In the first term defining \mathbf{r}_5 , we integrate by parts with respect to $\bar{\partial}_{\alpha}$ when $\bar{\partial}_{\beta}$ acts on \tilde{n}^j . This term is bounded by $\int_0^T \mathcal{R}$, since thanks to Lemma 2.1,

$$\int_0^T |\sqrt{\kappa} \tilde{v}_{tttt}|_0^2 \leq C \int_0^T |\sqrt{\kappa} \tilde{v}_{tttt}|_{0.25}^2 \leq \delta \int_0^T \|\sqrt{\kappa} \tilde{v}_{tttt}\|_1^2 + C_{\delta} T \sup_{t \in [0, T]} \|\tilde{v}_{tttt}(t)\|_0^2.$$

The remaining terms are similarly estimated. Hence,

$$(4.28) \quad \int_0^T \mathbf{r}_5 \leq \int_0^T \mathcal{R}.$$

Rewriting (4.15). Summing the inequalities (4.19), (4.24), (4.26), (4.27), (4.28) yields

$$(4.29) \quad \sum_{i=1}^5 \int_0^T \mathfrak{r}_i \leq \int_0^T \mathcal{R}.$$

Thanks to the inequality (4.29) and the identity (4.16), we equivalently write (4.15) as

$$(4.30) \quad \frac{1}{2} \frac{d}{dt} \int_{\Omega} \rho_0 |\partial_t^4 \tilde{v}|^2 + \frac{d}{dt} \int_{\Omega} \rho_0^2 \tilde{J}^{-3} |\partial_t^4 \tilde{J}|^2 + \frac{1}{2} \frac{d}{dt} \int_{\Gamma} \sqrt{\tilde{g}} \tilde{g}^{\alpha\beta} \tilde{v}_{ttt,\alpha} \cdot \tilde{n} \tilde{v}_{ttt,\beta} \cdot \tilde{n} \\ + \int_{\Gamma} \sqrt{\tilde{\kappa}} \tilde{v}_{ttt,\beta} \cdot \tilde{n} \sqrt{\tilde{g}} \tilde{g}^{\alpha\beta} \sqrt{\tilde{\kappa}} \tilde{v}_{ttt,\alpha} \cdot \tilde{n} + \int_{\Omega} \rho_0^2 \tilde{J}^{-1} |\sqrt{\tilde{\kappa}} \partial_t^5 \tilde{J}|^2 = \mathcal{R}.$$

The time-integral of (4.30) completes the proof. \square

Via Proposition 2.1, the estimates of Lemma 4.1 and Proposition 4.1 imply the following.

PROPOSITION 4.2 (the κ -independent estimates for \tilde{v}_{ttt} and $\sqrt{\tilde{\kappa}} \tilde{v}_{ttt}$).

$$\sup_{t \in [0, T]} \|\tilde{v}_{ttt}(t)\|_1^2 + \int_0^T \|\sqrt{\tilde{\kappa}} \tilde{v}_{ttt}\|_1^2 \leq \int_0^T \mathcal{R}.$$

Proof. The fundamental theorem of calculus provides for

$$(4.31) \quad \operatorname{div} \partial_t^a \tilde{\eta} = \tilde{a}_r^s \partial_t^a \tilde{\eta}^r{}_{,s} - \partial_t^a \tilde{\eta}^r{}_{,s} \int_0^t \tilde{a}_t^s, \quad a = 0, 1, 2, 3, 4.$$

The identity $\tilde{J}_t = \tilde{a}_r^s \tilde{v}^r{}_{,s}$ implies that for $a = 1, 2, 3$, $\partial_t^a \tilde{J}_t$ is equal to $\tilde{a}_r^s \partial_t^a \tilde{v}^r{}_{,s} + \tilde{j}$, where \tilde{j} scales like $D \partial_t^{a-1} \tilde{v}$. The estimate for $\partial_t^4 \tilde{J}$ stated in Proposition 4.1 therefore implies that

$$(4.32) \quad \sup_{t \in [0, T]} \|\operatorname{div} \tilde{v}_{ttt}(t)\|_0^2 \leq \int_0^T \mathcal{R}.$$

Using the identity $N = \tilde{n} - \int_0^t \partial_t \tilde{n}$ and the estimate

$$\left| \bar{\partial} \partial_t^a \tilde{v} \cdot \int_0^t \partial_t \tilde{n} \right|_{2.5-a}^2 \leq TC \|\partial_t^a \tilde{v}\|_{4-a}^2 \quad \text{for } a = 0, 1, 2, 3,$$

we infer from the trace-estimate stated in Proposition 4.1 that

$$\sup_{t \in [0, T]} |\bar{\partial} \tilde{v}_{ttt}(t) \cdot N|_{-0.5}^2 \leq \int_0^T \mathcal{R}.$$

Thanks to the curl-estimates of Lemma 4.1, Proposition 2.1 therefore establishes that

$$\sup_{t \in [0, T]} \|\tilde{v}_{ttt}(t)\|_1^2 \leq \int_0^T \mathcal{R}.$$

A similar analysis establishes the $L^2(0, T; H^1(\Omega))$ -estimate for $\sqrt{\tilde{\kappa}} \tilde{v}_{ttt}$. \square

We now improve the normal trace-estimates of Proposition 4.1.

PROPOSITION 4.3 (an improved κ -independent normal trace-estimate for \tilde{v}_{ttt}).

$$\sup_{t \in [0, T]} |\tilde{v}_{ttt} \cdot \tilde{n}(t)|_1^2 \leq \int_0^T \mathcal{R}.$$

Proof. The estimate of Proposition 4.2 together with the trace theorem implies that

$$\sup_{t \in [0, T]} |\tilde{v}_{ttt}|_{0.5}^2 \leq \int_0^T \mathcal{R}.$$

Combining this estimate with the normal trace-estimate of Proposition 4.1 completes the proof. \square

Step 3: The κ -independent divergence- and normal trace-estimates.

We equivalently write the approximate surface tension problem (4.13a) as

$$2\rho_0 \tilde{J}^{-2} \tilde{A}_i^k \tilde{J}_{,k} + \kappa \rho_0 \tilde{A}_i^k \tilde{J}_{t,k} = \tilde{v}_t^i + [2\tilde{J}^{-1} - \kappa \tilde{J}_t] \tilde{A}_i^k \rho_{0,k},$$

or, equivalently, setting $\kappa' = \kappa/2$,

$$(4.33) \quad \tilde{J}^{-2} \tilde{A}_i^k \tilde{J}_{,k} + \kappa' \tilde{A}_i^k \tilde{J}_{t,k} = \underbrace{\frac{1}{2\rho_0} [\tilde{v}_t + [2\tilde{J}^{-1} - \kappa \tilde{J}_t] \tilde{A}_i^k \rho_{0,k}]}_{\tilde{\mathcal{V}}_t}.$$

The fundamental theorem of calculus provides that (4.33) is equivalently written as

$$(4.34) \quad D\tilde{J} + \kappa' D\tilde{J}_t = \tilde{\mathcal{V}}_t - \underbrace{\left[\tilde{J}_{,k} \int_0^t \partial_t (\tilde{J}^{-2} \tilde{A}^k) + \kappa \tilde{J}_{t,k} \int_0^t \partial_t \tilde{A}^k \right]}_{\tilde{j}}.$$

LEMMA 4.2 (estimates for \tilde{J}_{ttt} and $\kappa \tilde{J}_{tttt}$ via Lemma 2.4).

$$\sup_{t \in [0, T]} \|\tilde{J}_{ttt}(t)\|_1^2 + \sup_{t \in [0, T]} \|\kappa \tilde{J}_{tttt}(t)\|_1^2 \leq \int_0^T \mathcal{R}.$$

Proof. Taking three time-derivatives of (4.34) produces an equation of the form $f + \kappa f_t = g$:

$$(4.35) \quad D\tilde{J}_{ttt} + \kappa' \partial_t (D\tilde{J}_{ttt}) = \partial_t^3 \tilde{\mathcal{V}}_t + \tilde{j}_{ttt}.$$

By (4.33) we have that $\partial_t^3 \tilde{\mathcal{V}}_t$ scales like $\tilde{v}_{tttt} + T D\tilde{v}_{ttt} + \kappa \partial_t^3 \tilde{J}$. Proposition 4.2 therefore implies that $\|\partial_t^3 \tilde{\mathcal{V}}_t(t)\|_0^2 \leq \int_0^T \mathcal{R}$. According to (4.34), we have that \tilde{j}_{ttt} scales like $T D\tilde{J}_{ttt} + T \kappa D\tilde{J}_{tttt} + T D\tilde{v}_{ttt}$. So, $\|\tilde{j}_{ttt}(t)\|_0^2 \leq \int_0^T \mathcal{R}$. The fundamental theorem of calculus provides a good estimate for \tilde{J}_{ttt} in $L^\infty(0, T; L^2(\Omega))$. Thus, Lemma 2.4 implies that

$$\sup_{t \in [0, T]} \|\tilde{J}_{ttt}(t)\|_1^2 \leq \int_0^T \mathcal{R}.$$

We similarly infer from (4.35) and the fundamental theorem of calculus that

$$\sup_{t \in [0, T]} \|\kappa \tilde{J}_{ttt}(t)\|_1^2 \leq \int_0^T \mathcal{R}.$$

This completes the proof. \square

LEMMA 4.3 (normal trace-estimates for \tilde{v}_{tt} and $\kappa \tilde{v}_{ttt}$).

$$\sup_{t \in [0, T]} |\bar{\partial} \tilde{v}_{tt} \cdot \tilde{n}(t)|_{0.5}^2 + \sup_{t \in [0, T]} |\kappa \bar{\partial} \tilde{v}_{ttt} \cdot \tilde{n}(t)|_{0.5}^2 \leq \int_0^T \mathcal{R}.$$

Proof. The fundamental theorem of calculus implies the desired estimates. For example,

$$\begin{aligned} |\bar{\partial} \tilde{v}_{tt} \cdot \tilde{n}(t)|_{0.5}^2 &\leq C |\tilde{v}_{tt} \cdot \tilde{n}(t)|_{0.5} |\bar{\partial} \tilde{v}_{tt} \cdot \tilde{n}(t)|_{1.5} \\ &\leq \mathcal{N}_0 + C_\delta \int_0^T \|\tilde{v}_{ttt}\|_1^2 + \delta \sup_{t \in [0, T]} |\bar{\partial} \tilde{v}_{tt} \cdot \tilde{n}(t)|_{1.5}^2 \leq \int_0^T \mathcal{R}. \quad \square \end{aligned}$$

Step 4: The κ -independent higher-order estimates via Proposition 2.1.

The divergence- and normal trace-estimates obtained in Step 3, together with the curl-estimates of Lemma 4.1, imply a good estimate for \tilde{v}_{tt} . Hence, successively repeating Step 3 yields

$$(4.36) \quad \sup_{t \in [0, T]} \sum_{a=0}^2 \|\partial_t^a \tilde{v}(t)\|_{4-a}^2 + \sup_{t \in [0, T]} \sum_{a=0}^2 \|\kappa \partial_t^a \tilde{v}_t(t)\|_{4-a}^2 \leq \int_0^T \mathcal{R}.$$

We recall that the boundary condition (3.1b) is

$$(4.37) \quad \tilde{g}^{\alpha\beta} \tilde{\eta}_{,\alpha\beta} \cdot \tilde{n} + \kappa \tilde{g}^{\alpha\beta} \tilde{v}_{,\alpha\beta} \cdot \tilde{n} = \underbrace{\beta(t) - \rho_0^2 \tilde{J}^{-2} + \kappa \rho_0^2 \tilde{J}^{-1} \tilde{J}_t}_{\tilde{j}}.$$

Writing the boundary condition (4.37) as an equation of the form $f + \kappa f_t = g$ yields

$$(4.38) \quad \tilde{g}^{\alpha\beta} \tilde{\eta}_{,\alpha\beta} \cdot \tilde{n} + \kappa \partial_t (\tilde{g}^{\alpha\beta} \tilde{\eta}_{,\alpha\beta} \cdot \tilde{n}) = \tilde{j} + \kappa \tilde{\eta}_{,\alpha\beta} \cdot (\tilde{n} \tilde{g}^{\alpha\beta})_t.$$

As $\kappa \tilde{\eta}$ is of the same regularity as $\tilde{\eta}$, the flow map for the κ -problem does not witness an improved boundary-regularity. The fundamental theorem of calculus, however, does imply a good estimate for the right-hand side of (4.38). Hence, we infer from Lemma 2.4 that the normal trace of $\tilde{\eta}$ and $\kappa \tilde{v}$ each has a good estimate in $L^\infty(0, T; H^{4.5}(\Gamma))$:

$$(4.39) \quad \sup_{t \in [0, T]} \|\tilde{\eta}(t)\|_5^2 + \sup_{t \in [0, T]} \|\kappa \tilde{v}(t)\|_5^2 \leq \int_0^T \mathcal{R}.$$

We collect the estimates (4.36) and (4.39) in the following

PROPOSITION 4.4 (higher-order κ -independent estimates).

$$\sup_{t \in [0, T]} \sum_{a=0}^3 \|\partial_t^a \tilde{\eta}(t)\|_{5-a}^2 + \sup_{t \in [0, T]} \sum_{a=0}^3 \|\kappa \partial_t^a \tilde{v}_t(t)\|_{5-a}^2 \leq \int_0^T \mathcal{R}.$$

Step 5: The κ -independent improved boundary-regularity estimates.

With the estimates provided by Proposition 4.4 for $\|\partial_t^a \tilde{\eta}\|_{5-a}^2$ and $\|\kappa \partial_t^a \tilde{v}\|_{5-a}^2$, $a = 0, 1, 2, 3$, we are in position to establish the improved boundary-regularity estimates.

PROPOSITION 4.5 (the κ -independent improved boundary-regularity via Lemma 2.4).

$$\sup_{t \in [0, T]} \sum_{a=0}^2 |\bar{\partial}^2 \partial_t^a \tilde{v} \cdot \tilde{n}(t)|_{2.5-a}^2 + \sup_{t \in [0, T]} \sum_{a=0}^2 |\bar{\partial}^2 \partial_t^a \tilde{v}_t \cdot \tilde{n}(t)|_{2.5-a}^2 \leq \int_0^T \mathcal{R}.$$

Proof. Taking three time-derivatives of (4.37), we obtain

$$(4.40) \quad \tilde{g}^{\alpha\beta} \tilde{v}_{tt, \alpha\beta} \cdot \tilde{n} + \kappa \partial_t (\tilde{g}^{\alpha\beta} \tilde{v}_{tt, \alpha\beta} \cdot \tilde{n}) = \partial_t^3 \tilde{j} - \underbrace{[\partial_t^3 (\tilde{g}^{\alpha\beta} \tilde{\eta}_{, \alpha\beta} \cdot \tilde{n}) - \tilde{g}^{\alpha\beta} \tilde{v}_{tt, \alpha\beta} \cdot \tilde{n}]}_{\tilde{j}_A} - \underbrace{[\kappa \partial_t^3 (\tilde{g}^{\alpha\beta} \tilde{v}_{, \alpha\beta} \cdot \tilde{n}) - \kappa \partial_t (\tilde{g}^{\alpha\beta} \tilde{v}_{tt, \alpha\beta} \cdot \tilde{n})]}_{\tilde{j}_B}.$$

By Proposition 4.4, the right-hand side of (4.40) has a good estimate in $L^\infty(0, T; H^{0.5}(\Gamma))$. Lemma 2.4 therefore provides that

$$\sup_{t \in [0, T]} |\bar{\partial}^2 \tilde{v}_{tt} \cdot \tilde{n}(t)|_{0.5}^2 \leq \int_0^T \mathcal{R}.$$

Since $\kappa \partial_t (\tilde{g}^{\alpha\beta} \tilde{v}_{tt, \alpha\beta} \cdot \tilde{n}) = \kappa \tilde{g}^{\alpha\beta} \tilde{v}_{ttt, \alpha\beta} \cdot \tilde{n} + \kappa \tilde{v}_{tt, \alpha\beta} \cdot (\tilde{n} \tilde{g}^{\alpha\beta})_t$, it follows from (4.40) that

$$\sup_{t \in [0, T]} |\kappa \bar{\partial}^2 \tilde{v}_{ttt} \cdot \tilde{n}(t)|_{0.5}^2 \leq \int_0^T \mathcal{R}.$$

The higher-order estimates are similarly established. \square

Step 6: Concluding the proof of Lemma 4.1. The sum of the estimates given in Propositions 4.1–4.5 competes the proof of Lemma 4.1. Taking δ sufficiently small in the inequality (4.5) yields a polynomial-type inequality of the form (2.20). Hence, for sufficiently small $T > 0$, independent of $\kappa > 0$,

$$(4.41) \quad \sup_{t \in [0, T]} E^\kappa(t) \leq 2\mathcal{N}_0,$$

where the higher-order energy function $E^\kappa(t)$ is defined in (4.1).

5. Construction of solutions to the κ -problem (3.1). In this section, we prove the following.

THEOREM 5.1 (existence theory for the κ -problem). *For C^∞ -class initial data (ρ_0, u_0, Ω) satisfying the conditions (1.13) and (1.14), and for some $T = T_\kappa > 0$, there exists a unique solution \tilde{v} to the κ -problem (3.1) verifying $(\tilde{v}, \tilde{v}_t, \dots, \tilde{v}_{tttt})|_{t=0} = (u_0, v_1, \dots, v_4)$ and*

$$(5.1) \quad \sup_{t \in [0, T]} \|\tilde{v}_{tttt}(t)\|_0^2 + \int_0^T |\tilde{v}_{tttt} \cdot \tilde{n}|_1^2 + \sum_{a=0}^4 \int_0^T \|\partial_t^a \tilde{v}\|_{9-2a}^2 < \infty.$$

Remark 9. We recall that the (time-differentiated) initial data v_a , $a = 1, 2, 3, 4$, are defined in section 3.3.

We establish Theorem 5.1 via a succession of two asymptotic estimates that correspond with two further approximations of our κ -problem.

Each of these further approximations involve the use of a convolution operator that smooths only in the direction tangent to the moving boundary. We use $\epsilon > 0$ and $\mu > 0$ as the two additional smoothing parameters.

Our first intermediate problem, which we call the $\kappa\epsilon$ -problem, is defined by smoothing the moving domain of the κ -problem. The overall structure of the $\kappa\epsilon$ -problem matches that of the κ -problem. It naturally follows that the ϵ -independent a priori estimates closely resemble the κ -independent a priori estimates of section 4. The $\epsilon = 0$ formal limit of the $\kappa\epsilon$ -problem is the κ -problem.

Our second intermediate problem, which we call the μ -problem, is defined by smoothing the κ -artificial viscosity term $\kappa g_\epsilon^{\alpha\beta} v_{,\alpha\beta} \cdot n_\epsilon$ appearing in the boundary condition of the $\kappa\epsilon$ -problem. The μ -problem is a nonlinear heat-type problem with Neumann-type boundary conditions. The $\mu = 0$ formal limit of the heat-type μ -problem is equivalent to the $\kappa\epsilon$ -problem. The key to obtaining the μ -independent a priori estimates is that the diffusive term of the heat-type μ -problem yields a trace-estimate on the boundary Γ .

5.1. Horizontal convolution by layers. For $\epsilon > 0$, we let $0 \leq \bar{\varphi}_\epsilon \in C_0^\infty(\mathbb{R}^2)$ with $\text{spt}(\bar{\varphi}_\epsilon) \subset \overline{B(0, \epsilon)}$ denote the family of standard mollifiers on \mathbb{R}^2 . With $x_h = (x_1, x_2)$, we define the operation of *horizontal convolution by layers* as follows:

$$\Lambda_\epsilon f(x_h, x_3) = \int_{\mathbb{R}^2} \bar{\varphi}_\epsilon(x_h - y_h) f(y_h, x_3) dy_h \quad \text{for } f \in L^1_{loc}(\mathbb{R}^2).$$

By standard properties of convolution, there exists a constant C which is independent of ϵ , such that for $s \geq 0$,

$$|\Lambda_\epsilon f|_s \leq C|f|_s \quad \forall f \in H^s(\Gamma).$$

Furthermore,

$$(5.2) \quad \epsilon |\Lambda_\epsilon f|_1 \leq C|f|_0 \quad \forall f \in L^2(\Gamma).$$

We recall the local coordinates near Γ are defined in section 2.1.5. We set

$$(5.3) \quad \epsilon_0 = \min_{l=1}^K \text{dist}(\text{spt } \xi_l \circ \theta_l, \partial B_l^+ \setminus \mathcal{D}_l).$$

We define the horizontally convolved vector field v_ϵ on Γ of a given vector v by

$$(5.4) \quad v_\epsilon = \sum_{l=1}^K \Lambda_\epsilon [(\xi_l v) \circ \theta_l] \circ \theta_l^{-1} \quad \text{on } \Gamma.$$

Given a sufficiently smooth vector field \check{v} , we set $\check{\eta} = e + \int_0^t \check{v}$ in Ω and $\check{\eta}_\epsilon = e + \int_0^t \check{v}_\epsilon$ on Γ . We define $\check{\zeta}_\epsilon$ to be the solution of the following time-dependent elliptic Dirichlet problem:

$$(5.5a) \quad \Delta \check{\zeta}_\epsilon = \Delta \check{\eta} \quad \text{in } \Omega,$$

$$(5.5b) \quad \check{\zeta}_\epsilon = \check{\eta}_\epsilon \quad \text{on } \Gamma.$$

We define the following ϵ -approximate Lagrangian variables:

$$\check{A}_\epsilon = [D\check{\zeta}_\epsilon]^{-1}, \quad \check{J}_\epsilon = \det D\check{\zeta}_\epsilon, \quad \check{a}_\epsilon = \check{J}_\epsilon \check{A}_\epsilon, \quad [\check{g}_\epsilon]_{\alpha\beta} = \check{\zeta}_{\epsilon,\alpha} \cdot \check{\zeta}_{\epsilon,\beta}, \quad \text{and } \sqrt{\check{g}_\epsilon} \check{n}_\epsilon = [\check{a}_\epsilon]^T N.$$

5.2. The $\kappa\epsilon$ -problem and its a priori estimates. We define an intermediate approximate problem, which we will refer to as the $\kappa\epsilon$ -problem, that is asymptotically consistent with the κ -problem (3.1). To indicate the dependence on the smoothing parameter ϵ of all the variables in the $\kappa\epsilon$ -problem, we place the symbol $\check{\cdot}$ over each of the variables. In the $\kappa\epsilon$ -problem, we smooth the moving boundary and use the corresponding twice-mean-curvature function

$$\sigma H(\check{\zeta}_\epsilon) = -\sigma \check{g}_\epsilon^{\alpha\beta} \check{\zeta}_{\epsilon,\alpha\beta} \cdot \check{n}_\epsilon.$$

DEFINITION 5.1 (the $\kappa\epsilon$ -problem). For $\kappa > 0$ and $\epsilon > 0$, we define \check{v} as the solution of

$$(5.6a) \quad \check{v}_t^i + 2[\check{A}_\epsilon]^k_i (\rho_0 \check{J}_\epsilon^{-1})_{,k} - \kappa [\check{A}_\epsilon]^k_i (\rho_0 \check{J}_t)_{,k} = 0 \quad \text{in } \Omega \times (0, T_\kappa(\epsilon)),$$

$$(5.6b) \quad \rho_0^2 \check{J}_\epsilon^{-2} - \kappa \rho_0^2 \check{J}_\epsilon^{-1} \check{J}_t = \check{H}_\epsilon \quad \text{on } \Gamma \times (0, T_\kappa(\epsilon)),$$

$$(5.6c) \quad (\check{\zeta}_\epsilon, \check{v})|_{t=0} = (e, u_0) \quad \text{on } \Omega.$$

The function \check{J}_t appearing in the left-hand sides of (5.6a) and (5.6b) is defined as

$$(5.7) \quad \check{J}_t = \check{J}_\epsilon \operatorname{div}_{\check{\zeta}_\epsilon} \check{v}.$$

The function \check{H}_ϵ appearing in the right-hand side of (5.6b) is defined as

$$(5.8) \quad \check{H}_\epsilon = \beta_\epsilon(t) - \sigma \check{g}_\epsilon^{\alpha\beta} \check{\zeta}_{\epsilon,\alpha\beta} \cdot \check{n}_\epsilon - \kappa \check{g}_\epsilon^{\alpha\beta} \check{v}_{,\alpha\beta} \cdot \check{n}_\epsilon.$$

The function $\beta_\epsilon(t)$ appearing in the right-hand side of (5.8) is defined as

$$(5.9) \quad \begin{aligned} \beta_\epsilon(t) = & \beta + \kappa \sum_{a=0}^3 \frac{t^a}{a!} \partial_t^a [-\rho_0^2 \check{J}_\epsilon^{-1} \check{J}_t + \check{g}_\epsilon^{\alpha\beta} \check{v}_{,\alpha\beta} \cdot \check{n}_\epsilon]|_{t=0} \\ & + \sigma \sum_{a=0}^3 \frac{t^a}{a!} \partial_t^a [H(\check{\zeta}_\epsilon) - H(\eta) + \rho_0^2 (\check{J}_\epsilon^{-2} - J^{-2})]|_{t=0}. \end{aligned}$$

Remark 10. Standard elliptic estimates show that the ϵ -approximate Lagrangian flow map $\check{\zeta}_\epsilon$ defined by (5.5) satisfies $\|\check{\zeta}_\epsilon\|_s \leq C\|\check{\eta}\|_s$ for a positive constant C independent of ϵ . By standard properties of convolution, $\lim_{\epsilon \rightarrow 0} \check{\zeta}_\epsilon = \check{\eta}$. It follows that the $\kappa\epsilon$ -problem (5.6) is asymptotically consistent with the κ -problem (3.1). The definition of $\beta_\epsilon(t)$ in (5.9) has been made to ensure that the compatibility conditions for the initial data are unchanged

5.2.1. The vectors \mathbf{v}_a at $t = 0$ for $a = 1, 2, 3, 4$. Letting \check{v} solve (5.6), the vector field $\partial_t^a \check{v}|_{t=0}$ for all $a \in \mathbb{N}$ is computed as follows:

$$\partial_t^a \check{v}|_{t=0} = \frac{\partial^{a-1}}{\partial t^{a-1}} (\kappa [\check{A}_\epsilon]^k (\rho_0 \check{J}_t)_{,k} - 2 [\check{A}_\epsilon]^k (\rho_0 \check{J}_\epsilon^{-1})_{,k}) \Big|_{t=0} \quad \text{on } \Omega.$$

This formula makes it clear that each $\partial_t^a \check{v}|_{t=0}$ is a function of space-derivatives of the initial data u_0 , $\Lambda_\epsilon u_0$, and ρ_0 . We define the constant-in-time vectors \mathbf{v}_a as

$$(5.10) \quad \mathbf{v}_a = \frac{\partial^{a-1}}{\partial t^{a-1}} (\kappa [\check{A}_\epsilon]^k (\rho_0 \check{J}_t)_{,k} - 2 [\check{A}_\epsilon]^k (\rho_0 \check{J}_\epsilon^{-1})_{,k}) \Big|_{t=0} \quad \text{for } a = 1, 2, 3, 4.$$

We have that $\mathbf{v}_a \rightarrow \mathbf{v}_a$ as $\epsilon \rightarrow 0$, where \mathbf{v}_a are defined in section 3.3. We use (5.6b) and the definition (5.9) of $\beta_\epsilon(t)$ to compute the following identities: for $a = 0, 1, 2, 3$,

$$(5.11) \quad \partial_t^a [\rho_0^2 \check{J}_\epsilon^{-2} - \kappa \rho_0^2 \check{J}_\epsilon^{-1} \check{J}_t]|_{t=0} = \partial_t^a [\beta_\epsilon(t)]|_{t=0} - \partial_t^a [\sigma \check{g}_\epsilon^{\alpha\beta} \check{\zeta}_{\epsilon,\alpha\beta} \cdot \check{n}_\epsilon + \kappa \check{g}_\epsilon^{\alpha\beta} \check{v}_{,\alpha\beta} \cdot \check{n}_\epsilon]|_{t=0}.$$

5.2.2. A priori estimates for the $\kappa\epsilon$ -problem (5.6). For $\epsilon > 0$, we define the following higher-order energy function:

$$(5.12) \quad E^\epsilon(t) = 1 + \|\check{v}_{tt}(t)\|_0^2 + \int_0^t |\check{v}_{tt} \cdot \check{n}_\epsilon|_1^2 + \sum_{a=0}^2 \int_0^t \|\partial_t^a \check{v}\|_{5-2a}^2.$$

DEFINITION 5.2 (notational convention for constants depending on $1/\delta\kappa > 0$). We let $\dot{\mathcal{P}}$ denote a generic polynomial with constant and coefficients depending on $1/\delta\kappa > 0$.

We define the constant $\dot{\mathcal{N}}_0 > 0$ by

$$\dot{\mathcal{N}}_0 = \dot{\mathcal{P}}(\|u_0\|_{100}, \|\rho_0\|_{100}).$$

We let $\dot{\mathcal{R}}$ denote generic lower-order terms satisfying

$$\int_0^T \dot{\mathcal{R}} \leq \dot{\mathcal{N}}_0 + \delta \sup_{t \in [0, T]} E^\epsilon(t) + T \dot{\mathcal{P}}\left(\sup_{t \in [0, T]} E^\epsilon(t)\right).$$

We assume $T > 0$ is taken sufficiently small to ensure that

$$\frac{1}{2} \leq \check{J} \leq \frac{3}{2} \quad \text{and} \quad \frac{1}{2} \leq \check{J}_\epsilon \leq \frac{3}{2} \quad \forall t \in [0, T] \text{ and } x \in \Omega.$$

Before proving existence of unique solutions to the $\kappa\epsilon$ -problem, we begin with the following.

LEMMA 5.1 (a priori estimates for the $\kappa\epsilon$ -problem). We let \check{v} solve the $\kappa\epsilon$ -problem (5.6) on a time-interval $[0, T]$ for some $T = T_\kappa(\epsilon) > 0$. Then independent of ϵ ,

$$(5.13) \quad \sup_{t \in [0, T]} E^\epsilon(t) \leq \int_0^T \dot{\mathcal{R}}.$$

We will establish Lemma 5.1 in the following four steps.

Step 1: The ϵ -independent curl-estimates. Taking the ϵ -approximate Lagrangian curl of (5.6a) yields

$$(5.14) \quad \text{curl}_{\check{\zeta}_\epsilon} \check{v}_t = 0.$$

Integrating the identity (5.14) in time from 0 to $t \in (0, T]$ provides that

$$(5.15) \quad \text{curl}_{\check{\zeta}_\epsilon} \check{v} = \text{curl} u_0 + \varepsilon_{.ji} \int_0^t \partial_t [\check{A}_\epsilon]_j^s \check{v}^i_{,s}.$$

We may therefore infer the curl estimates from Lemma 4.1. We record this fact as follows.

LEMMA 5.2 (the ϵ -independent curl-estimates for $\partial^a \check{v}$, for $a = 0, 1, 2$).

$$\sum_{a=0}^2 \int_0^T \|\text{curl} \partial_t^a \check{v}\|_{4-2a}^2 \leq \int_0^T \dot{\mathcal{R}}.$$

Step 2: The ϵ -independent estimate for \check{v}_{tt} . We equivalently write the momentum equations (5.6a) as

$$(5.16) \quad \rho_0 \check{v}_t^i + [\check{a}_\epsilon]_i^k (\rho_0^2 \check{J}_\epsilon^{-2})_{,k} - \kappa \rho_0 \check{J}_\epsilon^{-1} [\check{a}_\epsilon]_i^k (\rho_0 \check{J}_t)_{,k} = 0 \quad \text{in } \Omega \times (0, T_\kappa(\epsilon)).$$

LEMMA 5.3 (energy estimates for the action of ∂_t^2 in (5.16)).

$$\sup_{t \in [0, T]} \|\check{v}_{tt}(t)\|_0^2 + \int_0^T \|\check{J}_{ttt}\|_0^2 + \int_0^T |\bar{\partial} \check{v}_{tt} \cdot \check{n}_\epsilon|_0^2 \leq \int_0^T \check{\mathcal{R}}.$$

Proof. Testing the action of ∂_t^2 in (5.16) against \check{v}_{tt} in the $L^2(\Omega)$ -inner product and integrating by parts with respect to ∂_k in the interior integrals $\int_\Omega [\check{a}_\epsilon]_i^k \partial_t^2 (\rho_0^2 \check{J}_\epsilon^{-2})_{,k} \check{v}_{tt}^i$ and $-\kappa \int_\Omega [\check{a}_\epsilon]_i^k \partial_t^2 [\rho_0 \check{J}_\epsilon^{-1} (\rho_0 \check{J}_t)_{,k}] \check{v}_{tt}^i$ yields

$$(5.17) \quad \frac{1}{2} \frac{d}{dt} \int_\Omega \rho_0 |\check{v}_{tt}|^2 - \underbrace{\int_\Omega \partial_t^2 [\rho_0^2 \check{J}_\epsilon^{-2}] [\check{a}_\epsilon]_i^k \check{v}_{tt}^i}_{\check{\mathcal{R}}} + \kappa \underbrace{\int_\Omega \partial_t^2 [\rho_0^2 \check{J}_\epsilon^{-1} \check{J}_t] [\check{a}_\epsilon]_i^k \check{v}_{tt}^i}_{\mathcal{I}} + \underbrace{\int_\Gamma \partial_t^2 [\rho_0^2 \check{J}_\epsilon^{-2} - \kappa \rho_0^2 \check{J}_\epsilon^{-1} \check{J}_t] [\check{a}_\epsilon]_i^k \check{v}_{tt}^i N^k}_{\mathbf{i}} = \check{\mathcal{R}}.$$

It is convenient to write (5.17) as

$$(5.18) \quad \frac{1}{2} \frac{d}{dt} \int_\Omega \rho_0 |\check{v}_{tt}|^2 + \kappa \int_\Omega \rho_0^2 \check{J}_\epsilon^{-1} |\check{J}_{ttt}|^2 + \mathbf{i} = \check{\mathcal{R}},$$

where the error created in order to write $[\check{a}_\epsilon]_i^k \partial_t^2 \check{v}_{tt}^i$ as \check{J}_{ttt} in \mathcal{I} is of lower-order and so is absorbed in $\check{\mathcal{R}}$.

We write the boundary integral \mathbf{i} appearing in the left-hand side of (5.18) as

$$(5.19) \quad \mathbf{i} = \int_\Gamma \check{H}_{\epsilon tt} \sqrt{\check{g}_\epsilon} \check{v}_{tt} \cdot \check{n}_\epsilon,$$

where we have used the boundary condition (5.6b) and the formula (2.9). The definition (5.8) provides that $\check{H}_\epsilon = \beta_\epsilon(t) - \sigma \check{g}_\epsilon^{\alpha\beta} \check{\zeta}_{\epsilon, \alpha\beta} \cdot \check{n}_\epsilon - \kappa \check{g}_\epsilon^{\alpha\beta} \check{v}_{\epsilon, \alpha\beta} \cdot \check{n}_\epsilon$. Thus,

$$(5.20) \quad \mathbf{i} = \kappa \int_\Gamma \sqrt{\check{g}_\epsilon} \check{g}_\epsilon^{\alpha\beta} \check{v}_{tt, \beta} \cdot \check{n}_{\epsilon, \alpha} \check{v}_{tt, \alpha} \cdot \check{n}_\epsilon + \underbrace{\kappa \int_\Gamma \check{v}_{tt, \beta}^i (\check{n}_\epsilon^i \sqrt{\check{g}_\epsilon} \check{g}_\epsilon^{\alpha\beta} \check{n}_\epsilon^j)}_{\mathbf{j}} \check{v}_{tt}^j + \check{\mathcal{R}}.$$

Thanks to Lemma 2.1,

$$(5.21) \quad \int_0^T |\check{v}_{tt}|_0^2 \leq C \int_0^T |\check{v}_{tt}|_{0.25}^2 \leq \int_0^T \check{\mathcal{R}}.$$

Employing the Cauchy–Schwarz inequality and Young’s inequality with $\delta > 0$ thus yields

$$\int_0^T \mathbf{j} - \underbrace{\int_0^T \check{v}_{tt, \beta} \cdot \check{n}_{\epsilon, \alpha} \sqrt{\check{g}_\epsilon} \check{g}_\epsilon^{\alpha\beta} \check{v}_{tt} \cdot \check{n}_\epsilon}_{\mathbf{j}' } \leq \int_0^T \check{\mathcal{R}}.$$

Integrating by parts with respect to $\bar{\partial}_\beta$, we similarly have that $\int_0^T j' \leq \int_0^T \dot{\mathcal{R}}$. We have therefore established that the identity (5.20) is equivalently written as

$$i = \kappa \int_\Gamma \sqrt{\tilde{g}_\epsilon} \tilde{g}_\epsilon^{\alpha\beta} \tilde{v}_{tt,\beta} \cdot \tilde{n}_\epsilon \tilde{v}_{tt,\alpha} \cdot \tilde{n}_\epsilon + \dot{\mathcal{R}}.$$

Using this identity for i in the time-integral of (5.18) completes the proof. \square

PROPOSITION 5.1 (the ϵ -independent estimates for \tilde{v}_{tt}).

$$\sup_{t \in [0, T]} \|\tilde{v}_{tt}(t)\|_0^2 + \int_0^T |\tilde{v}_{tt} \cdot \tilde{n}_\epsilon|_1^2 + \int_0^T \|\tilde{v}_{tt}\|_1^2 \leq \dot{\mathcal{R}}.$$

Proof. The desired $L^\infty(0, T; L^2(\Omega))$ -estimate is provided by Lemma 5.3. The inequality (5.21) and the trace-estimate stated in Lemma 5.3 establish the desired $L^2(0, T; H^1(\Gamma))$ -estimate. We infer from the arguments proving Proposition 4.2 that Lemma 5.3 implies a divergence- and normal trace-estimate for \tilde{v}_{tt} . Thanks to the curl-estimate for \tilde{v}_{tt} stated in Lemma 5.2, we have by Proposition 2.1 that the proof is complete. \square

Step 3: The ϵ -independent estimate for \tilde{v}_t .

PROPOSITION 5.2 (the ϵ -independent estimate for \tilde{v}_t).

$$\int_0^T \|\tilde{v}_t\|_3^2 \leq \int_0^T \dot{\mathcal{R}}.$$

Proof. We write the boundary condition (5.6b) as

$$\kappa \tilde{g}_\epsilon^{\alpha\beta} \tilde{v}_{t,\alpha\beta} \cdot \tilde{n}_\epsilon = \underbrace{\kappa \rho_0^2 \check{J}_\epsilon^{-1} \check{J}_t - \rho_0^2 \check{J}_\epsilon^{-2} + \beta_\epsilon(t) - \sigma \tilde{g}_\epsilon^{\alpha\beta} \check{\zeta}_{\epsilon,\alpha\beta} \cdot \tilde{n}_\epsilon}_{\check{j}}.$$

Taking a time-derivative of this equation yields

$$(5.22) \quad \kappa \check{g}_\epsilon^{\alpha\beta} \tilde{v}_{t,\alpha\beta} \cdot \tilde{n}_\epsilon = \check{j}_t + \kappa \tilde{v}_{t,\alpha\beta} \cdot \partial_t (\tilde{n}_\epsilon \check{g}_\epsilon^{\alpha\beta}).$$

The right-hand side of (5.22) scales like $T \bar{\partial}^2 \tilde{v}_t$. We infer that

$$(5.23) \quad \int_0^T |\bar{\partial}^2 \tilde{v}_t \cdot \tilde{n}_\epsilon|_{0.5}^2 \leq \int_0^T \dot{\mathcal{R}}.$$

We equivalently write the momentum equations (5.16) as

$$[\check{A}_\epsilon]_i^k \check{J}_{t,k} = \underbrace{(\kappa \rho_0)^{-1} [\check{v}_t^i + 2[\check{A}_\epsilon]_i^k (\rho_0 \check{J}_\epsilon^{-1})_{,k} - \kappa \check{J}_t [\check{A}_\epsilon]_i^k \rho_{0,k}]}_{\check{v}_t}.$$

Using the fundamental theorem of calculus, we have that

$$(5.24) \quad D \check{J}_t = \check{V}_t + \check{J}_{t,k} \int_0^t \partial_t [\check{A}_\epsilon]_i^k.$$

Since the right-hand side of (5.24) scales like $\check{v}_t + D\check{v}$, a time-derivative of (5.24) yields

$$\int_0^T \|D \check{J}_{tt}\|_1^2 \leq \int_0^T \dot{\mathcal{R}}.$$

Lemma 5.3 provides a good estimate for \check{J}_{tt} in $L^2(0, T; L^2(\Omega))$. Thus,

$$(5.25) \quad \int_0^T \|\check{J}_{tt}\|_2^2 \leq \int_0^T \check{\mathcal{R}}.$$

Via Proposition 2.1, the curl-, normal trace-, and divergence-estimates for \check{v}_t , respectively, given in Lemma 5.2 and inequalities (5.23) and (5.25), complete the proof. \square

Step 4: Concluding the proof of Lemma 5.1. Repeating Step 3 establishes the following.

PROPOSITION 5.3 (the ϵ -independent estimate for \check{v}).

$$\int_0^T \|\check{v}\|_5^2 \leq \int_0^T \check{\mathcal{R}}.$$

The sum of the ϵ -independent estimates stated in Propositions 5.1, 5.2, and 5.3 completes the proof of Lemma 5.1.

As an intermediate step in proving Theorem 5.1, we will establish the following.

THEOREM 5.2 (solutions to the $\kappa\epsilon$ -problem). *For C^∞ -class initial data (ρ_0, u_0, Ω) satisfying the conditions (1.13) and (1.14), and for some $T = T_\kappa(\epsilon) > 0$, there exists a unique solution \check{v} to the $\kappa\epsilon$ -problem (5.6) verifying $(\check{v}, \check{v}_t, \dots, \check{v}_{tttt})|_{t=0} = (u_0, \mathbf{v}_1, \dots, \mathbf{v}_4)$ and*

$$(5.26) \quad \sup_{t \in [0, T]} \|\check{v}_{tttt}(t)\|_0^2 + \sum_{a=0}^4 \int_0^T \|\partial_t^a \check{v}\|_{9-2a}^2 + \sum_{a=0}^4 \int_0^T |\partial_t^a \check{v} \cdot \check{n}_\epsilon|_{9-2a}^2 < \infty.$$

Remark 11. We recall that the initial data \mathbf{v}_a , $a = 1, 2, 3, 4$, is defined in section 5.2.1.

5.3. Deriving a heat-type problem with Neumann-type boundary conditions. In order to establish an existence theory for the $\kappa\epsilon$ -problem, we will introduce an equivalent nonlinear heat equation. Specifically, we next derive a nonlinear heat-type problem with Neumann-type boundary conditions which is equivalent to the $\kappa\epsilon$ -problem (5.6).

5.3.1. Rewriting the momentum equations (5.6a) via (5.15). Setting

$$(5.27) \quad \check{\varrho} = \kappa\rho_0\check{J}_\epsilon$$

and using the definition (5.7), the momentum equations (5.6a) are equivalently written as

$$(5.28) \quad \check{v}_t - \check{\varrho}[\check{A}_\epsilon]^k(\operatorname{div}_{\check{\zeta}_\epsilon} \check{v})_{,k} = \operatorname{div}_{\check{\zeta}_\epsilon} \check{v}[\check{A}_\epsilon]^k \check{\varrho}_{,k} - 2[\check{A}_\epsilon]^k(\rho_0\check{J}_\epsilon^{-1})_{,k}.$$

Given a sufficiently smooth vector \check{v} , the identity $-\Delta\check{v} = \operatorname{curl}\operatorname{curl}\check{v} - D\operatorname{div}\check{v}$ in the ϵ -approximate Lagrangian variables is the identity

$$(5.29) \quad -[\check{A}_\epsilon]_r^j([\check{A}_\epsilon]_r^k \check{v}_{,k})_{,j} = \operatorname{curl}_{\check{\zeta}_\epsilon} \operatorname{curl}_{\check{\zeta}_\epsilon} \check{v} - [\check{A}_\epsilon]^s(\operatorname{div}_{\check{\zeta}_\epsilon} \check{v})_{,s}.$$

Using (5.29), the equations (5.28) are equivalently written as

$$(5.30) \quad \check{v}_t - \check{\varrho}[\check{A}_\epsilon]_r^j([\check{A}_\epsilon]_r^k \check{v}_{,k})_{,j} = \check{\varrho} \operatorname{curl}_{\check{\zeta}_\epsilon}(\operatorname{curl}_{\check{\zeta}_\epsilon} \check{v}) + \operatorname{div}_{\check{\zeta}_\epsilon} \check{v}[\check{A}_\epsilon]^k \check{\varrho}_{,k} - 2[\check{A}_\epsilon]^k(\rho_0\check{J}_\epsilon^{-1})_{,k}.$$

Thanks to the vorticity equation (5.15), we further have that (5.6a) is equivalent to

$$(5.31) \quad \check{v}_t - \check{\varrho}[\check{A}_\epsilon]_r^j([\check{A}_\epsilon]_r^k \check{v}_{,k}),_{,j} = \check{\mathcal{K}},$$

where the vector field $\check{\mathcal{K}}$ appearing in the right-hand side of (5.31) is defined as

$$(5.32) \quad \check{\mathcal{K}} = \check{\varrho} \operatorname{curl}_{\check{\zeta}_\epsilon} \left(\operatorname{curl} u_0 + \varepsilon_{,ji} \int_0^t \partial_t [\check{A}_\epsilon]_j^s \check{v}_{,s}^i \right) + \operatorname{div}_{\check{\zeta}_\epsilon} \check{v} [\check{A}_\epsilon]_{,k} \check{\varrho}_{,k} - 2[\check{A}_\epsilon]_{,k} (\rho_0 \check{J}_\epsilon^{-1})_{,k}.$$

5.3.2. Deriving a Neumann-type boundary condition for \check{v} . We decompose any vector field ξ on Γ into tangential and a normal components as

$$(5.33) \quad \xi = \xi^\alpha \check{g}_\epsilon^{\alpha\beta} \check{\zeta}_{\epsilon,\beta} + \xi^3 \check{n}_\epsilon,$$

where $\xi^\alpha = \xi \cdot \check{\zeta}_{\epsilon,\alpha}$ and $\xi^3 = \xi \cdot \check{n}_\epsilon$. In the special case of a flat boundary, we may use the standard orthogonal unit vectors e_k to write (5.33) as

$$(5.34) \quad \xi = \xi^\alpha e_\alpha + \xi^3 e_3.$$

For the special case where (5.33) takes the form (5.34), we have that

$$(5.35) \quad \frac{\partial \check{v}^i}{\partial N} = [(\operatorname{curl} \check{v}) \times N]^i + \check{v}^3_{, \alpha} e_\alpha^i + [\operatorname{div} \check{v} - \check{v}^\alpha_{, \alpha}] N^i.$$

This motivates the general decomposition; we decompose the vector \check{b} defined on Γ as follows:

$$(5.36) \quad \check{b} = \check{b}_{\operatorname{curl}} + \check{b}_{\operatorname{div}},$$

where $\check{b}_{\operatorname{curl}}$ is a tangent vector and $\check{b}_{\operatorname{div}}$ is a normal vector, respectively, defined as

$$(5.37a) \quad \check{b}_{\operatorname{curl}} = \frac{\sqrt{\check{g}_\epsilon}}{\check{J}_\epsilon} [\check{g}_\epsilon^{\alpha\beta} \check{v}^3_{, \beta} + \check{v}^\gamma \check{g}_\epsilon^{\gamma\delta} \check{g}_\epsilon^{\alpha\beta} \check{\zeta}_{\epsilon,\delta\beta} \cdot \check{n}_\epsilon] \check{\zeta}_{\epsilon,\alpha},$$

$$(5.37b) \quad \check{b}_{\operatorname{div}} = -\frac{\sqrt{\check{g}_\epsilon}}{\check{J}_\epsilon} [\check{g}_\epsilon^{\alpha\beta} \check{v}^\alpha_{, \beta} + \check{v}^\alpha (\sqrt{\check{g}_\epsilon} \check{g}_\epsilon^{\alpha\beta})_{, \beta} + \check{v}^3 H(\check{\zeta}_\epsilon)] \check{n}_\epsilon.$$

For the moving boundary $\Gamma_\epsilon(t) = \check{\zeta}_\epsilon(\Gamma)$, the identity (5.35) is written in general form as

$$(5.38) \quad \frac{\check{J}_\epsilon}{\sqrt{\check{g}_\epsilon}} N^j [\check{A}_\epsilon]_r^j [\check{A}_\epsilon]_r^k \check{v}_{,k}^i = ((\operatorname{curl}_{\check{\zeta}_\epsilon} \check{v}) \times \check{n}_\epsilon)^i + \check{n}_\epsilon^i \operatorname{div}_{\check{\zeta}_\epsilon} \check{v} + \frac{\check{J}_\epsilon}{\sqrt{\check{g}_\epsilon}} \check{b}^i,$$

or, equivalently,

$$(5.39) \quad N^j [\check{A}_\epsilon]_r^j [\check{A}_\epsilon]_r^k \check{v}_{,k} = \sqrt{\check{g}_\epsilon} \check{J}_\epsilon^{-1} (\operatorname{curl}_{\check{\zeta}_\epsilon} \check{v}) \times \check{n}_\epsilon + \sqrt{\check{g}_\epsilon} \check{J}_\epsilon^{-1} (\operatorname{div}_{\check{\zeta}_\epsilon} \check{v}) \check{n}_\epsilon + \check{b}.$$

Thanks to (5.15), we have that (5.39) multiplied by $\check{\varrho} = \kappa\rho_0\check{J}_\epsilon$ is equivalent to

$$(5.40) \quad \check{\varrho}N^j[\check{A}_\epsilon]^j_r[\check{A}_\epsilon]^k_r\check{v}_{,k} = \check{h}_{\text{curl}} + \frac{\sqrt{\check{g}_\epsilon}}{\rho_0}\kappa\rho_0^2(\text{div}_{\check{\zeta}_\epsilon}\check{v})\check{n}_\epsilon + \check{\varrho}\check{b},$$

where the tangential vector field \check{h}_{curl} appearing in the right-hand side of (5.40) is defined via the ϵ -approximate Lagrangian vorticity equation (5.15) as

$$(5.41) \quad \check{h}_{\text{curl}} = \check{\varrho}\sqrt{\check{g}_\epsilon}\check{J}_\epsilon^{-1}\left(\text{curl } u_0 + \varepsilon_{.j.i} \int_0^t \partial_t[\check{A}_\epsilon]^s_j\check{v}^i_{,s}\right) \times \check{n}_\epsilon.$$

We write the boundary condition (5.6b) as

$$(5.42) \quad \kappa\rho_0^2\text{div}_{\check{\zeta}_\epsilon}\check{v} = \rho_0^2\check{J}_\epsilon^{-2} - \beta_\epsilon(t) + \sigma\check{g}_\epsilon^{\alpha\beta}\check{\zeta}_{\epsilon,\alpha\beta}\cdot\check{n}_\epsilon + \kappa\check{g}_\epsilon^{\alpha\beta}\check{v}_{,\alpha\beta}\cdot\check{n}_\epsilon.$$

Setting

$$(5.43) \quad \check{\mathfrak{g}}_\epsilon^{\alpha\beta} = \kappa\frac{\sqrt{\check{g}_\epsilon}}{\rho_0}\check{g}_\epsilon^{\alpha\beta},$$

$$(5.44) \quad \check{h}_{\text{div}} = \frac{\sqrt{\check{g}_\epsilon}}{\rho_0}\left[\rho_0^2\check{J}_\epsilon^{-2} - \beta_\epsilon(t) + \sigma\check{g}_\epsilon^{\alpha\beta}\check{\zeta}_{\epsilon,\alpha\beta}\cdot\check{n}_\epsilon\right]\check{n}_\epsilon,$$

we find by using the identity (5.42) for $\kappa\rho_0^2\text{div}_{\check{\zeta}_\epsilon}\check{v}$ in (5.40) that

$$(5.45) \quad \check{\varrho}N^j[\check{A}_\epsilon]^j_r[\check{A}_\epsilon]^k_r\check{v}_{,k} = \check{h}_{\text{curl}} + \check{h}_{\text{div}} + \check{\mathfrak{g}}_\epsilon^{\alpha\beta}[\check{v}_{,\alpha\beta}\cdot\check{n}_\epsilon]\check{n}_\epsilon + \check{\varrho}\check{b}.$$

We define the vector field \check{h} as the sum

$$(5.46) \quad \check{h} = \check{h}_{\text{curl}} + \check{h}_{\text{div}} + \check{\mathfrak{g}}_\epsilon^{\alpha\beta}[\check{v}_{,\alpha\beta}\cdot\check{n}_\epsilon]\check{n}_\epsilon + \check{\varrho}[\check{b}_{\text{curl}} + \check{b}_{\text{div}}]$$

with the vectors \check{h}_{curl} , \check{h}_{div} , \check{b}_{curl} , and \check{b}_{div} , respectively, given in (5.41), (5.44), (5.37a), and (5.37b), and the function $\check{\varrho}$ defined in (5.27). We equivalently express the identity (5.45) as

$$(5.47) \quad \check{\varrho}N^j[\check{A}_\epsilon]^j_r[\check{A}_\epsilon]^k_r\check{v}_{,k} = \check{h}.$$

Remark 12. We record that the identities (5.42) and (5.44) show that

$$(5.48) \quad \check{\varrho}\sqrt{\check{g}_\epsilon}\check{J}_\epsilon^{-1}\text{div}_{\check{\zeta}_\epsilon}\check{v} = \check{h}_{\text{div}}\cdot\check{n}_\epsilon + \check{\mathfrak{g}}_\epsilon^{\alpha\beta}\check{v}_{,\alpha\beta}\cdot\check{n}_\epsilon.$$

5.3.3. The heat-type $\kappa\epsilon$ -problem with Neumann-type boundary conditions.

DEFINITION 5.3 (the heat-type $\kappa\epsilon$ -problem). *For $\kappa > 0$ and $\epsilon > 0$ given, we define \check{v} as the solution of the nonlinear heat-type system*

$$(5.49a) \quad \check{v}_t - \check{\varrho}[\check{A}_\epsilon]^j_r([\check{A}_\epsilon]^k_r\check{v}_{,k})_{,j} = \check{K} \quad \text{in } \Omega \times (0, T_\kappa(\epsilon)),$$

$$(5.49b) \quad \check{\varrho}N^j[\check{A}_\epsilon]^j_r[\check{A}_\epsilon]^s_r\check{v}_{,s} = \check{h} \quad \text{on } \Gamma \times (0, T_\kappa(\epsilon)),$$

$$(5.49c) \quad (\check{\zeta}_\epsilon, \check{v})|_{t=0} = (e, u_0) \quad \text{on } \Omega.$$

The bounded, positive function $\check{\varrho}$ is defined in (5.27). The vector fields \check{K} and \check{h} are defined in (5.32) and (5.46), respectively.

We will show in Appendix C that solutions to the heat-type $\kappa\epsilon$ -problem (5.49) are also solutions to the $\kappa\epsilon$ -problem (5.6).

Remark 13. We set $\mathbf{v}_a = \partial_t^a(\check{\varrho}[\check{A}_\epsilon]_r^j([\check{A}_\epsilon]_r^k \check{v},_k),_j + \check{\mathcal{K}})|_{t=0}$ for $a = 1, 2, 3$, and require the following compatibility conditions to be satisfied: for $a = 0, 1, 2, 3$,

$$(5.50) \quad \partial_t^a(\check{\varrho}N^j[\check{A}_\epsilon]_r^j[\check{A}_\epsilon]_r^s \check{v},_s - \check{h})|_{t=0} = 0.$$

PROPOSITION 5.4 (solutions to the heat-type $\kappa\epsilon$ -problem (5.49)). *For C^∞ -class initial data (ρ_0, u_0, Ω) satisfying (5.50), and for some $T = T_\kappa(\epsilon) > 0$, there exists a unique $\check{v} \in L^2(0, T; H^9(\Omega))$ with $\partial_t^a \check{v} \in L^2(0, T; H^{9-2a}(\Omega))$ for $a = 1, 2, 3, 4$, and $\check{v}_{tttt} \in L^\infty(0, T; L^2(\Omega))$ that solves the nonlinear heat-type $\kappa\epsilon$ -problem (5.49) on a time-interval $[0, T]$.*

The proof of Proposition 5.4 will rely on one further approximation, in which we smooth some of the terms in the right-hand side of the boundary condition (5.49b).

5.4. The μ -problem and its a priori estimates. We now define our second intermediate problem, which we term the μ -problem. The μ -problem is a system of nonlinear heat-type equations that is asymptotically consistent with the heat-type $\kappa\epsilon$ -problem (5.49). To indicate the dependence on the smoothing parameter μ of all the variables in the following problem, we place the symbol \circ over each of the variables.

Given a sufficiently smooth vector field \check{v} , we set $\check{\eta} = e + \int_0^t \check{v}$ in Ω and $\check{\eta}_\epsilon = e + \int_0^t \check{v}_\epsilon$ on Γ . We define $\check{\zeta}_\epsilon$ to be the solution of the following time-dependent elliptic Dirichlet problem:

$$(5.51a) \quad \Delta \check{\zeta}_\epsilon = \Delta \check{\eta} \quad \text{in } \Omega,$$

$$(5.51b) \quad \check{\zeta}_\epsilon = \check{\eta}_\epsilon \quad \text{on } \Gamma.$$

We define the following ϵ -approximate Lagrangian variables:

$$\mathring{A}_\epsilon = [D\check{\zeta}_\epsilon]^{-1}, \quad \mathring{J}_\epsilon = \det D\check{\zeta}_\epsilon, \quad \mathring{a}_\epsilon = \mathring{J}_\epsilon \mathring{A}_\epsilon, \quad [\mathring{g}_\epsilon]_{\alpha\beta} = \check{\zeta}_{\epsilon,\alpha} \cdot \check{\zeta}_{\epsilon,\beta}, \quad \text{and } \sqrt{\mathring{g}_\epsilon} \mathring{n}_\epsilon = [\mathring{a}_\epsilon]^T N.$$

We recall that the convolution operator Λ_μ for $\mu > 0$ is defined via section 5.1.

DEFINITION 5.4 (the μ -problem). *Given $\kappa > 0$ and $\epsilon > 0$, for $\mu > 0$ we define \check{v} as the solution of the nonlinear heat-type system*

$$(5.52a) \quad \check{v}_t - \check{\varrho}[\check{A}_\epsilon]_r^j([\check{A}_\epsilon]_r^k \check{v},_k),_j = \check{\mathcal{K}} \quad \text{in } \Omega \times (0, T_\kappa(\epsilon\mu)),$$

$$(5.52b) \quad \check{\varrho}N^j[\check{A}_\epsilon]_r^j[\check{A}_\epsilon]_r^s \check{v},_s = \mathring{h}^\mu + c^\mu(t) \quad \text{on } \Gamma \times (0, T_\kappa(\epsilon\mu)),$$

$$(5.52c) \quad (\check{\zeta}_\epsilon, \check{v})|_{t=0} = (e, u_0) \quad \text{on } \Omega.$$

The bounded, nonnegative function $\mathring{\varrho}$ is defined as

$$(5.53) \quad \mathring{\varrho} = \kappa\rho_0 \mathring{J}_\epsilon.$$

The vector field $\check{\mathcal{K}}$ appearing in the right-hand side of (5.52a) is defined as

$$(5.54) \quad \check{\mathcal{K}} = \mathring{\varrho} \operatorname{curl}_{\check{\zeta}_\epsilon} \left(\operatorname{curl} u_0 + \varepsilon_{,ji} \int_0^t \partial_t [\check{A}_\epsilon]_j^s \check{v}^i,_{,s} \right) + \operatorname{div}_{\check{\zeta}_\epsilon} \check{v} [\check{A}_\epsilon]_r^k \check{\varrho},_{,k} - 2[\check{A}_\epsilon]_r^k (\rho_0 \mathring{J}_\epsilon^{-1}),_{,k}.$$

The vector field \mathring{h}^μ appearing in the right-hand side of (5.52b) is given by

(5.55)

$$\mathring{h}^\mu = \mathring{h}_{\text{curl}} + \mathring{h}_{\text{div}} + \sum_{l=1}^K \sqrt{\xi_l} \left(\Lambda_\mu \left[\mathring{\mathfrak{g}}_{\epsilon,l}^{\alpha\beta} \Lambda_\mu \left[(\sqrt{\xi_l} \mathring{v}_{,\alpha\beta} \cdot \mathring{n}_\epsilon) \circ \theta_l \right] \right] \right) \circ \theta_l^{-1} \mathring{n}_\epsilon + \mathring{\varrho} \left[\mathring{b}_{\text{curl}}^\mu + \mathring{b}_{\text{div}}^\mu \right],$$

where $\mathring{\mathfrak{g}}_{\epsilon,l}^{\alpha\beta}$ appearing in the right-hand side of (5.55) is defined as

$$(5.56) \quad \mathring{\mathfrak{g}}_{\epsilon,l}^{\alpha\beta} = \kappa \frac{\sqrt{\mathring{g}_\epsilon}}{\rho_0} \mathring{g}_\epsilon^{\alpha\beta} \circ \theta_l,$$

and $\mathring{h}_{\text{curl}}, \mathring{h}_{\text{div}}, \mathring{b}_{\text{curl}}^\mu, \mathring{b}_{\text{div}}^\mu$ appearing in the right-hand side of (5.55) are defined as

(5.57a)

$$\mathring{h}_{\text{curl}} = \mathring{\varrho} \sqrt{\mathring{g}_\epsilon} \mathring{J}_\epsilon^{-1} \left(\text{curl } u_0 + \varepsilon_{.ji} \int_0^t \partial_t [\mathring{A}_\epsilon]_j^s \mathring{v}^i_{,s} \right) \times \mathring{n}_\epsilon,$$

(5.57b)

$$\mathring{h}_{\text{div}} = \frac{\sqrt{\mathring{g}_\epsilon}}{\rho_0} \left[\rho_0^2 \mathring{J}_\epsilon^{-2} - \beta_\epsilon(t) + \sigma \mathring{g}_\epsilon^{\alpha\beta} \mathring{\zeta}_{\epsilon,\alpha\beta} \cdot \mathring{n}_\epsilon \right] \mathring{n}_\epsilon,$$

(5.57c)

$$\mathring{b}_{\text{curl}}^\mu = \frac{\sqrt{\mathring{g}_\epsilon}}{\mathring{J}_\epsilon} \left[\sum_{l=1}^K \mathring{g}_\epsilon^{\alpha\beta} \sqrt{\xi_l} \left(\Lambda_\mu \left[(\sqrt{\xi_l} \mathring{v}_{,\alpha\beta} \cdot \mathring{n}_\epsilon)_{,\beta} \circ \theta_l \right] \right) \circ \theta_l^{-1} + \mathring{v}^\gamma \mathring{g}_\epsilon^{\gamma\delta} \mathring{g}_\epsilon^{\alpha\beta} \mathring{\zeta}_{\epsilon,\delta\beta} \cdot \mathring{n}_\epsilon \right] \mathring{\zeta}_{\epsilon,\alpha},$$

(5.57d)

$$\mathring{b}_{\text{div}}^\mu = - \sum_{l=1}^K \sqrt{\xi_l} \Lambda_\mu \left(\sqrt{\xi_l} \frac{\sqrt{\mathring{g}_\epsilon}}{\mathring{J}_\epsilon} \left[\mathring{g}_\epsilon^{\alpha\beta} \mathring{v}_{,\alpha\beta} + \mathring{v}^\alpha (\sqrt{\mathring{g}_\epsilon} \mathring{g}_\epsilon^{\alpha\beta})_{,\beta} + \mathring{v}^3 H(\mathring{\zeta}_\epsilon) \right] \circ \theta_l \right) \circ \theta_l^{-1} \mathring{n}_\epsilon.$$

The vector field $c^\mu(t)$ appearing in the right-hand side of (5.52b) is defined as

$$(5.58) \quad \begin{aligned} c^\mu(t) = & \sum_{a=0}^4 \frac{t^a}{a!} \partial_t^a \left\{ \mathring{\mathfrak{g}}_{\epsilon}^{\alpha\beta} [\mathring{v}_{,\alpha\beta} \cdot \mathring{n}_\epsilon] \mathring{n}_\epsilon \right. \\ & \left. - \sum_{l=1}^K \sqrt{\xi_l} \left(\Lambda_\mu \left[\mathring{\mathfrak{g}}_{\epsilon,l}^{\alpha\beta} \Lambda_\mu \left[(\sqrt{\xi_l} \mathring{v}_{,\alpha\beta} \cdot \mathring{n}_\epsilon) \circ \theta_l \right] \right] \right) \circ \theta_l^{-1} \mathring{n}_\epsilon \right\} \Big|_{t=0} \\ & + \sum_{a=0}^4 \frac{t^a}{a!} \partial_t^a \left\{ \frac{\sqrt{\mathring{g}_\epsilon}}{\mathring{J}_\epsilon} \mathring{g}_\epsilon^{\alpha\beta} \mathring{v}_{,\alpha\beta}^3 \mathring{\zeta}_{\epsilon,\alpha} \right. \\ & \left. - \frac{\sqrt{\mathring{g}_\epsilon}}{\mathring{J}_\epsilon} \left[\sum_{l=1}^K \mathring{g}_\epsilon^{\alpha\beta} \sqrt{\xi_l} \left(\Lambda_\mu \left[(\sqrt{\xi_l} \mathring{v}_{,\alpha\beta} \cdot \mathring{n}_\epsilon)_{,\beta} \circ \theta_l \right] \right) \circ \theta_l^{-1} \right] \mathring{\zeta}_{\epsilon,\alpha} \right\} \Big|_{t=0} \\ & + \sum_{a=0}^4 \frac{t^a}{a!} \partial_t^a \left\{ \mathring{b}_{\text{div}} - \mathring{b}_{\text{div}}^\mu \right\} \Big|_{t=0}. \end{aligned}$$

Again, the polynomial-in-time $c^\mu(t)$ is added to ensure that the compatibility conditions (5.50) continue to hold for the data.

Remark 14. The fixed-point solution to the μ -problem (5.52) is established in Appendix B.

Remark 15. For any h defined on Γ , we have that $h = \sum_{l=1}^K \xi_l h = \sum_{l=1}^K \sqrt{\xi_l}([\sqrt{\xi_l}h \circ \theta_l] \circ \theta_l^{-1})$.

For $\mu > 0$, we define the following higher-order energy function:

(5.59)

$$E^\mu(t) = 1 + \|\dot{v}_{tt}(t)\|_0^2 + \sum_{a=0}^2 \int_0^t \|\partial_t^a \dot{v}\|_{5-2a}^2 + \sum_{a=0}^2 \sum_{l=1}^K \int_0^t |\Lambda_\mu[(\sqrt{\xi_l} \partial_t^a \dot{v} \cdot \dot{n}_\epsilon) \circ \theta_l]|_{5-2a, \mathcal{D}_l}^2.$$

DEFINITION 5.5 (notational convention for constants depending on $1/\delta\kappa\epsilon > 0$). We let \ddot{P} denote a generic polynomial with constant and coefficients depending on $1/\delta\kappa\epsilon > 0$.

We define the constant $\check{N}_0 > 0$ by

(5.60)
$$\check{N}_0 = \ddot{P}(\|u_0\|_{100}, \|\rho_0\|_{100}).$$

We let $\ddot{\mathcal{R}}$ denote generic lower-order terms satisfying

$$\int_0^T \ddot{\mathcal{R}} \leq \check{N}_0 + \delta \sup_{t \in [0, T]} E^\mu(t) + T \ddot{P}(\sup_{t \in [0, T]} E^\mu(t)).$$

LEMMA 5.4 (a priori estimates for the μ -problem). We let \dot{v} solve the μ -problem (5.52) on a time-interval $[0, T]$ for some $T = T_\kappa(\epsilon\mu) > 0$. Then independent of μ ,

(5.61)
$$\sup_{t \in [0, T]} E^\mu(t) \leq \int_0^T \ddot{\mathcal{R}}.$$

We will establish Lemma 5.4 in the following four steps.

Step 1: The μ -independent estimates for \dot{v}_{tt} . Testing two time-derivatives of (5.52a) against \dot{v}_{tt} in the $L^2(\Omega)$ -inner product and integrating by parts in the interior integral $-\int_\Omega \partial_t^2(\varrho[\dot{A}_\epsilon]^j([\dot{A}_\epsilon]^k \dot{v}_{,k})_{,j}) \dot{v}_{tt}$ yields

(5.62)
$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \int_\Omega |\dot{v}_{tt}|^2 + \int_\Omega \varrho[\dot{A}_\epsilon]^k |\dot{v}_{tt,k}|^2 - \underbrace{\int_\Gamma [\dot{h}^\mu + c^\mu(t)]_{tt} \dot{v}_{tt}}_i \\ &= \underbrace{\int_\Omega \partial_t^2(\dot{\mathcal{K}} - (\varrho[\dot{A}_\epsilon]^j)_{,j} [\dot{A}_\epsilon]^k \dot{v}_{,k}) \dot{v}_{tt}}_{\ddot{\mathcal{R}}} + \ddot{\mathcal{R}}. \end{aligned}$$

The definition (5.55) of the vector field \dot{h}^μ provides that

$$\begin{aligned} \mathbf{i} &= - \underbrace{\sum_{l=1}^K \int_{\mathcal{D}_l} \Lambda_\mu[(\sqrt{\xi_l} \dot{v}_{tt, \alpha\beta} \cdot \dot{n}_\epsilon) \circ \theta_l] \mathfrak{g}_{\epsilon,l}^{\alpha\beta} \Lambda_\mu[(\sqrt{\xi_l} \dot{v}_{tt} \cdot \dot{n}_\epsilon) \circ \theta_l]}_{i'} \\ &\quad - \underbrace{\int_\Gamma (\varrho[\dot{b}_{\text{curl}}^\mu + \dot{b}_{\text{div}}^\mu])_{tt} \dot{v}_{tt}}_j + \ddot{\mathcal{R}}. \end{aligned}$$

We integrate by parts with respect to $\bar{\partial}_\alpha$ in i' to find that

$$\begin{aligned}
 i' &= \sum_{l=1}^K \int_{\mathcal{D}_l} \Lambda_\mu [(\sqrt{\xi_l} \dot{v}_{tt, \beta} \cdot \dot{n}_\epsilon) \circ \theta_l] \mathfrak{g}_{\epsilon, l}^{\alpha \beta} \Lambda_\mu [(\sqrt{\xi_l} \dot{v}_{tt, \alpha} \cdot \dot{n}_\epsilon) \circ \theta_l] \\
 &\quad + \underbrace{\sum_{l=1}^K \int_{\mathcal{D}_l} \Lambda_\mu [(\dot{v}_{tt, \beta} \cdot (\dot{n}_\epsilon \sqrt{\xi_l}), \alpha) \circ \theta_l] \mathfrak{g}_{\epsilon, l}^{\alpha \beta} \Lambda_\mu [(\sqrt{\xi_l} \dot{v}_{tt} \cdot \dot{n}_\epsilon) \circ \theta_l]}_{i'_A} \\
 &\quad + \underbrace{\sum_{l=1}^K \int_{\mathcal{D}_l} \Lambda_\mu [(\sqrt{\xi_l} \dot{v}_{tt, \beta} \cdot \dot{n}_\epsilon) \circ \theta_l] \bar{\partial}_\alpha \mathfrak{g}_{\epsilon, l}^{\alpha \beta} \Lambda_\mu [(\sqrt{\xi_l} \dot{v}_{tt} \cdot \dot{n}_\epsilon) \circ \theta_l]}_{i'_B} \\
 &\quad + \underbrace{\sum_{l=1}^K \int_{\mathcal{D}_l} \Lambda_\mu [(\sqrt{\xi_l} \dot{v}_{tt, \beta} \cdot \dot{n}_\epsilon) \circ \theta_l] \mathfrak{g}_{\epsilon, l}^{\alpha \beta} \Lambda_\mu [(\dot{v}_{tt} \cdot (\dot{n}_\epsilon \sqrt{\xi_l}), \alpha) \circ \theta_l]}_{i'_C}.
 \end{aligned}$$

We employ an $H^{-0.5}(\mathcal{D}_l)$ -duality pairing in the integrals i'_A , i'_B , and i'_C . For example,

$$\begin{aligned}
 i'_A &\leq C |\bar{\partial} \dot{v}_{tt} \circ \theta_l|_{-0.5, \mathcal{D}_l} |\Lambda_\mu [(\sqrt{\xi_l} \dot{v}_{tt} \cdot \dot{n}_\epsilon) \circ \theta_l]|_{0.5, \mathcal{D}_l} \\
 &\leq \delta \|\dot{v}_{tt}\|_1^2 + C_\delta |\Lambda_\mu [(\sqrt{\xi_l} \dot{v}_{tt} \cdot \dot{n}_\epsilon) \circ \theta_l]|_{0.5, \mathcal{D}_l}^2.
 \end{aligned}$$

Thanks to Lemma 2.1, $|\Lambda_\mu [(\sqrt{\xi_l} \dot{v}_{tt} \cdot \dot{n}_\epsilon) \circ \theta_l]|_{0.5, \mathcal{D}_l}^2 \leq C |\dot{v}_{tt}|_0 |\Lambda_\mu [(\sqrt{\xi_l} \dot{v}_{tt} \cdot \dot{n}_\epsilon) \circ \theta_l]|_{1, \mathcal{D}_l} \leq \ddot{\mathcal{R}} + \delta |\Lambda_\mu (\dot{v}_{tt} \cdot \dot{n}_\epsilon)|_{1, \mathcal{D}_l}^2$. Thus, we infer that

$$i = \sum_{l=1}^K \int_{\mathcal{D}_l} \Lambda_\mu [(\sqrt{\xi_l} \dot{v}_{tt} \cdot \dot{n}_\epsilon)_{, \beta} \circ \theta_l] \mathfrak{g}_{\epsilon, l}^{\alpha \beta} \Lambda_\mu [(\sqrt{\xi_l} \dot{v}_{tt} \cdot \dot{n}_\epsilon)_{, \alpha} \circ \theta_l] + \ddot{\mathcal{R}},$$

where we have used the definitions (5.57c) and (5.57d) in analyzing j. We infer from the time integral of (5.62) that

$$(5.63) \quad \sup_{t \in [0, T]} \|\dot{v}_{tt}(t)\|_0^2 + \int_0^T \|\dot{v}_{tt}\|_1^2 + \sum_{l=1}^K \int_0^t |\Lambda_\mu [(\sqrt{\xi_l} \dot{v}_{tt} \cdot \dot{n}_\epsilon) \circ \theta_l]|_{1, \mathcal{D}_l}^2 \leq \int_0^T \ddot{\mathcal{R}}.$$

Step 2: The μ -independent estimates for \dot{v}_t . Similar to Step 1, testing the action of $\bar{\partial}^2 \partial_t$ in (5.52a) against $\bar{\partial}^2 \dot{v}_t$ in the $L^2(\Omega)$ -inner product yields

$$(5.64) \quad \sup_{t \in [0, T]} \|\bar{\partial}^2 \dot{v}_t(t)\|_0^2 + \int_0^T \|\bar{\partial}^2 \dot{v}_t\|_1^2 + \sum_{l=1}^K \int_0^t |\Lambda_\mu [(\sqrt{\xi_l} \dot{v}_t \cdot \dot{n}_\epsilon) \circ \theta_l]|_{3, \mathcal{D}_l}^2 \leq \int_0^T \ddot{\mathcal{R}}.$$

The inequality (5.64) provides that $\int_0^T \|\dot{v}_t\|_{2.5}^2 \leq \int_0^T \ddot{\mathcal{R}}$. We infer from Step 2 of the proof of Lemma B.2 that by viewing a time-derivative of (5.52a) as an elliptic Dirichlet problem for \dot{v}_t that

$$(5.65) \quad \int_0^T \|\dot{v}_t\|_3^2 \leq \int_0^T \ddot{\mathcal{R}}.$$

Step 3: The μ -independent estimates for \hat{v} . Repeating Step 1, we test four tangential-derivatives of (5.52a) against $\bar{\partial}^4 \hat{v}$ in the $L^2(\Omega)$ -inner product and integrate by parts in the integral $-\int_{\Omega} \bar{\partial}^4 (\hat{\varrho} [\hat{A}_{\epsilon}]_r^j ([\hat{A}_{\epsilon}]_r^k \hat{v}_{,k})_{,j}) \bar{\partial}^4 \hat{v}$ to find that

$$\begin{aligned}
 (5.66) \quad & \frac{1}{2} \frac{d}{dt} \int_{\Omega} |\bar{\partial}^4 \hat{v}|^2 + \sum_{l=1}^K \int_{\mathcal{D}_l} \Lambda_{\mu} [\bar{\partial}^4 (\sqrt{\xi_l} \hat{v} \cdot \hat{n}_{\epsilon})_{,\beta} \circ \theta_l] \hat{g}_{\epsilon,l}^{\alpha\beta} \Lambda_{\mu} [\bar{\partial}^4 (\sqrt{\xi_l} \hat{v} \cdot \hat{n}_{\epsilon})_{,\alpha} \circ \theta_l] \\
 & + \int_{\Omega} \hat{\varrho} |[\hat{A}_{\epsilon}]_r^k \bar{\partial}^4 \hat{v}_{,k}|^2 = \underbrace{\int_{\Gamma} \left[\hat{\varrho} \sqrt{g_{\epsilon}} \hat{J}_{\epsilon}^{-1} \left(\varepsilon_{,ji} \int_0^t \partial_t [\hat{A}_{\epsilon}]_j^s \bar{\partial}^4 \hat{v}^i \right)_{,s} \times \hat{n}_{\epsilon} \right]}_{\mathcal{R}} \cdot \bar{\partial}^4 \hat{v} \\
 & + \underbrace{\int_{\Omega} \bar{\partial}^4 \left[\hat{\mathcal{K}} - (\hat{\varrho} [\hat{A}_{\epsilon}]_r^j)_{,j} [\hat{A}_{\epsilon}]_r^k \hat{v}_{,k} \right]}_{\mathcal{I}} \bar{\partial}^4 \hat{v} + \ddot{\mathcal{R}}.
 \end{aligned}$$

We use an $H^{-0.5}(\mathcal{D}_l)$ -duality pairing in analyzing the first term appearing in the right-hand side of (5.66) and conclude a good estimate thanks to the time-integral. Tangentially integrating by parts in \mathcal{I} , we find that

$$\int_0^T \mathcal{I} \leq \int_0^T \ddot{\mathcal{R}}.$$

Hence, the time-integral of (5.66) yields

$$(5.67) \quad \sup_{t \in [0, T]} \|\bar{\partial}^4 \hat{v}(t)\|_0^2 + \int_0^T \|\bar{\partial}^4 \hat{v}\|_1^2 + \sum_{l=1}^K \int_0^t |\Lambda_{\mu}[(\sqrt{\xi_l} \hat{v} \cdot \hat{n}_{\epsilon}) \circ \theta_l]|_{5, \mathcal{D}_l}^2 \leq \int_0^T \ddot{\mathcal{R}}.$$

The inequality (5.67) yields

$$(5.68) \quad \int_0^T |\hat{v}|_{4.5}^2 \leq \int_0^T \ddot{\mathcal{R}}.$$

Viewing (5.52a) as an elliptic Dirichlet problem for \hat{v} , we infer from elliptic regularity and the inequalities (5.65) and (5.68) that

$$(5.69) \quad \int_0^T \|\hat{v}\|_5^2 \leq \int_0^T \ddot{\mathcal{R}}.$$

Step 4: Concluding the proof of Lemma 5.4. The sum of the inequalities (5.63), (5.65), (5.67), and (5.69) completes the proof of Lemma 5.4. Taking δ sufficiently small in the inequality (5.61) yields a polynomial-type inequality of the form (2.20). Hence, for sufficiently small $T = T_{\kappa}(\epsilon) > 0$ and independently of $\mu > 0$,

$$(5.70) \quad \sup_{t \in [0, T]} E^{\mu}(t) \leq 2\check{\mathcal{N}}_0,$$

where the higher-order energy function $E^{\mu}(t)$ is defined in (5.59).

5.5. The proof of Proposition 5.4. Proposition B.1 establishes the existence and uniqueness of a solution \hat{v} to the μ -problem (5.52). Given the μ -independent

estimate (5.70), standard compactness arguments provide for the existence of the strong convergence, as μ tends to zero,

$$\begin{aligned} \check{\zeta}_\epsilon &\rightarrow \check{\zeta}_\epsilon \text{ in } L^2(0, T; H^4(\Omega)), & \check{\mathcal{K}} &\rightarrow \check{\mathcal{K}} \text{ in } L^2(0, T; H^2(\Omega)), \\ \check{v} &\rightarrow \check{v} \text{ in } L^2(0, T; H^4(\Omega)), & \check{h}^\mu &\rightarrow \check{h} \text{ in } L^2(0, T; H^{1.5}(\Gamma)), \\ \check{v}_t &\rightarrow \check{v}_t \text{ in } L^2(0, T; H^2(\Omega)), & c^\mu(t) &\rightarrow 0 \text{ in } L^2(0, T; H^{1.5}(\Gamma)). \end{aligned}$$

The definitions (5.54), (5.55), (5.58), respectively, define $\check{\mathcal{K}}, \check{h}^\mu, c^\mu(t)$. The limiting vectors $\check{\mathcal{K}}$ and \check{h} are, respectively, defined by (5.32) and (5.46). Letting $\phi \in L^2(0, T; H^1(\Omega))$, we have that the variational form of the μ -problem (5.52) is

$$\int_0^T \int_\Omega \check{v}_t \cdot \phi + \int_0^T \int_\Omega [\check{A}_\epsilon]_r^k \check{v}_{,k} (\varrho[\check{A}_\epsilon]_r^j \phi)_{,j} = \int_0^T \int_\Omega \check{\mathcal{K}} \cdot \phi + \int_0^T \int_\Gamma [\check{h}^\mu + c^\mu(t)] \cdot \phi.$$

We infer from the strong convergence of the sequences $(\check{\zeta}_\epsilon, \check{v}, \check{v}_t, \check{\mathcal{K}}, \check{h}^\mu, c^\mu(t))$ that the limit $(\check{\zeta}_\epsilon, \check{v}, \check{v}_t, \check{\mathcal{K}}, \check{h}, c(t))$ satisfies

$$\int_0^T \int_\Omega \check{v}_t \cdot \phi + \int_0^T \int_\Omega [\check{A}_\epsilon]_r^k \check{v}_{,k} (\varrho[\check{A}_\epsilon]_r^j \phi)_{,j} = \int_0^T \int_\Omega \check{\mathcal{K}} \cdot \phi + \int_0^T \int_\Gamma [\check{h} + c(t)] \cdot \phi.$$

Thus, \check{v} is a solution of the nonlinear heat-type $\kappa\epsilon$ -problem (5.49) on a time-interval $[0, T]$ for some $T = T_\kappa(\epsilon) > 0$. Standard arguments provide that $\check{\zeta}_\epsilon(0) = e$ and $(\check{v}, \check{v}_t, \dots, \check{v}_{tttt})|_{t=0} = (u_0, \mathbf{v}_1, \dots, \mathbf{v}_4)$. Furthermore, according to the inequality (5.70),

$$\sup_{t \in [0, T]} \|\check{v}_{tt}(t)\|_0^2 + \sum_{a=0}^2 \int_0^T \|\partial_t^a \check{v}\|_{5-2a}^2 + \sum_{a=0}^2 \int_0^T |\partial_t^a \check{v} \cdot \check{n}_\epsilon|_{5-2a}^2 \leq 2\check{\mathcal{N}}_0.$$

By the higher-order regularity stated in Proposition B.1, we infer that

$$(5.71) \quad \sup_{t \in [0, T]} \|\check{v}_{tttt}(t)\|_0^2 + \sum_{a=0}^4 \int_0^T \|\partial_t^a \check{v}\|_{9-2a}^2 + \sum_{a=0}^4 \int_0^T |\partial_t^a \check{v} \cdot \check{n}_\epsilon|_{9-2a}^2 < \infty.$$

5.6. The proof of Theorem 5.2. By Lemma C.1, the heat-type $\kappa\epsilon$ -problem (5.49) is equivalent to the $\kappa\epsilon$ -problem (5.6). Hence, Proposition 5.4 establishes the existence and uniqueness of a solution to the $\kappa\epsilon$ -problem on a time-interval $[0, T]$ for some $T = T_\kappa(\epsilon) > 0$ verifying $(\check{v}, \check{v}_t, \dots, \check{v}_{tttt})|_{t=0} = (u_0, \mathbf{v}_1, \dots, \mathbf{v}_4)$. The inequality (5.71) establishes the inequality (5.26).

5.7. The proof of Theorem 5.1. A unique solution to the $\kappa\epsilon$ -problem (5.6) exists by Theorem 5.2.

Taking δ sufficiently small in the inequality (5.13) yields a polynomial-type inequality of the form (2.20). Hence, for sufficiently small $T = T_\kappa > 0$ and independently of $\epsilon > 0$,

$$(5.72) \quad \sup_{t \in [0, T]} E^\epsilon(t) \leq 2\check{\mathcal{N}}_0,$$

where the higher-order energy function $E^\epsilon(t)$ is defined in (5.12).

For $\phi \in L^2(0, T; H^1(\Omega))$ such that $\phi \cdot \check{n}_\epsilon \in L^2(0, T; H^1(\Gamma))$, the variational equation for the $\kappa\epsilon$ -problem (5.6) is

$$(5.73) \quad \begin{aligned} &\int_0^T \int_\Omega \rho_0 \check{v}_t \cdot \phi - \int_0^T \int_\Omega \rho_0^2 \check{J}_\epsilon^{-2} [\check{a}_\epsilon]_i^k \phi^i_{,k} + \kappa \int_0^T \int_\Omega \check{J}_t [\check{a}_\epsilon]_i^k (\rho_0 \check{J}_\epsilon^{-1} \phi^i)_{,k} \\ &+ \int_0^T \int_\Gamma \beta_\epsilon(t) \sqrt{\check{g}_\epsilon} \phi \cdot \check{n}_\epsilon + \int_0^T \int_\Gamma [\sigma \check{\zeta}_{\epsilon, \beta} + \kappa \check{v}_{, \beta}]^i (\check{n}_\epsilon^i \sqrt{\check{g}_\epsilon} \check{g}_\epsilon^{\alpha \beta} \phi \cdot \check{n}_\epsilon)_{, \alpha} = 0. \end{aligned}$$

We infer from section 5.5 and the pointwise convergence $\beta_\epsilon(t) \rightarrow \beta(t)$ that the variational equation (5.73) converges to the variational form of equation (3.1b) as ϵ tends to zero. Hence, the $\epsilon = 0$ limit \tilde{v} of the solutions \check{v} to the $\kappa\epsilon$ -problem (5.6) solves the κ -problem (3.1). By section 5.2.1, the solution \tilde{v} of the κ -problem verifies $(\tilde{v}, \tilde{v}_t, \dots, \tilde{v}_{tttt})|_{t=0} = (u_0, v_1, \dots, v_4)$. According to (5.72),

$$\sup_{t \in [0, T]} \|\tilde{v}_{tt}(t)\|_0^2 + \int_0^T |\tilde{v}_{tt} \cdot \tilde{n}|_1^2 + \sum_{a=0}^2 \int_0^T \|\partial_t^a \tilde{v}\|_{5-2a}^2 \leq 2\mathcal{N}_0.$$

It follows from the proof of Lemma 5.1 that by including two more time-derivatives in the definition (5.12) of the energy function $E^\epsilon(t)$, the solution \tilde{v} of the κ -problem (3.1) satisfies

$$\sup_{t \in [0, T]} \|\tilde{v}_{tttt}(t)\|_0^2 + \int_0^T |\tilde{v}_{tttt} \cdot \tilde{n}|_1^2 + \sum_{a=0}^4 \int_0^T \|\partial_t^a \tilde{v}\|_{9-2a}^2 < \infty.$$

This establishes the inequality (5.1).

6. Well-posedness of the surface tension problem (1.7). In this section, we prove Theorem 1.1 via the κ -independent a priori estimates of section 4.

6.1. Existence. We obtain a solution v to surface tension problem (1.7) in the limit of \tilde{v} as the parabolic parameter κ tends to zero. According to Remark 7, $\beta(t) = \beta$ in the $\kappa = 0$ limit. Letting $\kappa = 0$ in section 4, we therefore conclude that the right-hand side of the inequality (4.41) depends only on $M_0 = P(E(0))$. That is, for sufficiently small $T > 0$, the energy function $E(t)$ defined in (1.9) satisfies

$$(6.1) \quad \sup_{t \in [0, T]} E(t) \leq 2M_0.$$

The assumption (4.3) on J remains valid by taking $T > 0$ even smaller if necessary. Hence,

$$f(t) = \rho_0 J^{-1}(t) \geq \frac{4}{3}\lambda.$$

Taking $T > 0$ even smaller if necessary, we ensure that $\rho(t) = f \circ \eta^{-1}(t)$ satisfies

$$\rho(t) \geq \lambda \quad \text{in } \bar{\Omega}(t).$$

Since $p(t) = \rho^2(t) - \beta > -\beta$ on $\Gamma(t)$, the boundary condition (1.1c) establishes that

$$\sigma H(t) > -\beta \quad \text{on } \Gamma(t).$$

6.2. Optimal regularity for the initial data. In order to construct solutions to the Euler equations via our parabolic regularization, it was convenient to smooth our initial data using the method described in Appendix A; in particular, we constructed data that was smooth (but still satisfied the compatibility conditions of our problem) by a combination of convolution operators with radius of convolution $\epsilon > 0$ and solutions to elliptic equations. In the limit as $\epsilon \rightarrow 0$, we recover the initial data with the Sobolev-class regularity.

The a priori estimate (6.1) thus allows us to pass to the limit $\epsilon \rightarrow 0$ and recover the Sobolev-class regularity stated in Theorems 1.1 and 1.2.

6.3. Uniqueness. We define

$$\mathbf{E}(v, t) = 1 + \sum_{a=0}^6 \|\partial_t^a \eta(t)\|_{6-a}^2 + |\partial_t^4 v \cdot n(t)|_1^2 + \sum_{a=0}^4 |\bar{\partial}^2 \partial_t^a \eta \cdot n(t)|_{4.5-a}^2.$$

We suppose that (η_1, v_1, f_1) and (η_2, v_2, f_2) are two solutions of the compressible surface tension problem (1.7) with $\mathbf{E}(v, 0)$ for $v = v_1, v_2$, bounded by some $\mathbf{M}_0 > 0$.

Then by setting

$$\zeta = \eta_1 - \eta_2, \quad w = v_1 - v_2, \quad \varrho = [f_1]^2 - [f_2]^2,$$

we have that (ζ, w, ϱ) satisfies

$$(6.2a) \quad \zeta = \int_0^t w \quad \text{in } \Omega \times (0, T],$$

$$(6.2b) \quad \rho_0 w_t^i + [a_1]_i^k \varrho_{,k} = [a_2 - a_1]_i^k ([f_2]^2)_{,k} \quad \text{in } \Omega \times (0, T],$$

$$(6.2c) \quad \begin{aligned} \varrho &= -\sigma [g_1]^{\alpha\beta} \zeta_{,\alpha\beta} \cdot n_1 - \sigma [g_1]^{\alpha\beta} \eta_{2,\alpha\beta} \cdot n_1 \\ &\quad + \sigma [g_2]^{\alpha\beta} \eta_{2,\alpha\beta} \cdot n_2 \quad \text{on } \Gamma \times (0, T], \end{aligned}$$

$$(6.2d) \quad (\zeta, w, \varrho)|_{t=0} = (0, 0, 0) \quad \text{on } \Omega.$$

We will establish that $w = 0$. Setting

$$(6.3) \quad E^\zeta(t) = \sum_{a=0}^5 \|\partial_t^a \zeta(t)\|_{5-a}^2,$$

we follow section 4 with κ set to zero.

The Lagrangian curl operator curl_{η_1} applied to

$$w_t^i + 2[A_1]_i^k f_{1,k} = 2[A_2]_i^k f_{2,k}$$

provides the following vorticity equation for the difference $w = v_1 - v_2$:

$$(6.4) \quad \text{curl}_{\eta_1} w_t = 2\varepsilon_{\cdot j i} [A_1]_j^s ([A_2]_i^k f_{2,k})_{,s}.$$

The time integral of the surface tension problem satisfied by v_2 yields

$$v_2 - u_0 = -2 \int_0^t [A_2]_i^k f_{2,k},$$

by which we infer that

$$\int_0^T \left\| D^2 \int_0^t [A_2]_i^k f_{2,k} \right\|_3^2 \leq T P \left(\sup_{t \in [0, T]} \|v_2(t)\|_5^2 \right) \leq C T \mathbf{M}_0.$$

The curl-estimates for w therefore follow from the analysis proving Lemma 4.1 with the vorticity equation (6.4) replacing the homogeneous vorticity equation (2.14).

Repeating the energy estimate for the fourth time-differentiated problem in Proposition 4.1, the highest-order term of the interior forcing term $\int_0^t \int_\Omega \partial_t^4 [a_2 - a_1]_i^k ([f_2]^2)_{,k} w_{tttt}^i$ obeys

$$\int_0^t \int_\Omega [a_2 - a_1]_i^k ([f_2]^2)_{tttt,k} w_{tttt}^i \leq C \int_0^T \|Dv_{2ttt}\|_1 \|w_{tttt}\|_0 \leq T P \left(\sup_{t \in [0, T]} E^\zeta(t) \right).$$

The boundary integral corresponding with

$$-\sigma[g_1]^{\alpha\beta}\eta_{2,\alpha\beta}\cdot n_1 + \sigma[g_2]^{\alpha\beta}\eta_{2,\alpha\beta}\cdot n_2 = -\sigma[g_1]^{\alpha\beta}\eta_{2,\alpha\beta}\cdot [n_1 - n_2] - \sigma[g_1 - g_2]^{\alpha\beta}\eta_{2,\alpha\beta}\cdot n_2$$

of (6.2c) is similarly bounded.

We notice that with

$$\int_{\Omega} \rho_0^2(J_1)^{-3}[a_1]_p^q v_{1ttt}^p, q [a_1]_r^s w_{tttt}^r, s = \int_{\Omega} \rho_0^2(J_1)^{-3}[a_1]_p^q w_{ttt}^p, q [a_1]_r^s w_{tttt}^r, s + \int_{\Omega} \rho_0^2(J_1)^{-3}[a_1]_p^q v_{2ttt}^p, q [a_1]_r^s w_{tttt}^r, s,$$

we preserve an energy estimate for $\|[a_1]_r^s w_{tttt}^r, s\|_0^2$ in Proposition 4.1.

Following the arguments in Step 3 of section 4 provides control of the divergence and normal trace of the functions of $E^\zeta(t)$. Proposition 2.1 and the initial condition (6.2d) imply that

$$\sup_{t \in [0, T]} E^\zeta(t) \leq TP \left(\sup_{t \in [0, T]} E^\zeta(t) \right)$$

for the higher-order energy function $E^\zeta(t)$ defined in (6.3). Using the polynomial-type inequality (2.20), we infer that $w = 0$ as desired.

7. The asymptotic limit as surface tension tends to zero. In this section, we establish an existence theory for the zero surface tension limit of (1.7) via a priori estimates that are independent of the surface tension parameter $\sigma > 0$. This asymptotic limit holds whenever the initial data satisfies the Taylor sign condition (1.15) and provides the following:

$$0 < \nu \int_{\Gamma} |\bar{\partial}^4 \eta \cdot n|^2 \leq - \int_{\Gamma} \frac{1}{\sqrt{g}} N^j a_\ell^j a_\ell^k (\rho_0^2 J^{-2})_{,k} |\bar{\partial}^4 \eta \cdot n|^2.$$

We recall that Theorem 1.1 provides solutions to (1.7) that satisfy

$$(7.1) \quad \sup_{t \in [0, T]} \sum_{a=0}^5 \|\partial_t^a \eta(t)\|_{5-a}^2 + |v_{ttt} \cdot n(t)|_1^2 + \sum_0^2 |\bar{\partial}^2 \partial_t^a v \cdot n(t)|_{2.5-a}^2 \leq C_\sigma$$

for a finite bound $C_\sigma > 0$ depending on $1/\sigma$. We will show that when the data satisfy the Taylor sign condition, as stated in Theorem 1.2, the estimates and the time-interval of existence are indeed independent of $\sigma > 0$.

7.1. The σ -problem. To indicate the dependence on the surface tension parameter $\sigma > 0$, we place the symbol \checkmark over each dependent variables. For $\sigma > 0$, Theorem 1.1 provides the existence (and uniqueness) to the following σ -problem.

DEFINITION 7.1 (the σ -problem). *For $\sigma > 0$, we define \check{v} as the solution of*

$$(7.2a) \quad \rho_0 \check{v}_t^i + \check{a}_i^k (\rho_0^2 \check{J}^{-2})_{,k} = 0 \quad \text{in } \Omega \times (0, T_\sigma],$$

$$(7.2b) \quad \rho_0^2 \check{J}^{-2} = \beta_\sigma(t) - \sigma \check{g}^{\alpha\beta} \check{\eta}_{,\alpha\beta} \cdot \check{n} \quad \text{on } \Gamma \times (0, T_\sigma],$$

$$(7.2c) \quad (\check{\eta}, \check{v})|_{t=0} = (e, u_0) \quad \text{on } \Omega.$$

The function $\beta_\sigma(t)$ appearing in the right-hand side of (7.2b) is defined as

$$(7.3) \quad \beta_\sigma(t) = \beta + \sum_{a=0}^6 \frac{t^a}{a!} \partial_t^a [\sigma \check{g}^{\alpha\beta} \check{\eta}_{,\alpha\beta} \cdot \check{n}]|_{t=0}.$$

Remark 16. The addition of the polynomial-in-time function in $\beta_\sigma(t)$ ensures that the correct compatibility conditions for the data are satisfied. The same existence theory that we developed for the surface tension problem clearly also holds with the inclusion of the $\beta_\sigma(t)$ function.

7.2. The higher-order energy function of the σ -independent estimates.

For $\sigma > 0$, we define

$$(7.4) \quad \begin{aligned} E^\sigma(t) = & 1 + \sum_{a=0}^7 \|\partial_t^a \check{\eta}(t)\|_{4.5-\frac{1}{2}a}^2 + \sum_{a=0}^5 \|\partial_t^a \check{J}(t)\|_{4.5-\frac{1}{2}a}^2 + \|\partial_t^6 \check{J}(t)\|_1^2 \\ & + \sum_{a=0}^5 \|\sqrt{\sigma} \partial_t^a \check{\eta}(t)\|_{5.5-\frac{1}{2}a}^2 + \|\sqrt{\sigma} \partial_t^5 \check{v}(t)\|_2^2 + \sum_{a=0}^1 |\sqrt{\sigma} \partial_t^{5+a} \check{v} \cdot \check{n}(t)|_{2-a}^2 \\ & + \sum_{a=0}^3 \|\sigma \partial_t^a \check{\eta}(t)\|_{6.5-\frac{1}{2}a}^2 + \|\sigma \check{v}_{ttt}(t)\|_4^2 \\ & + \sum_{a=0}^1 |\sigma \partial_t^{3+a} \check{v} \cdot \check{n}(t)|_{4-\frac{1}{2}a}^2 + |\sigma \partial_t^5 \check{v} \cdot \check{n}(t)|_{2.5}^2. \end{aligned}$$

Notice that when $\sigma = 0$, $E^{\sigma=0}(t) = \mathcal{E}(t)$, where $\mathcal{E}(t)$ is defined in (1.11).

7.3. Smoothing the initial data. Since $E^\sigma(t)$ has greater regularity requirements than $\mathcal{E}(t)$, for the purpose of establishing σ -independent estimates, we will smooth the initial data. More precisely, given data (Ω, ρ_0, u_0) satisfying the conditions (1.13), (1.15), and (1.16) stated in Theorem 1.2, we use the method of Appendix A to smooth this data so that $E^\sigma(0) < \infty$. Later, in section 7.5.2, we will recover the optimal regularity of the initial data stated in Theorem 1.2.

7.4. The a priori estimates for the σ -problem. We will allow constants to depend on $1/\delta > 0$.

DEFINITION 7.2 (notational convention for constants depending on $1/\delta > 0$). We let $\check{\mathcal{P}}$ denote a generic polynomial with constant and coefficients depending on $1/\delta > 0$.

We define the constant $\check{N}_0 > 0$ by

$$(7.5) \quad \check{N}_0 = \check{\mathcal{P}}(E^\sigma(0)).$$

We let $\check{\mathcal{R}}$ denote generic lower-order terms satisfying

$$\int_0^T \check{\mathcal{R}} \leq \check{N}_0 + \delta \sup_{t \in [0, T]} E^\sigma(t) + T \check{\mathcal{P}} \left(\sup_{t \in [0, T]} E^\sigma(t) \right).$$

We infer from the estimates (4.4) and (7.1) that for $T > 0$ taken sufficiently small,

$$(7.6a) \quad \frac{1}{2} \leq \check{J} \leq \frac{3}{2} \quad \forall t \in [0, T] \text{ and } x \in \Omega.$$

Since the initial data satisfy the Taylor sign condition (1.15), we also assume that

$$(7.6b) \quad 0 < \nu \leq -\frac{1}{\sqrt{g}} N^j \check{a}_\ell^j \check{a}_\ell^k (\rho_0^2 \check{J}^{-2})_{,k} \quad \forall t \in [0, T] \text{ and } x \in \Gamma.$$

LEMMA 7.1 (a priori estimates for the σ -problem). *We let \check{v} solve the σ -problem (7.2) on a time-interval $[0, T]$ for some $T = T_\sigma > 0$. Then independent of $1 \gg \sigma > 0$,*

$$(7.7) \quad \sup_{t \in [0, T]} E^\sigma(t) \leq \int_0^T \check{\mathcal{R}}.$$

We will establish Lemma 7.1 in the following nine steps.

Step 1: The σ -independent curl-estimates. We infer the following lemma from the arguments proving Lemma 4.1. (See also the proof of Proposition 3 in [13].)

LEMMA 7.2 (the σ -independent curl-estimates).

$$\begin{aligned} & \sup_{t \in [0, T]} \sum_{a=0}^7 \|\operatorname{curl} \partial_t^a \check{\eta}(t)\|_{3.5-\frac{1}{2}a}^2 + \sup_{t \in [0, T]} \sum_{a=0}^1 \|\sqrt{\sigma} \operatorname{curl} \partial_t^{3+a} \check{v}(t)\|_{2.5-\frac{1}{2}a}^2 \\ & + \sup_{t \in [0, T]} \|\sqrt{\sigma} \operatorname{curl} \partial_t^5 \check{v}(t)\|_1^2 + \sup_{t \in [0, T]} \sum_{a=0}^3 \|\sigma \operatorname{curl} \partial_t^a \check{\eta}(t)\|_{5.5-\frac{1}{2}a}^2 \\ & + \sup_{t \in [0, T]} \|\sigma \operatorname{curl} \partial_t^3 \check{v}(t)\|_3^2 \leq \int_0^T \check{\mathcal{R}}. \end{aligned}$$

Step 2: The σ -independent estimates for $\partial_t^7 \check{J}$, $\partial_t^6 \check{J}$ and $\sqrt{\sigma} \partial_t^6 \check{v} \cdot \check{n}$. We recall that

$$\check{f} = \rho_0 \check{J}^{-1}$$

is the Lagrangian density. Using the identity $\check{J}^{-1} \check{J}_t = \operatorname{div}_{\check{\eta}} \check{v}$, we have that

$$\begin{aligned} \partial_t \check{f}^2 &= -2 \check{f}^2 \operatorname{div}_{\check{\eta}} \check{v}, \\ \partial_t^2 \check{f}^2 &= -2 \check{f}^2 \operatorname{div}_{\check{\eta}} \check{v}_t - 2(\check{f}^2 \check{A}_r^s)_t \check{v}^r_{,s}. \end{aligned}$$

Letting the operator $-2[\check{f} \check{A}_i^j \partial_j] \check{J}^{-1}$ act in the Euler equations (7.2a) yields

$$-2 \check{f}^2 \operatorname{div}_{\check{\eta}} \check{v}_t - 2 \check{f} \check{A}_i^j [\check{A}_i^k (\rho_0^2 \check{J}^{-2})_{,k}]_{,j} = \check{v}_t^i \check{A}_i^j \check{f}^2_{,j}.$$

Using the Euler equations (7.2a) to write $\check{v}_t^i = -\rho_0^{-1} \check{a}_i^k \check{f}^2_{,k}$, we infer that \check{f}^2 satisfies

$$(7.8) \quad \partial_t^2 \check{f}^2 - 2 \check{f} \check{A}_i^j [\check{A}_i^k \check{f}^2_{,k}]_{,j} = \underbrace{-\rho_0^{-1} \check{a}_i^k \check{f}^2_{,k} \check{A}_i^j \check{f}^2_{,j} - 2(\check{f}^2 \check{A}_i^j)_t \check{v}^i}_{\check{F}}_{,j}.$$

Since \check{F} scales like $D\check{J} + D\check{v}$, it follows that $\partial_t^6 \check{F}$ is in $L^2(\Omega)$.

Similar to the tangential identity (4.14),

$$(7.9) \quad \rho_0 \check{J}^{-1} \check{v}_t \cdot \check{\eta}_{,\gamma} = \bar{\partial}_\gamma \left[\sigma \check{g}^{\mu\nu} \check{\eta}_{,\mu\nu} \cdot \check{n} - \beta_\sigma(t) \right] \quad \text{on } \Gamma,$$

thanks to the Euler equations (7.2a) and the Laplace–Young boundary condition (7.2b).

PROPOSITION 7.1 (energy estimates for the action of ∂_t^6 in the wave-type equation (7.8)).

$$\sup_{t \in [0, T]} \|\partial_t^7 \check{J}(t)\|_0^2 + \sup_{t \in [0, T]} \|\partial_t^6 \check{J}(t)\|_1^2 + \sup_{t \in [0, T]} |\sqrt{\sigma} \partial_t^6 \check{v} \cdot \check{n}(t)|_1^2 \leq \int_0^T \check{\mathcal{R}}.$$

Proof. Testing six time-derivatives of (7.8) against $\rho_0^{-2} \check{J}^3 \partial_t^7 \check{f}^2$ in the $L^2(\Omega)$ -inner product and integrating by parts with respect to ∂_j in the integral $-2 \int_{\Omega} \rho_0^{-1} \check{a}_i^j \partial_t^6 [\check{a}_i^k \check{f}^2, k]_{,j} \partial_t^7 \check{f}^2$ yields

$$\begin{aligned} (7.10) \quad & \frac{1}{2} \frac{d}{dt} \int_{\Omega} \rho_0^{-2} \check{J}^3 |\partial_t^7 \check{f}^2|^2 + 2 \underbrace{\int_{\Omega} \rho_0^{-1} \partial_t^6 [\check{a}_i^k \check{f}^2, k] \check{a}_i^j \partial_t^7 \check{f}^2}_{\mathcal{I}} - 2 \underbrace{\int_{\Gamma} \rho_0^{-1} \partial_t^6 [\check{a}_i^k \check{f}^2, k] \check{a}_i^j N^j \partial_t^7 \check{f}^2}_{\mathbf{i}} \\ & = \underbrace{\frac{1}{2} \int_{\Omega} (\rho_0^{-2} \check{J}^3)_t |\partial_t^7 \check{f}^2|^2}_{\check{\mathcal{R}}} + 2 \underbrace{\sum_{l=1}^6 c_l \int_{\Omega} \check{f} \check{A}_i^j (\partial_t^l \check{J}^{-1} \partial_t^{6-l} [\check{a}_i^k \check{f}^2, k])_{,j} \rho_0^{-2} \check{J}^3 \partial_t^7 \check{f}^2}_{\check{\mathcal{R}}} \\ & \quad + 2 \underbrace{\sum_{l=1}^6 c_l \int_{\Omega} \partial_t^l (\check{f} \check{A}_i^j) \partial_t^{6-l} (\check{A}_i^k \check{f}^2, k)_{,j} \rho_0^{-2} \check{J}^3 \partial_t^7 \check{f}^2}_{\check{\mathcal{R}}} + \underbrace{\int_{\Omega} \partial_t^6 \check{F} \rho_0^{-2} \check{J}^3 \partial_t^7 \check{f}^2}_{\check{\mathcal{R}}} \\ & \quad + 2 \underbrace{\int_{\Omega} \rho_0^{-1} \check{a}_i^j (\check{J}^{-1})_{,j} \partial_t^6 [\check{a}_i^k \check{f}^2, k] \check{J} \partial_t^7 \check{f}^2}_{\check{\mathcal{R}}}. \end{aligned}$$

We have used the Cauchy–Schwarz inequality to analyze all of the terms in the right-hand side of (7.10) except for the highest-order terms of $\int_{\Omega} \partial_t^4 (\check{f} \check{A}_i^j) \partial_t^2 (\check{A}_i^k \check{f}^2, k)_{,j} \rho_0^{-2} \check{J}^3 \partial_t^7 \check{f}^2$, where we have used an L^4 – L^4 – L^2 Hölder inequality.

Writing $\partial_t \check{f}^2 = -2\rho_0^2 \check{J}^{-3} \check{J}_t$, it follows that

$$(7.11) \quad \partial_t^7 \check{f}^2 = -2\rho_0^2 \check{J}^{-3} \partial_t^7 \check{J} - 2 \sum_{l=1}^6 c_l \partial_t^l (\rho_0^2 \check{J}^{-3}) \partial_t^{7-l} \check{J}.$$

The identity (7.11) provides that (7.10) multiplied by $\frac{1}{2}$ is equivalent to

$$(7.12) \quad \frac{d}{dt} \int_{\Omega} \rho_0^2 \check{J}^{-3} |\partial_t^7 \check{J}|^2 + \mathcal{I} - \mathbf{i} = \check{\mathcal{R}}.$$

Analysis of \mathcal{I} in (7.12). We equivalently write

$$\mathcal{I} = \underbrace{\int_{\Omega} \rho_0^{-1} \check{a}_i^k \partial_t^6 \check{f}^2}_{\mathcal{I}_a} + \sum_{l=0}^5 c_l \underbrace{\int_{\Omega} \rho_0^{-1} \partial_t^{6-l} \check{a}_i^k \partial_t^l \check{f}^2, k \check{a}_i^j \partial_t^7 \check{f}^2, j}_{\mathcal{I}_{b,l}}.$$

We have that

$$\mathcal{I}_a = \frac{1}{2} \frac{d}{dt} \int_{\Omega} \rho_0^{-1} |\check{a}_i^k \partial_t^6 \check{f}^2, k|^2 - \int_{\Omega} \rho_0^{-1} \check{a}_i^k \partial_t^6 \check{f}^2, k \partial_t \check{a}_i^j \partial_t^6 \check{f}^2, j.$$

Similar to (7.11), we have that $\partial_t^6 \check{f}^2$ is equal to $-2\check{f}^2 \check{J}^{-1} \partial_t^6 \check{J}$ plus lower-order terms. Thus,

$$\mathcal{I}_a = 2 \frac{d}{dt} \int_{\Omega} \rho_0^{-1} \check{f}^4 |\check{A}^k \partial_t^6 \check{J}_{,k}|^2 + \check{\mathcal{R}}.$$

For $\int_0^T \mathcal{I}_{b,l}$, we integrate by parts with respect to a time-derivative of $\partial_t^l \check{f}^2_{,j}$. For example,

$$\int_0^T \mathcal{I}_{b,0} = \int_{\Omega} \rho_0^{-1} \partial_t^6 \check{a}_i^k \check{f}^2_{,k} \check{a}_i^j \partial_t^6 \check{f}^2_{,j} \Big|_0^T - \int_0^T \int_{\Omega} \rho_0^{-1} \partial_t [\partial_t^6 \check{a}_i^k \check{f}^2_{,k} \check{a}_i^j] \partial_t^6 \check{f}^2_{,j} = \int_0^T \check{\mathcal{R}}.$$

Since $\partial_t^6 \check{a}$ scales like $D \partial_t^5 \check{v}$, the fundamental theorem of calculus ensures a good estimate for the term evaluated at time $t = T$. The terms $\int_0^T \mathcal{I}_{b,l}$, $l = 1, \dots, 5$, are similarly analyzed:

$$\sum_{l=0}^5 \mathcal{I}_{b,l} = \check{\mathcal{R}}.$$

This establishes that

$$(7.13) \quad \mathcal{I} = 2 \frac{d}{dt} \int_{\Omega} \rho_0^{-1} \check{f}^4 |\check{A}^k \partial_t^6 \check{J}_{,k}|^2 + \check{\mathcal{R}}.$$

Rewriting the boundary integral in (7.12). We use the trace of the action of ∂_t^6 in the Euler equations (7.2a) to write $\mathfrak{i} = \int_{\Gamma} \partial_t^7 \check{f}^2 \partial_t^7 \check{v}^i \check{a}_i^j N^j$, or, equivalently,

$$\mathfrak{i} = \int_{\Gamma} \partial_t^7 \check{f}^2 \sqrt{\check{g}} \partial_t^7 \check{v} \cdot \check{n}.$$

Using the Laplace–Young boundary condition (7.2b), we find that

$$(7.14) \quad \begin{aligned} \mathfrak{i} &= \frac{1}{2} \frac{d}{dt} \int_{\Gamma} \sqrt{\check{g}} \check{g}^{\alpha\beta} \sqrt{\sigma} \partial_t^6 \check{v}_{,\alpha} \cdot \check{n} \sqrt{\sigma} \partial_t^6 \check{v}_{,\beta} \cdot \check{n} \\ &\quad - \underbrace{\sigma \int_{\Gamma} \sqrt{\check{g}} \check{g}^{\alpha\beta} \partial_t^6 \check{v}_{,\alpha} \cdot \check{n}_t \partial_t^6 \check{v}_{,\beta} \cdot \check{n}}_{\mathfrak{j}_1} - \underbrace{\frac{1}{2} \int_{\Gamma} (\sqrt{\check{g}} \check{g}^{\alpha\beta})_t \sqrt{\sigma} \partial_t^6 \check{v}_{,\alpha} \cdot \check{n} \sqrt{\sigma} \partial_t^6 \check{v}_{,\beta} \cdot \check{n}}_{\check{\mathcal{R}}} \\ &\quad + \underbrace{\sigma \int_{\Gamma} \partial_t^6 \check{v}_{,\beta} \cdot \check{n} \sqrt{\check{g}} \check{g}^{\alpha\beta} \partial_t^7 \check{v}_{,\alpha} + \sigma \int_{\Gamma} \partial_t^6 \check{v}_{,\beta} \cdot \check{n}_{,\alpha} \sqrt{\check{g}} \check{g}^{\alpha\beta} \partial_t^7 \check{v} \cdot \check{n}}_{\mathfrak{j}_2} \\ &\quad + \underbrace{\sigma \int_{\Gamma} \partial_t^6 \check{v}_{,\beta} \cdot \check{n} (\sqrt{\check{g}} \check{g}^{\alpha\beta})_{,\alpha} \partial_t^7 \check{v} \cdot \check{n} - \sigma \sum_{l=0}^6 c_l \int_{\Gamma} \check{g}^{\alpha\beta} \partial_t^l \check{\eta}^j_{,\alpha\beta} \partial_t^{7-l} \check{n}^j \sqrt{\check{g}} \partial_t^7 \check{v} \cdot \check{n}}_{\mathfrak{j}_3} \\ &\quad - \underbrace{\sigma \sum_{l=1}^7 c_l \int_{\Gamma} \partial_t^l \check{g}^{\alpha\beta} \partial_t^{7-l} [\check{\eta}_{,\alpha\beta} \cdot \check{n}]}_{\mathfrak{j}_4} \sqrt{\check{g}} \partial_t^7 \check{v} \cdot \check{n} + \underbrace{\int_{\Gamma} \partial_t^7 \beta_{\sigma}(t) \sqrt{\check{g}} \partial_t^7 \check{v} \cdot \check{n}}_0. \end{aligned}$$

Analysis of $\int_0^T \mathbf{j}_1$ in the time-integral of (7.14). The action of $\bar{\partial}_\alpha \partial_t^5$ in the tangential identity (7.9) provides that

$$(7.15a) \quad \partial_t^6 \check{v}_{,\alpha} \cdot \check{\eta}_{,\gamma} = \rho_0^{-1} \check{J} [\bar{\partial}_{\alpha\gamma} \partial_t^5 (\sigma \check{g}^{\mu\nu} \check{\eta}_{,\mu\nu} \cdot \check{n} - \partial_t^5 \beta_\sigma(t)) - \check{l}_{\gamma\alpha}],$$

where $\check{l}_{\gamma\alpha}$ is such that $\check{l}_{\gamma\alpha} \in H^{-0.5}(\Gamma)$, $\sqrt{\sigma} \check{l}_{\gamma\alpha} \in H^{0.5}(\Gamma)$ and is given by

$$(7.15b) \quad \check{l}_{\gamma\alpha} = \partial_t^6 \check{v} \cdot (\check{\eta}_{,\gamma} \rho_0 \check{J}^{-1})_{,\alpha} + \sum_{l=0}^4 c_l [\partial_t^a \check{v}_t \cdot \partial_t^{5-l} (\check{\eta}_{,\gamma} \rho_0 \check{J}^{-1})]_{,\alpha}.$$

Setting $\check{\ell}_\gamma^{\alpha\beta} = \rho_0^{-1} \check{J} \sqrt{\check{g}} \check{g}^{\alpha\beta} \check{g}^{\gamma\delta} \check{v}_{,\delta} \cdot \check{n}$ we use the tangential identity (7.15), together with the outward normal differentiation formula (2.11c), to find that

$$(7.16) \quad \mathbf{j}_1 = \underbrace{\sigma \int_\Gamma \check{\ell}_\gamma^{\alpha\beta} \bar{\partial}_{\gamma\alpha} \partial_t^5 (\check{g}^{\mu\nu} \check{\eta}_{,\mu\nu} \cdot \check{n}) \partial_t^6 \check{v}_{,\beta} \cdot \check{n}}_{\mathbf{j}'_1} + \underbrace{\int_\Gamma \check{\ell}_\gamma^{\alpha\beta} \sqrt{\sigma} \check{l}_{\gamma\alpha} \sqrt{\sigma} \partial_t^6 \check{v}_{,\beta} \cdot \check{n} + \check{\mathcal{R}}}_{\check{\mathcal{R}}}.$$

We integrate by parts with respect to a time-derivative of $\partial_t^6 \check{v}_{,\beta} \cdot \check{n}$ to write

$$\begin{aligned} \int_0^T \mathbf{j}'_1 &= - \underbrace{\int_\Gamma \bar{\partial}_\gamma \partial_t^5 [\check{g}^{\mu\nu} \check{\eta}_{,\mu\nu} \cdot \check{n}] \sigma^2 \bar{\partial}_\alpha [\check{\ell}_\gamma^{\alpha\beta} \partial_t^5 \check{v}_{,\beta} \cdot \check{n}] \Big|_0^T}_{\mathbf{j}'_{1A}} \\ &+ \underbrace{\int_0^T \sigma^2 \int_\Gamma \bar{\partial}_\gamma \partial_t^5 [\check{g}^{\mu\nu} \check{\eta}_{,\mu\nu} \cdot \check{n}] \bar{\partial}_\alpha [\check{\ell}_\gamma^{\alpha\beta} \partial_t^5 \check{v}_{,\beta} \cdot \check{n}_t]}_{\mathbf{j}'_{1B}} \\ &+ \underbrace{\int_0^T \sigma^2 \int_\Gamma \bar{\partial}_\gamma \partial_t^5 [\check{g}^{\mu\nu} \check{\eta}_{,\mu\nu} \cdot \check{n}] \bar{\partial}_\alpha [\partial_t \check{\ell}_\gamma^{\alpha\beta} \partial_t^5 \check{v}_{,\beta} \cdot \check{n}]}_{\mathbf{j}'_{1C}} \\ &+ \underbrace{\int_0^T \sigma^2 \int_\Gamma \bar{\partial}_\gamma \partial_t^6 [\check{g}^{\mu\nu} \check{\eta}_{,\mu\nu} \cdot \check{n}] \bar{\partial}_\alpha [\check{\ell}_\gamma^{\alpha\beta} \partial_t^5 \check{v}_{,\beta} \cdot \check{n}]}_{\mathbf{j}'_{1D}}. \end{aligned}$$

Since $\sigma \partial_t^5 \check{v} \cdot \check{n}$ is in $H^{2.5}(\Gamma)$, we may take σ sufficiently small so that

$$|\sigma \partial_t^5 \check{v} \cdot \check{n}(T)|_2^2 \leq \sqrt{\sigma} C |\sqrt{\sigma} \partial_t^5 \check{v}(T)|_{1.5} |\sigma \partial_t^5 \check{v} \cdot \check{n}(T)|_{2.5} \leq \int_0^T \check{\mathcal{R}}.$$

Thus, the Cauchy–Schwarz inequality provides that

$$(7.17) \quad \mathbf{j}'_{1A} \Big|_0^T = - \int_\Gamma \check{g}^{\mu\nu} \bar{\partial}_\gamma [\partial_t^4 \check{v}_{,\mu\nu} \cdot \check{n}] \sigma^2 \check{\ell}_\gamma^{\alpha\beta} \bar{\partial}_\alpha [\partial_t^5 \check{v}_{,\beta} \cdot \check{n}] \Big|_0^T + \int_0^T \check{\mathcal{R}} = \int_0^T \check{\mathcal{R}}.$$

Since $\sigma \partial_t^5 \check{v} \cdot \check{n}$ is in $H^{3.5}(\Gamma)$, we find by use of an $H^{-0.5}(\Gamma)$ -duality pairing that

$$(7.18) \quad \mathbf{j}'_{1B} = \underbrace{\sqrt{\sigma} \int_\Gamma \bar{\partial}_\gamma [\check{g}^{\mu\nu} \sigma \partial_t^4 \check{v}_{,\mu\nu} \cdot \check{n}] \check{\ell}_\gamma^{\alpha\beta} \sqrt{\sigma} \partial_t^5 \check{v}_{,\alpha\beta} \cdot \check{n}_t}_{\check{\mathcal{R}}} + \int_0^T \check{\mathcal{R}}.$$

We employ the Cauchy–Schwarz inequality to conclude that

$$(7.19) \quad \int_0^T j_{1C}' \leq \int_0^T \ddot{\mathcal{R}}.$$

We employ the Cauchy–Schwarz inequality or an $H^{-0.5}(\Gamma)$ -duality pairing to conclude that

$$(7.20) \quad \int_0^T j_{1D}' \leq \int_0^T \ddot{\mathcal{R}}.$$

The inequalities (7.17), (7.18), (7.19), (7.20) establish that

$$(7.21) \quad \int_0^T j_1 \leq \int_0^T \ddot{\mathcal{R}}.$$

Analysis of $\int_0^T j_2$ in the time-integral of (7.14). We equivalently write j_2 as

$$(7.22) \quad j_2 = \underbrace{\sigma \int_{\Gamma} \partial_t^6 \check{v}_{,\beta} \cdot \check{n} \sqrt{\check{g}} \check{g}^{\alpha\beta} \partial_t^7 \check{v} \cdot \check{n}_{,\alpha}}_{j_{2a}} + \underbrace{\sigma \int_{\Gamma} \partial_t^6 \check{v}_{,\beta} \cdot \check{n}_{,\alpha} \sqrt{\check{g}} \check{g}^{\alpha\beta} \partial_t^7 \check{v} \cdot \check{n}}_{j_{2b}}.$$

The action of ∂_t^6 in the tangential identity (7.9) yields

$$(7.23a) \quad \partial_t^7 \check{v} \cdot \check{\eta}_{,\gamma} = \rho_0^{-1} \check{J} [\bar{\partial}_{\gamma} \partial_t^6 (\sigma \check{g}^{\mu\nu} \check{\eta}_{,\mu\nu} \cdot \check{n} - \partial_t^6 \beta_{\sigma}(t)) - \check{l}_{\gamma}],$$

where \check{l}_{γ} is such that $\sqrt{\sigma} \check{l}_{\gamma}$ is in $H^{0.5}(\Gamma)$ and is given by

$$(7.23b) \quad \check{l}_{\gamma} = \sum_{l=0}^5 c_l \partial_t^l \check{v}_t \cdot \partial_t^{6-l} (\check{\eta}_{,\gamma} \rho_0 \check{J}^{-1}).$$

Letting $\check{\ell}_{\gamma}^{\beta} = -\rho_0^{-1} \check{J} \sqrt{\check{g}} \check{g}^{\alpha\beta} \check{g}^{\gamma\delta} \check{\eta}_{,\delta\alpha} \cdot \check{n}$ and using the tangential identity (7.23), we have that

$$j_{2a} = \underbrace{-\sigma^2 \int_{\Gamma} [\partial_t^6 \check{v}_{,\beta} \cdot \check{n} \check{\ell}_{\gamma}^{\beta}]_{,\gamma} \partial_t^6 [\check{g}^{\mu\nu} \check{\eta}_{,\mu\nu} \cdot \check{n}]}_{j_{2a'}} + \ddot{\mathcal{R}}.$$

We integrate by parts with respect to a time-derivative of $\partial_t^6 \check{v}_{,\beta}$ in order to write

$$\begin{aligned} & - \int_0^T j_{2a}' \\ &= -\sigma^2 \int_{\Gamma} [\check{\ell}_{\gamma}^{\beta} \partial_t^5 \check{v}_{,\beta} \cdot \check{n}]_{,\gamma} \partial_t^6 [\check{g}^{\mu\nu} \check{\eta}_{,\mu\nu} \cdot \check{n}] \Big|_0^T + \int_0^T \sigma^2 \int_{\Gamma} [\check{\ell}_{\gamma}^{\beta} \partial_t^5 \check{v}_{,\beta} \cdot \check{n}]_{,\gamma} \partial_t^6 [\check{g}^{\mu\nu} \check{\eta}_{,\mu\nu} \cdot \check{n}] \\ & \quad + \int_0^T \sigma^2 \int_{\Gamma} \partial_t [\check{\ell}_{\gamma}^{\beta} \partial_t^5 \check{v}_{,\beta} \cdot \check{n}]_{,\gamma} \partial_t^6 [\check{g}^{\mu\nu} \check{\eta}_{,\mu\nu} \cdot \check{n}] + \int_0^T \sigma^2 \int_{\Gamma} [\check{\ell}_{\gamma}^{\beta} \partial_t^5 \check{v}_{,\beta} \cdot \check{n}]_{,\gamma} \partial_t^7 [\check{g}^{\mu\nu} \check{\eta}_{,\mu\nu} \cdot \check{n}] \\ &= \int_0^T \underbrace{\sigma^2 \int_{\Gamma} [\check{\ell}_{\gamma}^{\beta} \partial_t^5 \check{v}_{,\beta} \cdot \check{n}]_{,\gamma} \check{g}^{\mu\nu} \partial_t^6 \check{v}_{,\mu\nu} \cdot \check{n}}_{j_{2a}''} + \int_0^T \ddot{\mathcal{R}}. \end{aligned}$$

We write

$$\int_0^T j_{2a}'' = \int_0^T \underbrace{\sigma^2 \int_{\Gamma} \check{\ell}^{\beta} \partial_t^5 \check{v}_{,\beta\gamma} \cdot \check{n} \check{g}^{\mu\nu} \partial_t^6 \check{v}_{,\mu\nu} \cdot \check{n}}_{j_{2a}'''} + \int_0^T \underbrace{\sigma^2 \int_{\Gamma} \partial_t^5 \check{v}_{,\beta} \cdot [\check{n} \check{\ell}^{\beta}]_{,\gamma} \check{g}^{\mu\nu} \partial_t^6 \check{v}_{,\mu\nu} \cdot \check{n}}_j.$$

Using integration by parts with respect to a time-derivative of $\partial_t^6 \check{v}_{,\mu\nu}$, we conclude that $\int_0^T j = \int_0^T \check{\mathcal{R}}$. Letting $\check{\ell} = \rho_0^{-1} \check{J} \sqrt{\check{g}} \check{g}^{\gamma\delta} \check{\eta}_{,\gamma\delta} \cdot \check{n}$, we utilize the symmetry of $\check{\ell}^{\beta}_{\gamma}$ to exchange $\bar{\partial}_{\alpha}$ and $\bar{\partial}_{\gamma}$ via integration by parts for

$$(7.24) \quad \begin{aligned} \int_0^T j_{2a}''' &= \frac{1}{2} \int_{\Gamma} \check{\ell} |\check{g}^{\mu\nu} \sigma \partial_t^5 \check{v}_{,\mu\nu} \cdot \check{n}|^2 \Big|_0^T - \int_0^T \sigma^2 \int_{\Gamma} \check{\ell} \partial_t^5 \check{v}_{,\alpha\beta} \cdot \check{n} \partial_t^5 \check{v}_{,\mu\nu} \cdot (n \check{g}^{\mu\nu})_t \\ &\quad - \frac{1}{2} \int_0^T \int_{\Gamma} \check{\ell}_t |\check{g}^{\mu\nu} \sigma \partial_t^5 \check{v}_{,\mu\nu} \cdot \check{n}|^2 = \int_0^T \check{\mathcal{R}}. \end{aligned}$$

We have thus established that

$$\int_0^T j_{2a} = \int_0^T \check{\mathcal{R}}.$$

The analysis of $\int_0^T j_{2b}$ is similar. We set $\check{\ell}^{\gamma\beta} = \sqrt{\check{g}} \check{g}^{\alpha\beta} \check{g}^{\gamma\delta} \check{\eta}_{,\delta\alpha} \cdot \check{n}$ and write

$$\begin{aligned} \int_0^T j_{2b} &= -\sigma^{\frac{3}{2}} \int_{\Gamma} \partial_t^6 \check{v} \cdot [\check{n}_{,\alpha} \sqrt{\check{g}} \check{g}^{\alpha\beta} \sqrt{\sigma} \partial_t^6 \check{v} \cdot \check{n}]_{,\beta} \Big|_0^T \\ &\quad - \int_0^T \sigma^2 \int_{\Gamma} \partial_t^6 \check{v}^j_{,\beta} [\check{n}^j_{,\alpha} \sqrt{\check{g}} \check{g}^{\alpha\beta} \check{n}^i]_t \partial_t^6 \check{v}^i \\ &\quad + \int_0^T \underbrace{\sigma^2 \int_{\Gamma} \check{\ell}^{\gamma\beta} \partial_t^7 \check{v}_{,\beta} \cdot \check{\eta}_{,\gamma} \partial_t^6 \check{v} \cdot \check{n}}_{j_{2b}'}} = \int_0^T \check{\mathcal{R}} + \int_0^T j_{2b}'. \end{aligned}$$

Regarding $\int_0^T j_{2b}'$, we use the identity

$$(7.25) \quad \partial_t^7 \check{v}_{,\beta} \cdot \check{\eta}_{,\gamma} = \rho_0^{-1} \check{J} [\bar{\partial}_{\beta\gamma} \partial_t^6 (\check{g}^{\mu\nu} \check{\eta}_{,\mu\nu} \cdot \check{n} - \beta_{\sigma}(t)) - \check{l}'_{\beta\gamma}],$$

where $\check{l}'_{\beta\gamma} = \partial_t^7 \check{v} \cdot \check{\eta}_{,\gamma\beta} \rho_0 \check{J}^{-1} + \partial_t^7 \check{v} \cdot \check{\eta}_{,\gamma} (\rho_0 \check{J}^{-1})_{,\beta} + \bar{\partial}_{\beta} \check{l}_{\gamma}$ with \check{l}_{γ} given by (7.23b). We integrate by parts with respect to $\bar{\partial}_{\beta\gamma}$ in $\sigma^3 \int_0^T \int_{\Gamma} \rho_0^{-1} \check{J} \bar{\partial}_{\beta\gamma} \partial_t^6 [\check{g}^{\mu\nu} \check{\eta}_{,\mu\nu} \cdot \check{n}] \check{\ell}^{\gamma\beta} \partial_t^6 \check{v} \cdot \check{n}$, where the highest-order term produced by $\bar{\partial}_{\beta\gamma}$ -integration by parts is (7.24). To estimate the integral where $\bar{\partial}_{\beta} \check{l}_{\gamma}$ appears, we have the choice of integration by parts with respect to $\bar{\partial}_{\beta}$ or an $H^{-0.5}(\Gamma)$ -duality pairing. In the integral where $\partial_t^7 \check{v} \cdot \check{\eta}_{,\gamma} (\rho_0 \check{J}^{-1})_{,\beta}$ appears, we use the identity (7.23). Thus, we conclude that $\int_0^T j_{2b}' = \int_0^T \check{\mathcal{R}}$ and

$$(7.26) \quad \int_0^T j_2 \leq \int_0^T \check{\mathcal{R}}.$$

Analysis of $\int_0^T j_3$ in the time-integral of (7.14). We equivalently write j_3 as

$$j_3 = \underbrace{\sigma \int_{\Gamma} \partial_t^6 \check{v}_{,\beta} \cdot \check{n} (\sqrt{\check{g}} \check{g}^{\alpha\beta})_{,\alpha} \partial_t^7 \check{v} \cdot \check{n}}_{j_{3a}} - \sum_{l=0}^6 c_l \underbrace{\sigma \int_{\Gamma} \sqrt{\check{g}} \check{g}^{\alpha\beta} \partial_t^l \check{\eta}^j_{,\alpha\beta} \partial_t^{7-l} \check{n}^j \partial_t^7 \check{v} \cdot \check{n}}_{j_{3b,l}}.$$

We infer from our analysis of (4.25) that

$$-\int_0^T j_{3b,0} = \int_0^T \underbrace{\sigma \int_{\Gamma} \sqrt{\check{g}} \check{g}^{\alpha\beta} \check{g}^{\gamma\delta} \check{\eta}_{,\alpha\beta} \cdot \check{\eta}_{,\gamma} \partial_t^6 \check{v}_{,\delta} \cdot \check{n} \partial_t^7 \check{v} \cdot \check{n}}_{-j_{3a}} + \int_0^T \check{\mathcal{R}}.$$

Hence,

$$j_3 = -\sum_{l=1}^6 c_l j_{3b,l} + \check{\mathcal{R}}.$$

The terms $\int_0^T j_{3b,l}$ for $l = 1, \dots, 5$ are analyzed by integrating by parts with respect to a time-derivative of $\partial_t^7 \check{v}$ and then using elementary estimates. Thus,

$$j_3 = -j_{3b,6} + \check{\mathcal{R}}.$$

Integration by parts with respect to a time-derivative of $\partial_t^7 \check{v}$ yields

$$\begin{aligned} \int_0^T j_{3b,6} &= -\int_0^T \sigma \int_{\Gamma} \sqrt{\check{g}} \check{g}^{\alpha\beta} \partial_t^6 \check{v}_{,\alpha\beta} \cdot \check{n}_t \partial_t^6 \check{v} \cdot \check{n} + \int_0^T \check{\mathcal{R}} \\ &= \int_0^T \underbrace{\sigma \int_{\Gamma} \sqrt{\check{g}} \check{g}^{\alpha\beta} \partial_t^6 \check{v}_{,\beta} \cdot \check{n}_t \partial_t^6 \check{v}_{,\alpha} \cdot \check{n}}_{j_{3b,6'}} + \int_0^T \check{\mathcal{R}}. \end{aligned}$$

Letting $\check{\ell}_{\alpha\beta}^{\gamma} = \sqrt{\check{g}} \check{g}^{\alpha\beta} \check{g}^{\gamma\delta} \check{v}_{,\delta} \cdot \check{n}$, we once again integrate by parts with respect to time:

$$\begin{aligned} \int_0^T j_{3b,6'} &= -\int_{\Gamma} \partial_t^6 \check{v}_{,\beta} \cdot \left[\check{\eta}_{,\gamma} \check{\ell}_{\alpha\beta}^{\gamma} \sigma \partial_t^5 \check{v}_{,\alpha} \cdot \check{n} \right]_{,\beta} \Big|_0^T \\ &\quad - \int_0^T \sigma \int_{\Gamma} \partial_t^6 \check{v}_{,\beta} \cdot \partial_t^5 \check{v}_{,\alpha} \cdot (\check{n}^i \check{\eta}_{,\beta}^j \check{\ell}_{\alpha\beta}^{\gamma})_t - \underbrace{\int_0^T \sigma \int_{\Gamma} \check{\ell}_{\alpha\beta}^{\gamma} \partial_t^7 \check{v}_{,\beta} \cdot \check{\eta}_{,\gamma} \partial_t^5 \check{v}_{,\alpha} \cdot \check{n}}_j \\ &= -\int_0^T j + \int_0^T \check{\mathcal{R}}. \end{aligned}$$

We conclude via the tangential identity (7.25) that $j = \check{\mathcal{R}}$. Hence,

$$(7.27) \quad \int_0^T j_3 \leq \int_0^T \check{\mathcal{R}}.$$

Analysis of $\int_0^T j_4$ in the time-integral of (7.14). We integrate by parts with respect to a time-derivative of $\partial_t^7 \check{v}$ in the $\int_0^T j_4$ -terms and, if need be, spatially integrate by parts. For example, letting $j_4 = \sum_{l=1}^7 j_{4,l}$, we find that after integration by parts with respect to time,

$$\begin{aligned} \int_0^T j_{4,1} &= \int_0^T \sigma \int_{\Gamma} (\sqrt{\check{g}} \check{g}^{\alpha\beta})_t \partial_t^7 (\check{\eta}_{,\alpha\beta} \cdot \check{n}) \partial_t^6 \check{v} \cdot \check{n} + \int_0^T \check{\mathcal{R}} \\ &= -\int_0^T \sigma \int_{\Gamma} \partial_t^6 \check{v}_{,\alpha} \cdot \bar{\partial}_{\beta} [\check{n} (\sqrt{\check{g}} \check{g}^{\alpha\beta})_t \partial_t^6 \check{v} \cdot \check{n}] + \int_0^T \check{\mathcal{R}} = \int_0^T \check{\mathcal{R}}. \end{aligned}$$

Similarly, integration by parts with respect to time provides for the expression

$$\sum_{l=2}^6 \int_0^T j_{4,l} = \int_0^T \ddot{\mathcal{R}}.$$

Finally, using the differentiation formulas (2.10a) and (2.10b),

$$\int_0^T j_{4,7} = \int_0^T \int_{\Gamma} \sqrt{g}(2\check{g}^{\alpha\mu}\check{g}^{\nu\beta} - \check{g}^{\alpha\beta}\check{g}^{\mu\nu}) \partial_t^6 \check{v}_{,\mu} \cdot \check{\eta}_{,\nu} \check{\eta}_{,\alpha\beta} \cdot \check{n} \partial_t^7 \check{v} \cdot \check{n} + \int_0^T \ddot{\mathcal{R}} = \int_0^T \ddot{\mathcal{R}},$$

where the second equality follows from our above analysis of $\int_0^T j_{2b}$.

Hence,

$$(7.28) \quad \int_0^T j_4 \leq \int_0^T \ddot{\mathcal{R}}.$$

Rewriting (7.12). The inequalities (7.21), (7.26), (7.27), (7.28) provide that the boundary integral i expressed as (7.14) satisfies

$$(7.29) \quad i = \frac{1}{2} \frac{d}{dt} \int_{\Gamma} \sqrt{g} \check{g}^{\alpha\beta} \sqrt{\sigma} \partial_t^6 \check{v}_{,\alpha} \cdot \check{n} \sqrt{\sigma} \partial_t^6 \check{v}_{,\beta} \cdot \check{n} + \ddot{\mathcal{R}}.$$

Using the identities (7.13) and (7.29), we have that (7.12) is equivalently written as

$$(7.30) \quad \frac{d}{dt} \int_{\Omega} \rho_0^2 \check{J}^{-3} |\partial_t^7 \check{J}|^2 + 2 \frac{d}{dt} \int_{\Omega} \rho_0^{-1} \check{f}^4 |A^k \partial_t^6 \check{J}_{,k}|^2 + \frac{1}{2} \frac{d}{dt} \int_{\Gamma} \sqrt{g} \check{g}^{\alpha\beta} \sqrt{\sigma} \partial_t^6 \check{v}_{,\alpha} \cdot \check{n} \sqrt{\sigma} \partial_t^6 \check{v}_{,\beta} \cdot \check{n} = \ddot{\mathcal{R}}.$$

The time-integral of (7.30) completes the proof. \square

Step 3: The energy estimates for the action of $\bar{\partial}^a \partial_t^{8-2a}$, $a = 1, 2, 3, 4$.

We define the vector \check{a}_{α}^k as $\check{\eta}_{,\alpha} \cdot \check{a}^k$ and let the vector $\check{a}_{\check{n}}^k$ be defined as $\check{n} \cdot \check{a}^k$. Using these vector identities, we decompose the cofactor matrix \check{a} as

$$(7.31) \quad \check{a}_r^k = \check{a}_{\alpha}^k \check{g}^{\alpha\beta} \check{\eta}^r_{,\beta} + \check{a}_{\check{n}}^k \check{n}^r \quad \text{on } \Gamma.$$

Since $\check{a}^k N^k = \sqrt{g} \check{n}$ by the formula (2.9), it follows that

$$(7.32) \quad \check{a}_{\check{n}}^k = \frac{1}{\sqrt{g}} N^j \check{a}_{\ell}^j \check{a}_{\ell}^k.$$

According to the identity (7.32), the lower-bound (7.6b) is equivalently stated as

$$(7.33) \quad 0 < \frac{\nu}{2} \leq -\check{a}_{\check{n}}^k (\rho_0^2 \check{J}^{-2})_{,k}.$$

PROPOSITION 7.2 (energy estimates for the action of $\bar{\partial} \partial_t^6$ in the Euler equations (7.2a)).

$$\sup_{t \in [0, T]} \|\bar{\partial} \partial_t^6 \check{v}(t)\|_0^2 + \sup_{t \in [0, T]} |\sqrt{\sigma} \bar{\partial} \partial_t^5 \check{v} \cdot \check{n}(t)|_1^2 + \sup_{t \in [0, T]} |\bar{\partial} \partial_t^5 \check{v} \cdot \check{n}(t)|_0^2 \leq \int_0^T \ddot{\mathcal{R}}.$$

Proof. Testing the action of $\bar{\partial}\partial_t^6$ in the Euler equations (7.2a) against $\bar{\partial}\partial_t^6\check{v}$ in the $L^2(\Omega)$ -inner product and integrating by parts in the integral $\int_{\Omega} \check{a}_i^k \bar{\partial}\partial_t^6(\rho_0^2 \check{J}^{-2})_{,k} \bar{\partial}\partial_t^6\check{v}^i$ yields

$$(7.34) \quad \int_{\Omega} \bar{\partial}\partial_t^6[\rho_0\check{v}_t^i] \bar{\partial}\partial_t^6\check{v}^i + \underbrace{\int_{\Omega} \bar{\partial}\partial_t^6\check{a}_i^k[\rho_0^2\check{J}^{-2}]_{,k} \bar{\partial}\partial_t^6\check{v}^i}_{\mathcal{K}} - \underbrace{\int_{\Omega} \bar{\partial}\partial_t^6[\rho_0^2\check{J}^{-2}]\check{a}_i^k \bar{\partial}\partial_t^6\check{v}^i_{,k}}_{\mathcal{I}} + \underbrace{\int_{\Gamma} \bar{\partial}\partial_t^6[\rho_0^2\check{J}^{-2}]\check{a}_i^k \bar{\partial}\partial_t^6\check{v}^i N^k}_{\mathcal{I}} = \ddot{\mathcal{R}}.$$

We used the Cauchy–Schwarz inequality or an L^4 – L^4 – L^2 Hölder inequality to analyze the lower-order terms in (7.34).

Analysis of \mathcal{I} in (7.34). We have that

$$-\mathcal{I} = \underbrace{\frac{d}{dt} \int_{\Omega} \rho_0^2 \check{J}^{-3} |\bar{\partial}\partial_t^6 \check{J}|^2}_{\int_0^T \ddot{\mathcal{R}}} + \underbrace{\int_{\Omega} \bar{\partial}\partial_t^6(\rho_0^2 \check{J}^{-2}) \bar{\partial}\partial_t^6 \check{a}_i^k \check{v}^i}_{\mathcal{I}_1} + \underbrace{\int_{\Omega} \bar{\partial}\partial_t^6(\rho_0^2 \check{J}^{-2}) \partial_t \check{a}_i^k \bar{\partial}\partial_t^5 \check{v}^i}_{\mathcal{I}_2} + \ddot{\mathcal{R}}.$$

Integration by parts with respect to a time-derivative of $\partial_t^6(\rho_0^2 \check{J}^{-2})$ yields

$$(7.35) \quad \int_0^T \mathcal{I}_1 = - \underbrace{\int_{\Omega} \bar{\partial} [\bar{\partial}\partial_t^5(\rho_0^2 \check{J}^{-2}) \check{v}^i_{,k}] \partial_t^6 \check{a}_i^k|_0^T}_{\mathcal{I}_{1A}} - \underbrace{\int_0^T \int_{\Omega} \bar{\partial}\partial_t^5(\rho_0^2 \check{J}^{-2}) \partial_t (\bar{\partial}\partial_t^6 \check{a}_i^k \check{v}^i_{,k})}_{\mathcal{I}_{1B}}.$$

Since $\partial_t^5 \check{J}$ is in $H^2(\Omega)$ and the highest-order term of $\partial_t^6 \check{a}_i^k$ scales like $D\partial_t^5 \check{v}$, the term \mathcal{I}_{1A} is bounded by $\int_0^T \ddot{\mathcal{R}}$ thanks to the fundamental theorem of calculus. Since the analysis of \mathcal{I}_{1B} is similar, we have established that $\mathcal{I}_1 = \ddot{\mathcal{R}}$. Similarly, $\mathcal{I}_2 = \ddot{\mathcal{R}}$. Thus,

$$(7.36) \quad \mathcal{I} = \ddot{\mathcal{R}}.$$

Analysis of \mathcal{K} in (7.34). From the differentiation formula (2.6) we infer that

$$\mathcal{K} = \int_{\Omega} \check{J}^{-1} (\check{a}_r^s \check{a}_i^k - \check{a}_i^s \check{a}_r^k) \bar{\partial}\partial_t^5 \check{v}^r_{,s} (\rho_0^2 \check{J}^{-2})_{,k} \bar{\partial}\partial_t^6 \check{v}^i + \ddot{\mathcal{R}}.$$

Since $\check{a}_r^s \bar{\partial}\partial_t^5 \check{v}^r_{,s}$ is equal to $\bar{\partial}\partial_t^6 \check{J}$ plus lower-order terms, we write

$$\begin{aligned} \mathcal{K} &= - \int_{\Omega} \check{J}^{-1} \check{a}_r^k (\rho_0^2 \check{J}^{-2})_{,k} \bar{\partial}\partial_t^5 \check{v}^r_{,s} \check{a}_i^s \bar{\partial}\partial_t^6 \check{v}^i + \ddot{\mathcal{R}} \\ &= \underbrace{\int_{\Omega} \check{J}^{-1} \check{a}_r^k (\rho_0^2 \check{J}^{-2})_{,k} \bar{\partial}\partial_t^5 \check{v}^r \check{a}_i^s \bar{\partial}\partial_t^6 \check{v}^i}_{\mathcal{K}'} - \underbrace{\int_{\Gamma} \check{J}^{-1} \check{a}_r^k (\rho_0^2 \check{J}^{-2})_{,k} \bar{\partial}\partial_t^5 \check{v}^r \check{a}_i^s \bar{\partial}\partial_t^6 \check{v}^i N^s}_{\mathcal{K}} + \ddot{\mathcal{R}}. \end{aligned}$$

Integrating by parts with respect to a time-derivative of $\bar{\partial}\partial_t^6\check{v}^i$,_s yields

$$\begin{aligned} \int_0^T \mathcal{K}' &= \underbrace{\int_{\Omega} \check{J}^{-1}\check{a}_r^k(\rho_0^2\check{J}^{-2})_{,k} \bar{\partial}\partial_t^5\check{v}^r\check{a}_i^s\bar{\partial}\partial_t^5\check{v}^i}_{\int_0^T \check{\mathcal{K}}} \Big|_0^T \\ &\quad - \int_0^T \underbrace{\int_{\Omega} \check{J}^{-1}\check{a}_r^k(\rho_0^2\check{J}^{-2})_{,k} \bar{\partial}\partial_t^5\check{v}^r\check{a}_i^s\bar{\partial}\partial_t^5\check{v}^i}_{\check{\mathcal{K}}} \Big|_0^T \\ &\quad - \int_0^T \underbrace{\int_{\Omega} \check{J}^{-1}\check{a}_r^k(\rho_0^2\check{J}^{-2})_{,k} \bar{\partial}\partial_t^5\check{v}^r(\check{a}_i^s)_t\bar{\partial}\partial_t^5\check{v}^i}_{\mathcal{K}''} \Big|_0^T, \end{aligned}$$

where we have again used that $\check{a}_i^s\bar{\partial}\partial_t^5\check{v}^i$,_s is equal to $\bar{\partial}\partial_t^6\check{J}$ plus lower-order terms. Lemma 2.5 provides that

$$\int_0^T \mathcal{K}'' \leq C \int_0^T \|\bar{\partial}\partial_t^5\check{v}\|_{0,5} \|\bar{\partial}D\partial_t^5\check{v}\|_{H^{0.5}(\Omega)'} \leq C \int_0^T \|\partial_t^5\check{v}\|_{1.5}^2 \leq \int_0^T \check{\mathcal{R}}.$$

The decomposition (7.31) provides that

$$\begin{aligned} -\mathbf{k} &= - \underbrace{\int_{\Gamma} \sqrt{\check{g}}(\rho_0^2\check{J}^{-2})_{,\alpha} \bar{\partial}\partial_t^5\check{v} \cdot \check{\eta}_{,\beta} \check{g}^{\alpha\beta} \bar{\partial}\partial_t^6\check{v} \cdot \check{n}}_{\mathbf{k}_1} \\ &\quad - \underbrace{\int_{\Gamma} \sqrt{\check{g}}\check{J}^{-1}\check{a}_{\check{n}}^k(\rho_0^2\check{J}^{-2})_{,k} \bar{\partial}\partial_t^5\check{v} \cdot \check{n} \bar{\partial}\partial_t^6\check{v} \cdot \check{n}}_{\mathbf{k}_2}. \end{aligned}$$

We have used the identities $\check{J}^{-1}\check{a} = \check{A}$ and $\check{\eta}_{,\alpha} \cdot \check{A}^k = \delta_{\alpha}^k$, where δ_{α}^k is the Kronecker delta, in writing \mathbf{k}_1 . Thanks to the Laplace–Young boundary condition (7.2b), we have that

$$\mathbf{k}_1 = \underbrace{\int_{\Gamma} \sqrt{\check{g}}(\beta_{\sigma}(t))_{,\alpha} \bar{\partial}\partial_t^5\check{v} \cdot \check{\eta}_{,\beta} \check{g}^{\alpha\beta} \bar{\partial}\partial_t^6\check{v} \cdot \check{n} - \int_{\Gamma} \sqrt{\check{g}}(\check{g}^{\mu\nu}\check{\eta}_{,\mu\nu} \cdot \check{n})_{,\alpha} \sqrt{\sigma} \bar{\partial}\partial_t^5\check{v} \cdot \check{\eta}_{,\beta} \check{g}^{\alpha\beta} \sqrt{\sigma} \bar{\partial}\partial_t^6\check{v} \cdot \check{n}}_{\check{\mathcal{R}}}.$$

We have used the identity (7.3) to deduce that $\bar{\partial}\beta_{\sigma}(t)$ scales like σ in $L^{\infty}(\Gamma)$ in the first term on the right-hand side, and we have used an L^4 – L^4 – L^2 Hölder inequality in the second term on the right-hand side. We also have that

$$\begin{aligned} -\mathbf{k}_2 &= \frac{1}{2} \frac{d}{dt} \int_{\Gamma} \sqrt{\check{g}}\check{J}^{-1}[-\check{a}_{\check{n}}^k(\rho_0^2\check{J}^{-2})_{,k}] |\bar{\partial}\partial_t^5\check{v} \cdot \check{n}|^2 \\ &\quad + \frac{1}{2} \int_{\Gamma} \underbrace{\partial_t(\sqrt{\check{g}}\check{J}^{-1}\check{a}_{\check{n}}^k(\rho_0^2\check{J}^{-2})_{,k})}_{\check{\mathcal{K}}} |\bar{\partial}\partial_t^5\check{v} \cdot \check{n}|^2 \\ &\quad + \int_{\Gamma} \underbrace{\sqrt{\check{g}}\check{J}^{-1}\check{a}_{\check{n}}^k(\rho_0^2\check{J}^{-2})_{,k} \bar{\partial}\partial_t^5\check{v} \cdot \check{n} \bar{\partial}\partial_t^5\check{v} \cdot \check{n}_t}_{\check{\mathcal{K}}}. \end{aligned}$$

Hence,

$$(7.37) \quad \mathcal{K} = \frac{1}{2} \frac{d}{dt} \int_{\Gamma} \sqrt{\check{g}} \check{J}^{-1} [-\check{a}_{\check{n}}^k (\rho_0^2 \check{J}^{-2})_{,k}] |\bar{\partial} \partial_t^5 \check{v} \cdot \check{n}|^2 + \check{\mathcal{R}}.$$

Analysis of \mathbf{i} in (7.34). Using the Laplace–Young boundary condition (7.2b), we find that

$$(7.38) \quad \begin{aligned} \mathbf{i} &= \frac{1}{2} \frac{d}{dt} \int_{\Gamma} \sqrt{\check{g}} \check{g}^{\alpha\beta} \sqrt{\sigma} \bar{\partial} \partial_t^5 \check{v}_{,\alpha} \cdot \check{n} \sqrt{\sigma} \bar{\partial} \partial_t^5 \check{v}_{,\beta} \cdot \check{n} - \underbrace{\sigma \int_{\Gamma} \sqrt{\check{g}} \partial_t^5 \check{v}_{,\alpha\beta} \cdot \bar{\partial} (\check{n} \check{g}^{\alpha\beta}) \bar{\partial} \partial_t^6 \check{v} \cdot \check{n}}_{\mathbf{j}_1} \\ &\quad - \underbrace{\sigma \int_{\Gamma} \sqrt{\check{g}} \check{g}^{\alpha\beta} \bar{\partial} \partial_t^5 \check{v}_{,\alpha} \cdot \check{n}_t \bar{\partial} \partial_t^5 \check{v}_{,\beta} \cdot \check{n}}_{\mathbf{j}_2} - \frac{1}{2} \underbrace{\int_{\Gamma} (\sqrt{\check{g}} \check{g}^{\alpha\beta})_t \sqrt{\sigma} \bar{\partial} \partial_t^5 \check{v}_{,\alpha} \cdot \check{n} \sqrt{\sigma} \bar{\partial} \partial_t^5 \check{v}_{,\beta} \cdot \check{n}}_{\check{\mathcal{R}}} \\ &\quad + \underbrace{\int_{\Gamma} \sigma \bar{\partial} \partial_t^5 \check{v}_{,\beta} \cdot \check{n} \sqrt{\check{g}} \check{g}^{\alpha\beta} \bar{\partial} \partial_t^6 \check{v} \cdot \check{n}_{,\alpha}}_{\check{\mathcal{R}}} + \underbrace{\sigma \int_{\Gamma} \bar{\partial} \partial_t^5 \check{v}_{,\beta} \cdot \check{n}_{,\alpha} \sqrt{\check{g}} \check{g}^{\alpha\beta} \bar{\partial} \partial_t^6 \check{v} \cdot \check{n}}_{\mathbf{j}_3} \\ &\quad + \underbrace{\sigma \int_{\Gamma} \bar{\partial} \partial_t^5 \check{v}_{,\beta} \cdot \check{n} (\sqrt{\check{g}} \check{g}^{\alpha\beta})_{,\alpha} \bar{\partial} \partial_t^6 \check{v} \cdot \check{n}}_{\check{\mathcal{R}}} + \underbrace{\sum_{l=0}^5 c_l \int_{\Gamma} \check{g}^{\alpha\beta} \partial_t^l \check{\eta}^j_{,\alpha\beta} \partial_t^{6-l} \check{n}^j \bar{\partial} (\sigma \sqrt{\check{g}} \bar{\partial} \partial_t^6 \check{v} \cdot \check{n})}_{\mathbf{j}_4} \\ &\quad + \underbrace{\sum_{l=1}^6 c_l \int_{\Gamma} \partial_t^l \check{g}^{\alpha\beta} \partial_t^{6-l} [\check{\eta}_{,\alpha\beta} \cdot \check{n}] \bar{\partial} (\sigma \sqrt{\check{g}} \bar{\partial} \partial_t^6 \check{v} \cdot \check{n})}_{\mathbf{j}_5} + \underbrace{\int_{\Gamma} \bar{\partial} \partial_t^6 \beta_{\sigma}(t) \bar{\partial} \partial_t^6 \check{v} \cdot \check{n}}_{\check{\mathcal{R}}}. \end{aligned}$$

We integrate by parts with respect to a time-derivative of $\bar{\partial} \partial_t^6 \check{v} \cdot \check{n}$ to write

$$\begin{aligned} \int_0^T \mathbf{j}_1 &= - \int_{\Gamma} \sqrt{\check{g}} \partial_t^5 \check{v}_{,\alpha} \cdot \left[\bar{\partial} (\check{n} \check{g}^{\alpha\beta}) \sigma \bar{\partial} \partial_t^5 \check{v} \cdot \check{n} \right]_{,\beta} \Big|_0^T \\ &\quad - \int_0^T \int_{\Gamma} \sqrt{\check{g}} \sqrt{\sigma} \partial_t^5 \check{v}_{,\alpha\beta} \cdot \bar{\partial} (\check{n} \check{g}^{\alpha\beta}) \sqrt{\sigma} \bar{\partial} \partial_t^5 \check{v} \cdot \check{n}_t \\ &\quad + \int_0^T \int_{\Gamma} \partial_t^5 \check{v}_{,\alpha} \cdot \left[[\bar{\partial} (\check{n} \check{g}^{\alpha\beta}) \sqrt{\check{g}}]_t \sigma \bar{\partial} \partial_t^5 \check{v} \cdot \check{n} \right]_{,\beta} \\ &\quad + \int_0^T \int_{\Gamma} \sqrt{\check{g}} \partial_t^6 \check{v}_{,\alpha} \cdot \left[\bar{\partial} (\check{n} \check{g}^{\alpha\beta}) \sigma \bar{\partial} \partial_t^5 \check{v} \cdot \check{n} \right]_{,\beta}. \end{aligned}$$

Since $\sigma \partial_t^5 \check{v} \cdot \check{n} \in H^{2.5}(\Gamma)$, it follows that

$$\int_0^T \mathbf{j}_1 = \int_0^T \check{\mathcal{R}}.$$

Similarly, integration by parts with respect to a time-derivative of $\bar{\partial} \partial_t^5 \check{v}_{,\beta} \cdot \check{n}$ yields

$$\int_0^T \mathbf{j}_2 = \int_0^T \check{\mathcal{R}}.$$

Since $\sqrt{\sigma}\partial_t^5\check{v}\cdot\check{n}\in H^2(\Gamma)$ and $\sqrt{\sigma}\partial_t^4\check{v}\in H^{2.5}(\Gamma)$, for the $l=5$ term of j_4 , we have that

$$\begin{aligned} & \int_0^T \int_\Gamma \check{g}^{\alpha\beta} \partial_t^4 \check{v}^j{}_{,\alpha\beta} \partial_t \check{n}^j \bar{\partial}(\sigma\sqrt{\check{g}}\bar{\partial}\partial_t^6\check{v}\cdot\check{n}) \\ &= \int_\Gamma \check{g}^{\alpha\beta} \sqrt{\sigma} \partial_t^4 \check{v}^j{}_{,\alpha\beta} \partial_t \check{n}^j \bar{\partial}(\sqrt{\sigma}\sqrt{\check{g}}\bar{\partial}\partial_t^5\check{v}\cdot\check{n}) \Big|_0^T \\ & \quad - \int_0^T \int_\Gamma \check{g}^{\alpha\beta} \sqrt{\sigma} \partial_t^4 \check{v}^j{}_{,\alpha\beta} \partial_t \check{n}^j \bar{\partial}(\sqrt{\sigma}\sqrt{\check{g}}\bar{\partial}\partial_t^5\check{v}\cdot\check{n}_t) \\ & \quad - \int_0^T \int_\Gamma \partial_t^4 \check{v}^j{}_{,\alpha\beta} (\partial_t \check{n}^j \check{g}^{\alpha\beta})_t \bar{\partial}(\sigma\sqrt{\check{g}}\bar{\partial}\partial_t^5\check{v}\cdot\check{n}) \\ & \quad - \int_0^T \int_\Gamma \check{g}^{\alpha\beta} \partial_t^5 \check{v}^j{}_{,\alpha\beta} \partial_t \check{n}^j \bar{\partial}(\sigma\sqrt{\check{g}}\bar{\partial}\partial_t^5\check{v}\cdot\check{n}) \\ &= \int_0^T \ddot{\mathcal{R}} - \int_0^T \int_\Gamma \check{g}^{\alpha\beta} \partial_t^5 \check{v}^j{}_{,\alpha\beta} \partial_t \check{n}^j \bar{\partial}(\sigma\sqrt{\check{g}}\bar{\partial}\partial_t^5\check{v}\cdot\check{n}) = \int_0^T \ddot{\mathcal{R}}. \end{aligned}$$

The last equality follows from the analysis of $\int_0^T j_2$. Since the analysis of j_5 and the $l=0, \dots, 4$ terms of j_4 are similarly established, we infer that (7.38) is equally written as

$$(7.39) \quad \mathbf{i} = \frac{1}{2} \frac{d}{dt} \int_\Gamma \sqrt{\check{g}} \check{g}^{\alpha\beta} \sqrt{\sigma} \bar{\partial} \partial_t^5 \check{v}_{,\alpha} \cdot \check{n} \sqrt{\sigma} \bar{\partial} \partial_t^5 \check{v}_{,\beta} \cdot \check{n} + \ddot{\mathcal{R}}.$$

The time-integral of (7.34). Using the identities (7.36), (7.37), and (7.39) in the time-integral of (7.34) completes the proof. \square

PROPOSITION 7.3 (energy estimates for the action of $\bar{\partial}^2\partial_t^4$ in the Euler equations (7.2a)).

$$\sup_{t\in[0,T]} \|\bar{\partial}^2\partial_t^4\check{v}(t)\|_0^2 + \sup_{t\in[0,T]} |\sqrt{\sigma}\bar{\partial}^2\check{v}_{ttt}\cdot\check{n}(t)|_1^2 + \sup_{t\in[0,T]} |\bar{\partial}^2\check{v}_{ttt}\cdot\check{n}(t)|_0^2 \leq \int_0^T \ddot{\mathcal{R}}.$$

Proof. Testing the action of $\bar{\partial}^2\partial_t^4$ in the Euler equations (7.2a) against $\bar{\partial}^2\partial_t^4\check{v}$ in the $L^2(\Omega)$ -inner product and integrating by parts in the integral $\int_\Omega \check{a}_i^k \bar{\partial}^2\partial_t^4(\rho_0^2 \check{J}^{-2})_{,k} \bar{\partial}^2\partial_t^4\check{v}^i$ yields

$$\begin{aligned} & \int_\Omega \bar{\partial}^2\partial_t^4[\rho_0\check{v}^i] \bar{\partial}^2\partial_t^4\check{v}^i + \underbrace{\int_\Omega \bar{\partial}^2\partial_t^4\check{a}_i^k[\rho_0^2\check{J}^{-2}]_{,k} \bar{\partial}^2\partial_t^4\check{v}^i}_{\mathcal{K}} - \underbrace{\int_\Omega \bar{\partial}^2\partial_t^4[\rho_0^2\check{J}^{-2}]\check{a}_i^k \bar{\partial}^2\partial_t^4\check{v}^i}_{\mathcal{I},k} \\ & \quad + \underbrace{\int_\Gamma \bar{\partial}^2\partial_t^4[\rho_0^2\check{J}^{-2}]\check{a}_i^k \bar{\partial}^2\partial_t^4\check{v}^i N^k}_{\mathbf{i}} = \ddot{\mathcal{R}}. \end{aligned}$$

The analysis of \mathcal{K} and \mathbf{i} follows from the proof of Proposition 7.2. Hence,

$$(7.40) \quad \begin{aligned} & \frac{1}{2} \frac{d}{dt} \int_\Omega \rho_0 |\bar{\partial}^2\partial_t^4\check{v}|^2 + \frac{1}{2} \frac{d}{dt} \int_\Gamma \sqrt{\check{g}} \check{J}^{-1} [-\check{a}_n^k(\rho_0^2\check{J}^{-2})_{,k}] |\bar{\partial}^2\check{v}_{ttt}\cdot\check{n}|^2 \\ & \quad + \frac{1}{2} \frac{d}{dt} \int_\Gamma \sqrt{\check{g}} \check{g}^{\alpha\beta} \sqrt{\sigma} \bar{\partial}^2\check{v}_{ttt,\alpha} \cdot \check{n} \sqrt{\sigma} \bar{\partial}^2\check{v}_{ttt,\beta} \cdot \check{n} = \ddot{\mathcal{R}} + \mathcal{I}. \end{aligned}$$

We have that

$$\begin{aligned}
 -\mathcal{I} &= \underbrace{\frac{d}{dt} \int_{\Omega} \rho_0^2 \check{J}^{-3} |\bar{\partial}^2 \partial_t^4 \check{J}|^2}_{\int_0^T \check{\mathcal{R}}} + \int_{\Omega} \bar{\partial}^2 \partial_t^4 (\rho_0^2 \check{J}^{-2}) \bar{\partial}^2 \partial_t^4 a_i^k v^i \\
 &\quad + \int_{\Omega} \bar{\partial}^2 \partial_t^4 (\rho_0^2 \check{J}^{-2}) \partial_t a_i^k \bar{\partial}^2 \partial_t^3 v^i_{,k} + \check{\mathcal{R}}.
 \end{aligned}$$

Since $\bar{\partial}^2 \partial_t^4 \check{J}$ is in $H^{0.5}(\Omega)$, we use Lemma 2.5 to conclude that

$$(7.41) \quad \mathcal{I} = \check{\mathcal{R}}.$$

Using (7.41) in the time-integral of (7.40) completes the proof. \square

We infer the following two propositions from the proof of Proposition 7.3.

PROPOSITION 7.4 (energy estimates for the action of $\bar{\partial}^3 \partial_t^2$ in the Euler equations (7.2a)).

$$\sup_{t \in [0, T]} \|\bar{\partial}^3 \check{v}_{tt}(t)\|_0^2 + \sup_{t \in [0, T]} |\sqrt{\sigma} \bar{\partial}^3 \check{v}_t \cdot \check{n}(t)|_1^2 + \sup_{t \in [0, T]} |\bar{\partial}^3 \check{v}_t \cdot \check{n}(t)|_0^2 \leq \int_0^T \check{\mathcal{R}}.$$

PROPOSITION 7.5 (energy estimates for the action of $\bar{\partial}^4$ in the Euler equations (7.2a)).

$$\sup_{t \in [0, T]} \|\bar{\partial}^4 \check{v}(t)\|_0^2 + \sup_{t \in [0, T]} |\sqrt{\sigma} \bar{\partial}^4 \check{\eta} \cdot \check{n}(t)|_1^2 + \sup_{t \in [0, T]} |\bar{\partial}^4 \check{\eta} \cdot \check{n}(t)|_0^2 \leq \int_0^T \check{\mathcal{R}}.$$

Step 4: The σ -independent higher-order estimates via Proposition 2.1.

LEMMA 7.3 (the σ -independent lower-order estimates for $\partial_t^a \check{J}$, $a = 0, \dots, 7$).

$$\sup_{t \in [0, T]} \sum_{a=0}^7 \|\partial_t^a \check{J}(t)\|_{3.5-\frac{1}{2}a}^2 \leq \int_0^T \check{\mathcal{R}}.$$

Proof. The $a = 6$ and $a = 7$ cases are provided by Proposition 7.1. The higher-order estimates for $a = 0, \dots, 5$ are established by interpolation and the fundamental theorem of calculus. For example, using that $\partial_t^6 \check{J}$ is in $H^1(\Omega)$

$$\sup_{t \in [0, T]} \|\partial_t^5 \check{J}(t)\|_{1.5}^2 \leq C_{\delta} \sup_{t \in [0, T]} \|\partial_t^5 \check{J}(t)\|_1^2 + \delta \sup_{t \in [0, T]} \|\partial_t^5 \check{J}(t)\|_2^2 \leq \int_0^T \check{\mathcal{R}}. \quad \square$$

LEMMA 7.4 (the σ -independent normal trace-estimates for $\partial_t^a \check{\eta}$, $a = 0, \dots, 7$).

$$\sup_{t \in [0, T]} \sum_{a=0}^7 |\partial_t^a \check{\eta}(t) \cdot N|_{4-\frac{1}{2}a}^2 \leq \int_0^T \check{\mathcal{R}}.$$

Proof. By Lemma 2.2,

$$\sup_{t \in [0, T]} |\bar{\partial} \partial_t^6 \check{v}(t) \cdot N|_{-0.5}^2 \leq C \sup_{t \in [0, T]} \left(\|\bar{\partial} \partial_t^6 \check{v}(t)\|_0^2 + \|\operatorname{div} \partial_t^6 \check{v}(t)\|_0^2 \right) \leq \int_0^T \check{\mathcal{R}},$$

thanks to the estimates stated in Proposition 7.2 and Lemma 7.3. Using the same argument, the $L^2(\Omega)$ -estimates for $\bar{\partial}^3 \check{v}_{tt}$ and $\bar{\partial}^4 \check{v}$, respectively, given in Propositions 7.4 and 7.5, and the divergence-estimates given by Lemma 7.3 provide that

$$\sup_{t \in [0, T]} \sum_{a=0}^3 |\partial_t^{2a} \check{v}(t) \cdot N|_{3.5-a}^2 \leq \int_0^T \ddot{\mathcal{R}}.$$

Using the fundamental theorem of calculus, the normal trace-estimates stated in Propositions 7.2, 7.3 and 7.5 complete the proof. \square

Via Proposition 2.1, the estimates stated in Lemmas 7.2, 7.3, and 7.4 establish the following.

PROPOSITION 7.6 (the σ -independent estimates for $\partial_t^a \check{\eta}$, $a = 0, \dots, 7$).

$$\sup_{t \in [0, T]} \sum_{a=0}^7 \|\partial_t^a \check{\eta}(t)\|_{4.5-\frac{1}{2}a}^2 \leq \int_0^T \ddot{\mathcal{R}}.$$

Via Proposition 2.1, the estimates stated in Lemma 7.2 and Proposition 7.2 establish the following.

PROPOSITION 7.7 (the σ -independent estimate for $\sqrt{\sigma} \partial_t^5 \check{v}$).

$$\|\sqrt{\sigma} \partial_t^5 \check{v}(t)\|_2^2 \leq \int_0^T \ddot{\mathcal{R}}.$$

Step 5: The σ -independent higher-order estimates via interpolation.

PROPOSITION 7.8 (the σ -independent estimates for $\sqrt{\sigma} \partial_t^a \check{\eta}$, $a = 0, \dots, 3$).

$$\sup_{t \in [0, T]} \sum_{a=0}^3 \|\sqrt{\sigma} \partial_t^a \check{\eta}(t)\|_{5.5-\frac{1}{2}a}^2 \leq \int_0^T \ddot{\mathcal{R}}.$$

Proof. We note that by use of interpolation and Young’s inequality,

$$\sup_{t \in [0, T]} \|\sqrt{\sigma} \check{\eta}(t)\|_{5.5}^2 \leq C_\delta \sup_{t \in [0, T]} \|\check{\eta}(t)\|_{4.5}^2 + \delta \sup_{t \in [0, T]} \|\sigma \check{\eta}(t)\|_{5.5}^2.$$

It follows that the estimates given in Proposition 7.6 complete the proof. \square

Step 6: The σ -independent higher-order estimates via the Euler equations (7.2a).

PROPOSITION 7.9 (the σ -independent estimates for $\partial_t^a \check{J}$, $a = 0, \dots, 5$).

$$\sup_{t \in [0, T]} \sum_{a=0}^5 \|\partial_t^a \check{J}(t)\|_{4.5-\frac{1}{2}a}^2 \leq \int_0^T \ddot{\mathcal{R}}.$$

Proof. We infer from the a th time-derivative of the Euler equations (7.2a) that

$$\sup_{t \in [0, T]} \|\partial_t^a \check{J}(t)\|_{4.5-\frac{1}{2}a}^2 \leq C \sup_{t \in [0, T]} \|\partial_t^a \check{A}(t)\|_{3.5-\frac{1}{2}a}^2 + C \sup_{t \in [0, T]} \|\partial_t^a \check{v}_t(t)\|_{3.5-\frac{1}{2}a}^2 + \int_0^T \ddot{\mathcal{R}}.$$

The highest-order term of $\partial_t^a \check{A}$ scales like $\check{A} \check{A} D \partial_t^a \check{\eta}$. Hence, the estimates stated in Proposition 7.6 complete the proof. \square

Given Proposition 7.9, we now establish the following.

PROPOSITION 7.10 (the σ -independent estimates for $\sqrt{\sigma}\partial_t^4\check{v}$ and $\sqrt{\sigma}\check{v}_{ttt}$ via Proposition 2.1).

$$\sup_{t \in [0, T]} \sum_{a=0}^1 \|\sqrt{\sigma}\partial_t^{3+a}\check{v}(t)\|_{3.5-\frac{1}{2}a}^2 \leq \int_0^T \ddot{\mathcal{R}}.$$

Proof. Via Proposition 2.1, the estimates stated in Lemma 7.2 and Propositions 7.3 and 7.9 establish the estimate for $\sqrt{\sigma}\check{v}_{ttt}$. For $\sqrt{\sigma}\partial_t^4\check{v}$, we have that

$$|\sqrt{\sigma}\partial_t^4\check{v} \cdot \check{n}(t)|_{2.5}^2 \leq C_\delta |\partial_t^4\check{v} \cdot \check{n}(t)|_{1.5}^2 + \delta |\sigma\partial_t^4\check{v} \cdot \check{n}(t)|_{3.5}^2 \leq \int_0^T \ddot{\mathcal{R}},$$

thanks to the estimate for $\partial_t^4\check{v}$ stated in Proposition 7.6. Hence, we infer via Proposition 2.1 the estimate for $\sqrt{\sigma}\partial_t^4\check{v}$ from the estimates stated in Lemma 7.2 and Proposition 7.9. \square

Step 7: The σ -independent higher-order estimates for $\sigma\partial_t^a\check{\eta}$, $a = 0, \dots, 4$.

LEMMA 7.5 (the σ -independent normal trace-estimates for $\sigma\partial_t^a\check{\eta}$, $a = 0, \dots, 4$).

$$\sup_{t \in [0, T]} \sum_{a=0}^3 |\sigma\partial_t^2\partial_t^a\check{\eta} \cdot \check{n}(t)|_{4-\frac{1}{2}a}^2 + \sup_{t \in [0, T]} |\sigma\partial_t^3\check{v} \cdot \check{n}(t)|_{3.5}^2 \leq \int_0^T \ddot{\mathcal{R}}.$$

Proof. The estimates for $\partial_t^a\check{J}$ given in Proposition 7.9 and the fundamental theorem of calculus provide that the a th time-derivative of the Laplace–Young boundary condition (7.2b) yields the desired estimates. \square

PROPOSITION 7.11 (the σ -independent estimates for $\sigma\partial_t^a\check{\eta}$, $a = 0, \dots, 4$).

$$\sup_{t \in [0, T]} \sum_{a=0}^3 \|\sigma\partial_t^a\check{\eta}(t)\|_{6.5-\frac{1}{2}a}^2 + \sup_{t \in [0, T]} \|\sigma\partial_t^3\check{v}(t)\|_4^2 \leq \int_0^T \ddot{\mathcal{R}}.$$

Proof. Since $\sigma < \sqrt{\sigma}$, we infer from the proof of Proposition 7.9 that the estimates for $\sqrt{\sigma}\partial_t^a\check{v}$, $a = 3, 4, 5$, stated in Propositions 7.7 and 7.10 establish the divergence-estimates for $\sigma\check{v}_t$, $\sigma\check{v}_{tt}$, and $\sigma\check{v}_{ttt}$. Similarly, the estimates for $\sqrt{\sigma}\partial_t^a\check{v}$, $a = 1, 2$, stated in Proposition 7.8 establish the divergence-estimates for $\sigma\check{v}$ and $\sigma\check{\eta}$. Via Proposition 2.1, the estimates stated in Lemmas 7.2 and 7.5 therefore complete the proof. \square

Step 8: The σ -independent improved boundary-regularity estimates.

Considering the action of ∂_t^a , $a = 4, 5, 6$, in the Laplace–Young boundary condition (7.2b), we notice that the estimates stated in Propositions 7.9 and 7.11 establish the following.

PROPOSITION 7.12 (the σ -independent estimates for $\sigma\partial_t^a\check{v} \cdot \check{n}$, $a = 3, 4, 5$).

$$\sup_{t \in [0, T]} \sum_{a=0}^1 |\sigma\partial_t^{3+a}\check{v} \cdot \check{n}(t)|_{4-\frac{1}{2}a}^2 + \sup_{t \in [0, T]} |\sigma\partial_t^5\check{v} \cdot \check{n}(t)|_{2.5}^2 \leq \int_0^T \ddot{\mathcal{R}}.$$

Step 9: Concluding the proof of Lemma 7.1. The sum of the estimates given in Propositions 7.1–7.12 completes the proof of Lemma 7.1. Taking δ sufficiently small in the inequality (7.7) yields a polynomial-type inequality of the form (2.20). Hence, there exists $T > 0$ that is independent of $\sigma > 0$ and verifies

$$(7.42) \quad \sup_{t \in [0, T]} E^\sigma(t) \leq 2\ddot{\mathcal{N}}_0.$$

7.5. The proof of Theorem 1.2.

7.5.1. Existence. Since $E^\sigma(0) < \infty$, Theorem 1.1 provides us with a sequence of solutions \check{v} to the σ -problem (7.2). The σ -independent estimate (7.42) and standard compactness arguments establish the strong convergence, as σ tends to zero,

$$\begin{aligned} \check{\eta} &\rightarrow \eta \text{ in } L^2(0, T; H^{3.5}(\Omega)), \\ \check{v}_t &\rightarrow v_t \text{ in } L^2(0, T; H^{2.5}(\Omega)). \end{aligned}$$

Letting $\phi \in L^2(0, T; H^1(\Omega))$, we have that the variational form of (7.2) is

$$\begin{aligned} \int_0^T \int_\Omega \rho_0 \check{v}_t \cdot \phi - \int_0^T \int_\Omega \rho_0^2 \check{J}^{-2} \check{a}_i^k \phi^i_{,k} + \int_0^T \int_\Gamma \beta_\sigma(t) \sqrt{\check{g}} \phi \cdot \check{n} \\ = \sigma \int_0^T \int_\Gamma \sqrt{\check{g}} \check{g}^{\mu\nu} \check{\eta}_{, \mu\nu} \cdot \check{n} \phi \cdot \check{n}. \end{aligned}$$

The strong convergence of the sequences $(\check{\eta}, \check{v}_t)$ and the pointwise convergence $\beta_\sigma(t) \rightarrow \beta$ provide that the limit (η, v_t, β) satisfies

$$\int_0^T \int_\Omega \rho_0 v_t \cdot \phi - \int_0^T \int_\Omega \rho_0^2 J^{-2} a_i^k \phi^i_{,k} + \int_0^T \int_\Gamma \beta \sqrt{g} \phi \cdot n = 0.$$

Thus, v is a solution of the zero surface tension limit of (1.7) on a nonempty time-interval $[0, T]$. Standard arguments provide that $v(0) = u_0$ and $\eta(0) = e$. Furthermore, letting $\sigma = 0$ in the a priori estimates in section 7.4, we conclude that the right-hand side of inequality (7.42) depends only on $\mathcal{M}_0 = P(\mathcal{E}(0))$. Hence, for sufficiently small $T > 0$, the higher-order energy function $\mathcal{E}(t)$ defined in (1.11) satisfies

$$(7.43) \quad \sup_{t \in [0, T]} \mathcal{E}(t) \leq 2\mathcal{M}_0.$$

The bounds (7.6) remain valid by taking $T > 0$ even smaller if necessary. By the arguments in section 6.1, the boundedness of J in assumption (7.6a) implies that

$$\rho(t) \geq \lambda \text{ in } \bar{\Omega}(t).$$

Similarly, the lower-bound (7.6b) provides that $\rho(t) = f \circ \eta^{-1}(t)$ satisfies

$$0 < \nu \leq -\frac{\partial \rho^2(t)}{\partial n(t)} \text{ on } \Gamma(t).$$

7.5.2. Optimal regularity for the initial data. As we just noted,

$$\sup_{t \in [0, T]} \mathcal{E}(t) \leq P(\mathcal{E}(0)).$$

We can then let the convolution parameter (used in Appendix A) tend to zero, and Theorem 1.2 then holds for the regularity class $\mathcal{E}(0) < \infty$.

7.5.3. Uniqueness. We infer the uniqueness of the solution to the zero surface tension limit of (1.7) by repeating the arguments given in section 6.3.

Appendix A. Constructing C^∞ -class initial data. We demonstrate how given initial data (ρ_0, u_0, Ω) of finite Sobolev regularity (the optimal regularity stated in the existence theorem) satisfying the conditions (1.13) and (1.14) (or the conditions (1.13), (1.15), and (1.16)), we generate smoothed C^∞ -class data (ρ_0, v_0, Ω) which satisfy similar statements of the conditions (1.13) and (1.14) (or the conditions (1.13), (1.15), and (1.16)).

A.1. The C^∞ -class data for the surface tension problem (1.7). We suppose the initial data (ρ_0, u_0, Ω) is of the optimal regularity stated in Theorem 1.1 and satisfy the conditions (1.13) and (1.14). Letting

$$(A.1) \quad \mathbf{J}_a = \partial_t^a (J^{-2})|_{t=0} \quad \text{and} \quad \mathbf{H}_a = \partial_t^a H(\eta)|_{t=0},$$

we will construct C^∞ -class data (ρ_0, v_0, Ω) which satisfy

$$(A.2a) \quad \sigma \mathbf{H}_0 \geq 4\lambda^2 - \beta \quad \text{for } \sigma, \beta > 0,$$

$$(A.2b) \quad \rho_0 \geq 2\lambda > 0 \quad \text{in } \overline{\Omega},$$

$$(A.2c) \quad \rho_0^2 \mathbf{J}_a = \partial_t^a \beta + \sigma \mathbf{H}_a \quad \text{on } \Gamma \quad \text{for } a = 0, 1, 2, 3.$$

A.1.1. Defining the C^∞ -class initial domain Ω . By Whitney's approximation theorem, every C^1 manifold is diffeomorphic to a C^∞ manifold. In particular, for Ω of class H^s , $s \geq 4$, there is a C^∞ domain Ω which is arbitrarily close to Ω with respect to the H^s -norm.

Setting $\Gamma = \partial\Omega$ defines a C^∞ -class surface. We let η denote the $C^\infty(\Gamma)$ -vector describing the geometry of Γ . In other words, $\bar{\partial}\eta$ spans the tangent space of Γ and, by letting \mathbf{N} denote the outward-pointing unit normal vector to the surface Γ , we have that $\mathbf{N} = \eta_{,1} \times \eta_{,2} / |\eta_{,1} \times \eta_{,2}|$. We let \mathbf{g} denote the surface metric induced by η and let \mathbf{H}_0 denote twice the mean curvature of Γ . Thus,

$$\mathbf{H}_0 = -\mathbf{g}^{\alpha\beta} \eta_{,\alpha\beta} \cdot \mathbf{N} \quad \text{in } C^\infty(\Gamma).$$

We assume that the given H^5 -class domain $\Omega \subset \mathbb{R}^3$ is such that (1.14) is satisfied. Thus, for $\epsilon > 0$ taken sufficiently small, standard properties of convolution provide that

$$(A.3) \quad \beta + \sigma \mathbf{H}_0 \geq 2\lambda^2 \quad \text{for } \sigma, \beta > 0.$$

Hence, (A.2a) is satisfied.

A.1.2. Defining the C^∞ -class data $(\varrho_0, \mathbf{V}_0, \Omega)$ to satisfy (A.2) for $a = 0$. Given $u_0 \in H^4(\Omega)$, we define u_0 in the C^∞ -class domain Ω via the equation

$$(A.4) \quad u_0 = \sum_{l=1}^L [\xi_l u_0 \circ \theta_l] \circ \theta_l^{-1} \quad \text{in } H^4(\Omega).$$

Using the operator Λ_ϵ defined in section 5.1, for $\epsilon > 0$ we define the vector field $\mathbf{V}_0 \in C^\infty(\Omega)$ as the solution of the following elliptic Dirichlet problem:

$$(A.5a) \quad \mathbf{V}_0 - \epsilon \Delta \mathbf{V}_0 = \varphi^\epsilon * \mathcal{E}_\Omega(u_0) \quad \text{in } \Omega,$$

$$(A.5b) \quad \mathbf{V}_0 = \sum_{l=1}^K \Lambda_\epsilon [\xi_l u_0 \circ \theta_l] \circ \theta_l^{-1} \quad \text{on } \Gamma.$$

Given $\rho_0 \in H^4(\Omega)$, we define ϱ_0 in the C^∞ -class domain Ω via the equation

$$(A.6) \quad \varrho_0 = \sum_{l=1}^L [\xi_l \rho_0 \circ \theta_l] \circ \theta_l^{-1} \quad \text{in } H^4(\Omega).$$

Assuming that $\rho_0 \geq 2\lambda > 0$ in $\bar{\Omega}$, we let $\lambda > 0$ be such that $2\lambda^2 \geq 9\lambda^2$. It follows for ϱ_0 defined by (A.6) that

$$(A.7) \quad \varrho_0 \geq 3\lambda \text{ in } \bar{\Omega}.$$

For $\mu > 0$ and $\epsilon > 0$, we define $\varrho_0 \in C^\infty(\Omega)$ to be the solution of

$$(A.8a) \quad \varrho_0 - \mu\Delta\varrho_0 = \varphi^\mu * \mathcal{E}_\Omega(\varrho_0) \text{ in } \Omega,$$

$$(A.8b) \quad \varrho_0 = \sqrt{\beta + \sigma H_0} \text{ on } \Gamma.$$

The boundary condition (A.8b) implies for $a = 0$ that (A.2c) is satisfied, since (A.8b) is equivalently stated as

$$(A.9) \quad \varrho_0^2 J_0 = \beta + \sigma H_0 \text{ on } \Gamma.$$

Elliptic regularity provides for $s \geq 2$ and a positive constant C independent of μ that

$$\|\sqrt{\mu}\varrho_0\|_{H^s(\Omega)}^2 \leq C \left(\|\varphi^\mu * \mathcal{E}_\Omega(\varrho_0)\|_{H^{s-2}(\Omega)}^2 + \|\varrho_0\|_{H^{s-\frac{1}{2}}(\Gamma)}^2 \right).$$

The boundary forcing function used in defining ϱ_0 is in $C^\infty(\Gamma)$. Since the function ϱ_0 appearing in the right-hand side of (A.8a) is in $H^4(\Omega)$, there exists a constant C which is independent of μ verifying

$$\|\sqrt{\mu}\varrho_0\|_{H^6(\Omega)}^2 \leq C.$$

Equation (A.8a) provides that $\varrho_0 = \mu\Delta\varrho_0 + \varphi^\mu * \mathcal{E}_\Omega(\varrho_0)$ in Ω . Hence, by standard properties of convolution and taking μ sufficiently small, we conclude that

$$\|\varrho_0 - \varrho_0\|_{L^\infty(\Omega)} \leq \sqrt{\mu}\|\sqrt{\mu}\Delta\varrho_0\|_{H^2(\Omega)} + \|\varphi^\mu * \mathcal{E}_\Omega(\varrho_0) - \varrho_0\|_{L^\infty(\Omega)} \leq \lambda.$$

Recalling the lower-bound (A.7), we conclude that $\varrho_0 \geq 2\lambda > 0$ in Ω . Letting ϵ be sufficiently small, the boundary condition (A.8b) provides that

$$(A.10) \quad \varrho_0 \geq 2\lambda > 0 \text{ in } \bar{\Omega}.$$

By (A.10) and (A.3), the boundary condition (A.8b) provides that

$$(A.11) \quad \beta + \sigma H_0 \geq 4\lambda^2 \text{ for } \sigma, \beta > 0.$$

Hence, the C^∞ -class data $(\varrho_0, \mathbf{V}_0, \Omega)$ satisfy the conditions (A.2) for $a = 0$, as verified by (A.9), (A.10), and (A.11).

A.1.3. Formal definitions. We formally set ρ_0 equal to ϱ_0 defined by (A.8). We define

$$(A.12) \quad \mathbf{v}_{a+1} = -2\partial_t^a (A^k(\rho_0 J^{-1}))|_{t=0} \text{ for } a = 0, \dots, 5$$

and assume that (A.12) yields $\mathbf{v}_0 = \mathbf{V}_0$ defined by (A.5). We make the following definitions:

$$(A.13) \quad \mathbf{j}_a = \text{div } \mathbf{v}_a \text{ and } \mathbf{k}_a = \partial_t^a (A_r^s v^r, s)|_{t=0} \text{ for } a = 0, \dots, 6.$$

Using the notation

$$(A.14) \quad [\mathcal{A}_a]_r^s = \partial_t^a A_r^s|_{t=0} \text{ for } a = 0, \dots, 5,$$

it follows that \mathbf{k}_a defined in (A.13) satisfies

$$(A.15) \quad \mathbf{k}_0 = \mathbf{j}_0, \quad \mathbf{k}_1 = [\mathcal{A}_1]_r^s \mathbf{v}_{0,s}^r + \mathbf{j}_1, \quad \text{and} \quad \mathbf{k}_2 = [\mathcal{A}_2]_r^s \mathbf{v}_{0,s}^r + 2[\mathcal{A}_1]_r^s \mathbf{v}_{1,s}^r + \mathbf{j}_2.$$

We also have that \mathbf{J}_a defined by (A.1) is equivalently given by

$$(A.16) \quad \mathbf{J}_0 = 1, \quad \mathbf{J}_1 = -2\mathbf{k}_0, \quad \mathbf{J}_2 = 4\mathbf{k}_0^2 - 2\mathbf{k}_1, \quad \text{and} \quad \mathbf{J}_3 = -8\mathbf{k}_0^3 + 12\mathbf{k}_0\mathbf{k}_1 - 2\mathbf{k}_2.$$

Using (A.15) and (A.16), the condition (A.2c) that $\rho_0^2 \mathbf{J}_a = \sigma \mathbf{H}_a$ is equivalently stated as

$$(A.17) \quad \operatorname{div} \mathbf{v}_{a-1} = \mathbf{j}_{a-1} = \underbrace{-\frac{1}{2} [\sigma \mathbf{H}_a - \rho_0^2 \mathbf{J} - 2\mathbf{k}_{a-1}]}_{\mathbf{f}_{a-1}} + \mathbf{d}_{a-1} \quad \text{for } a = 1, 2, 3,$$

where \mathbf{d}_{a-1} represents the lower-order terms defining \mathbf{k}_{a-1} . We notice that \mathbf{v}_a , $a = 1, 2$, defined by (A.12) is equivalently given by $\mathbf{v}_1 = -2D\rho_0$ and $\mathbf{v}_2 = -2[\mathcal{A}_1]_r^k \rho_{0,k} - 2D(\rho_0 \mathbf{j}_0)$. Thus,

$$(A.18) \quad \operatorname{div} \mathbf{v}_0 = \mathbf{f}_0, \quad \operatorname{div} \mathbf{v}_1 = \mathbf{f}_1 = -2\Delta \rho_0, \quad \text{and} \quad \operatorname{div} \mathbf{v}_2 = \mathbf{f}_2 \approx -2\Delta \operatorname{div} \mathbf{v}_0.$$

A.1.4. Defining \mathbf{v}_0 so that the C^∞ -class data $(\rho_0, \mathbf{v}_0, \Omega)$ satisfy (A.2) for $\mathbf{a} = \mathbf{0}, \mathbf{1}$. Extending N and $\tau_\alpha = \eta_{,\alpha} / |\eta_{,\alpha}|$ for $\alpha = 1, 2$ into Ω , we decompose any vector $\xi \in \mathbb{R}^3$ into normal and tangential components as

$$\xi = \xi^\alpha \tau_\alpha + \xi^3 N \quad \text{in } \Omega.$$

It follows for sufficiently regular ξ that $\operatorname{div} \xi = \xi^i_{,i}$ is equivalently written as

$$(A.19) \quad \operatorname{div} \xi = \underbrace{\xi^{\alpha, \alpha} + \xi^{3, 3} + \xi^\alpha \operatorname{div} \tau_\alpha + \xi^3 \operatorname{div} N}_{j_0} \quad \text{on } \Gamma.$$

This provides the following identity for $(N \cdot D)\xi^3 = \xi^{3, 3}$:

$$(A.20) \quad \frac{\partial \xi^3}{\partial N} = \underbrace{j_0 - [\xi^{\alpha, \alpha} + \xi^\alpha \operatorname{div} \tau_\alpha + \xi^3 \operatorname{div} N]}_{\varphi_0} \quad \text{on } \Gamma.$$

We define $\mathbf{v}_0^3 \in C^\infty(\Omega)$ by

$$(A.21a) \quad \mathbf{v}_0^3 + \epsilon \Delta^2 \mathbf{v}_0^3 = \varphi^\epsilon * \mathcal{E}_\Omega(\mathbf{u}_0^3) \quad \text{in } \Omega,$$

$$(A.21b) \quad \frac{\partial \mathbf{v}_0^3}{\partial N} = \varphi_0 \quad \text{on } \Gamma,$$

$$(A.21c) \quad \mathbf{v}_0^3 = \mathbf{V}_0^3 \quad \text{on } \Gamma,$$

where the boundary forcing function φ_0 is defined as

$$(A.22) \quad \varphi_0 = \mathbf{f}_0 - [\mathbf{V}_0^{\alpha, \alpha} + \mathbf{V}_0^\alpha \operatorname{div} \tau_\alpha + \mathbf{V}_0^3 \operatorname{div} N]$$

with the function \mathbf{f}_0 appearing in the right-hand side of (A.22) defined by (A.17). With \mathbf{V}_0^α denoting the tangential component of the vector defined by (A.5), we define the vector $\mathbf{v}_0 \in C^\infty(\Omega)$ as

$$(A.23) \quad \mathbf{v}_0 = \mathbf{V}_0^\alpha \tau_\alpha + \mathbf{v}_0^3 N \quad \text{in } \Omega.$$

The boundary condition (A.21c) and the definition (A.23) imply that

$$(A.24) \quad \mathbf{v}_0 = \mathbf{V}_0 \quad \text{on } \Gamma.$$

Thanks to (A.19), the boundary condition (A.21b) and the definition (A.22) yield

$$(A.25) \quad \operatorname{div} \mathbf{v}_0 = \mathbf{f}_0.$$

Thanks to the definition (A.17) of \mathbf{f}_0 , we equivalently write (A.25) as

$$(A.26) \quad \varrho_0^2 \mathbf{J}_1 = \sigma \mathbf{H}_1 \quad \text{on } \Gamma.$$

Hence, the C^∞ -class data $(\varrho_0, \mathbf{v}_0, \Omega)$ satisfy the conditions (A.2) for $a = 0, 1$, as verified by (A.11), (A.9), (A.10), and (A.26).

A.1.5. Defining the C^∞ -class data $(\rho_0, \mathbf{v}_0, \Omega)$ to satisfy (A.2) for $a = 0, 1, 2$. According to (A.18), the condition (A.2c) for $a = 2$ may be imposed by prescribing a Dirichlet boundary condition for $\Delta \rho_0$. We define $\rho_0 \in C^\infty(\Omega)$ to be the solution of the polyharmonic problem

$$(A.27a) \quad \rho_0 - \mu \Delta^3 \rho_0 = \varphi^\mu * \mathcal{E}_\Omega(\varrho_0) \quad \text{in } \Omega,$$

$$(A.27b) \quad -2\Delta \rho_0 = \mathbf{f}_1 \quad \text{on } \Gamma,$$

$$(A.27c) \quad -2 \frac{\partial \rho_0}{\partial \mathbf{N}} = \mathbf{V}_1^3 \quad \text{on } \Gamma,$$

$$(A.27d) \quad \rho_0 = \varrho_0 \quad \text{on } \Gamma.$$

The boundary-forcing function \mathbf{f}_1 appearing in the right-hand side of (A.27b) is defined as

$$(A.28) \quad \mathbf{f}_1 = -\frac{\sigma \mathbf{H}_2}{2\varrho_0^2} + 2\mathbf{k}_0^2 - [\mathcal{A}_1]_r^s \mathbf{v}_{0,s}^r.$$

Using the boundary conditions of the polyharmonic problem (A.21), we may express the right-hand side of (A.28) in terms of \mathbf{V}_0 , \mathbf{V}_1 , and φ_0 , where

$$(A.29) \quad \mathbf{V}_1 = -2D\varrho_0 \quad \text{on } \Gamma.$$

The boundary-forcing function \mathbf{V}_1^3 appearing in (A.27c) is the normal component of \mathbf{V}_1 , and the function ϱ_0 is the solution of the elliptic Dirichlet problem (A.8). The boundary conditions (A.27c) and (A.27d) imply that $\mathbf{v}_1 = -2D\rho_0$ defined by (A.12) satisfies

$$(A.30) \quad \mathbf{v}_1 = \mathbf{V}_1 \quad \text{on } \Gamma.$$

Polyharmonic regularity provides for $s \geq 0$ a positive constant C independent of μ that

$$\begin{aligned} & \|\sqrt{\mu} \rho_0\|_{H^{s+6}(\Omega)}^2 \\ & \leq C \left(\|\varphi^\mu * \mathcal{E}_\Omega(\varrho_0)\|_{H^s(\Omega)}^2 + \|\Delta \rho_0\|_{H^{s+3.5}(\Gamma)}^2 + \left\| \frac{\partial \rho_0}{\partial \mathbf{N}} \right\|_{H^{s+4.5}(\Gamma)}^2 + \|\rho_0\|_{H^{s+5.5}(\Gamma)}^2 \right). \end{aligned}$$

Recalling section A.1.2, the arguments establishing the lower-bound (A.10) provide that

$$(A.31) \quad \rho_0 \geq 2\lambda > 0 \quad \text{in } \overline{\Omega}.$$

Similarly, we infer from the arguments given in section A.1.4 that the boundary condition (A.27b) and the identity (A.30) establish that

$$(A.32) \quad \rho_0^2 J_2 = \sigma H_2 \quad \text{on } \Gamma.$$

Hence, the C^∞ -class data $(\rho_0, \mathbf{v}_0, \Omega)$, where ρ_0 is the solution of (A.27) and \mathbf{v}_0 is given by (A.23), satisfy the conditions (A.2) for $a = 0, 1, 2$ as verified by (A.11), (A.9), (A.26), (A.31), and (A.32).

Remark 17. According to (A.18), the condition (A.2c) for $a = 3$ may be imposed by prescribing a Dirichlet boundary condition for $\frac{\partial}{\partial N} \Delta \mathbf{v}_0^3$ in a polyharmonic problem for \mathbf{v}_0^3 satisfying $\mathbf{v}_0^3 + \epsilon \Delta^4 \mathbf{v}_0^3 = \varphi^\epsilon * \mathcal{E}_\Omega(u_0^3)$.

A.2. The C^∞ -class data for the zero surface tension limit of (1.7). We suppose the initial data (ρ_0, u_0, Ω) is of the optimal regularity stated in Theorem 1.2 and satisfy the conditions (1.13), (1.15), and (1.16). Then setting $\sigma = 0$ in the construction of the C^∞ -class data of section A.1, $\mathbf{J}_a = \partial_t^a (J^{-2})|_{t=0}$ satisfies

$$J_0 = \beta, \quad J_1 = 0, \quad J_2 = 0, \quad \text{and} \quad J_3 = 0 \quad \text{on } \Gamma.$$

For $\mu > 0$, and with ϱ_0 defined by (A.6), ϱ_0 solving (A.8), we define $\rho_0 \in C^\infty(\Omega)$ to be the solution of the polyharmonic problem

$$(A.33a) \quad \rho_0 - \mu \Delta^5 \rho_0 = \varphi^\mu * \mathcal{E}_\Omega(\varrho_0) \quad \text{in } \Omega,$$

$$(A.33b) \quad \Delta^2 \rho_0 = \mathbf{f}_3 \quad \text{on } \Gamma,$$

$$(A.33c) \quad \frac{\partial \Delta \rho_0}{\partial N} = \frac{\partial \Delta \varrho_0}{\partial N} \quad \text{on } \Gamma,$$

$$(A.33d) \quad -2\Delta \rho_0 = \mathbf{f}_1 \quad \text{on } \Gamma,$$

$$(A.33e) \quad -2 \frac{\partial \rho_0}{\partial N} = \mathbf{V}_1^3 \quad \text{on } \Gamma,$$

$$(A.33f) \quad \rho_0 = \sqrt{\beta} \quad \text{on } \Gamma,$$

where the boundary-forcing function \mathbf{f}_3 is defined below in (A.36). By section A.1.5,

$$(A.34) \quad \rho_0 \geq \lambda > 0 \quad \text{in } \bar{\Omega}.$$

According to the definition (A.12) of \mathbf{v}_a ,

$$\mathbf{v}_3 = -2[\mathcal{A}_2]^k \rho_{0,k} - 4[\mathcal{A}_1]^k (\rho_0 \mathbf{K}_0)_{,k} - 2D(\rho_0 \mathbf{K}_1) \quad \text{in } \Omega,$$

where $\mathbf{K}_a = \partial_t^a (J^{-1}(J^{-1} J_t))|_{t=0} = \partial_t^a (J^{-1}(A_r^s v^r, s))|_{t=0}$. We compute

$$\mathbf{J}_4 = 16\mathbf{k}_0^4 - 48\mathbf{k}_0^2 \mathbf{k}_1 + 12\mathbf{k}_1^2 + 16\mathbf{k}_0 \mathbf{k}_2 - 2\mathbf{k}_3.$$

Setting $\mathbf{J}_4 = 0$ yields an equation for \mathbf{k}_3 . Since $\mathbf{k}_3 = [\mathcal{A}_2]_r^s v_{0,s}^r + 3[\mathcal{A}_2]_r^s v_{1,s}^r + 3[\mathcal{A}_1]_r^s v_{2,s}^r + \mathbf{j}_3$,

$$\mathbf{j}_3 = \frac{1}{2} [16\mathbf{k}_0^4 - 48\mathbf{k}_0^2 \mathbf{k}_1 + 12\mathbf{k}_1^2 + 16\mathbf{k}_0 \mathbf{k}_2] - [\mathcal{A}_2]_r^s v_{0,s}^r + 3[\mathcal{A}_2]_r^s v_{1,s}^r + 3[\mathcal{A}_1]_r^s v_{2,s}^r.$$

On the other hand, the divergence of $\mathbf{v}_3 = -2[\mathcal{A}_2]^k \rho_{0,k} - 4[\mathcal{A}_1]^k (\rho_0 \mathbf{K}_0)_{,k} - 2D(\rho_0 \mathbf{K}_1)$ yields

$$\text{div } \mathbf{v}_3 = \text{div} [-2[\mathcal{A}_2]^k \rho_{0,k} - 4[\mathcal{A}_1]^k (\rho_0 \mathbf{K}_0)_{,k} - 2D\rho_0 \mathbf{K}_1] - 2\rho_{0,i} \mathbf{K}_{1,i} - 2\rho_0 \Delta \mathbf{K}_1.$$

We solve for $\Delta \mathbf{K}_1$ to write

$$\Delta \mathbf{K}_1 = \frac{1}{2\rho_0} \underbrace{\left[-\mathbf{j}_3 + \operatorname{div} \left[-2[\mathcal{A}_2]^k \rho_{0,k} - 4[\mathcal{A}_1]^k (\rho_0 \mathbf{K}_0)_{,k} - 2D\rho_0 \mathbf{K}_1 \right] - 2\rho_{0,i} \mathbf{K}_{1,i} \right]}_{\mathfrak{J}_3}.$$

Using that $\mathbf{K}_1 = -\mathbf{k}_0^2 + \mathbf{k}_1$ and $\mathbf{k}_1 = [\mathcal{A}_1]_r^s \mathbf{v}_{0,s}^r + \mathbf{j}_1$, we find that

$$(A.35) \quad \Delta \mathbf{j}_1 = \mathfrak{J}_3 + \Delta \mathbf{k}_0^2 - \Delta([\mathcal{A}_1]_r^s \mathbf{v}_{0,s}^r).$$

Since $\mathbf{v}_1 = -2D\rho_0$ implies that $\mathbf{j}_1 = -2\Delta\rho_0$, we equivalently write (A.35) as

$$(A.36) \quad \Delta^2 \rho_0 = -\frac{1}{2} \underbrace{\left[\mathfrak{J}_3 + \Delta \mathbf{k}_0^2 - \Delta([\mathcal{A}_1]_r^s \mathbf{v}_{0,s}^r) \right]}_{f_3} \quad \text{on } \Gamma.$$

The quantity $D\mathbf{K}_1$ may be expressed using the boundary conditions (A.33c)–(A.33f) and $D^a \mathbf{v}_0$, $a = 0, 1, 2, 3$, may be expressed using the boundary conditions of the polyharmonic problems defining \mathbf{v}_0 . It follows from (A.12) and (A.36) that $\mathbf{J}_a = \partial_t^a (J^{-2})|_{t=0}$ satisfies

$$(A.37) \quad \mathbf{J}_a = \partial_t^a \beta \quad \text{on } \Gamma \text{ for } a = 0, 1, 2, 3, 4.$$

With $0 < 2\nu \leq -\frac{\partial \rho_0^2}{\partial N}$, we may take ϵ sufficiently small so that $\boldsymbol{\varrho}_0$ defined in (A.8) satisfies

$$0 < \nu \leq -\frac{\partial \boldsymbol{\varrho}_0^2}{\partial N} \quad \text{on } \Gamma.$$

We then have that (A.29) defining the vector field $\mathbf{V}_1 \in C^\infty(\Gamma)$ is

$$\mathbf{V}_1 = -\frac{\partial \boldsymbol{\varrho}_0^2}{\partial N} \mathbf{N} \quad \text{on } \Gamma.$$

The boundary condition (A.33e) of the polyharmonic problem (A.33) defining ρ_0 provides that

$$0 < \nu \leq -\frac{\partial \rho_0^2}{\partial N} \quad \text{on } \Gamma.$$

The conditions (A.33e), (A.34), and (A.37) establish that the C^∞ -class initial data $(\rho_0, \mathbf{v}_0, \boldsymbol{\Omega})$ satisfy

$$(A.38a) \quad \rho_0 \geq \lambda > 0 \quad \text{in } \overline{\Omega},$$

$$(A.38b) \quad -\frac{\partial \rho_0^2}{\partial N} \geq \nu > 0 \quad \text{on } \Gamma,$$

$$(A.38c) \quad \rho_0^2 \mathbf{J}_a = \partial_t^a \beta \quad \text{on } \Gamma \text{ for } a = 0, 1, 2, 3, 4.$$

The conditions (A.38) are analogous statements of the conditions (1.13), (1.15), and (1.16).

Remark 18. According to (A.12),

$$\operatorname{div} \mathbf{v}_4 = \operatorname{div} \left[-2[\mathcal{A}_3]^k \rho_{0,k} - 6[\mathcal{A}_2]^k (\rho_0 \mathbf{K}_0)_{,k} - 2[\mathcal{A}_1]^k (\rho_0 \mathbf{K}_1)_{,k} - 2D\rho_0 \mathbf{K}_2 \right] - 2\rho_{0,i} \mathbf{K}_{2,i} - 2\rho_0 \Delta \mathbf{K}_2.$$

Since $\Delta \mathbf{K}_2$ is approximately $\Delta^2 \operatorname{div} \mathbf{v}_0$, the boundary condition $\mathbf{J}_5 = 0$ may be imposed by prescribing a Dirichlet boundary condition for $\frac{\partial}{\partial N} \Delta^2 \mathbf{v}_0^3$ in a polyharmonic problem for \mathbf{v}_0^3 satisfying $\mathbf{v}_0^3 + \epsilon \Delta^6 \mathbf{v}_0^3 = \varphi^\epsilon * \mathcal{E}_\Omega(\mathbf{u}_0^3)$.

The boundary condition $\mathbf{J}_6 = 0$ is obtained by defining a Dirichlet boundary condition for $\Delta^3 \boldsymbol{\rho}_0$ in a polyharmonic problem for $\boldsymbol{\rho}_0$ satisfying $\boldsymbol{\rho}_0 - \mu \Delta^7 \boldsymbol{\rho}_0 = \varphi^\mu * \mathcal{E}_\Omega(\boldsymbol{\rho}_0)$.

Appendix B. Solutions to the μ -problem (5.52). In this appendix, we construct a fixed-point solution \hat{v} to the μ -problem (5.52).

PROPOSITION B.1 (solutions to the μ -problem). *For C^∞ -class initial data (ρ_0, u_0, Ω) satisfying the conditions (5.50), and for some $T = T_\kappa(\epsilon\mu) > 0$, there exists a unique $\hat{v} \in L^2(0, T; H^9(\Omega))$ solving the μ -problem (5.52) on a time-interval $[0, T]$ with $\partial_t^a \hat{v} \in L^2(0, T; H^{9-2a}(\Omega))$ for $a = 1, 2, 3, 4$, $\hat{v}_{ttt} \in L^\infty(0, T; L^2(\Omega))$, and $(\hat{v}, \hat{v}_t, \dots, \hat{v}_{tttt})|_{t=0} = (u_0, \mathbf{v}_1, \dots, \mathbf{v}_4)$.*

B.1. The functional framework for the fixed-point scheme. To establish the existence and uniqueness of solutions to the μ -problem (5.52), we define for $T > 0$ the Hilbert space

$$\mathbf{X}_T = \{v \in L^2(0, T; H^5(\Omega)) \mid \partial_t^a v \in L^2(0, T; H^{5-2a}(\Omega)) \text{ for } a = 1, 2\},$$

endowed with the natural Hilbert norm

$$\|v\|_{\mathbf{X}_T}^2 = \sum_{a=0}^2 \|\partial_t^a v\|_{L^2(0, T; H^{5-2a}(\Omega))}^2.$$

For $\mathcal{M} > 0$ (where the particular value of \mathcal{M} is specified later), we define the closed, bounded, convex subset $\mathcal{C}_T(\mathcal{M})$ of \mathbf{X}_T

$$(B.1) \quad v \in \mathcal{C}_T(\mathcal{M}) \subset \mathbf{X}_T$$

to be all $v \in \mathbf{X}_T$ that satisfy each of the following conditions:

$$(X1) \quad (v, v_t, v_{tt})|_{t=0} = (u_0, \mathbf{v}_1, \mathbf{v}_2) \text{ and}$$

$$(X2) \quad \|v\|_{\mathbf{X}_T}^2 \leq \mathcal{M}.$$

LEMMA B.1 (solutions to the μ -problem in \mathbf{X}_T). *For C^∞ -class initial data (ρ_0, u_0, Ω) satisfying the conditions (1.13) and (1.14), and for some $T = T_\kappa(\epsilon\mu) > 0$, there exists a unique $\hat{v} \in \mathbf{X}_T$ that solves the μ -problem (5.52) on a time-interval $[0, T]$ and verifies $(\hat{v}, \hat{v}_t, \hat{v}_{tt})|_{t=0} = (u_0, \mathbf{v}_1, \mathbf{v}_2)$.*

B.2. Linearizing the μ -problem (5.52). Letting $\bar{v} \in \mathcal{C}_T(\mathcal{M})$, we set $\bar{\eta} = e + \int_0^t \bar{v}$ and $\bar{\eta}_\epsilon = e + \int_0^t \bar{v}_\epsilon$ on Γ . We define $\bar{\zeta}_\epsilon$ to be the solution of the following time-dependent elliptic Dirichlet problem:

$$(B.2a) \quad \Delta \bar{\zeta}_\epsilon = \Delta \bar{\eta} \quad \text{in } \Omega,$$

$$(B.2b) \quad \bar{\zeta}_\epsilon = \bar{\eta}_\epsilon \quad \text{on } \Gamma.$$

We define the following ϵ -approximate Lagrangian variables:

$$\bar{A}_\epsilon = [D\bar{\zeta}_\epsilon]^{-1}, \quad \bar{J}_\epsilon = \det D\bar{\zeta}_\epsilon, \quad \bar{a}_\epsilon = \bar{J}_\epsilon \bar{A}_\epsilon, \quad [\bar{g}_\epsilon]_{\alpha\beta} = \bar{\zeta}_{\epsilon,\alpha} \cdot \bar{\zeta}_{\epsilon,\beta}, \quad \text{and } \sqrt{\bar{g}_\epsilon} \bar{n}_\epsilon = [\bar{a}_\epsilon]^T N.$$

We assume that $T > 0$ is given such that independently of the choice of $\bar{v} \in \mathcal{C}_T(\mathcal{M})$,

the ϵ -approximate Lagrangian map $\bar{\zeta}_\epsilon$ is injective for all $t \in [0, T]$ and that

$$(B.3) \quad \frac{1}{2} \leq \bar{J}(t) \leq \frac{3}{2} \quad \text{and} \quad \frac{1}{2} \leq \bar{J}_\epsilon(t) \leq \frac{3}{2} \quad \forall t \in [0, T] \quad \text{and} \quad x \in \bar{\Omega}.$$

This is possible by the inequality (4.4) as $\bar{v} \in C_T(\mathcal{M})$ satisfies $\|\bar{v}\|_{\mathbf{X}_T}^2 \leq \mathcal{M}$.

DEFINITION B.1 (the system of linear heat-equations for v). For $\bar{v} \in C_T(\mathcal{M})$, $\kappa > 0$, $\epsilon > 0$, and $\mu > 0$ given, we define v to be the solution of the system of linear equations

$$(B.4a) \quad v_t - \bar{\varrho} [\bar{A}_\epsilon]_r^j ([\bar{A}_\epsilon]_r^k v, k)_{,j} = \bar{K} \quad \text{in } \Omega \times (0, T_\kappa(\epsilon\mu)),$$

$$(B.4b) \quad \bar{\varrho} N^j [\bar{A}_\epsilon]_r^j [\bar{A}_\epsilon]_r^s v, s = \bar{h}^\mu + c^\mu(t) \quad \text{on } \Gamma \times (0, T_\kappa(\epsilon\mu)),$$

$$(B.4c) \quad v|_{t=0} = u_0 \quad \text{on } \Omega.$$

The bounded, nonnegative function $\bar{\varrho}$ is defined as

$$(B.5) \quad \bar{\varrho} = \kappa \rho_0 \bar{J}_\epsilon.$$

The vector field \bar{K} appearing in the right-hand side of (B.4a) is given by

$$(B.6) \quad \bar{K} = \bar{\varrho} \operatorname{curl}_{\bar{\zeta}_\epsilon} \left(\operatorname{curl} u_0 + \varepsilon_{,ji} \int_0^t \partial_t [\bar{A}_\epsilon]_j^s \bar{v}^i, s \right) + \operatorname{div}_{\bar{\zeta}_\epsilon} \bar{v} [\bar{A}_\epsilon]_r^k \bar{\varrho}_{,k} - 2 [\bar{A}_\epsilon]_r^k (\rho_0 \bar{J}^{-1})_{,k}.$$

The vector field \bar{h}^μ appearing in the right-hand side of (B.4b) is given by

$$(B.7) \quad \begin{aligned} \bar{h}^\mu = & \bar{h}_{\operatorname{curl}} + \bar{h}_{\operatorname{div}} + \sum_{l=1}^K \sqrt{\xi_l} \left(\Lambda_\mu \left[\bar{\mathfrak{g}}_{\epsilon,l}^{\alpha\beta} \Lambda_\mu \left[(\sqrt{\xi_l} \bar{v}_{,\alpha\beta} \cdot \bar{n}_\epsilon) \circ \theta_l \right] \right] \right) \\ & \circ \theta_l^{-1} \bar{n}_\epsilon + \bar{\varrho} \left[\bar{b}_{\operatorname{curl}}^\mu + \bar{b}_{\operatorname{div}}^\mu \right], \end{aligned}$$

where

$$(B.8) \quad \bar{\mathfrak{g}}_{\epsilon,l}^{\alpha\beta} = \kappa \frac{\sqrt{g_\epsilon}}{\rho_0} g_\epsilon^{\alpha\beta} \circ \theta_l,$$

and the vectors $\bar{h}_{\operatorname{curl}}$, $\bar{h}_{\operatorname{div}}$, $\bar{b}_{\operatorname{curl}}^\mu$, and $\bar{b}_{\operatorname{div}}^\mu$ are defined as

$$(B.9a) \quad \bar{h}_{\operatorname{curl}} = \bar{\varrho} \sqrt{g_\epsilon} (\bar{J}_\epsilon)^{-1} \left(\operatorname{curl} u_0 + \varepsilon_{,ji} \int_0^t \partial_t [\bar{A}_\epsilon]_j^s \bar{v}^i, s \right) \times \bar{n}_\epsilon,$$

$$(B.9b) \quad \bar{h}_{\operatorname{div}} = \frac{\sqrt{g_\epsilon}}{\rho_0} \left[\rho_0^2 (\bar{J}_\epsilon)^{-2} - \beta_\epsilon(t) + \sigma \bar{g}_\epsilon^{\alpha\beta} \bar{\zeta}_{\epsilon,\alpha\beta} \cdot \bar{n}_\epsilon \right] \bar{n}_\epsilon,$$

$$(B.9c) \quad \bar{b}_{\operatorname{curl}}^\mu = \frac{\sqrt{g_\epsilon}}{\bar{J}_\epsilon} \left[\sum_{l=1}^K \bar{g}_\epsilon^{\alpha\beta} \sqrt{\xi_l} \Lambda_\mu \left[(\sqrt{\xi_l} \bar{v} \cdot \bar{n}_\epsilon)_{,\beta} \circ \theta_l \right] \circ \theta_l^{-1} + \bar{v}^\gamma \bar{g}_\epsilon^{\gamma\delta} \bar{g}_\epsilon^{\alpha\beta} \bar{\zeta}_{\epsilon,\delta\beta} \cdot \bar{n}_\epsilon \right] \bar{\zeta}_{\epsilon,\alpha},$$

$$(B.9d) \quad \bar{b}_{\operatorname{div}}^\mu = - \sum_{l=1}^K \sqrt{\xi_l} \left(\Lambda_\mu \left[\sqrt{\xi_l} \frac{\sqrt{g_\epsilon}}{\bar{J}_\epsilon} \left[\bar{g}_\epsilon^{\alpha\beta} \bar{v}_{,\beta}^\alpha + \bar{v}^\alpha (\sqrt{g_\epsilon} \bar{g}_\epsilon^{\alpha\beta})_{,\beta} + \bar{v}^3 H(\bar{\zeta}_\epsilon) \right] \circ \theta_l \right] \right) \circ \theta_l^{-1} \bar{n}_\epsilon.$$

The vector field $c^\mu(t)$ appearing in the right-hand side of (B.4b) is given by (5.58).

B.3. Implementation of the fixed-point scheme to solve the μ -problem (5.52). We will allow constants to depend on $1/\delta\kappa\epsilon\mu$ in our fixed-point scheme.

DEFINITION B.2 (notational convention for constants depending on $1/\delta\kappa\epsilon\mu > 0$). Given $\bar{v} \in C_T(\mathcal{M})$, we let $\ddot{\mathcal{P}}$ denote a generic polynomial with constant and coefficients depending on $1/\delta\kappa\epsilon\mu > 0$. We define the constant $\ddot{N}_0 > 0$ by

$$\ddot{N}_0 = \ddot{\mathcal{P}}(\|u_0\|_{100}, \|\rho_0\|_{100}).$$

We let $\ddot{\mathcal{R}}$ denote generic lower-order terms satisfying

$$\int_0^T \ddot{\mathcal{R}} \leq \ddot{N}_0 + \delta \|\bar{v}\|_{\mathbf{X}_T}^2 + T \ddot{\mathcal{P}}(\|\bar{v}\|_{\mathbf{X}_T}^2).$$

LEMMA B.2. Given $\bar{v} \in C_T(\mathcal{M})$, $\kappa > 0$, $\epsilon > 0$, and $\mu > 0$, there exists a unique $v \in \mathbf{X}_T$ solving (B.4) and satisfying 1. Furthermore,

$$(B.10) \quad \|v\|_{\mathbf{X}_T}^2 \leq \int_0^T \ddot{\mathcal{R}}.$$

Proof. Standard parabolic theory provides for the existence and uniqueness of $v \in \mathbf{X}_T$ solving (B.4) and satisfying 1. For the purpose of establishing the estimate (B.10), it is useful to note the scaling relations

$$\bar{\mathcal{K}} \sim D\bar{v} + \int_0^t D^2\bar{v} \quad \text{and} \quad \bar{h}^\mu \sim \bar{v} + \int_0^t D\bar{v} + \Lambda_\mu \bar{\partial}^2\bar{v} + \bar{h}_\epsilon,$$

where $\bar{h}_\epsilon \in C^\infty(\Gamma)$ with $|\bar{h}_\epsilon|_s \leq C_\epsilon |\bar{v}|_s$ thanks to the inequality (5.2). We will establish the inequality (B.10) in the following three steps.

Step 1: Parabolic estimates for v_{tt} . Testing two time-derivatives of (B.4a) against v_{tt} in the $L^2(\Omega)$ -inner product and integrating by parts with respect to ∂_j in the integral $-\int_\Omega \partial_t^2 (\bar{\varrho} [\bar{A}_\epsilon]^j ([\bar{A}_\epsilon]^k v_{t,k})_{,j}) v_{tt}$ yields

$$(B.11) \quad \frac{1}{2} \frac{d}{dt} \int_\Omega |v_{tt}|^2 + \int_\Omega \bar{\varrho} |[\bar{A}_\epsilon]^k v_{tt,k}|^2 = \underbrace{\int_\Omega \bar{\mathcal{K}}_{tt} v_{tt}}_{\ddot{\mathcal{R}}} + \underbrace{\int_\Gamma \bar{h}_{tt}^\mu v_{tt}}_{\ddot{\mathcal{R}}} + \ddot{\mathcal{R}}.$$

We used the boundary condition (B.4b) in writing the boundary integral \mathbf{i} . Thanks to Lemma 2.1, we have that $|v_{tt}|_0^2 = \ddot{\mathcal{R}}$. Hence, $\mathbf{i} = \ddot{\mathcal{R}}$. Taking the time-integral of (B.11) and using that $\bar{\varrho} \geq \lambda_\epsilon > 0$,

$$(B.12) \quad \sup_{t \in [0, T]} \|v_{tt}(t)\|_0^2 + \int_0^T \|v_{tt}\|_1^2 \leq \int_0^T \ddot{\mathcal{R}}.$$

Step 2: Elliptic estimates for v_t . A time-derivative of (B.4) yields the following linear Neumann-type elliptic problem for v_t :

$$(B.13a) \quad -\bar{\varrho} [\bar{A}_\epsilon]^j ([\bar{A}_\epsilon]^k v_{t,k})_{,j} = \bar{\mathcal{G}}_1 \quad \text{in } \Omega,$$

$$(B.13b) \quad \bar{\varrho} N^j [\bar{A}_\epsilon]^j [\bar{A}_\epsilon]^s v_{t,s} = \bar{\mathcal{J}}_1 \quad \text{on } \Gamma,$$

where the forcing functions $\bar{\mathcal{G}}_1$ and $\bar{\mathcal{J}}_1$ are defined as

$$\begin{aligned} \bar{\mathcal{G}}_1 &= \bar{\mathcal{K}}_t - v_{tt} + (\bar{\varrho} [\bar{A}_\epsilon]^j)_t ([\bar{A}_\epsilon]^k v_{t,k})_{,j} + \bar{\varrho} [\bar{A}_\epsilon]^j (\partial_t [\bar{A}_\epsilon]^k v_{t,k})_{,j}, \\ \bar{\mathcal{J}}_1 &= \partial_t [\bar{h}^\mu + c^\mu(t)] - (\bar{\varrho} N^j [\bar{A}_\epsilon]^j [\bar{A}_\epsilon]^s)_t v_{t,s}. \end{aligned}$$

Since $\bar{\varrho}[\bar{A}_\epsilon]_r^j \partial_j [[\bar{A}_\epsilon]_r^k \partial_k]$ is a uniformly elliptic operator, standard elliptic regularity theory provides that for $s \geq 2$

$$(B.14) \quad \|v_t\|_s^2 \leq C(\|v_t\|_0^2 + \|\bar{\mathcal{G}}_1\|_{s-2}^2 + |\bar{j}_1|_{s-\frac{3}{2}}^2)$$

with the constant C depending on the coefficients of the elliptic and Neumann-type operators. The fundamental theorem of calculus provides that

$$\|v_t\|_0^2 \leq \ddot{\mathcal{N}}_0 + CT \sup_{t \in [0, T]} \|v_{tt}(t)\|_0^2 \leq \int_0^T \ddot{\mathcal{R}}.$$

Thanks to the estimate (B.12), we conclude that

$$\sup_{t \in [0, T]} \|\bar{\mathcal{G}}_1(t)\|_0^2 + \int_0^T \|\bar{\mathcal{G}}_1\|_1^2 \leq \int_0^T \ddot{\mathcal{R}}.$$

Since \bar{j}_1 scales like $Dv + D\bar{v} + \bar{v}_t$, the fundamental theorem of calculus provides that

$$\sup_{t \in [0, T]} |\bar{j}_1(t)|_{0.5}^2 + \int_0^T |\bar{j}_1|_{1.5}^2 \leq \int_0^T \ddot{\mathcal{R}} + \sup_{t \in [0, T]} \|\bar{v}_t(t)\|_1^2.$$

Since interpolation and Young’s inequality provide that $\|\bar{v}_t\|_1^2 \leq C_\delta \|\bar{v}_t\|_0^2 + \delta \|\bar{v}_t\|_2^2$, we conclude by use of the fundamental theorem of calculus that

$$\sup_{t \in [0, T]} |\bar{j}_1(t)|_{0.5}^2 + \int_0^T |\bar{j}_1|_{1.5}^2 \leq \int_0^T \ddot{\mathcal{R}}.$$

It therefore follows from the inequality (B.14) that

$$(B.15) \quad \sup_{t \in [0, T]} \|v_t(t)\|_2^2 + \int_0^T \|v_t\|_3^2 \leq \int_0^T \ddot{\mathcal{R}}.$$

Step 3: Concluding the proof of Lemma B.2. By repeating Step 2, we infer the following elliptic estimate for v :

$$(B.16) \quad \sup_{t \in [0, T]} \|v(t)\|_4^2 + \int_0^T \|v\|_5^2 \leq \int_0^T \ddot{\mathcal{R}}.$$

The sum of the inequalities (B.12), (B.15), and (B.16) establishes the inequality (B.10). \square

LEMMA B.3. *Let v be the solution of (B.4) determined by $\bar{v} \in \mathcal{C}_T(\mathcal{M})$. The solutions map*

$$\mathfrak{S}: \mathcal{C}_T(\mathcal{M}) \rightarrow \mathcal{C}_T(\mathcal{M}): \bar{v} \mapsto v$$

is a well-defined map for some $T = T_\kappa(\epsilon\mu)$ and possesses a unique fixed-point.

Proof. We recall that $v \in \mathbf{X}_T$ is unique and satisfies 1 by Lemma B.2 . Setting $\mathcal{M} = \ddot{\mathcal{N}}_0 + 1$, we take δ so that $\delta \|\bar{v}\|_{\mathbf{X}_T}^2 < \frac{1}{2}$ and let $T = T_\kappa(\epsilon\mu)$ be sufficiently small so that the inequality given in Lemma B.2 reads

$$\|v\|_{\mathbf{X}_T}^2 \leq \mathcal{M}.$$

Hence, for some $T = T_\kappa(\epsilon\mu) > 0$, the solutions map \mathfrak{S} is well defined.

Letting $\bar{v}_l \in \mathcal{C}_T(\mathcal{M})$ for $l = 1, 2$, we set $\bar{\eta}_l = e + \int_0^t \bar{v}_l$ in Ω and $\bar{\eta}_{\epsilon l} = e + \int_0^t \bar{v}_{\epsilon l}$ on Γ . We define $\bar{\zeta}_{\epsilon l}$ to be the solution of the following time-dependent elliptic Dirichlet problem:

$$\begin{aligned} \text{(B.17a)} \quad & \Delta \bar{\zeta}_{\epsilon l} = \Delta \bar{\eta}_l \quad \text{in } \Omega, \\ \text{(B.17b)} \quad & \bar{\zeta}_{\epsilon l} = \bar{\eta}_{\epsilon l} \quad \text{on } \Gamma. \end{aligned}$$

We define the following ϵ -approximate Lagrangian variables:

$$\bar{A}_{\epsilon l} = [D\bar{\zeta}_{\epsilon l}]^{-1}, \quad \bar{J}_l = \det D\bar{\zeta}_{\epsilon l}, \quad \bar{a}_{\epsilon l} = \bar{J}_{\epsilon l} \bar{A}_{\epsilon l}, \quad [\bar{g}_{\epsilon l}]_{\alpha\beta} = \bar{\zeta}_{\epsilon l, \alpha} \cdot \bar{\zeta}_{\epsilon l, \beta}, \quad \sqrt{\bar{g}_{\epsilon l}} \bar{n}_{\epsilon l} = [\bar{a}_{\epsilon l}]^T N.$$

For $l = 1, 2$, we also take the quantities $\bar{\varrho}_l, \bar{\mathcal{K}}_l, \bar{h}_l^\mu$ to be, respectively, formed via the definitions (B.5), (B.6), (B.7) with $\bar{v} = \bar{v}_l$. Letting v_l denote the solution of (B.4) formed using $\bar{v} = \bar{v}_l$, we have that the difference

$$w = v_1 - v_2$$

satisfies

$$\begin{aligned} \text{(B.18a)} \quad & w_t - \bar{\varrho}_1 [\bar{A}_{\epsilon 1}]_r^j ([\bar{A}_{\epsilon 1}]_r^k w_{,k})_{,j} = F \quad \text{in } \Omega \times (0, T_\kappa(\epsilon\mu)), \\ \text{(B.18b)} \quad & \bar{\varrho}_1 N^j [\bar{A}_{\epsilon 1}]_r^j [\bar{A}_{\epsilon 1}]_r^s w_{,s} = G \quad \text{on } \Gamma \times (0, T_\kappa(\epsilon\mu)), \\ \text{(B.18c)} \quad & w|_{t=0} = 0 \quad \text{on } \Omega, \end{aligned}$$

where the vector field F appearing in the right-hand side of (B.18a) is given by

$$\text{(B.19)} \quad F = \bar{\mathcal{K}}_1 - \bar{\mathcal{K}}_2 + \underbrace{\bar{\varrho}_1 [\bar{A}_{\epsilon 1}]_r^j ([\bar{A}_{\epsilon 1}]_r^k v_{2,k})_{,j} - \bar{\varrho}_2 [\bar{A}_{\epsilon 2}]_r^j ([\bar{A}_{\epsilon 2}]_r^k v_{2,k})_{,j}}_j,$$

and the vector field G appearing in the right-hand side of (B.18b) is given by

$$\text{(B.20)} \quad G = \bar{h}_1^\mu - \bar{h}_2^\mu - \underbrace{[\bar{\varrho}_1 N^j [\bar{A}_{\epsilon 1}]_r^j [\bar{A}_{\epsilon 1}]_r^k v_{2,k} + \bar{\varrho}_2 N^j [\bar{A}_{\epsilon 2}]_r^j [\bar{A}_{\epsilon 2}]_r^k v_{2,k}]}_j.$$

Testing two time-derivatives of (B.18a) against w_{tt} in the $L^2(\Omega)$ -inner product and integrating by parts in the integrals $-\int_\Omega \partial_t^2 (\bar{\varrho}_1 [\bar{A}_{\epsilon 1}]_r^j ([\bar{A}_{\epsilon 1}]_r^k w_{,k})_{,j}) w_{tt}$ and $\int_\Omega \partial_{tt} w_{tt}$ yields

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \int_\Omega |w_{tt}|^2 + \int_\Omega \partial_t^2 (\bar{\varrho}_1 [\bar{A}_{\epsilon 1}]_r^j [\bar{A}_{\epsilon 1}]_r^k w_{,k}) w_{tt, j} \\ & \quad + \int_\Omega \partial_t^2 ((\bar{\varrho}_1 [\bar{A}_{\epsilon 1}]_r^j)_{,j} [\bar{A}_{\epsilon 1}]_r^k w_{,k}) w_{tt} - \int_\Gamma (\bar{h}_1^\mu - \bar{h}_2^\mu)_{tt} w_{tt} \\ & = \int_\Omega \partial_t^2 [\bar{\mathcal{K}}_1 - \bar{\mathcal{K}}_2 - ((\bar{\varrho}_1 [\bar{A}_{\epsilon 1}]_r^j)_{,j} [\bar{A}_{\epsilon 1}]_r^k - (\bar{\varrho}_2 [\bar{A}_{\epsilon 2}]_r^j)_{,j} [\bar{A}_{\epsilon 2}]_r^k) v_{2,k}] w_{tt} \\ \text{(B.21)} \quad & - \int_\Omega \partial_t^2 [(\bar{\varrho}_1 [\bar{A}_{\epsilon 1}]_r^j [\bar{A}_{\epsilon 1}]_r^k - \bar{\varrho}_2 [\bar{A}_{\epsilon 2}]_r^j [\bar{A}_{\epsilon 2}]_r^k) v_{2,k}] w_{tt, j}. \end{aligned}$$

We have used that two time-derivatives of the boundary condition (B.18b) is equivalent to

$$\partial_t^2 [\bar{\varrho}_1 N^j [\bar{A}_{\epsilon 1}]_r^j [\bar{A}_{\epsilon 1}]_r^s w_{,s} + j] = (\bar{h}_1^\mu - \bar{h}_2^\mu)_{tt}.$$

Since $\partial_t^2(\bar{\varrho}_1 - \bar{\varrho}_2) = \int_0^t \partial_t^3(\bar{\varrho}_1 - \bar{\varrho}_2)$ scales like $\int_0^T D(\bar{v}_1 - \bar{v}_2)_{tt}$, we infer that the time-integral of (B.21) yields

$$(B.22) \quad \sup_{t \in [0, T]} \|w_{tt}(t)\|_0^2 + \int_0^T \|w_{tt}\|_1^2 \leq \delta \int_0^T \|\bar{v}_1 - \bar{v}_2\|_{\mathbf{X}_T}^2 + C_{\delta\kappa\mu} T \|\bar{v}_1 - \bar{v}_2\|_{\mathbf{X}_T}^2.$$

We use the estimate (B.22) and follow Steps 2 and 3 of the proof of Lemma B.2 to obtain estimates for w_t in $L^2(0, T; H^3(\Omega))$ and w in $L^2(0, T; H^5(\Omega))$. Thus,

$$(B.23) \quad \|w\|_{\mathbf{X}_T}^2 \leq \delta \int_0^T \|\bar{v}_1 - \bar{v}_2\|_{\mathbf{X}_T}^2 + C_{\delta\kappa\mu} T \|\bar{v}_1 - \bar{v}_2\|_{\mathbf{X}_T}^2.$$

We recall that $w = v_1 - v_2$. The inequality (B.23) therefore provides that \mathfrak{S} is a contraction mapping for sufficiently small $\delta > 0$ and $T = T_\kappa(\epsilon\mu) > 0$. Hence, the solutions mapping \mathfrak{S} possesses a unique fixed-point $\check{v} \in \mathcal{C}_T(\mathcal{M})$. \square

B.4. The proof of Lemma B.1. The proof of Lemma B.1 is complete since the fixed-point $\check{v} \in \mathcal{C}_T(\mathcal{M})$ of Lemma B.3 verifies 1 and is the unique solution of the μ -problem (5.52).

B.5. The proof of Proposition B.1. We infer from the proof of Lemma B.1 that considering two more time-derivatives in the functional framework of our fixed-point scheme establishes Proposition B.1.

Appendix C. Equivalence of the $\kappa\epsilon$ -problem (5.6) and the heat-type $\kappa\epsilon$ -problem (5.49). In this appendix, we prove the following.

LEMMA C.1. *The $\kappa\epsilon$ -problem (5.6) and the heat-type $\kappa\epsilon$ -problem (5.49) are equivalent.*

Proof. Sections 5.3.1 and 5.3.2 provide that a solution of the $\kappa\epsilon$ -problem (5.6) satisfies the heat-type $\kappa\epsilon$ -problem (5.49). We now establish the converse. Using the identity (5.29) stating $-[\check{A}_\epsilon]_r^j [[\check{A}_\epsilon]_r^k \check{v}_{,k}]_{,j} = \text{curl}_{\check{\zeta}_\epsilon} \text{curl}_{\check{\zeta}_\epsilon} \check{v} - [\check{A}_\epsilon]^s (\text{div}_{\check{\zeta}_\epsilon} \check{v})_{,s}$, the nonlinear heat-type equations (5.49a) are equivalently written as

$$\check{v}_t + \check{\varrho} \text{curl}_{\check{\zeta}_\epsilon} \text{curl}_{\check{\zeta}_\epsilon} \check{v} - \check{\varrho} [\check{A}_\epsilon]^s (\text{div}_{\check{\zeta}_\epsilon} \check{v})_{,s} = \check{\mathcal{K}}.$$

The identity

$$-\check{\varrho} [\check{A}_\epsilon]^s (\text{div}_{\check{\zeta}_\epsilon} \check{v})_{,s} = -[\check{A}_\epsilon]^k (\check{\varrho} \text{div}_{\check{\zeta}_\epsilon} \check{v})_{,k} + \text{div}_{\check{\zeta}_\epsilon} \check{v} [\check{A}_\epsilon]^k \check{\varrho}_{,k}$$

and the definition (5.32) of $\check{\mathcal{K}}$ imply that the equations (5.49a) are equivalently written as

$$(C.1) \quad \check{v}_t - [\check{A}_\epsilon]^k (\check{\varrho} \text{div}_{\check{\zeta}_\epsilon} \check{v} - 2\rho_0 \check{J}_\epsilon^{-1})_{,k} = -\check{\varrho} \text{curl}_{\check{\zeta}_\epsilon} \left[\text{curl}_{\check{\zeta}_\epsilon} \check{v} - \left(\text{curl } u_0 + \varepsilon_{.ji} \int_0^t \partial_t [\check{A}_\epsilon]_j^s \check{v}^i_{,s} \right) \right].$$

Using the identity $\text{curl}_{\check{\zeta}_\epsilon} \check{v} = \text{curl } u_0 + \int_0^t \partial_t (\text{curl}_{\check{\zeta}_\epsilon} \check{v}) = \text{curl } u_0 + \int_0^t \partial_t (\varepsilon_{.ji} [\check{A}_\epsilon]_j^s \check{v}^i_{,s})$,

$$(C.2) \quad \int_0^t \text{curl}_{\check{\zeta}_\epsilon} \check{v}_t = \text{curl}_{\check{\zeta}_\epsilon} \check{v} - \left(\text{curl } u_0 + \varepsilon_{.ji} \int_0^t \partial_t [\check{A}_\epsilon]_j^s \check{v}^i_{,s} \right).$$

The identity (C.2) implies that the equations (C.1) are equivalently written as

$$(C.3) \quad \check{v}_t - [\check{A}_\epsilon]^k (\check{\varrho} \text{div}_{\check{\zeta}_\epsilon} \check{v} - 2\rho_0 \check{J}_\epsilon^{-1})_{,k} = -\check{\varrho} \text{curl}_{\check{\zeta}_\epsilon} \left[\int_0^t \text{curl}_{\check{\zeta}_\epsilon} \check{v}_t \right].$$

Applying the ϵ -approximate Lagrangian curl operator $\text{curl}_{\zeta_\epsilon}$ to (C.3) yields

$$(C.4) \quad \text{curl}_{\zeta_\epsilon} \check{v}_t + \text{curl}_{\zeta_\epsilon} \left(\check{\rho} \text{curl}_{\zeta_\epsilon} \int_0^t \text{curl}_{\zeta_\epsilon} \check{v}_t \right) = 0 \quad \text{in } \Omega.$$

The boundary condition (5.49b) is written as

$$(C.5) \quad \begin{aligned} & \check{\rho} N^j [\check{A}_\epsilon]_r^j [\check{A}_\epsilon]_r^s \check{v}_{,s} \\ &= \check{h}_{\text{curl}} + \check{\rho} \check{b} + \frac{\sqrt{\check{g}_\epsilon}}{\rho_0} \left[\rho_0^2 \check{J}_\epsilon^{-2} - \beta_\epsilon(t) + \sigma \check{g}_\epsilon^{\alpha\beta} \check{\zeta}_{\epsilon,\alpha\beta} \cdot \check{n}_\epsilon + \check{\mathfrak{g}}_\epsilon^{\alpha\beta} \check{v}_{,\alpha\beta} \cdot \check{n}_\epsilon \right] \check{n}_\epsilon. \end{aligned}$$

Taking the scalar product of (C.5) with $\check{\rho}^{-1} \check{\partial} \check{\zeta}_\epsilon$ yields

$$\underbrace{\sqrt{\check{g}_\epsilon} \check{J}_\epsilon^{-1} [(\text{curl}_{\zeta_\epsilon} \check{v}) \times \check{n}_\epsilon] \cdot \check{\partial} \check{\zeta}_\epsilon}_{N^j [\check{A}_\epsilon]_r^j [\check{A}_\epsilon]_r^k \check{v}_{,k} \cdot \check{\partial} \check{\zeta}_\epsilon} = \underbrace{\sqrt{\check{g}_\epsilon} \check{J}_\epsilon^{-1} \left[\left(\text{curl } u_0 + \varepsilon_{\cdot ji} \int_0^t \partial_t [\check{A}_\epsilon]_j^s \check{v}^i_{,s} \right) \times \check{n}_\epsilon \right] \cdot \check{\partial} \check{\zeta}_\epsilon}_{\check{\rho}^{-1} \check{h}_{\text{curl}} \cdot \check{\partial} \check{\zeta}_\epsilon}.$$

Using that $[(\text{curl}_{\zeta_\epsilon} \check{v}) \times \check{n}_\epsilon] \cdot \check{n}_\epsilon = 0$ and $\check{h}_{\text{curl}} \cdot \check{n}_\epsilon = 0$, we have that

$$(C.6) \quad (\text{curl}_{\zeta_\epsilon} \check{v}) \times \check{n}_\epsilon = \left(\text{curl } u_0 + \varepsilon_{\cdot ji} \int_0^t \partial_t [\check{A}_\epsilon]_j^s \check{v}^i_{,s} \right) \times \check{n}_\epsilon \quad \text{on } \Gamma.$$

We note that (C.6) is equivalently stated as $(\int_0^t \text{curl}_{\zeta_\epsilon} \check{v}_t) \times \check{n}_\epsilon = 0$. Hence, by (C.4), we record that $\int_0^t \text{curl}_{\zeta_\epsilon} \check{v}_t$ satisfies

$$(C.7a) \quad \text{curl}_{\zeta_\epsilon} \check{v}_t + \text{curl}_{\zeta_\epsilon} \left(\check{\rho} \text{curl}_{\zeta_\epsilon} \int_0^t \text{curl}_{\zeta_\epsilon} \check{v}_t \right) = 0 \quad \text{in } \Omega \times [0, T],$$

$$(C.7b) \quad \left(\int_0^t \text{curl}_{\zeta_\epsilon} \check{v}_t \right) \times \check{n}_\epsilon = 0 \quad \text{on } \Gamma \times [0, T],$$

$$(C.7c) \quad \left(\int_0^t \text{curl}_{\zeta_\epsilon} \check{v}_t \right) |_{t=0} = 0 \quad \text{on } \Omega.$$

Testing (C.7a) against $\int_0^t \text{curl}_{\zeta_\epsilon} \check{v}_t$ in the $L^2(\Omega)$ -inner product and integrating by parts with respect to the operator $\text{curl}_{\zeta_\epsilon}$ in $\int_\Omega \text{curl}_{\zeta_\epsilon} (\check{\rho} \text{curl}_{\zeta_\epsilon} \int_0^t \text{curl}_{\zeta_\epsilon} \check{v}_t) \int_0^t \text{curl}_{\zeta_\epsilon} \check{v}_t$ yields

$$(C.8) \quad \begin{aligned} & \frac{1}{2} \frac{d}{dt} \left\| \int_0^t \text{curl}_{\zeta_\epsilon} \check{v}_t \right\|_0^2 + \left\| \sqrt{\check{\rho}} \text{curl}_{\zeta_\epsilon} \int_0^t \text{curl}_{\zeta_\epsilon} \check{v}_t \right\|_0^2 \\ & + \int_\Gamma N^s [\check{A}_\epsilon]_j^s \varepsilon_{ijk} \left(\check{\rho} \text{curl}_{\zeta_\epsilon} \int_0^t \text{curl}_{\zeta_\epsilon} \check{v}_t \right)^k \left(\int_0^t \text{curl}_{\zeta_\epsilon} \check{v}_t \right)^i = 0. \end{aligned}$$

With the boundary condition (C.7b), $\int_\Gamma N^s [\check{A}_\epsilon]_j^s \varepsilon_{ijk} (\check{\rho} \text{curl}_{\zeta_\epsilon} \int_0^t \text{curl}_{\zeta_\epsilon} \check{v}_t)^k (\int_0^t \text{curl}_{\zeta_\epsilon} \check{v}_t)^i = \int_\Gamma \sqrt{\check{g}_\epsilon} \check{J}_\epsilon^{-1} [(\int_0^t \text{curl}_{\zeta_\epsilon} \check{v}_t) \times \check{n}] \cdot (\check{\rho} \text{curl}_{\zeta_\epsilon} \int_0^t \text{curl}_{\zeta_\epsilon} \check{v}_t) = 0$. Integrating the identity (C.8) in time from 0 to $t \in (0, T]$ produces $\|\int_0^t \text{curl}_{\zeta_\epsilon} \check{v}_t\|_0^2 = 0$. So according to (C.7a), $\|\text{curl}_{\zeta_\epsilon} \check{v}_t\|_0 = 0$. The regularity of \check{v} stated in Proposition 5.4 then provides that almost everywhere in Ω ,

$$\text{curl}_{\zeta_\epsilon} \check{v}_t = 0.$$

Equations (C.3) are thus written as

$$(C.9a) \quad \check{v}_t - [\check{A}_\epsilon]^k (\kappa \rho_0 \check{J}_\epsilon \operatorname{div}_{\check{\zeta}_\epsilon} \check{v} - 2\rho_0 \check{J}_\epsilon^{-1})_{,k} = 0.$$

As the solution \check{v} of the heat-type $\kappa\epsilon$ -problem (5.49) verifies the ϵ -approximate Lagrangian vorticity equation (5.15), we infer that the boundary condition (C.5) of the heat-type problem is equivalently written as

$$(C.9b) \quad \kappa \rho_0^2 \operatorname{div}_{\check{\zeta}_\epsilon} \check{v} = \rho_0^2 \check{J}_\epsilon^{-2} - \beta_\epsilon(t) + \sigma \check{g}_\epsilon^{\alpha\beta} \check{\zeta}_{\epsilon,\alpha\beta} \cdot \check{n}_\epsilon + \kappa \check{g}_\epsilon^{\alpha\beta} \check{v}_{,\alpha\beta} \cdot \check{n}_\epsilon.$$

Equations (C.9) are equivalent to the momentum equations and boundary condition of the $\kappa\epsilon$ -problem (5.6). \square

Appendix D. Notation.

u	Eulerian velocity	3691
p	Eulerian pressure	3691
ρ	Eulerian density	3691
u_0	Initial velocity	3691
ρ_0	Initial density	3691
σ	Surface tension	3691
β	A parameter to the equation of state $p = \alpha\rho^\gamma - \beta$ for $\gamma > 1 \dots$	3691
η	The particle flow map	3692
v	Lagrangian velocity	3692
f	Lagrangian density	3692
A	Inverse of the deformation tensor $D\eta$	3692
J	Jacobian determinant of the deformation tensor $D\eta$	3692
a	Cofactor matrix of the deformation tensor $D\eta$	3692
e	The identity function $e(x) = x$	3692
D	The three-dimensional gradient operator	3696
$\bar{\partial}$	The surface gradient operator	3696
$\operatorname{div}_\eta, \operatorname{curl}_\eta$	The Lagrangian divergence and curl operators	3697
n	The outward-pointing unit normal to the moving surface $\eta(\Gamma)$..	3698
g	The surface metric induced by the moving surface $\eta(\Gamma)$	3698
g_0	The surface metric of the initial surface Γ	3698
\sqrt{g}	The square root of the determinant of the metric g	3698
Δ_g	The Laplace–Beltrami operator	3698
$\ \cdot\ _s$	The norm of the Hilbert space $H^s(\Omega)$	3699
$ \cdot _s$	The norm of the Hilbert space $H^s(\Gamma)$	3699
$H(\eta)$	Twice the mean curvature of the moving surface $\eta(\Gamma)$	3700
N	The outward-pointing unit normal to Γ	3700
\tilde{v}	A solution of the κ -problem (3.1)	3704
$\tilde{\eta}$	The Lagrangian map of the solution \tilde{v} to the κ -problem	3704
$E^\kappa(t)$	The higher-order energy function for solutions \tilde{v} to the κ -problem	3704
$\mathcal{P}, \mathcal{N}_0, \mathcal{R}$	The generic system of constants used in the κ -independent estimates	3704
Λ_ϵ	The horizontal-convolution operator for $\epsilon > 0$	3718
\check{v}	A solution of the $\kappa\epsilon$ -problem (5.6)	3720

$E^\epsilon(t)$	The higher-order energy function for solutions \tilde{v} to the $\kappa\epsilon$ -problem.....	3720
$\dot{\mathcal{P}}, \dot{\mathcal{N}}_0, \dot{\mathcal{R}}$	The generic system of constants used in the ϵ -independent estimates.....	3720
\dot{v}	A solution of the μ -problem (5.52).....	3728
$E^\mu(t)$	The higher-order energy function for solutions \dot{v} to the μ -problem (5.52).....	3728
$\ddot{\mathcal{P}}, \ddot{\mathcal{N}}_0, \ddot{\mathcal{R}}$	The generic system of constants used in the μ -independent estimates.....	3728
\ddot{v}	A solution of the σ -problem (7.2).....	3735
$E^\sigma(t)$	The higher-order energy function for solutions \ddot{v} to the surface tension problem.....	3735
$\ddot{\mathcal{P}}, \ddot{\mathcal{N}}_0, \ddot{\mathcal{R}}$	The generic system of constants used in the σ -independent estimates.....	3735
\mathbf{X}_T	The Hilbert space used in the fixed-point scheme.....	3758
$\mathcal{C}_T(\mathcal{M})$	A closed, bounded, convex subset of \mathbf{X}_T	3758
\bar{v}	An arbitrary vector in $\mathcal{C}_T(\mathcal{M})$	3758
$\ddot{\mathcal{P}}, \ddot{\mathcal{N}}_0, \ddot{\mathcal{R}}$	The generic system of constants used in the fixed-point scheme.....	3760

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