ON THE IMPOSSIBILITY OF FINITE-TIME SPLASH SINGULARITIES FOR VORTEX SHEETS

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ABSTRACT. In fluid dynamics, an interface *splash* singularity occurs when a locally smooth interface self-intersects in finite time. By means of elementary arguments, we prove that such a singularity cannot occur in finite time for vortex sheet evolution, i.e. for the two-phase incompressible Euler equations. We prove this by contradiction; we assume that a splash singularity does indeed occur in finite time. Based on this assumption, we find precise blow-up rates for the components of the velocity gradient which, in turn, allow us to characterize the geometry of the evolving interface just prior to self-intersection. The constraints on the geometry then lead to an impossible outcome, showing that our assumption of a finite-time splash singularity was false.

CONTENTS

1. INTRODUCTION

1.1. **The interface splash singularity.** The fluid interface *splash singularity* was introduced by Castro, Córdoba, Fefferman, Gancedo, & Gómez-Serrano in [13]. A *splash singularity* occurs when

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a fluid interface remains locally smooth but self-intersects in finite time. For the two-dimensional water waves problem, Castro, Córdoba, Fefferman, Gancedo, & Gómez-Serrano [13] showed that a splash singularity occurs in finite time using methods from complex analysis together with a clever transformation of the equations. In Coutand & Shkoller [20], we showed the existence of a finite-time splash singularity for the water waves equations in two or three-dimensions (and, more generally, for the one-phase Euler equations), using a very different approach, founded upon an approximation of the self-intersecting fluid domain by a sequence of smooth fluid domains, each with non self-intersecting boundary.

1.2. **The two-fluid incompressible Euler equations.** A natural question, then, is whether a splash singularity can occur for vortex sheet evolution, in which two phases of the fluid are present. Consider the two-phase incompressible Euler equations: Let $\mathcal{D} \subseteq \mathbb{R}^2$ denote an open, bounded set, which comprises the volume occupied by two incompressible and inviscid fluids with different densities. At the initial time $t = 0$, we let Ω^+ denote the volume occupied by the *lower* fluid with density ρ^+ and we let Ω^- denote the volume occupied by the *upper* fluid with density ρ^-. Mathematically, the sets Ω^+ and Ω^- denote two disjoint open bounded subsets of $\mathcal{D}$ such that $\overline{\mathcal{D}} = \overline{\Omega^+} \cup \overline{\Omega^-}$ and $\Omega^+ \cap \Omega^- = \emptyset$. The *material interface* at time $t = 0$ is given by $\Gamma := \overline{\Omega^+} \cap \overline{\Omega^-}$, and $\partial\mathcal{D} = \partial(\Omega^- \cap \Omega^+)/\Gamma$. (We can also consider the case that $\Omega^+ = \mathbb{T} \times (0, -1)$, $\Omega^- = \mathbb{T} \times (0, 1)$, and $\Gamma = \mathbb{T} \times \{0\}$.)

For time $t \in [0, T]$ for some $T > 0$ fixed, $\Omega^+(t)$ and $\Omega^-(t)$ denote the time-dependent volumes of the two fluids, respectively, separated by the moving material interface $\Gamma(t)$. Let $u^\pm$ and $p^\pm$

FIGURE 1. Two examples of the evolution of a vortex sheet $\Gamma(t)$ by the Euler equations. The two fluid regions are denoted by $\Omega^+(t)$ and $\Omega^-(t)$.

denote the velocity field and pressure function, respectively, in $\Omega^\pm(t)$. Then, planar vortex sheet $\Gamma(t)$ evolves according to the incompressible and irrotational Euler equations:

$$\rho^\pm(u_t^\pm + u^\pm \cdot Du^\pm) + Dp^\pm = \rho^\pm g \mathrm{e}_2 \qquad \text{in} \quad \Omega^\pm(t)\,, \tag{1.1a}$$

$$\operatorname{curl} u^\pm = 0, \quad \operatorname{div} u^\pm = 0 \qquad \text{in} \quad \Omega^\pm(t)\,, \tag{1.1b}$$

$$p^+ - p^- = \sigma H \qquad \text{on} \quad \Gamma(t)\,, \tag{1.1c}$$

$$(u^+ - u^-) \cdot \mathcal{N} = 0 \qquad \text{on} \quad \Gamma(t)\,, \tag{1.1d}$$

$$u(0) = u_0 \qquad \text{on} \quad \{t = 0\} \times \mathcal{D}\,, \tag{1.1e}$$

$$\mathcal{V}(\Gamma(t)) = u^+(t) \cdot \mathcal{N}(t)\,, \tag{1.1f}$$

where $\mathcal{V}(\Gamma(t))$ denotes the speed of the moving interface $\Gamma(t)$ in the normal direction, and $\mathcal{N}(\cdot, t)$ denotes the outward-pointing unit normal to $\partial\Omega^+(t)$, g denotes gravity, and e_2 is the vertical unit vector $(0, 1)$. Equation (1.1f) indicates that $\Gamma(t)$ moves with the normal component of the fluid velocity. The variables $0 < \rho^\pm$ denote the densities of the two fluids occupying $\Omega^\pm(t)$, respectively, $H(t)$ is twice the mean curvature of $\Gamma(t)$, and $\sigma > 0$ is the surface tension parameter which we will henceforth set to one. For notational simplicity, we will also set $\rho^+ = 1$ and $\rho^- = 1$. On the fixed boundary $\partial\mathcal{D}$, we set $u^\pm \cdot \mathcal{N} = 0$.

Via an elementary proof by contradiction, we prove that a finite-time splash singularity cannot occur for vortex sheets governed by (1.1). We rule-out a single splash singularity in which one self-intersection occurs, as well as the case that many (finite or infinite) simultaneous self-intersections occur. We also rule out a *splat* singularity, wherein the interface $\Gamma(t)$ self-intersects along a curve (see [13] and [20] for a precise definition).

1.3. **Outline of the paper.** In Section 2, we introduce Lagrangian coordinates (using the flow of u^-) for the purpose of fixing the domain and the material interface. Rather than using an arbitrary parameterization of the evolving interface $\Gamma(t)$, we specifically use the Lagrangian parameterization which has some important features for our analysis that general parameterizations do not. With this parameterization defined, we state the main theorem of the paper in Section 3 which states that a finite-time splash singularity cannot occur in this setting. In Section 4, we derive the evolution equations for the vorticity along the interface as well as the evolution equation for the tangential derivative of the vorticity; the latter plays a fundamental role in our analysis. In particular, under the assumption that the tangential derivative of vorticity blows-up in finite time, we find the precise blow-up rates for the components of $\nabla u^-(\cdot, t)$. Letting $\eta : \Gamma \to \Gamma(t)$ denote the Lagrangian parameterization of the vortex sheet, and supposing that the two reference points x_0 and x_1 in Γ evolve towards one another so that $|\eta(x_0, t) - \eta(x_1, t)| \to 0$ as $t \to T$, in Section 6, we find the evolution equation for the distance $\boldsymbol{\delta}\eta(t) = \eta(x_0, t) - \eta(x_1, t)$ between the two contact points. We can determine that the two portions of the curve $\Gamma(t)$ converge towards self-intersection in an essentially horizontal approach.

Finally, using the evolution equation for $\boldsymbol{\delta}\eta(t)$, we prove our main theorem in Section 7; in particular, we show that our assumption of a finite-time self-intersection of the curve $\Gamma(t)$ as $t \to T$ leads to the following contradiction: we first show that $u^-(\eta(x_0, T), T) - u^-(\eta(x_1, T), T) = 0$, and then we proceed to show that $u^-(\eta(x_0, T), T) - u^-(\eta(x_1, T), T) \neq 0$. We first arrive at this contradiction for a single splash singularity, meaning that one self-intersection point exists for $\Gamma(T)$; then, we proceed to prove that a finite (or even infinite) number of self-intersections also cannot occur. We conclude by showing that a *splat* singularity, wherein $\Gamma(T)$ self-intersects along a curve rather than a point, also cannot occur.

1.4. **A brief history of prior results.**

1.4.1. *Local-in-time well-posedness.* We begin with a short history of the local-in-time existence theory for the free-boundary incompressible Euler equations. For the *irrotational* case of the water waves problem, and for 2-D fluids (and hence 1-D interfaces), the earliest local existence results were obtained by Nalimov [34], Yosihara [44], and Craig [14] for initial data near equilibrium. Beale, Hou, & Lowengrub [9] proved that the linearization of the 2-D water wave problem is well-posed if the Rayleigh-Taylor sign condition $\frac{\partial p}{\partial n} < 0$ on $\Gamma \times \{t = 0\}$ is satisfied by the initial data (see [36] and [39]). Wu [40] established local well-posedness for the 2-D water waves problem and showed that, due to irrotationality, the Taylor sign condition is satisfied. Later Ambrose & Masmoudi [6], proved local well-posedness of the 2-D water waves problem as the limit of zero surface tension. For 3-D fluids (and 2-D interfaces), Wu [41] used Clifford analysis to prove local existence of the water waves problem with *infinite depth*, again showing that the Rayleigh-Taylor sign condition is always satisfied in the irrotational case by virtue of the maximum principle holding for the potential flow. Lannes [32] provided a proof for the *finite depth case with varying bottom*. Recently, Alazard, Burq & Zuily [2] have established low regularity solutions (below the Sobolev embedding) for the water waves equations.

The first local well-posedness result for the 3-D incompressible Euler equations without the irrotationality assumption was obtained by Lindblad [33] in the case that the domain is diffeomorphic to the unit ball using a Nash-Moser iteration. Coutand & Shkoller [18] proved local well-posedness for arbitrary initial geometries that have H^3-class boundaries without derivative loss. Shatah & Zeng [38] established a priori estimates for this problem using an infinite-dimensional geometric formulation, and Zhang and Zhang proved well-poseness by extending the complex-analytic method of Wu [41] to allow for vorticity. Again, in the latter case the domain was with infinite depth.

1.4.2. *Long-time existence.* It is of great interest to understand if solutions to the Euler equations can be extended for all time when the data is sufficiently smooth and small, or if a finite-time singularity can be predicted for other types of initial conditions.

Because of irrotationality, the water waves problem does not suffer from vorticity concentration; therefore, singularity formation involves only the loss of regularity of the interface or interface collision. In the case that the irrotational fluid is infinite in the horizontal directions, certain dispersive-type properties can be made use of. For sufficiently smooth and small data, Alvarez-Samaniego and Lannes [5] proved existence of solutions to the water waves problem on large time-intervals (larger than predicted by energy estimates), and provided a rigorous justification for a variety of asymptotic regimes. By constructing a transformation to remove the quadratic nonlinearity, combined with decay estimates for the linearized problem (on the infinite half-space domain), Wu [42] established an almost global existence result (existence on time intervals which are exponential in the size of the data) for the 2-D water waves problem with sufficiently small data. In a different framework, Alazard, Burq & Zuily [2] have also proven this result. Using position-velocity potential holomorphic coordinates, Hunter, Ifrim, & Tataru [26] have also proved almost global existence of the 2-D water waves problem.

Wu [43] then proved global existence in 3-D for small data. Using the method of spacetime resonances, Germain, Masmoudi, and Shatah [25] also established global existence for the 3-D irrotational problem for sufficiently small data. More recently, global existence for the 2-D water waves problem with small data was established by Ionescu & Pusateri [31], Alazard & Delort [3, 4], and Ifrim & Tataru [28, 29].

1.4.3. *The finite-time splash and splat singularity.* The finite-time *splash* and *splat singularities* were introduce by Castro, Córdoba, Fefferman, Gancedo, and Gómez-Serrano [13]; therein, using methods from complex analysis, they proved that a locally smooth interface can self-intersect in finite time for the 2-D water waves equations and hence established the existence of finite-time splash and splat singularites (see also [11] and [12]). In Coutand & Shkoller [20], we established the existence of finite-time splash and splat singularities for the 2-D and 3-D water waves and Euler equations (with vorticity) using an approximation of the self-intersecting domain by a sequence of standard Sobolev-class domains, each with non self-intersecting boundary. Our approach can be applied to many one-phase hyperbolic free-boundary problems, and shows that splash singularities can occur with surface tension, with compressibility, with magnetic fields, and for many one-phase hyperbolic free-boundary problems.

Recently, Ionsecu, Fefferman, and Lie [30] have proven that a splash singularity cannot occur for planar vortex sheets (or two-fluid interfaces) with surface tension. Their proof relies on a very sophisticated harmonic analysis of the integral kernel of the Birkhoff-Rott equation, and shows that the distance between the two evolving curves has a double exponential bound. Other than vortex sheet evolution for the two-phase Euler equations, it is of interest to determine the possibility of finite-time splash singularities for other fluids models. In this regard, Gancedo & Strain [24] have recently shown that a finite-time splash singularity cannot occur for the three-phase Muskat equations. In addition to the study of other fluids models, it is also of great interest to determine a mechanism for the loss of regularity of the evolving interface, which, in turn, could allow for finite-time self-intersection.

2. Fixing the fluid domains using the Lagrangian flow of $u-$

Let $\tilde{\eta}$ denote the Lagrangian flow map of u^- in Ω^- so that $\tilde{\eta}_t(x,t) = u^-(\tilde{\eta}(x,t),t)$ for $x \in \Omega^-$ and $t \in (0,T)$, with initial condition $\tilde{\eta}(x,0) = x$. Since $\operatorname{div} u^- = 0$, it follows that $\det \nabla\tilde{\eta} = 1$. By a theorem of [21], we define $\Psi : \Omega^+ \to \Omega^+(t)$ as incompressible extension of $\tilde{\eta}$, satisfying $\det \nabla\Psi = 1$ and $\|\Psi\|_{H^s(\Omega^+)} \le C\|\eta^-|_\Gamma\|_{H^{s-1/2}(\Gamma)}$ for $s > 2$. We then set

$$\eta(x,t) = \begin{cases} \tilde{\eta}(x,t), & x \in \overline{\Omega^-} \\ \Psi(x,t), & x \in \Omega^+ \end{cases} .$$

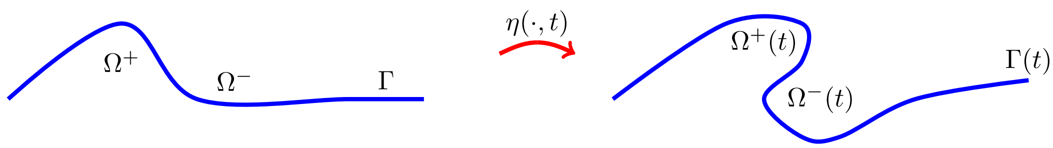


Figure 2. The mapping $\eta(\cdot,t)$ fixes the two fluid domains and the interface. The moving interface $\Gamma(t)$ is the image of Γ by $\eta(\cdot,t)$.

We define the following quantities set on the fixed domains and boundary:

$$\begin{aligned} v^+ &= u^- \circ \eta\,, && \text{in} \quad \Omega^- \times [0,T]\,, \\ v^- &= u^+ \circ \eta\,, && \text{in} \quad \Omega^+ \times [0,T]\,, \\ q^\pm &= p^\pm \circ \eta\,, && \text{in} \quad \Omega^\pm \times [0,T]\,, \\ A &= [\nabla\eta]^{-1}\,, && \text{in} \quad \mathcal{D} \times [0,T]\,, \\ \mathcal{H} &= H \circ \eta\,, && \text{on} \quad \Gamma \times [0,T]\,, \\ \delta v &= v^+ - v^-\,, && \text{on} \quad \Gamma \times [0,T]\,, \end{aligned}$$

The Eulerian momentum equations (1.1a) can then be written on the fixed domains as

$$v_t^+ + \nabla v^+ A\,(v^+ - \Psi_t) + A^T \nabla q^+ = g e_2 \qquad \text{in} \quad \Omega^+ \times [0,T]\,, \tag{2.1a}$$

$$v_t^- + A^T \nabla q^- = g e_2 \qquad \text{in} \quad \Omega^- \times [0,T]\,, \tag{2.1b}$$

and the pressure jump condition (1.1c) is $\delta q = \mathcal{H}$ on $\Gamma \times [0,T]$, where $\delta q = q^+ - q^-$.

Using the Einstein summation convention, $[\nabla v^+ A\,(v^+ - \Psi_t)]^i = v^i{,_r}\, A^r_j (v^+_j - \partial_t \Psi_j)$. This is the advection term; when Ψ is the identity map, we recover the Eulerian description, while if Ψ is the Lagrangian flow map, then we recover the Lagrangian description. The form (2.1.a) is called the Arbitrary Lagrangian Eulerian (ALE) description of the fluid flow in Ω^+

3. The main result

In [15, 16], we proved that if at time $t = 0$, $u_0^\pm \in H^k(\Omega^\pm)$ and Γ of class H^{k+1} for integers $k \ge 3$, then there exists a solution $(u^\pm(\cdot,t), \Gamma(t))$ of the system (1.1) satisfying $u^\pm \in L^\infty(0,T; H^k(\Omega^\pm(t)))$ and $\Gamma(t) \in H^{k+1}$.

Theorem 3.1 (No finite-time splash singularity). *Given a solution to (1.1) such that*

$$v^+ \in L^\infty(0,T; H^3(\Omega^+) \cap W^{2,\infty}(\Gamma))\,,\ \ \eta|_\Gamma \in L^\infty(0,T; W^{4,\infty}(\Gamma))\,,$$

a splash (or splat) singularity cannot occur in finite time. In particular, suppose that for a constant $1 \le \mathcal{M} < \infty$*,*

$$\sup_{t\in[0,T]} \Big(\|v^+(\cdot,t)\|_{H^3(\Omega^+)} + \big\|v^+(\cdot,t)\big\|_{W^{2,\infty}(\Gamma)} + \|\eta(\cdot,t)\|_{W^{4,\infty}(\Gamma)} \Big) < \mathcal{M}\,, \tag{3.1}$$

then $\Gamma(t)$ *cannot self-intersect.*

Remark 1. *The regularity assumptions (3.1) are reasonable; in the event that a splash singularity occurs at time* $t = T$*, only the domain* $\Omega^-(T)$ *would form a cusp, thus allowing for the possibility of blow-up for* $\|\nabla u^-(\cdot,t)\|_{L^\infty(\Omega^-(t))}$ *as* $t \to T$*. The domain* $\Omega(t)$ *remains smooth even up to the contact time* $t = T$*.*

4. Evolution equations on Γ for the vorticity and its tangential derivative

4.1. Geometric quantities defined on Γ and Γ(t).

We set

$$\begin{aligned} \mathcal{N}(x,t) &= \text{ unit normal vector field on } \Gamma(t)\,, \qquad n = \mathcal{N}\circ\eta \\ \mathcal{T}(x,t) &= \text{ unit tangent vector field on } \Gamma(t)\,, \qquad \tau = \mathcal{T}\circ\eta\,. \end{aligned}$$

We choose the unit-normal $\mathcal{N}$ to point into Ω^-. In a local coordinate (x_1, x_2), we set

$$G(x,t) = |\eta'(x,t)|^{-1}\,,\ \text{where } (\cdot)' = \partial(\cdot)/\partial x_1\,.$$

Hence,

$$\tau(x,t) = G\eta'(x,t)\,,\ n(x,t) = G\eta'^{\perp}(x,t)\,,\ x^\perp = (-x_2, x_1)\,. \tag{4.1}$$

4.2. Evolution equation for the vorticity on Γ.

Equation (2.1a) is $v^+_t + \nabla v^+\, A\,(v^+ - \Psi_t) + A^T\nabla q^+ = g e_2$. By definition, on Γ, $\Psi_t = v^-$, os that $v^+ - \Psi_t = \delta v$. Since $\delta v \cdot n = 0$ on Γ, we see that $\delta v = (\delta v\cdot\tau)\tau$. Hence, the advection term can be written (using the Einstein summation convention) as $\frac{\partial v^+}{\partial x_r} A^r_j \tau_j (\delta v\cdot\tau)$. From (4.1), $\tau_j = G\eta'_j$ which in our local coordinate system is the same as $G\frac{\eta_j}{\partial x_1}$. Since $A = [\nabla\eta]^{-1}$, we see that $A^r_j\ \frac{\eta_j}{\partial x_1} = \delta^r_1$, where δ^r_1 denotes the Kronecker delta.

It follows that on Γ, (2.1a) takes the form

$$v^+_t + G{v^+}'\,\delta v\cdot\tau + A^T\nabla q^+ = g e_2\,. \tag{4.2}$$

Equation (2.1b) does not have the advection term, and remains the same on Γ. Subtracting (2.1b) from (4.2a), taking the scalar product of this difference with τ, and using that $\delta q = \mathcal{H}$, yields

$$\delta v_t\cdot\tau + G{v^+}'\cdot\tau(\delta v\cdot\tau) + G\mathcal{H}' = 0\,,$$

from which it follows that

$$(\delta v\cdot\tau)_t + G{v^+}'\cdot\tau(\delta v\cdot\tau) + G\mathcal{H}' = 0 \text{ on } \Gamma\times[0,T)\,. \tag{4.3}$$

4.3. Evolution equation for derivative of vorticity $\nabla_{\mathcal{T}}\delta u\cdot\mathcal{T}$.

On Γ, we denote the tangential derivative by $\nabla_{\mathcal{T}}$. The chain-rule shows that the vorticity along particle trajectories can be written as

$$[\nabla_{\mathcal{T}}\delta u\cdot\mathcal{T}]\circ\eta = G\delta v'\cdot\tau \tag{4.4}$$

(which also provides an intrinsic definition for the derivative v'). Our analysis will rely on the evolution equation satisfied by $G\delta v'\cdot\tau$. By differentiating (4.3), we find that

$$(\delta v'\cdot\tau)_t + G{v^+}'\cdot\tau(\delta v'\cdot\tau) + (\delta v\cdot\tau)'[G{v^+}''\cdot\tau - G\mathcal{H}{v^+}'\cdot n - g^{-1}\,\eta''\cdot\tau\,{v^+}'\cdot\tau] + (G\mathcal{H}')' = 0 \tag{4.5}$$

Define our "forcing function" $\mathcal{A}$ to be

$$\mathcal{A} = G(\delta v\cdot\tau)'[G{v^+}''\cdot\tau - G\mathcal{H}{v^+}'\cdot n - g^{-1}\,\eta''\cdot\tau\,{v^+}'\cdot\tau] + G(G\mathcal{H}')'\,,$$

we see that equation (4.5) is simply

$$(\delta v' \cdot \tau)_t + G{v^+}' \cdot \tau(\delta v' \cdot \tau) + G^{-1}\mathcal{A} = 0\,. \tag{4.6}$$

Multiplying (4.6) by G and commuting G with the time-derivative shows that

$$(G\delta v' \cdot \tau)_t + G({v^-}' \cdot \tau + {v^+}' \cdot \tau)(G\delta v' \cdot \tau) + \mathcal{A} = 0\,.$$

Writing ${v^-}' \cdot \tau = -\delta v' \cdot \tau + {v^+}' \cdot \tau$, we arrive at the desired evolution equation

$$(G\delta v' \cdot \tau)_t - (G\delta v' \cdot \tau)^2 + 2G{v^+}' \cdot \tau(G\delta v' \cdot \tau) + \mathcal{A} = 0\,. \tag{4.7}$$

Notice that the coefficient $2G{v^+}' \cdot \tau$ as well as the forcing function $\mathcal{A}$ are both smooth by our assumed bounds (3.1).

Remark 2. *In [30], Ionsecu, Fefferman, and Lie use the notation $z(\alpha, t)$ to denote a smooth parameterization of $\Gamma(t)$, whereas we use the Lagrangian parameterization $\eta(x,t)$ of $\Gamma(t)$ for points x in the reference curve Γ. Our notation η' corresponds to $\partial_\alpha z$ in [30]. Furthermore, our $\delta v \cdot \tau$ is the same as $\frac{\omega}{|\partial_\alpha z|}$ in [30]. The tangential derivative of vorticity $[\nabla_\tau \delta u \cdot \tau] \circ \eta$ corresponds to $\partial_\alpha \left(\frac{\omega}{|\partial_\alpha z|}\right)/|\partial_\alpha z|$ in [30].*

5. Bounds for ∇u^- and the rate of blow-up

Lemma 5.1. *Assuming (3.1),*

$$\sup_{t \in [0,T]} \|v^-(\cdot, t)\|_{W^{1,\infty}(\Gamma)} \lesssim \mathcal{M}\,. \tag{5.1}$$

Proof. For notational convenience, we again denote v^+ simply by v. With $\tau_0 = \tau(x, 0)$, solving (4.3) using an integrating factor, we find that

$$\delta v \cdot \tau = \delta u_0 \cdot \tau_0 \, \exp\left(-\int_0^t Gv' \cdot \tau\right) - \exp\left(-\int_0^t Gv' \cdot \tau\right)\int_0^t G\mathcal{H}' \exp\left(\int_0^s Gv' \cdot \tau\right)\, ds\,. \tag{5.2}$$

We set $\mathcal{I}(t) = \exp\left(\int_0^t \|Gv' \cdot \tau\|_{L^\infty(\Gamma)}\right)$. Notice that by the chain-rule and as in formula (4.4), $G\delta v' \cdot \tau = [\nabla_\tau u^+ \cdot \tau] \circ \eta$ so by (3.1), $\mathcal{I}(t)$ is bounded. It follows from (5.2) that

$$\|\delta v \cdot \tau(\cdot, t)\|_{L^\infty(\Gamma)} \le \mathcal{I}(t)\|\delta u_0\|_{L^\infty(\Gamma)} + \mathcal{I}(t)\int_0^t \|G\mathcal{H}'\|_{L^\infty(\Gamma)}\,.$$

Again from (3.1), the derivative of the mean curvature $\mathcal{H}' \in W^{1,\infty}(\Gamma)$ so we see that $\|\delta v \cdot \tau(\cdot, t)\|_{L^\infty(\Gamma)}$ is bounded.

Next, as $\delta v \cdot n = 0$, and $v^+ \cdot n$ is bounded according to (3.1), we find that $\|v^-(\cdot, t)\|_{L^\infty(\Gamma)} \lesssim \mathcal{M}$ for all $t \in [0, T]$. Then, from (4.6),

$$\delta v' \cdot \tau = \delta u_0' \cdot \tau_0 \, \exp\left(-\int_0^t Gv' \cdot \tau\right) - \exp\left(-\int_0^t Gv' \cdot \tau\right)\int_0^t G^{-1}\mathcal{A} \exp\left(\int_0^s Gv' \cdot \tau\right)\, ds\,.$$

so that

$$\|\delta v' \cdot \tau(\cdot, t)\|_{L^\infty(\Gamma)} \le \mathcal{I}(t)\|\delta u_0' \cdot \tau_0\|_{L^\infty(\Gamma)} + \mathcal{I}(t)\int_0^t \|G^{-1}\mathcal{A}\|_{L^\infty(\Gamma)}\,,$$

from which we may conclude that $\|{v^-}' \cdot \tau(\cdot, t)\|_{L^\infty(\Gamma)} \lesssim \mathcal{M}$ for all $t \in [0, T]$. Then, since $\delta v' \cdot n = -\delta v \cdot n'$, the bound for δv together with (3.1) completes the proof. □

Lemma 5.2. *Assuming (3.1),*

$$\sup_{t \in [0,T]} \|\nabla u^-(\cdot, t)\|_{L^\infty(\eta(\Omega^-, t))} \lesssim \frac{\mathcal{M}}{\min_\Gamma |\eta'(\cdot, t)|}\,. \tag{5.3}$$

Proof. From (4.4) and Lemma 5.1, $\|[\nabla_\mathcal{T} \delta u \cdot \mathcal{T}] \circ \eta\|_{L^\infty(\Gamma)} \lesssim \mathcal{M}/\min_\Gamma |\eta'(\cdot,t)|$. Then, we see that $\max_{y\in\eta(\Gamma,t)} |\nabla_\mathcal{T} \delta u \cdot \mathcal{T}| \lesssim \mathcal{M}/\min_\Gamma |\eta'(\cdot,t)|$. Hence, with our assumed bounds (3.1),

$$\max_{y\in\eta(\Gamma,t)} \left|\nabla_\mathcal{T} u^- \cdot \mathcal{T}\right| \lesssim \frac{\mathcal{M}}{\min_\Gamma |\eta'(\cdot,t)|}\,. \tag{5.4}$$

Next, as $\delta u \cdot \mathcal{N} = 0$ (where recall that $\delta u = u^+ - u^-$ on $\Gamma(t)$), we have the identity $0 = \nabla_\mathcal{T}(\delta u \cdot \mathcal{N}) = (\nabla_\mathcal{T} \delta u) \cdot \mathcal{N} + \delta u \cdot \nabla_\mathcal{T} \mathcal{N}$; hence, we see that

$$\nabla_\mathcal{T} u^- \cdot \mathcal{N} = \nabla_\mathcal{T} u^+ \cdot \mathcal{N} + \delta u \cdot \nabla_\mathcal{T} \mathcal{N}\,.$$

Lemma 5.1 provides us with $L^\infty(\Gamma)$ control of u^-; hence, with (3.1), it follows that

$$\max_{y\in\eta(\Gamma,t)} \left|[\nabla_\mathcal{T} u^- \cdot \mathcal{N}](y)\right| \lesssim \mathcal{M}\,. \tag{5.5}$$

The inequalities (5.4) and (5.5) together with the fact that $\operatorname{div} u^- = \operatorname{curl} u^- = 0$ in $\eta(\Omega^-,t)$ implies that for any $t < T$,

$$\|\nabla u^-(\cdot,t)\|_{L^\infty(\eta(\Gamma,t))} \lesssim \frac{\mathcal{M}}{\min_\Gamma |\eta'(\cdot,t)|}\,. \tag{5.6}$$

As $\Delta \nabla u^- = 0$ in $\eta(\Omega^-,t)$, the maximum and minimum principle applied to each component of ∇u^-, together with (5.6), provide the inequality (5.3). $\square$

Hence, $\lim_{t\to T} \sup_{y\in\Gamma(t)} \|\nabla u^-(y,t)\|_{L^\infty(\eta(\Omega^-,t))} = \infty$ iff $\lim_{t\to T} g(x,t) \to 0^+$ for some $x \in \Gamma$.

THEOREM 5.1. *With the assumed bounds (3.1), if there is a sequence $t_n \to T$ such that*

$$\max_{x\in\Gamma} |[\nabla_\mathcal{T} \delta u \cdot \mathcal{T}](\eta(x,t_n),t_n)| \to \infty\,, \tag{5.7}$$

then there exists t_0 sufficiently close to T such that for $0 < \epsilon \ll 1$,

$$\max_{y\in\eta(\overline{\Omega^-},t)} |\nabla u^-(y,t)| \le \frac{1+\epsilon}{T-t} \qquad \forall t \in [t_0,T)\,. \tag{5.8}$$

Furthermore, if there exist two points $x_0, x_1 \in \Gamma$ such that $\eta(x_0,T) = \eta(x_1,T)$ with tangent vector to $\Gamma(T)$ at $\eta(x_0,T)$ given by e_1, then for $0 < \epsilon \ll 1$,

$$\max_{y\in\eta(\overline{\Omega^-},t)} \left|\frac{\partial u_2^-}{\partial x_1}(y,t)\right| \le \frac{\epsilon}{T-t} \qquad \forall t \in [t_0,T)\,. \tag{5.9}$$

Proof. Step 1. Blow-up rate for vorticity $[\nabla_\mathcal{T} \delta u \cdot \mathcal{T}](\eta(x_0,t),t)$ as $t \to T$. We first suppose that for some $x_0 \in \Gamma$, $|[\nabla_\mathcal{T} \delta u \cdot \mathcal{T}](\eta(x_0,t_n),t_n)| \to \infty$, and establish that $[\nabla_\mathcal{T} \delta u \cdot \mathcal{T}](\eta(x_0,t),t)$ (which, recall, equals $G\delta v' \cdot \tau(x_0,t)$) has a precise blow-up rate under the assumption (5.7).

We set

$$\chi(t) = G\delta v' \cdot \tau(x_0,t)\,,$$

and define the coefficient function

$$\mathfrak{a}(t) = 2G{v^+}' \cdot \tau(x_0,t)\,.$$

Then, (4.7) reads

$$\chi_t - \chi^2 + \mathfrak{a}\,\chi = -\mathcal{A}\,. \tag{5.10}$$

This equation can be written as

$$\left[\exp \int_0^t \mathfrak{a}(s)ds\; \chi\right]_t - \exp \int_0^t \mathfrak{a}(s)ds\; \chi^2 = -\exp \int_0^t \mathfrak{a}(s)ds\; \mathcal{A}$$

so that

$$\int_0^t \exp\left(-\int_s^t \mathfrak{a}(r)dr\right)\chi(s)^2 ds = \exp\left(\int_0^t \mathfrak{a}(s)ds\right)\chi(t) - \chi(0) + \int_0^t \exp\left(\int_0^s \mathfrak{a}(r)dr\right)\mathcal{A}(s)ds\,. \tag{5.11}$$

Thanks to (3.1), $\mathfrak{a}(t)$ has a minimum and maximum on $[0,T]$. Hence, there are positive constants c_1, c_2, c_3 such that for any $t \in [0,T)$,

$$c_1 \int_0^t \chi^2(s)ds - c_3 \le \chi(t) \le c_2 \int_0^t \chi^2(s)ds + c_3\,,$$

and by (5.7), the limit as $t \to T$ is well-defined and

$$\lim_{t\to T} \chi(t) = \infty\,, \tag{5.12}$$

For $t > t_0$ sufficiently close to T, we can then divide (5.10) by χ^2, and integrate from t_0 to t, to find that

$$-\frac{1}{\chi(t)} + \frac{1}{\chi(t_0)} - t + t_0 + \int_{t_0}^t \left(\frac{\mathfrak{a}(s)}{\chi(s)} + \frac{\mathcal{A}(s)}{\chi^2(s)}\right)ds = 0\,.$$

Using the limit in (5.12),

$$\frac{1}{\chi(t_0)} - T + t_0 + \int_{t_0}^T \left(\frac{\mathfrak{a}(s)}{\chi(s)} + \frac{\mathcal{A}(s)}{\chi^2(s)}\right)ds = 0\,,$$

from which we obtain the following identity: for $t \in [t_0, T)$,

$$\chi(t) = \left[T - t - \int_t^T \left(\frac{\mathfrak{a}(s)}{\chi(s)} + \frac{\mathcal{A}(s)}{\chi^2(s)}\right)ds\right]^{-1}\,. \tag{5.13}$$

From (5.12), this formula implies that the integrand is small as t is close to T, and then provides the rate of blow-up:

$$\lim_{t\to T} \chi(t)(T-t) = 1\,.$$

Using (3.1), we see that

$$\lim_{t\to T} [\nabla_\tau u^- \cdot \tau](\eta(x_0,t),t)\ (T-t) = -1\,. \tag{5.14}$$

Step 2. Maximum of vorticity derivative blows-up on $\Gamma(t)$*.* Having established the blow-up rate for $[\nabla_\tau \delta u \cdot \tau](\eta(x_0,t)$, we shall next prove that for any $t \in [0,T)$, the quantity $\max_{x\in\Gamma}[\nabla_\tau \delta u \cdot \tau](\eta(x,t))$ (which equals $\max_{x\in\Gamma} G\delta v' \cdot \tau(x,t)$) has the same blow-up rate. For each $x \in \Gamma$ and $t \in [0,T)$, we now set

$$\mathfrak{A}(x,t) = 2G{v^+}' \cdot \tau(x,t) \text{ and } \mathfrak{X}(x,t) = G\delta v' \cdot \tau(x,t)\,. \tag{5.15}$$

Following (5.11), we see that

$$\mathfrak{X}(x,t) \ge \exp\left(-\int_0^t \mathfrak{A}(x,s)ds\right)\mathfrak{X}(x,0) - \exp\left(-\int_0^t \mathfrak{A}(x,s)ds\right)\int_0^t \exp\left(\int_0^s \mathfrak{a}(x,r)dr\right)\mathcal{A}(x,s)ds\,; \tag{5.16}$$

hence, there exists a positive constant c_4 such that $\mathfrak{X}(x,t) > -c_4$. Since $\mathfrak{X}_t = \mathfrak{X}^2 - \mathfrak{A}\,\mathfrak{X} - \mathcal{A}$, there is a positive constant c_5,

$$\mathfrak{X}_t > \mathfrak{X}^2/2 - c_5\,.$$

It follows that if $\mathfrak{X}(x,t_0) \ge \sqrt{2c_5}$, then $\mathfrak{X}(x,\cdot)$ is increasing on $[t_0,T)$. For $x \in \Gamma$ we choose $t_0 < T$ sufficiently close to T so that for $0 < \epsilon \ll 1$ fixed,

$$\mathfrak{X}(x,t_0) > \sqrt{2c_5} + 1 + \frac{8c_6}{\epsilon}\,, \quad c_6 = \sup_{(t,x)\in[0,T]\times\Gamma} \left(|\mathfrak{A}(x,t)| + \ \mathcal{A}(x,t)|\right)\,, \tag{5.17}$$

with c_6 denoting a bounded constant thanks to (3.1). Since $\mathcal{X}(x,\cdot)$ is increasing for such an x, for $t\in[t_0,T)$, the limit of $\mathcal{X}(x,t)$ as $t\to T$ is well-defined in the interval $(1+\sqrt{2c_5}+8c_6/\epsilon,\infty]$, and thus so is the limit of $\frac{1}{\mathcal{X}(x,t)}$. Analogous to (5.13), we obtain that

$$\mathcal{X}(x,t)=\left[\frac{1}{\lim_{t\to T}\mathcal{X}(x,t)}+T-t+\int_T^t\left(\frac{\mathfrak{A}(s)}{\mathcal{X}(s)}+\frac{\mathcal{A}(s)}{\mathcal{X}^2(s)}\right)ds\right]^{-1}.$$

From (5.17), we then have that for all $t\in[t_0,T)$,

$$\mathcal{X}(x,t)\le\left[\frac{1}{\lim_{t\to T}\mathcal{X}(x,t)}+(T-t)(1-\epsilon)\right]^{-1}$$

and since $\lim_{t\to T}\mathcal{X}(x,t)\ge 0$, then for all $t<T$,

$$\mathcal{X}(x,t)\le\frac{1}{(T-t)(1-\epsilon)}\,. \tag{5.18}$$

Step 3. Blow-up rate for ∇u^- in $\overline{\Omega^-(t)}$ as $t\to T$. From (5.18), for any $t\in[t_0,T)$,

$$\max_{y\in\eta(\Gamma,t)}|[\nabla_{\mathcal{T}}\delta u\cdot\mathcal{T}](y,t)|\le\frac{1+2\epsilon}{(T-t)}\,. \tag{5.19}$$

The inequalities (5.5) and (5.19), together with the fact that $\operatorname{div}u^-=\operatorname{curl}u^-=0$ in $\eta(\Omega^-,t)$, show that

$$\max_{y\in\eta(\Gamma,t)}|\nabla u^-(y,t)|\le\frac{1+2\epsilon}{T-t}\,, \tag{5.20}$$

where $\max_{y\in\eta(\Gamma,t)}|\nabla u^-(y,t)|$ denotes the maximum over all of the components of the matrix ∇u^-. Now, for any fixed $t\in[0,T)$, since each component of ∇u^- is harmonic in the domain $\eta(\Omega^-,t)$, the maximum and minimum principles together with the boundary estimate (5.20) shows that (5.8) holds.

Step 4. Asymptotic estimates for the components of ∇u^- as $t\to T$ in an ϵ-neighborhood of the splash. Since

$$\frac{\partial u^-}{\partial x_1}:=\nabla_{\mathrm{e}_1}u^-=(\mathcal{T}\cdot\mathrm{e}_1)\nabla_{\mathcal{T}}u^-+(\mathcal{N}\cdot\mathrm{e}_1)\nabla_{\mathcal{N}}u^-\,,$$

we have that

$$\begin{aligned}\frac{\partial u_2^-}{\partial x_1}&=(\mathcal{T}\cdot\mathrm{e}_1)\nabla_{\mathcal{T}}u^-\cdot(\mathcal{T}\cdot\mathrm{e}_2\ \mathcal{T}+\mathcal{N}\cdot\mathrm{e}_2\ \mathcal{N})+(\mathcal{N}\cdot\mathrm{e}_1)\nabla_{\mathcal{N}}u^-\cdot(\mathcal{T}\cdot\mathrm{e}_2\ \mathcal{T}+\mathcal{N}\cdot\mathrm{e}_2\ \mathcal{N})\\&=(\mathcal{T}\cdot\mathrm{e}_1)(\mathcal{T}\cdot\mathrm{e}_2)\nabla_{\mathcal{T}}u^-\cdot\mathcal{T}+(\mathcal{T}\cdot\mathrm{e}_1)(\mathcal{N}\cdot\mathrm{e}_2)\nabla_{\mathcal{T}}u^-\cdot\mathcal{N}+(\mathcal{T}\cdot\mathrm{e}_2)(\mathcal{N}\cdot\mathrm{e}_1)\nabla_{\mathcal{N}}u^-\cdot\mathcal{T}\\&\quad+(\mathcal{N}\cdot\mathrm{e}_1)(\mathcal{N}\cdot\mathrm{e}_2)\nabla_{\mathcal{N}}u^-\cdot\mathcal{N}\,.\end{aligned} \tag{5.21}$$

By rotating our co-ordinate system, if necessary, we suppose that the tangent and normal directions to $\Gamma(T)$ at $\eta(x_0,T)$ are given by the standard basis vectors $\mathrm{e}_1=(1,0)$ and $\mathrm{e}_2=(0,1)$, respectively.

Next, choose a point $\eta(x,t)\in\Gamma(t)$ in a small neighborhood of $\eta(x_0,t)$, and let the curve $\mathcal{S}(t)$ denote that portion of $\Gamma(t)$ that connects $\eta(x_0,t)$ to $\eta(x,t)$. Let $\vec{l}(t):[0,1]\to\mathcal{S}(t)$ denote a unit-speed parameterization such that $\vec{l}(t)(1)=\eta(x,t)$ and $\vec{l}(t)(0)=\eta(x_0,t)$. Then,

$$\begin{aligned}\mathcal{N}(\eta(x,t),t)\cdot\mathrm{e}_1-\mathcal{N}(\eta(x_0,t),t)\cdot\mathrm{e}_1&=\int_{\mathcal{S}(t)}\nabla(\mathcal{N}\cdot\mathrm{e}_1)\cdot d\vec{l}\\ \mathcal{T}(\eta(x,t),t)\cdot\mathrm{e}_2-\mathcal{T}(\eta(x_0,t),t)\cdot\mathrm{e}_2&=\int_{\mathcal{S}(t)}\nabla(\mathcal{T}\cdot\mathrm{e}_2)\cdot d\vec{l}.\end{aligned}$$

From our assumed bounds (3.1), there is a constant $c_7 > 0$ such that for $t \le T$

$$|\mathcal{N}(\eta(x,t),t)\cdot \mathrm{e}_1 - \mathcal{N}(\eta(x_0,t),t)\cdot \mathrm{e}_1| + |\mathcal{T}(\eta(x,t),t)\cdot \mathrm{e}_2 - \mathcal{T}(\eta(x_0,t),t)\cdot \mathrm{e}_2| \\ \le c_7|\eta(x,t)-\eta(x_0,t)|\,.$$

Similarly,

$$\mathcal{N}(\eta(x_0,t),t)\cdot \mathrm{e}_1 = \mathcal{N}(\eta(x_0,t),t)\cdot \mathrm{e}_1 - \mathcal{N}(\eta(x_0,T),T)\cdot \mathrm{e}_1 = \int_T^t \partial_t n(x_0,s)\cdot \mathrm{e}_1 ds$$

$$\mathcal{T}(\eta(x_0,t),t)\cdot \mathrm{e}_2 = \mathcal{T}(\eta(x_0,t),t)\cdot \mathrm{e}_2 - \mathcal{T}(\eta(x_0,T),T)\cdot \mathrm{e}_2 \ = \int_T^t \partial_t \tau(x_0,s)\cdot \mathrm{e}_2 ds$$

so that (by readjusting the constant c_7 if necessary), we have that

$$|\mathcal{N}(\eta(x_0,t),t)\cdot \mathrm{e}_1| + |\mathcal{T}(\eta(x_0,t),t)\cdot \mathrm{e}_2| \le c_7(T-t)\,.$$

Next, choose $t_0 \in [0,T)$ and $x \in \gamma_0 \subset \Gamma$, with γ_0 a sufficiently small neighborhood of x_0, such that

$$c_7(T-t) < \frac{\epsilon}{2} \ \text{ and } \ c_7|\eta(x,t)-\eta(x_0,t)| < \frac{\epsilon}{2} \ \ \forall\, x\in\gamma_0,\ t\in[t_0,T)\,. \tag{5.22}$$

It follows that

$$|\mathcal{N}(\eta(x,t),t)\cdot \mathrm{e}_1| + |\mathcal{T}(\eta(x,t),t)\cdot \mathrm{e}_2| < \epsilon \ \ \forall\, x\in\gamma_0,\ t\in[t_0,T)\,. \tag{5.23}$$

Consequently, from (5.5), (5.20) and (5.21), we see that

$$\left|\frac{\partial u_2^-}{\partial x_1}(\eta(x,t),t)\right| \le \frac{3\epsilon}{T-t} + |\nabla_{\mathcal{T}} u^- \cdot \mathcal{N}|(\eta(x,t),t) + |\nabla_{\mathcal{N}} u^- \cdot \mathcal{T}|(\eta(x,t),t)\,,$$

which thanks to (5.5) and the fact that $\operatorname{curl} u^- = \nabla_{\mathcal{T}} u^- \cdot \mathcal{N} - \nabla_{\mathcal{N}} u^- \cdot \mathcal{T} = 0$, provides us with

$$\left|\frac{\partial u_2^-}{\partial x_1}(\eta(x,t),t)\right| \le \frac{3\epsilon}{T-t} + c_8\mathcal{M} \ \ \forall\, x\in\gamma_0,\ t\in[t_0,T)\,,$$

for a constant $c_8 > 0$. Thus , by choosing t_0 closer to T if necessary, we have that

$$\left|\frac{\partial u_2^-}{\partial x_1}(\eta(x,t),t)\right| \le \frac{3\epsilon}{T-t} \ \ \forall\, x\in\gamma_0,\ t\in[t_0,T)\,. \tag{5.24}$$

In the identical fashion, we can show that there exists a sufficiently small neighbourhood $\gamma_1 \subset \Gamma$ of x_1, such that for all $x \in \gamma_1$ and $t \in [t_0,T)$ (again taking $T-t_0$ even smaller if necessary), the inequality (5.24) holds. Now, for x in Γ in the complement of γ_0 and γ_1, we have that $\nabla u^-(\eta(x,t),t)$ is bounded by a constant $\mathcal{M}_\epsilon$ (by a standard elliptic regularity argument) and is, therefore, less than $\frac{\epsilon}{T-t}$ for t close enough to T. Hence, for $T-t_0$ sufficiently small,

$$\max_{y\in\eta(\Gamma,t)}\left|\frac{\partial u_2^-}{\partial x_1}(y,t)\right| \le \frac{3\epsilon}{T-t} \quad \forall t\in[t_0,T)\,,$$

which, thanks to the maximum and minimum principles applied to the harmonic function $\frac{\partial u_2^-}{\partial x_1}$, provides us with (5.9). Since $0 < \epsilon \ll 1$, we replace 3ϵ by ϵ, and replace $1+2\epsilon$ by $1+\epsilon$. This completes the proof. $\square$

Corollary 5.1. *With (3.1) and (5.7) holding, for $T-t_0$ sufficiently small,*

$$\|\nabla_{\mathcal{T}} u^- \cdot \mathcal{T}(\cdot,t)\|_{L^\infty(\Gamma(t))} \le \frac{1+3(T-t)}{T-t} \ \ \forall t\in[t_0,T)\,. \tag{5.25}$$

Proof. Using the notation from the proof of Theorem 5.1,

$$\mathcal{X}(x,t) = \nabla_\tau \delta u \cdot \tau(\eta(x,t),t)\,.$$

and we recall that $\mathcal{X}(x_0,t)=\chi(t)$ and that $\mathcal{X}(x,t)$ satisfies

$$\mathcal{X}_t(x,t) - \mathcal{X}^2(x_0,t) + \mathfrak{A}(x,t)\,\mathcal{X}(x,t) = -\mathcal{A}(x,t)\,. \tag{5.26}$$

We let $\delta t = T-t$, and fix $0<\epsilon \ll 1$.

Since $\lim_{t\to T}\mathcal{X}(x_0,t)(T-t)=1$, for δt sufficiently small, we have that

$$(1-\epsilon)\delta t^{-1} \le \mathcal{X}(x_0,t) \le (1+\epsilon)\delta t^{-1}.$$

Substituting this inequality into (5.13), we see that

$$\mathcal{X}(x_0,t) \le \frac{1}{(1-\delta t)\delta t} \le \frac{1+2\delta t}{\delta t}\,. \tag{5.27}$$

If we replace x_0 with x_1, then (5.27) continues to hold.

Now, for the sake of contradiction, we will assume that there exists a sequence of points (x^*,t^*), with t^* converging to T, such that

$$\mathcal{X}(x^*,t^*) > \frac{1+3(T-t^*)}{T-t^*}\,. \tag{5.28}$$

Then, for comparison, we choose the point x_0 or x_1 which is strictly closest to the point x^* corresponding to a subsequence of t^*. We assume this point is x_0 (for otherwise we reverse the labels on the two points x_0 and x_1). Notice that (5.28) implies that for $T-t_0$ sufficiently small,

$$|x^*-x_0| \le \epsilon\,. \tag{5.29}$$

We define

$$\mathcal{Y}(t) = \mathcal{X}(x^*,t) - \mathcal{X}(x_0,t) \text{ and } \mathcal{Z}(t) = \mathcal{X}(x^*,t) + \mathcal{X}(x_0,t)\,,$$
$$\delta\mathfrak{A}(t) = \mathfrak{A}(x^*,t) - \mathfrak{A}(x_0,t) \text{ and } \delta\mathcal{A}(t) = \mathcal{A}(x^*,t) - \mathcal{A}(x_0,t)\,.$$

Then, setting $\mathcal{P}(t) = \mathcal{Z}(t) - \mathfrak{A}(x^*,t)$, from (5.26), $\mathcal{Y}(t)$ satisfies

$$\mathcal{Y}_t(t) - \mathcal{P}(t)\mathcal{Y}(t) = -\delta\mathfrak{A}(t)\mathcal{X}(x_0,t) - \delta\mathcal{A}(t)\,,$$

and hence

$$\left[e^{-\int_{t^*}^t \mathcal{P}(s)ds}\mathcal{Y}(t)\right]_t = -e^{-\int_{t^*}^t \mathcal{P}(s)ds}\left[\delta\mathfrak{A}(t)\mathcal{X}(x_0,t) + \delta\mathcal{A}(t)\right]\,.$$

Integrating from t_0 to t, we see that

$$\mathcal{Y}(t) = e^{\int_{t^*}^t \mathcal{P}(s)ds}\left(\mathcal{Y}(t^*) - \int_{t^*}^t e^{-\int_{t^*}^s \mathcal{P}(r)dr}\left[\delta\mathfrak{A}(s)\mathcal{X}(x_0,s) + \delta\mathcal{A}(s)\right]ds\right)\,. \tag{5.30}$$

Our goal is to show that $\mathcal{Y}(t)\ge 0$, for all $t\ge t^*$. Since $\mathcal{X}(x_0,t)<1$ by (5.27), we see that (5.28) shows that

$$\mathcal{Y}(t^*) > 1\,, \tag{5.31}$$

so all we need to prove is that the second term on the right-hand side of (5.30),

$$\kappa(t^*,t) = -\int_{t^*}^t e^{-\int_{t^*}^s \mathcal{P}(r)dr}\left[\delta\mathfrak{A}(s)\mathcal{X}(x_0,s) + \delta\mathcal{A}(s)\right]ds\,, \tag{5.32}$$

is very small for t^* close to T.

We first consider $-\int_{t^*}^s \mathcal{P}(r)dr$ which is equal to $-\int_{t^*}^s \mathcal{Z}(r)dr + \int_{t^*}^s \mathfrak{A}(x^*,r)dr$. Since $\mathcal{X}(x^*,t)$ is positive, we see that $\mathcal{Z}(t) > \mathcal{X}(x_0,t)$ and so $-\mathcal{Z}(t) < -\mathcal{X}(x_0,t)$, and as we noted above, $\mathcal{X}(x_0,t) > (1-\epsilon)\delta t^{-1}$. Hence $-\int_{t^*}^s \mathcal{Z}(r)dr < -\int_{t^*}^s \mathcal{X}(x_0,r)dr$, so that

$$e^{-\int_{t^*}^s \mathcal{Z}(r)dr} < e^{-\int_{t^*}^s \mathcal{X}(x_0,r)dr} \le e^{-\int_{t^*}^s \frac{1-\epsilon}{T-r}dr} = \left[\frac{T-s}{T-t^*}\right]^{1-\epsilon}$$

and since $e^{\int_{t^*}^s \mathfrak{A}(x^*,r)dr} \le C$, then

$$e^{-\int_{t^*}^s \mathcal{P}(r)dr} < C\left[\frac{T-s}{T-t^*}\right]^{1-\epsilon}.$$

From (5.32), we see that

$$\begin{aligned}|\kappa(t^*,t)| &\le C\int_{t^*}^t \left[\frac{(T-s)}{T-t^*}\right]^{1-\epsilon}\left(\frac{1+\epsilon}{T-s}\delta\mathfrak{A}(s)+\delta\mathcal{A}(s)\right)ds\\ &\le \frac{C(1+\epsilon)}{(T-t^*)^{1-\epsilon}}\int_{t^*}^t (T-s)^{-\epsilon}\delta\mathfrak{A}(s)ds + \frac{C}{(T-t^*)^{1-\epsilon}}\int_{t^*}^t (T-s)^{1-\epsilon}\delta\mathcal{A}(s)ds\,.\end{aligned}$$

Let $\vec{r}$ denote a unit-speed parameterization of the path $\gamma\subset\Gamma$ starting at x_0 and ending at x^*. From (5.15), $\mathfrak{A}(x,t) = 2Gv^{+'}\cdot\tau(x,t)$, so that thanks to our assumed bounds (3.1), we see that

$$\delta\mathfrak{A}(t) = \int_\gamma \nabla\mathfrak{A}\cdot d\vec{r} \le C|x^*-x_0| \le C\epsilon\,,$$

the last inequality following from (5.29). It follows that

$$\begin{aligned}|\kappa(t^*,t)| &\le \frac{\epsilon C(T-t)^{1-\epsilon}}{(T-t^*)^{1-\epsilon}} + \epsilon C + \frac{C(T-t)^{2-\epsilon}}{(T-t^*)^{1-\epsilon}} + C(T-t^*)\\ &\le C\left[\epsilon + (T-t^*)\right].\end{aligned}$$

Hence, for $T-t^*$ sufficiently small, and $t\in[t^*,T)$, we have $|\kappa(t^*,t)|<1$. Thanks to (5.31), this implies that for such any such t^*, and for all $t\in[t^*,T)$, $\mathcal{Y}(t)\ge 0$, which by the definition of $\mathcal{Y}(t)$, implies that

$$\mathcal{X}(x^*,t) \ge \mathcal{X}(x_0,t)\,,$$

and thus $\lim_{t\to T}\mathcal{X}(x^*,t)=\infty$. Now, from our assumption of a single splash contact in this section, this implies that either $x^*=x_0$ or $x^*=x_1$. Since x^* is closer to x_0, we then have $x^*=x_0$. Thus, by (5.27) and (5.28), we then have

$$3<2\,,$$

which is the contradiction needed to establish that our assumption (5.28) was wrong.

By definition of $\mathcal{X}(x,t)$, this then shows that $\sup_{y\in\Gamma(t)}|\nabla_\tau\delta u\cdot\tau(\cdot,t)| \le \frac{1+3(T-t)}{T-t}$ for all $t\in[t_0,T)$ with $T-t_0$ taken sufficiently small. Together with Lemma 5.1, this, then, completes the proof. □

6. The interface geometry near the assumed blow-up

We let x_0 and x_1 in Γ denote the two reference points which are moving toward one another. If a splash singularity occurs at time T, then $\lim_{t\to T}|\eta(x_0,t)-\eta(x_1,t)|=0$. In this section, we find the evolution equation for the distance between the two contact points $\eta(x_0,t)$ and $\eta(x_1,t)$.

For $T-t_0$ sufficiently small (so that $\eta(x_0,t)$ is very close to $\eta(x_1,t)$), and in a sufficiently small (space) neighborhood of $\eta(x_0,t)$, the interface $\Gamma(t)$ locally consists of two subsets, each containing one of the two points that will come into contact at time $t=T$; specifically, we let $\Gamma_0(t)\subset\Gamma(t)$ be the subset containing $\eta(x_0,t)$, and we let $\Gamma_1(t)\subset\Gamma(t)$ denote the subset containing $\eta(x_1,t)$ (see Figure 3).

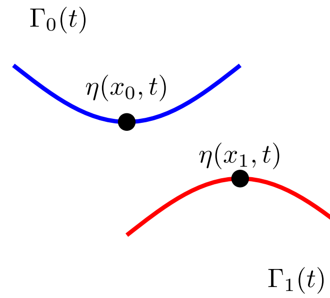


FIGURE 3. For t sufficiently close to T, the interface $\Gamma(t)$ has a local neighborhood of $\eta(x_0,t)$ called $\Gamma_0(t)$ and a local neighborhood of $\eta(x_1,t)$ called $\Gamma_1(t)$. The two subset $\Gamma_0(t)$ and $\Gamma_1(t)$ are, by definition, connected to one another, but we only draw the two subsets that are moving toward each.

By Lemma 5.1,
$$\sup_{t\in[0,T]} \left(\|\tau_t(\cdot,t)\|_{L^\infty(\Gamma)} + \|n_t(\cdot,t)\|_{L^\infty(\Gamma)}\right) \lesssim \mathcal{M}\,,$$
so that $n,\tau \in C([0,T],L^\infty(\Gamma))$. Recall that the tangent and normal directions to $\Gamma(T)$ at $\eta(x_0,T)$ and $\eta(x_1,T)$ are given by the standard basis vectors $\mathrm{e}_1=(1,0)$ and $\mathrm{e}_2=(0,1)$, respectively.

Next, we define
$$\boldsymbol{\delta}\eta(t)=\eta(x_0,t)-\eta(x_1,t) \ \text{ and } \ \boldsymbol{\delta}u^-(t)=u^-(\eta(x_0,t),t)-u^-(\eta(x_1,t),t)\,,$$
and
$$\boldsymbol{\delta}\eta_1=\boldsymbol{\delta}\eta\cdot\mathrm{e}_1\,,\ \boldsymbol{\delta}\eta_2=\boldsymbol{\delta}\eta\cdot\mathrm{e}_2 \ \text{ and } \ \boldsymbol{\delta}u_1^-=\boldsymbol{\delta}u^-\cdot\mathrm{e}_1\,,\ \boldsymbol{\delta}u_2^-=\boldsymbol{\delta}u^-\cdot\mathrm{e}_2\,.$$
We choose $0\le t_0<T$ such that t_0 is infinitesimally close to T. Since η is the flow of the velocity u^-, we see that for any $t\in[t_0,T)$,
$$\partial_t\boldsymbol{\delta}\eta = u^-(\eta(x_0,t),t)-u^-(\eta(x_1,t),t)\,. \tag{6.1}$$
Our next result establishes the evolution equation for $\boldsymbol{\delta}\eta(t)$.

THEOREM 6.1 (Evolution equation for $\boldsymbol{\delta}\eta(t)$). *With the assumed bounds (3.1), and for $x_0,x_1\in\Gamma$ such that $|\eta(x_0,t)-\eta(x_1,t)|\to 0$ as $t\to T$, if $|[\nabla_\tau\delta u\cdot\tau](\eta(x_0,t),t)|\to\infty$ as $t\to T$, then for $0<\epsilon\ll 1$ and $0<T-t_0$ sufficiently small, we have that for all $t\in[t_0,T)$,*
$$\partial_t\boldsymbol{\delta}\eta(t)=\mathcal{M}(t)\boldsymbol{\delta}\eta(t) \ \text{where}\ \mathcal{M}(t)=\frac{1}{T-t}\left[\begin{array}{cc} -\beta_1(t) & \varepsilon_1(t)\\ \mathcal{E}_2(t) & \alpha_2(t)\end{array}\right], \tag{6.2}$$
where the coefficients $\beta_1(t),\alpha_2(t)\in[-2\epsilon,1+6(T-t)]$ and $\varepsilon_1(t),\mathcal{E}_2(t)\in[-2\epsilon,2\epsilon]$.

Proof. Step 1. The geometric set-up. Figure 4 shows the geometry of the two approaching curves at some instant of time $t\in[t_0,T)$: the left side of the figure shows the case that $\eta_2(x_0,t)\le\eta(x_1,t)$ and the right side of the figure shows the case that $\eta_2(x_0,t)>\eta(x_1,t)$. Of course, both $\Gamma_0(t)$ and $\Gamma_1(t)$ can have very small oscillations in near the contact points, but this does not effect the qualitative picture in any way. Our idea is to connect $\eta(x_0,t)$ with $\eta(x_1,t)$ using a specially chosen path. The tangent vector to both $\eta(x_0,t)$ and $\eta(x_1,t)$ is horizontal, and by the continuity of the tangent vector, in a sufficiently small neighborhood of the contact points, the tangent vector to the interface is very close to horizontal. Thus, for $T-t_0$ sufficiently small, the two approaching curves $\Gamma_0(t)$ and $\Gamma_1(t)$ are nearly flat, and when $\eta(x_1,t)$ is sufficiently close to $\eta(x_0,t)$, a portion of $\Gamma_1(t)$ must lie below $\eta(x_0,t)$. We define the point $z(t)\in\Gamma$ such that $\eta(z(t),t)$ is the vertical projection of $\eta(x_0,t)$ onto the curve $\Gamma_1(t)$ (as shown in Figure 4). Finally, we define $\gamma_1(t)$ to be the vertical line

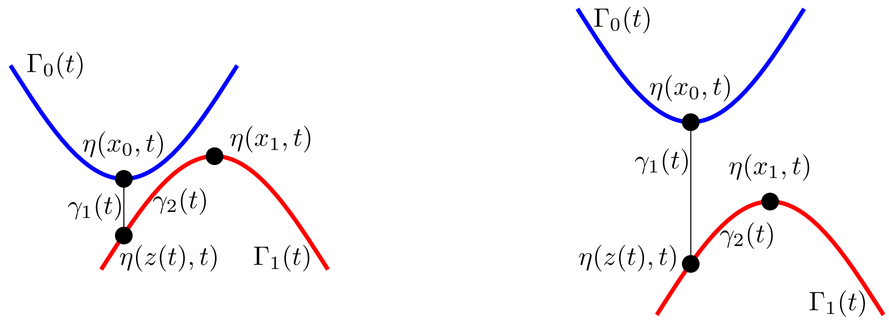


FIGURE 4. Left: $\eta_2(x_0,t) \le \eta_2(x_1,t)$. Right: $\eta_2(x_0,t) > \eta_2(x_1,t)$.

segment connecting $\eta(x_0,t) \in \Gamma_0(t)$ to $\eta(z(t),t) \in \Gamma_0(t)$, and we define $\gamma_2(t)$ to be the portion of $\Gamma_1(t)$ connecting $\eta(z(t),t)$ to $\eta(x_1,t)$.

We will rely on the following two claims:

Claim 1. $|\eta_2(x_1,t) - \eta_2(z(t),t)| = b(t)\boldsymbol{\delta}\eta_1(t)^2$ for a bounded function $b(t)$.

Proof. Near the point $\eta(x_1,t)$, we consider $\Gamma_1(t)$ as a graph $(X, h(X,t))$ above the horizontal X-axis, such that $h(0,t) = \eta(x_1,t)$ with tangent vector $(1, h'(X,t))$, which at $X = 0$ must be horizontal horizontal, so that $h'(0,t) = 0$. Since h is a C^2 function, we can write the Taylor series for $h(X,t)$ about $X = 0$ as

$$h(X,t) = h(0,t) + \frac{h''(\xi)}{2}X^2 \text{ for some } \xi \in (0,X)\,.$$

If $\eta(z(t),t) = h(X,t)$ for some X close to 0, then $X = \boldsymbol{\delta}\eta_1(t)$. By setting $b(t) = \frac{h''(\xi)}{2}$, the proof is complete. □

Claim 2. $|\boldsymbol{\delta}\eta_1(t)| \lesssim \mathcal{M}(T-t)$, and for $T - t_0$ sufficiently small, and with $0 < \epsilon \ll 1$ introduced in Theorem 5.1, $|\boldsymbol{\delta}\eta_1(t)| \le \epsilon$ for all $t \in [t_0, T)$.

Proof. By the fundamental theorem of calculus, $|\boldsymbol{\delta}\eta_1(t)| \le \int_T^t |\delta v(s)|ds \lesssim \mathcal{M}(T-t)$ by Lemma 5.1. Then, we choose $T - t_0$ sufficiently small. □

Step 2. The case that $\eta_2(x_0,t) > \eta_2(x_1,t)$. We will first consider the geometry displayed on the right side of Figure 4. With $\vec{r}_1(t)$ and $\vec{r}_2(t)$ denoting unit-speed parameterizations for $\gamma_1(t)$ and $\gamma_2(t)$,

$$\begin{aligned} u_1^-(\eta(x_0,t),t) - u_1^-(\eta(x_1,t),t) &= \left[u_1^-(\eta(x_0,t),t) - u_1^-(\eta(z(t),t),t)\right] \\ &\qquad + \left[u_1^-(\eta(z(t),t),t) - u_1^-(\eta(x_1,t),t)\right] \\ &= \int_{\gamma_1(t)} \nabla u_1^- \cdot d\vec{r}_1 + \int_{\gamma_2(t)} \nabla u_1^- \cdot d\vec{r}_2 \\ &= \int_{\gamma_1(t)} \frac{\partial u_2^-}{\partial x_1}\, dx_2 + \int_{\gamma_2(t)} \nabla_\tau u_1^-\, ds\,, \end{aligned}$$

where we have used the fact that $\frac{\partial u_1^-}{\partial x_2} = \frac{\partial u_2^-}{\partial x_1}$ in the last equality, as $\operatorname{curl} u^- = 0$. We will evaluate these two integrals using the mean value theorem for integrals, together with our estimate (5.25) for

$\nabla_{\mathcal{T}} u^- \cdot \mathcal{T}$, and hence for $\frac{\partial u_1^-}{\partial x_1}$ (which is close to $\nabla_{\mathcal{T}} u^-$ for $T - t_0$ sufficiently small), and estimate (5.9) for $\frac{\partial u_2^-}{\partial x_1}$. In particular,

$$\begin{aligned} u_1^-(\eta(x_0,t),t) - u_1^-(\eta(x_1,t),t) \\ &= \frac{\varepsilon_1(t)}{T-t}\left(\eta_2(x_0,t) - \eta_2(z(t),t)\right) - \varrho(t)\frac{\alpha_1(t)}{T-t}\boldsymbol{\delta}\eta_1(t) - \nu(t)\frac{\alpha_1(t)}{T-t}\left(\eta_2(x_1,t) - \eta_2(z(t),t)\right), \\ &= \frac{\varepsilon_1(t)}{T-t}\boldsymbol{\delta}\eta_2(t) + \frac{\varepsilon_1(t)}{T-t}\left(\eta_2(x_1,t) - \eta_2(z(t),t)\right) - \varrho(t)\frac{\alpha_1(t)}{T-t}\boldsymbol{\delta}\eta_1(t) \\ &\qquad - \nu(t)\frac{\alpha_1(t)}{T-t}\left(\eta_2(x_1,t) - \eta_2(z(t),t)\right), \end{aligned} \tag{6.3}$$

where $\varepsilon_1(t) \in [-\epsilon, \epsilon]$, and where we choose $\alpha_1(t) \in [-\epsilon, 1 + 3(T-t)]$, where $0 < \epsilon \ll 1$ is defined in Step 4 of the proof of Theorem 5.1. The functions $\varrho(t)$ and $\nu(t)$ satisfy $|1 - \varrho(t)| \ll 1$ and $0 \leq \nu(t) \ll 1$; this follows since $\Gamma_1(t)$ is nearly flat near $\eta(x_0,t)$, so the vertical distance $|\eta_2(x_1,t) - \eta_2(z(t),t)|$ is nearly zero, while the horizontal distance $|\eta_1(x_1,t) - \eta_1(z(t),t)|$ is nearly the total distance $|\eta(x_1,t) - \eta(z(t),t)|$.

The negative sign in front of $\alpha_1(t)$ is determined by the limiting behavior of $\frac{\partial u_1^-}{\partial x_1}$ given by (5.14). From Claim 1 above, we then see that

$$\begin{aligned} u_1^-(\eta(x_0,t),t) - u_1^-(\eta(x_1,t),t) \\ &= \frac{\varepsilon_1(t)}{T-t}\boldsymbol{\delta}\eta_2(t) + \frac{b(t)\boldsymbol{\delta}\eta_1(t)\varepsilon_1(t)}{T-t}\boldsymbol{\delta}\eta_1(t) - \frac{\varrho\alpha_1(t)}{T-t}\boldsymbol{\delta}\eta_1(t) - \frac{\nu b(t)\boldsymbol{\delta}\eta_1(t)\alpha_1(t)}{T-t}\boldsymbol{\delta}\eta_1(t). \end{aligned}$$

We set

$$\beta_1(t) = \left[\varrho(t) + \nu(t)b(t)\boldsymbol{\delta}\eta_1(t)\right]\alpha_1(t) - b(t)\boldsymbol{\delta}\eta_1(t)\varepsilon_1(t).$$

Then, with Claim 2, we see that $\beta_1(t) \in [-2\epsilon, 1 + 6(T-t)]$, and that

$$u_1^-(\eta(x_0,t),t) - u_1^-(\eta(x_1,t),t) = -\frac{\beta_1(t)}{T-t}\boldsymbol{\delta}\eta_1(t) + \frac{\varepsilon_1(t)}{T-t}\boldsymbol{\delta}\eta_2(t)\,. \tag{6.4}$$

Similarly, for u_2^-, we have that

$$\begin{aligned} u_2^-(\eta(x_0,t),t) - u_2^-(\eta(x_1,t),t) &= \left[u_2^-(\eta(x_0,t),t) - u_2^-(\eta(z(t),t),t)\right] \\ &\qquad + \left[u_2^-(\eta(z(t),t),t) - u_2^-(\eta(x_1,t),t)\right] \\ &= \int_{\gamma_1(t)} \nabla u_2^- \cdot d\vec{r}_1 + \int_{\gamma_2(t)} \nabla u_2^- \cdot d\vec{r}_2 \\ &= \int_{\gamma_1(t)} \frac{\partial u_2^-}{\partial x_2}\, dx_2 + \int_{\gamma_2(t)} \nabla_{\mathcal{T}} u_2^-\, ds\,, \\ &= \frac{\alpha_2(t)}{T-t}\left(\eta_2(x_0,t) - \eta_2(z(t),t)\right) + \varrho(t)\frac{\varepsilon_2(t)}{T-t}\boldsymbol{\delta}\eta_1(t) + \nu(t)\frac{\varepsilon_2(t)}{T-t}\left(\eta_2(x_1,t) - \eta_2(z(t),t)\right), \\ &= \frac{\alpha_2(t)}{T-t}\boldsymbol{\delta}\eta_2(t) + \frac{b(t)\boldsymbol{\delta}\eta_1(t)\alpha_2(t)}{T-t}\boldsymbol{\delta}\eta_1(t) + \frac{\varrho(t)\varepsilon_2(t)}{T-t}\boldsymbol{\delta}\eta_1(t) + \frac{\nu(t)b(t)\boldsymbol{\delta}\eta_1(t)\varepsilon_2(t)}{T-t}\boldsymbol{\delta}\eta_1(t)\,, \end{aligned}$$

with $\varepsilon_2(t) \in [-\epsilon, \epsilon]$ and $\alpha_2(t) \in [-\epsilon, 1 + 3(T-t)]$, and where $0 \leq 1 - \varrho(t) \ll 1$ and $0 \leq \nu(t) \ll 1$. Setting

$$\mathcal{E}_2(t) = b(t)\boldsymbol{\delta}\eta_1(t)\alpha_2(t) + \left[\varrho(t) + \nu(t)b(t)\boldsymbol{\delta}\eta_1(t)\right]\varepsilon_2(t)\,, \tag{6.5}$$

we see that by Claim 2, $\mathcal{E}_2(t) \in [-2\epsilon, 2\epsilon]$, and

$$u_2^-(\eta(x_0,t),t) - u_2^-(\eta(x_1,t),t) = \frac{\mathcal{E}_2(t)}{T-t}\boldsymbol{\delta}\eta_1(t) + \frac{\alpha_2(t)}{T-t}\boldsymbol{\delta}\eta_2(t)\,. \tag{6.6}$$

Equations (6.1), (6.4) and (6.6), then give the desired relation (6.2). We can now also establish the estimate

$$|\mathcal{E}_2(t)| \le \mathcal{M}(T-t)\,. \tag{6.7}$$

The first term on the right-hand side of (6.5) satisfies this inequality by Claim 2 above. To show that the second term on the right-hand side of (6.5) satisfies this inequality, we explain why the function $|\varepsilon_2(t)|$ has this bound. In fact, we have already proven this in obtaining the inequality (5.24), where ϵ can be replaced with $C(T-t)$.

Step 3. The case that $\eta_2(x_0,t) \le \eta_2(x_1,t)$. We next consider the geometry displayed on the left side of Figure 4. Again, using $\vec r_1(t)$ and $\vec r_2(t)$ to denote unit-speed parameterisations for $\gamma_1(t)$ and $\gamma_2(t)$, we see that once again

$$\begin{aligned} u_1^-(\eta(x_0,t),t) - u_1^-(\eta(x_1,t),t) &= \big[u_1^-(\eta(x_0,t),t) - u_1^-(\eta(z(t),t),t)\big] \\ &\quad + \big[u_1^-(\eta(z(t),t),t) - u_1^-(\eta(x_1,t),t)\big] \\ &= \int_{\gamma_1(t)} \frac{\partial u_2^-}{\partial x_1}\,dx_2 + \int_{\gamma_2(t)} \nabla_\tau u_1^-\,ds\,, \end{aligned}$$

where s denotes arc length. We again evaluate these two integrals using the mean value theorem for integrals:

$$\begin{aligned} &u_1^-(\eta(x_0,t),t) - u_1^-(\eta(x_1,t),t) \\ &\quad = \frac{\varepsilon_1(t)}{T-t}\left(\eta_2(x_0,t) - \eta_2(z(t),t)\right) - \frac{\varrho(t)\alpha_1(t)}{T-t}\boldsymbol{\delta}\eta_1(t) - \frac{\nu(t)\alpha_1(t)}{T-t}\left(\eta_2(x_1,t) - \eta_2(z(t),t)\right), \end{aligned}$$

where once again $\alpha_1(t) \in [-\epsilon, 1+\epsilon]$ and $\varepsilon_1(t) \in [-\epsilon,\epsilon]$. For some $\theta(t) \in (0,1]$, $|\eta_2(x_0,t) - \eta_2(z(t),t)| = \theta(t)\,|\eta_2(x_1,t) - \eta_2(z(t),t)|$. Hence, by Claim 1,

$$\begin{aligned} &u_1^-(\eta(x_0,t),t) - u_1^-(\eta(x_1,t),t) \\ &\quad = \frac{\theta(t)b(t)\boldsymbol{\delta}\eta_1(t)\varepsilon_1(t)}{T-t}\boldsymbol{\delta}\eta_1(t) - \frac{\varrho(t)\alpha_1(t)}{T-t}\boldsymbol{\delta}\eta_1(t) - \frac{b(t)\boldsymbol{\delta}\eta_1(t)\nu(t)\alpha_1(t)}{T-t}\boldsymbol{\delta}\eta_1(t). \end{aligned}$$

With

$$\beta_1(t) = [\varrho(t) + b(t)\boldsymbol{\delta}\eta_1(t)\nu(t)]\alpha_1(t) - \theta(t)b(t)\boldsymbol{\delta}\eta_1(t)\varepsilon_1(t)\,,$$

then $\beta_1(t) \in [-2\epsilon, 1+6(T-t)]$ and

$$u_1^-(\eta(x_0,t),t) - u_1^-(\eta(x_1,t),t) = -\frac{\beta_1(t)}{T-t}\boldsymbol{\delta}\eta_1(t)\,.$$

Similarly, for u_2^-, we have that

$$\begin{aligned} u_2^-(\eta(x_0,t),t) - u_2^-(\eta(x_1,t),t) &= \big[u_2^-(\eta(x_0,t),t) - u_2^-(\eta(z(t),t),t)\big] \\ &\quad + \big[u_2^-(\eta(z(t),t),t) - u_2^-(\eta(x_1,t),t)\big] \\ = \frac{\alpha_2(t)}{T-t}\left(\eta_2(x_0,t) - \eta_2(z(t),t)\right) &+ \frac{\varrho(t)\varepsilon_2(t)}{T-t}\boldsymbol{\delta}\eta_1(t) + \frac{\nu(t)\varepsilon_2(t)}{T-t}\left(\eta_2(x_1,t) - \eta_2(z(t),t)\right)\,, \end{aligned}$$

with $\varepsilon_2(t) \in [-\epsilon,\epsilon]$ and $\alpha_2(t) \in [-\epsilon, 1+3(T-t)]$. Hence, from Claim 1, we see that

$$\begin{aligned} &u_2^-(\eta(x_0,t),t) - u_2^-(\eta(x_1,t),t) \\ &\qquad = \frac{\theta(t)b(t)\boldsymbol{\delta}\eta_1(t)\alpha_2(t)}{T-t}\boldsymbol{\delta}\eta_1(t) + \frac{\varrho(t)\varepsilon_2(t)}{T-t}\boldsymbol{\delta}\eta_1(t) + \frac{\nu(t)b(t)\boldsymbol{\delta}\eta_1(t)\varepsilon_2(t)}{T-t}\boldsymbol{\delta}\eta_1(t)\,. \end{aligned}$$

Setting

$$\mathcal{E}_2(t) = \theta(t)b(t)\boldsymbol{\delta}\eta_1(t)\alpha_2(t) + \varrho(t)\varepsilon_2(t) + \nu(t)b(t)\boldsymbol{\delta}\eta_1(t)\varepsilon_2(t)\,,$$

we see that by Claim 2, $\mathcal{E}_2(t) \in [-2\epsilon, 2\epsilon]$, and

$$u_2^-(\eta(x_0,t),t) - u_2^-(\eta(x_1,t),t) = \frac{\mathcal{E}_2(t)}{T-t}\boldsymbol{\delta}\eta_1(t)\,.$$

In this case, $\boldsymbol{\delta}\eta_t = \mathcal{M}\,\boldsymbol{\delta}\eta$ with

$$\mathcal{M}(t) = \frac{1}{T-t}\begin{bmatrix} -\beta_1(t) & 0 \\ \mathcal{E}_2(t) & 0 \end{bmatrix}.$$

which is a special case of the matrix given (6.2) with $\epsilon_1(t) = 0$ and $\alpha_2(t) = 0$. This completes the proof. □

7. Proof of the Main Theorem

We now give a proof of Theorem 3.1. We assume that either a splash or splat singularity does indeed occur, and then show that this leads to a contradiction.

We begin the proof with the case that a single splash singularity occurs at time $t = T$ and that there exist two points x_0 and x_1 in Γ, such that $\eta(x_0,T) = \eta(x_1,T)$, as we assumed in Section 6. (In Sections 7.2 and 7.3, we will also rule-out the case of multiple simultaneous splash singularities, as well as the splat singularity.)

7.1. **A single splash singularity cannot occur in finite time.** As we stated above, for $T - t_o$ sufficiently small and in a small neighborhood of $\eta(x_0,T)$, the interface $\Gamma(t)$, $t \in [t_0,T)$, consists of two curves $\Gamma_0(t)$ and $\Gamma_1(t)$ evolving towards one another, with $\eta(x_0,t) \in \Gamma_0(t)$ and $\eta(x_1,t) \in \Gamma_1(t)$. We consider the two cases that either $|\nabla u^-(\cdot,t)|$ remains bounded or blows-up as $t \to T$.

7.1.1. *The case that $|\nabla u^-(\eta(x_0,t),t)| \to \infty$ as $t \to T$.* We prove that both $\boldsymbol{\delta}u_1^-(\eta(x_0,T),T) \neq 0$ and $\boldsymbol{\delta}u_1^-(\eta(x_0,T),T) = 0$.

Step 1. $\boldsymbol{\delta}u_1^- \neq 0$ at the assumed splash singularity $\eta(x_0,T)$. From (6.2), $\boldsymbol{\delta}\eta_2(t) = \mathcal{M}_{21}\boldsymbol{\delta}\eta_1(t) + \mathcal{M}_{22}\boldsymbol{\delta}\eta_2(t)$. Using the integrating factor

$$I(t_0,t) = e^{\int_{t_0}^t \frac{\alpha_2(s)}{T-s}\ ds}\,,$$

we have that

$$\boldsymbol{\delta}\eta_2(t) = I(t_0,t)\left(\boldsymbol{\delta}\eta_2(t_0) + \int_{t_0}^t I(t_0,s)^{-1}\frac{\mathcal{E}_2(s)}{T-s}\boldsymbol{\delta}\eta_1(s)\ ds\right). \tag{7.1}$$

By Theorem 6.1, the function $\alpha_2(t)$ has lower bound given by $\alpha_2(t) \geq -\epsilon$, from which it follows that

$$I(t_0,t) \geq \left(\frac{T-t}{T-t_0}\right)^{\epsilon} \text{ and hence } I(t_0,t)^{-1} \leq \left(\frac{T-t}{T-t_0}\right)^{-\epsilon}. \tag{7.2}$$

Using (7.2) and (7.1) together with $|\delta\eta(t)| \leq \mathcal{M}(T-t)$, we find that

$$\left|\boldsymbol{\delta}\eta_2(t_0) + \int_{t_0}^t I(t_0,s)^{-1}\frac{\mathcal{E}_2(s)}{T-s}\boldsymbol{\delta}\eta_1(s)ds\right| \leq \mathcal{M}(T-t)^{1-\epsilon}(T-t_0)^{\epsilon}\,;$$

therefore, in the limit as $t \to T$ we must have that

$$\boldsymbol{\delta}\eta_2(t_0) + \int_{t_0}^T I(t_0,s)^{-1}\frac{\mathcal{E}_2(s)}{T-s}\boldsymbol{\delta}\eta_1(s)\ ds = 0\,.$$

Since this holds for arbitrary $t_0 < T$, it follows that for any $t < T$ we have the same relation

$$\boldsymbol{\delta}\eta_2(t) + \int_t^T I(t,s)^{-1}\frac{\mathcal{E}_2(s)}{T-s}\boldsymbol{\delta}\eta_1(s)\ ds = 0\,. \tag{7.3}$$

Using (7.2) and the inequality $|\boldsymbol{\delta}\eta_1| \leq \mathcal{M}(T-s)$ in (7.3) and the more precise estimate (6.7), we then obtain that

$$|\boldsymbol{\delta}\eta_2(t)| \leq \mathcal{M}(T-t)^{-\epsilon} \int_t^T (T-s)^{1+\epsilon}\, ds \leq \mathcal{M}(T-t)^2\,. \tag{7.4}$$

We now notice that, thanks to (6.2) used for both components $\boldsymbol{\delta}\eta_1$ and $\boldsymbol{\delta}\eta_2$, we have that

$$\partial_t |\boldsymbol{\delta}\eta|^2 = -2\frac{\beta_1(t)}{T-t}|\boldsymbol{\delta}\eta_1|^2 + 2\frac{\varepsilon_1(t)+\mathcal{E}_2(t)}{T-t}\boldsymbol{\delta}\eta_1\ \boldsymbol{\delta}\eta_2 + 2\frac{\alpha_2(t)}{T-t}|\boldsymbol{\delta}\eta_2|^2\,. \tag{7.5}$$

Therefore,

$$\partial_t|\boldsymbol{\delta}\eta|^2 \geq -\frac{2+C\epsilon}{T-t}|\boldsymbol{\delta}\eta|^2\,,$$

from which we infer that

$$|\boldsymbol{\delta}\eta(t)|^2 \geq |\boldsymbol{\delta}\eta(0)|^2 \frac{(T-t)^{2+C\epsilon}}{T^{2+C\epsilon}}\,. \tag{7.6}$$

Since $\boldsymbol{\delta}\eta(0) \neq 0$, (7.6) and (7.4) allow us to conclude that as $t \to T$, $|\boldsymbol{\delta}\eta_2(t)|$ is much smaller than $|\boldsymbol{\delta}\eta_1(t)|$, and that

$$|\boldsymbol{\delta}\eta_2(t)| = \lambda(t)|\boldsymbol{\delta}\eta_1(t)|\,. \tag{7.7}$$

with

$$|\lambda(t)| \leq C(\mathcal{M})(T-t)^{1-\frac{C\epsilon}{2}}\,. \tag{7.8}$$

We now use (7.8) in (7.5), and conclude that for $t \geq t_0$ and $T-t_0$ sufficiently small,

$$\partial_t|\boldsymbol{\delta}\eta|^2 \geq -\frac{2+3(T-t)}{T-t}|\boldsymbol{\delta}\eta_1|^2 - \frac{\sqrt{T-t}}{T-t}|\boldsymbol{\delta}\eta_1|^2 \geq -\frac{2+2\sqrt{T-t}}{T-t}|\boldsymbol{\delta}\eta|^2\,,$$

and in particular,

$$\partial_t\left(|\boldsymbol{\delta}\eta|^2 e^{\int_{t_0}^t \frac{2}{T-s}\ ds} e^{\int_{t_0}^t \frac{2}{\sqrt{T-s}}\ ds}\right) \geq 0\,.$$

Therefore,

$$|\boldsymbol{\delta}\eta(t)|^2 \geq |\boldsymbol{\delta}\eta(t_0)|^2 \frac{(T-t)^2}{(T-t_0)^2} e^{-\int_{t_0}^T \frac{2}{T-s}\ ds} \geq C(t_0)|\boldsymbol{\delta}\eta(t_0)|^2 (T-t)^2\,,$$

with $C(t_0) > 0$. From (7.7), we then infer from the previous relation that

$$|\boldsymbol{\delta}\eta_1(t)| \geq D(t_0)|\boldsymbol{\delta}\eta(t_0)|(T-t)\,,$$

with $D(t_0) > 0$. Since $\boldsymbol{\delta}\eta(T) = 0$, this also means that for all $t \in (t_0, T)$,

$$\left|\frac{\eta_1(x_0,t)-\eta_1(x_0,T)}{t-T} - \frac{\eta_1(x_1,t)-\eta_1(x_1,T)}{t-T}\right| \geq D(t_0)|\boldsymbol{\delta}\eta(t_0)|\,.$$

Then, taking the limit as $t \to T$ provides us with the inequality

$$|v_1^-(x_0,T) - v_1^-(x_1,T)| \geq D(t_0)|\boldsymbol{\delta}\eta(t_0)|\,,$$

which is the same as

$$|\boldsymbol{\delta}u_1^-(\eta(x_0,T),T)| \geq D(t_0)|\boldsymbol{\delta}\eta(t_0)| > 0\,. \tag{7.9}$$

Step 2. $\boldsymbol{\delta}u_1^- = 0$ *at the assumed splash singularity* $\eta(x_0,T)$. Having shown that $\boldsymbol{\delta}u_1^- \neq 0$ at the splash singularity, in order to arrive at a contradiction, we shall next prove that we also have $\boldsymbol{\delta}u_1^- = 0$ at the splash singularity.

We now define the following two curves. The first curve $\gamma_1(t)$ is the vertical segment joining $\eta(x_1,t) \in \Gamma_1(t)$ to a point $\eta(z(t),t) \in \Gamma_0(t)$. This segment is contained in full in the closure of $\Omega^-(t)$ (for $T-t$ sufficiently small), since the vertical direction is close to the normal direction and no part of the interface which is horizontally close to $\eta(x_1,t)$ can intersect it (other than $\eta(x_1,t)$). The second curve $\gamma_2(t)$ is the portion of $\Gamma_0(t)$ linking $\eta(z(t),t)$ to $\eta(x_0,t)$.

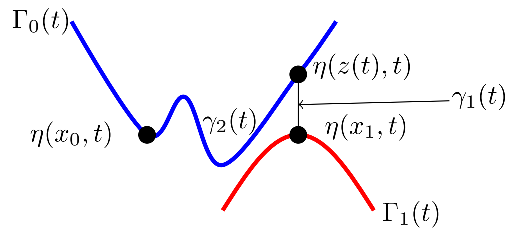


FIGURE 5. Since $\boldsymbol{\delta}\eta_2(t)=0$, the $\eta(x_0,t)$ is approaching $\eta(x_1,t)$ horizontally. The portion of the interface $\Gamma_0(t)$, near $\eta(x_0,t)$, is shown to have an oscillation that may only disappear in the limit as $t\to T$.

We now simply write

$$\begin{aligned}\boldsymbol{\delta}u_1^-(t)&=u_1^-(\eta(x_0,t),t)-u_1^-(\eta(z(t),t)+u_1^-(\eta(z(t),t),t)-u_1^-(\eta(x_1,t)\\&=u_1^-(\eta(x_0,t),t)-u_1^-(\eta(z(t),t)+\int_{\gamma_1(t)}\nabla u_1^-\cdot\tau\;dl\\&=u_1^-(\eta(x_0,t),t)-u_1^-(\eta(z(t),t)+\int_{\gamma_1(t)}\frac{\partial u_1^-}{\partial x_2}\;dx_2\,,\end{aligned}\tag{7.10}$$

where we have used that e_2 is the tangent vector to $\gamma_1(t)$ in the last equality of (7.10).

Next, we estimate the length of the vertical segment $\gamma_1(t)$, by simply noticing that

$$\begin{aligned}|\eta(x_0,t)-\eta(x_1,t)|^2&=|\eta(x_0,t)-\eta(z(t),t)|^2+|\eta(z(t),t)-\eta(x_1,t)|^2\\&\quad+2|\eta(x_0,t)-\eta(z(t),t)||\eta(z(t),t)-\eta(x_1,t)|\cos\theta\,,\end{aligned}\tag{7.11}$$

where θ denotes the angle between the two vectors $\eta(x_0,t)-\eta(z(t),t)$ and $\eta(z(t),t)-\eta(x_1,t)$. By continuity, the direction of the tangent vector τ on $\Gamma_0(t)$ in a small neighborhood of $\eta(x_0,t)$ is very close to e_1; hence, we have that $\eta(x_0,t)-\eta(z(t),t)$ is in direction close to e_1. On the other hand, $\eta(z(t),t)-\eta(x_1,t)$ is in the direction e_2. Therefore, θ is very close to $\frac{\pi}{2}$ which then, in turn, implies from (7.11) that

$$\begin{aligned}|\eta(x_0,t)-\eta(x_1,t)|^2&\geq|\eta(x_0,t)-\eta(z(t),t)|^2+|\eta(z(t),t)-\eta(x_1,t)|^2\\&\quad-\frac{1}{2}|\eta(x_0,t)-\eta(z(t),t)||\eta(z(t),t)-\eta(x_1,t)|\\&\geq\frac{3}{4}|\eta(x_0,t)-\eta(z(t),t)|^2+\frac{3}{4}|\eta(z(t),t)-\eta(x_1,t)|^2\,,\end{aligned}$$

which shows that the square of the length of the vertical segment satisfies

$$\begin{aligned}|\eta(x_0,t)-\eta(z(t),t)|^2&\leq\frac{4}{3}|\eta(x_0,t)-\eta(x_1,t)|^2\\&\leq\frac{4}{3}|\eta(x_0,t)-\eta(x_0,T)-\eta(x_1,t)+\eta(x_1,T)|^2\\&\leq\frac{4}{3}\left|\int_T^t v^-(x_0,s)\;ds-\int_T^t v^-(x_1,s)\;ds\right|^2\\&\leq\frac{16}{3}(T-t)^2\|v^-\|^2_{L^\infty(\Gamma)}\\&\lesssim\mathcal{M}(T-t)^2\,.\end{aligned}\tag{7.12}$$

thanks to Lemma 5.1.

Then, with our estimate (5.9) on $\frac{\partial u_2^-}{\partial x_1}$ and the fact that $\operatorname{curl} u^- = 0$, we then have with (7.12) that

$$\left| \int_{\gamma_1(t)} \nabla u_1^- \cdot \tau \, dl \right| \lesssim \sqrt{\mathcal{M}} \, (T-t) \frac{\epsilon}{T-t} = \epsilon \sqrt{\mathcal{M}} \le \epsilon \mathcal{M} \, . \tag{7.13}$$

It remains to estimate the difference $u_1^-(\eta(x_0,t),t) - u_1^-(\eta(z(t),t)$ appearing on the right-hand side of (7.10). Let $\Gamma_0 \subset \Gamma$ denote that portion of the reference interface such that $\eta(\cdot,t) : \Gamma_0 \to \Gamma_0(t)$. From Lemma 5.1, v^- is continuous along Γ_0. Next, we have that η is continuous and injective from $\gamma_0 \times [0,T]$, γ_0 being a closed part of Γ_0 next to x_0, into a closed subset $\mathcal{K}$ of the space containing $\cup_{t\in[t_0,T]}[\gamma_2(t) \times \{t\}]$. As a result, η^{-1} is also continuous and injective from $\mathcal{K}$ into $\gamma_0 \times [t_0,T]$. By composition, $u^- = v^- \circ \eta^{-1}$ is also continuous on $\mathcal{K}$. Therefore, $u_1(\eta(x_0,t),t) - u_1(\eta(z_0(t),t)$ converges to zero as $t \to T$.

With this fact, we can infer from (7.10) and (7.12) that as $t \to T$

$$|\boldsymbol{\delta} u_1^-(T)| \le \epsilon \mathcal{M} \, ,$$

this being true for any $\epsilon > 0$. Therefore,

$$|\boldsymbol{\delta} u_1^-(T)| = 0 \, ,$$

which is a contradiction with (7.9).

We shall next explain why a non-singular gradient of the velocity u^- also does not allow for a splash singularity, which will finish the proof of our main result in the case of a single self-intersection.

7.1.2. *The case that $|\nabla u^-(x,t)|$ remains bounded.* If $\|\nabla u^-(\cdot,t)\|_{L^\infty(\Omega^-(t))}$ is bounded on $[0,T]$, we can still obtain the differential equation $\boldsymbol{\delta}\eta_t(t) = \mathcal{M}(t)\boldsymbol{\delta}\eta(t)$ using the same path integral that we used in the proof of Theorem 6.1, with paths shown in Figure 4; in this case, however, the components of the matrix $\mathcal{M}$ are bounded on $[0,T]$. The two components of $\boldsymbol{\delta}\eta_t(t) = \mathcal{M}(t)\boldsymbol{\delta}\eta(t)$ are given by $\boldsymbol{\delta}\eta_1(t) = \mathcal{M}_{11}\boldsymbol{\delta}\eta_1(t) + \mathcal{M}_{12}\boldsymbol{\delta}\eta_2(t)$ and $\boldsymbol{\delta}\eta_2(t) = \mathcal{M}_{21}\boldsymbol{\delta}\eta_1(t) + \mathcal{M}_{22}\boldsymbol{\delta}\eta_2(t)$.

Hence, we see that

$$\partial_t |\boldsymbol{\delta}\eta|^2 = 2\mathcal{M}_{11}|\boldsymbol{\delta}\eta_1|^2 + 2\mathcal{M}_{12}(t) + \mathcal{M}_{21}(t)\boldsymbol{\delta}\eta_1 \; \boldsymbol{\delta}\eta_2 + 2\mathcal{M}_{22}|\boldsymbol{\delta}\eta_2|^2 \, .$$

with $\mathcal{M}_{ij}$ bounded for $i,j = 1,2$. Therefore,

$$\partial_t |\boldsymbol{\delta}\eta|^2 \ge -C(\mathcal{M})|\boldsymbol{\delta}\eta|^2 \, ,$$

which then provides

$$|\boldsymbol{\delta}\eta(t)|^2 \ge |\boldsymbol{\delta}\eta(0)|^2 e^{-C(\mathcal{M})t} \, .$$

Since $\boldsymbol{\delta}\eta(0) \ne 0$, we then cannot have $\boldsymbol{\delta}\eta(T) = 0$ for any finite T.

7.2. **An arbitrary number (finite or infinite) of splash singularities at time T is not possible.** We assume that an arbitrary number of simultaneous splash singularities occur at time $T > 0$. We now focus one of the many possible self-intersection points. To this end, let x_0 and x_1 be two points in Γ such that $\eta(x_0,T) = \eta(x_1,T)$. Let $\Gamma_0 \subset \Gamma$ be a local neighborhood of x_0 and let $\Gamma_1 \subset \Gamma$ be a local neighborhood of x_1.

Then, there exists a sequence of points $x_0^n \in \Gamma_0$ converging to x_0, and of a sequence of points $x_1^n \in \Gamma_1$ converging to x_1 such that

$$d_0^n := d(\eta(x_0^n,T), \eta(\Gamma_1,T)) \ne 0 \, , \qquad d_1^n := d(\eta(x_1^n,T), \eta(\Gamma_0,T)) \ne 0 \;\; \forall n \in \mathbb{N} \, , \tag{7.14}$$

where d denotes the distance function; otherwise, if (7.14) did not hold, then we would have non trivial neighborhoods γ_0 of x_0 and γ_1 of x_1 such that $\eta(\gamma_0,T) = \eta(\gamma_1,T)$, which means a splat singularity occurs at $t = T$, and we treat that case below in Section 7.3.

We continue to let e_1 denote the tangent direction to $\Gamma(T)$ at the splash contact point $\eta(x_0,t)$. We then have, by the continuity of the tangent vector τ to the interface, that for both sequences of points,

$$|\mathrm{e}_1 - \tau(\eta(x_0^n,T),T)| \leq \epsilon\,, \tag{7.15}$$

for $\epsilon > 0$ fixed and n large enough. We now call z_1^n the orthogonal projection of $\eta(x_0^n,T)$ onto $\eta(\Gamma_1,T)$. We then have from (7.14) that

$$|\eta(x_0^n,T) - z_1^n| = d_0^n > 0\,. \tag{7.16}$$

Furthermore, we denote by the unit vector e_0^n the direction of the vector $\eta(x_0^n,T) - z_1^n$ (with base point at z_1^n and "arrow" at $\eta(x_0^n,T)$). By definition, e_0^n points in the normal direction to $\eta(\Gamma_1,T)$ at z_1^n and by (7.15), e_0^n is close to e_2. For each point x_0^n, the segment $(\eta(x_0^n,T), z_1^n)$ is contained in $\eta(\Omega^-,T)$.

By continuity of η on $\Gamma\times[0,T]$ we also infer from (7.16) that there exists a connected neighborhood γ_0^n of x_0^n on Γ, of length $L_n > 0$, such that for any $x\in\gamma_0^n$ we have

$$d(\eta(x,T),\eta(\Gamma_1,T)) \geq \frac{d_0^n}{2}; \tag{7.17}$$

moreover, the direction of the vector $\eta(x,T) - P_{\eta(\Gamma_1,T)}(\eta(x,T))$, normal to $\eta(\Gamma_1,T)$ at $P_{\eta(\Gamma_1,T)}(\eta(x,T))$, is close to e_2, where $P_{\eta(\Gamma_1,T)}$ denotes the orthogonal projection onto $\eta(\Gamma_1,T)$.

Note that for each $x\in\gamma_0^n$, the segment $(\eta(x,T), P_{\eta(\Gamma_1,T)}(\eta(x,T)))$ is contained in $\eta(\Omega^-,T)$. By continuity of the direction of these vectors, we then have that

$$\omega_n = \cup_{x\in\gamma_0^n}(\eta(x,T), P_{\eta(\Gamma_1,T)}(\eta(x,T)))\,, \tag{7.18}$$

is an open set contained in $\eta(\Omega^-,T)$. Furthermore, $\partial\omega_n$ contains the set $\eta(\gamma_0^n,T)$ of length $L_n > 0$

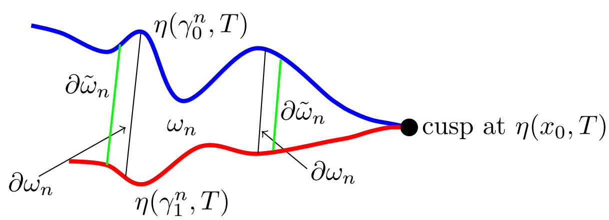


FIGURE 6. The open set ω_n is contained in the larger open set $\tilde{\omega}_n$

(as its top boundary), and by continuity of the directions, $\partial\omega_n$ also contain a connected subset $\eta(\gamma_1^n,T)$ of $\eta(\Gamma_1,T)$, of length greater than $\frac{L_n}{2}$ (as its bottom boundary). Because ω_n does not intersect the cusp which occurs at the contact point, we define the open set $\tilde{\omega}_n \supset \omega_n$, such that the lateral part of $\partial\tilde{\omega}_n$ is parallel to the lateral part of $\partial\omega_n$ and connects $\eta(\Gamma_0,T)$ and $\eta(\Gamma_1,T)$ as shown in Figure 6.

Next, we introduce the stream functions $\psi^\pm$ such that $u^\pm(\cdot,T) = \nabla^\perp\psi^\pm$, and we recall that u^+ (and hence ψ^+) has the good regularity in $\overline{\Omega^+(T)}$ given by (3.1). Let $\mathcal{W}_n$ be an open set such that $\omega_n \subset \mathcal{W}_n \subset \tilde{\omega}_n$. Let $0\leq\vartheta_n\leq 1$ denote a C^∞ cut-off function which is equal to 1 in $\overline{\omega_n}$ and equal to 0 on $\overline{\tilde{\omega}_n}/\mathcal{W}_n$.

We have that ψ^- is an $H^1(\Omega^-(T))$ weak solution of $\Delta\psi^- = 0$ in $\Omega^-(T)$ and $\psi^- = \psi^+$ on $\partial\Omega^-(T)$. Then $\vartheta_n\psi^-$ satisfies

$$\begin{aligned} -\Delta(\vartheta_n\psi^-) &= -\psi^-\Delta\vartheta_n - 2\nabla\vartheta_n\cdot\nabla\psi^-\,, && \text{in } \tilde{\omega}_n\,,\\ \vartheta_n\psi^- &= \psi^+ && \text{on } \eta(\Gamma_0,T)\cup\eta(\Gamma_1,T)\cap\partial\tilde{\omega}_n\,, \end{aligned}$$

and as $\psi^+ \in H^{3.5}(\eta(\Gamma_0,T)) \cup H^{3.5}(\eta(\Gamma_1,T))$, standard elliptic regularity shows that

$$\psi^- \in H^4(\omega_n)\,,$$

and therefore that

$$\nabla u^-(\cdot,T) \in H^3(\omega_n) \subset L^\infty(\omega_n)\,. \tag{7.19}$$

Let $\mathcal{D}_n^r$ denote the pre-image of ω_n under the map $\eta(\cdot,T)$. Let us assume that $\partial\mathcal{D}_n^r \cap \Gamma_0$ lies to the right of x_0. Since $\overline{\omega_n}$ does not intersect the splash singularity at time T, $\eta(\cdot,T)$ is bijective and continuous from $\mathcal{D}_n^r$ into ω_n, and therefore $\mathcal{D}_n^r$ is an open connected set.

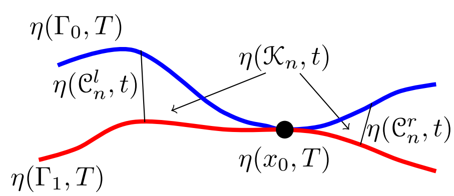
Furthermore, $\nabla u^- \circ \eta$ is also continuous on $\overline{\mathcal{D}_n^r} \times [0,T]$ which, thanks to (7.19), shows that for all $t \in [0,T]$,

$$\|\nabla u^-(\cdot,t)\|_{L^\infty(\eta(\mathcal{D}_n^r,t))} \le \mathcal{M}_n^r\,. \tag{7.20}$$

We can also choose the sequence x_0^n to lie on the left of x_0 (otherwise, we would have a splat singularity). This similarly gives an open neighborhood $\mathcal{D}_n^l$ of the same type as $\mathcal{D}_n^r$ satisfying for all $t \in [0,T]$,

$$\|\nabla u^-(\cdot,t)\|_{L^\infty(\eta(\mathcal{D}_n^l,t))} \le \mathcal{M}_n^l\,. \tag{7.21}$$

We now denote by $\mathcal{C}_n^r$ (respectively $\mathcal{C}_n^l$) the lateral part of $\partial\mathcal{D}_n^r$ (respectively $\partial\mathcal{D}_n^l$) joining Γ_0 to Γ_1, and we denote by $\mathcal{K}_n$ the open set delimited by $\mathcal{C}_n^r$; the subset of Γ_0 containing x_0 linking $\mathcal{C}_n^r$ to $\mathcal{C}_n^l$; $\mathcal{C}_n^l$; and the subset of Γ_1 containing x_1 linking $\mathcal{C}_n^l$ to $\mathcal{C}_n^r$.



For n large enough, we will have estimate (7.15) satisfied at any point of $\partial\mathcal{K}_n \cap \Gamma$, with moreover the length of $\partial\mathcal{K}_n \cap \Gamma$ being of order ϵ. This then implies in a way similar to Step 4 of Theorem 5.1, that

$$\left\|\frac{\partial u_2^-}{\partial x_1}(\cdot,t)\right\|_{L^\infty(\eta(\partial\mathcal{K}_n\cap\Gamma,t))} \le \frac{\epsilon}{T-t}\,, \tag{7.22}$$

for any $t<T$. Moreover, for t close enough to T, the maximum of the two constants $\mathcal{M}_n^r$ and $\mathcal{M}_n^l$ of (7.20) and (7.21) will become smaller than $\frac{\epsilon}{T-t}$. Thus, for any $t<T$ close enough to T,

$$\left\|\frac{\partial u_2^-}{\partial x_1}(\cdot,t)\right\|_{L^\infty(\eta(\partial\mathcal{K}_n,t))} \le \frac{\epsilon}{T-t}\,,$$

which by application (for each fixed $t<T$ close enough to T) of the maximum and minimum principle for the harmonic function $\frac{\partial u_2^-}{\partial x_1}(\cdot,t)$ on the open set $\eta(\mathcal{K}_n,t)$ provides

$$\left\|\frac{\partial u_2^-}{\partial x_1}(\cdot,t)\right\|_{L^\infty(\eta(\mathcal{K}_n,t))} \le \frac{\epsilon}{T-t}\,. \tag{7.23}$$

We can then apply the same arguments as in the Sections 6 and 7.1 to exclude a splash singularity associated with x_0 and x_1 simply by working in the neighborhood of size $C\epsilon$ (C bounded from below away from 0) where (7.23) holds.

7.3. **A splat singularity is not possible.** We now assume the existence of a splat singlarity: there exists two disjoint closed subsets of Γ, which we denote by Γ_0 and Γ_1, with non zero measure, such that contact occurs at time T and $\eta(\Gamma_0,T)=\eta(\Gamma_1,T)$. We furthermore assume that the set

$$\mathcal{S}_0=\{x\in\Gamma_0\ :\lim_{t\to T^-}|\nabla u^-(\eta(x,t),t)|=\infty\}\,, \tag{7.24}$$

has a non-empty interior, and denote by x_0 and y_0 two distinct points on $\mathcal{S}_0$ such that the part γ_0 on Γ_0 linking x_0 to y_0 is contained in $\mathcal{S}_0$. We denote by $L(t)$ the length of the curve $\eta(\gamma_0)(t)$, which is given by

$$L(t)=\int_{\gamma_0}|\eta'(x,t)|\,dl\,. \tag{7.25}$$

Now, for any $x\in\mathcal{S}_0$, $\lim_{t\to T^-}\eta'(x,t)=0$. We also have the uniform bound $|\eta'|_{L^\infty(\Gamma,t)}\le\mathcal{M}$ where $\mathcal{M}$ is independent of $t<T$. Therefore, by the dominated convergence theorem,

$$\lim_{t\to T^-}L(t)=0\,, \tag{7.26}$$

which shows that $\eta(x_0,T)=\eta(y_0,T)$, which is a contradiction with the fact that η is injective on $\Gamma_0\times[0,T]$. Therefore our assumption that $\mathcal{S}_0$ has non-empty interior was wrong, which shows that this set has empty interior. Therefore the set

$$\mathcal{B}_0=\{x\in\Gamma_0\ :\lim_{t\to T^-}|\nabla u^-(\eta(x,t),t)|<\infty\}\,, \tag{7.27}$$

is dense in Γ_0. Furthermore, by Lemma 5.1, $|v'(\cdot,t)|_{L^\infty(\Gamma)}\le\mathcal{M}$ where $\mathcal{M}$ is independent of $t<T$. Hence, by Lemma 5.2, $\mathcal{B}_0$ is defined equivalently as

$$\mathcal{B}_0=\{x\in\Gamma_0\ :|\eta'(x,T)|>0\}\,,$$

which shows that this set is open on Γ_0. Therefore, $\mathcal{B}_0$ is an open and dense subset of Γ_0.

Now since η is continuous and injective from $\Gamma_0\times[0,T]$ onto its image, it also is a homeomorphism from $\Gamma_0\times[0,T]$ onto its image, which shows that $\eta(\mathcal{B}_0,T)$ is open and dense in $\eta(\Gamma_0,T)$. In a similar way, we can show that

$$\mathcal{B}_1=\{x\in\Gamma_1\ :\lim_{t\to T}|\nabla u^-(\eta(x,t),t)|<\infty\}\,, \tag{7.28}$$

is also open and dense in $\eta(\Gamma_1,T)$. Now our splat singularity assumption means that $\eta(\Gamma_0,T)=\eta(\Gamma_1,T)$, showing that $\eta(\mathcal{B}_0,T)$ and $\eta(\mathcal{B}_1,T)$ are two open and dense sets in $\eta(\Gamma_0,T)=\eta(\Gamma_1,T)$. They, therefore, have an open and dense intersection.

Let z be a point in this intersection. By definition, there exists $z_0\in\mathcal{B}_0$ and $z_1\in\mathcal{B}_1$ such that $\eta(z_0,T)=\eta(z_1,T)$. We are therefore back to the case where interface self-intersection occurs with non-singular ∇u^- (from the definition of the sets $\mathcal{B}_0$ and $\mathcal{B}_1$), except that we do not have an estimate for ∇u^- valid for the entire interface $\Gamma(t)$.

We now consider two open connected curves $\gamma_0\subset\mathcal{B}_0$ and $\gamma_1\subset\mathcal{B}_1$ such that for any point $z_0\in\gamma_0$ there exist a point $z_1\in\gamma_1$ such that $\eta(z_0,T)=\eta(z_1,T)$. For $t\in[T_0,T)$, T_0 being very close to T, these two curves are close to each other, and have tangent vector close to e_1, the unit tangent vector at $\eta(x_0,T)$ (by taking a subset of each curve if necessary). For $\epsilon\in(0,1)$ fixed, we have that for any $z\in\gamma_0\cup\gamma_1$ and any $t\in[T_0,T)$,

$$|\tau(\eta(z,t),t)-\mathrm{e}_1|\le\epsilon\,. \tag{7.29}$$

We now define the following two curves. From $\eta(z_0,t)$ we move along the vertical segment $C_1(t)$ joining $\eta(z_0,t)$ to $\eta(\mathcal{Z}_1(t),t)\in\eta(\gamma_1,t)$. This vertical segment is contained in $\Omega^-(t)$ since the tangent vector to the boundary is near horizontal in the neighborhood we are considering. We next call $C_2(t)$ the curve joining $\eta(\mathcal{Z}_1(t),t)$ to $\eta(z_1,t)$ on $\eta(\gamma_1,t)$. Then

$$
\begin{aligned}
u_1^-(\eta(z_0,t),t) - u_1^-(\eta(z_1,t),t) &= \left[u_1^-(\eta(z_0,t),t) - u_1^-(\eta(\mathcal{Z}_1(t),t),t)\right] \\
&\quad + \left[u_1^-(\eta(\mathcal{Z}_1(t),t),t) - u_1^-(\eta(z_1,t),t)\right] \\
&= \int_{C_1(t)} \nabla u_1^- \cdot d\vec{r}_1 + \int_{C_2(t)} \nabla u_1^- \cdot d\vec{r}_2 \\
&= \int_{C_1(t)} \frac{\partial u_2^-}{\partial x_1} dx_2 + \int_{C_2(t)} \tau_1 \frac{\partial u_1^-}{\partial x_1} + \tau_2 \frac{\partial u_1^-}{\partial x_2}\, ds\,.
\end{aligned}
$$

We again evaluate these two integrals using the mean value theorem for integrals, together with our estimates (5.19)–(5.20) for ∇u^-. Thanks to (7.29) and the fact that (by restricting our neighbourhood if necessary)

$$
\left|\int_{C_2(t)} ds\right| \le (1+\epsilon)\ |\boldsymbol{\delta}\eta_1(t)|\,, \tag{7.30}
$$

we see that

$$
u_1^-(\eta(z_0,t),t) - u_1^-(\eta(z_1,t),t) = \frac{\alpha_{12}(t)}{T-t}\boldsymbol{\delta}\eta_2(t) + \alpha_{11}(t)\boldsymbol{\delta}\eta_1(t)\,, \tag{7.31}
$$

with $\alpha_{12}(t) \in [-1-\epsilon, 1+\epsilon]$ thanks to our estimates on ∇u^- and $|\alpha_{11}(t)| \le \mathcal{M}$ thanks to the fact that ∇u^- on $C_2(t)$ is controlled in L^∞ (independently of $t \in [T_0, T)$) due to the fact that $\gamma_1 \subset \mathcal{B}_1$ and our definition (7.28). Similarly,

$$
u_2^-(\eta(z_0,t),t) - u_2^-(\eta(z_1,t),t) = \frac{\alpha_{22}(t)}{T-t}\boldsymbol{\delta}\eta_2(t) + \alpha_{21}(t)\boldsymbol{\delta}\eta_1(t)\,, \tag{7.32}
$$

with $\alpha_{22}(t) \in [-1-\epsilon, 1+\epsilon]$ and $|\alpha_{21}(t)| \le \mathcal{M}$.

By solving equation (6.1), we have that

$$
\boldsymbol{\delta}\eta(t) = \mathcal{S}(\mathcal{M}(t))\boldsymbol{\delta}\eta(T_0)\,, \tag{7.33}
$$

with

$$
\mathcal{M}(t) = \begin{pmatrix} \alpha_{11}(t) & \frac{\alpha_{12}(t)}{T-t} \\ \alpha_{21}(t) & \frac{\alpha_{22}(t)}{T-t} \end{pmatrix},
$$

and $\mathcal{S}(\mathcal{M}(t))$ denotes the solution operator. Using the mean value theorem, we have that

$$
\int_{T_0}^t \mathcal{M}(s)\ ds = \begin{pmatrix} \alpha_{11}(t_1(t))(T-t) & -\alpha_{12}(t_2(t))\log(\frac{T-t}{T-T_0}) \\ \alpha_{21}(t_3(t))(T-t) & -\alpha_{22}(t_4(t))\log(\frac{T-t}{T-T_0}) \end{pmatrix},
$$

where each $t_i(t) \in [0,t]$. Since $\delta\eta(T) = 0$, with $\delta\eta(T_0) \neq 0$, we must have

$$
\lim_{t\to T} \det e^{\int_{T_0}^t \mathcal{M}(s)\ ds} = 0\,,
$$

which then means

$$
\lim_{t\to T} \operatorname{Tr} \int_{T_0}^t \mathcal{M}(s)\ ds = -\infty\,.
$$

Since the first row of the matrix has bounded coefficients as $t \to T$, we see that

$$
\lim_{t\to T} -\alpha_{22}(t_4(t))\log(\frac{T-t}{T-T_0}) = -\infty\,. \tag{7.34}
$$

Next, by forming the characteristic polynomial of the matrix, we see that the product of the (eventually complex conjugate) eigenvalues is

$$
p(t) = -(\alpha_{11}(t_1(t))\alpha_{22}(t_4(t)) - \alpha_{12}(t_2(t))\alpha_{21}(t_3(t)))(T-t)\log(\frac{T-t}{T-T_0}) \to 0\,,
$$

as $t \to T$ since each α_{ij} term is bounded, whereas the sum of the eigenvalues is

$$s(t) = \alpha_{11}(t_1(t))(T-t) - \alpha_{22}(t_4(t))\log(\frac{T-t}{T-T_0}) \to -\infty\,,$$

as $t \to T$ due to (7.34). Therefore, $s^2 - 4p > 0$ for t close enough to T, which implies that the eigenvalues are real with the asymptotic behaviour

$$\lim_{t\to T}\left(\lambda_1(t) - \frac{p(t)}{s(t)}\right) = 0\,, \quad \lim_{t\to T}\left(\lambda_2(t) + \alpha_{22}(t_4(t))\log(\frac{T-t}{T-T_0})\right) = 0\,, \tag{7.35}$$

and with associated corresponding eigenvectors satisfying

$$\lim_{t\to T}(e_1(t) - \mathrm{e}_1) = 0\,, \quad \lim_{t\to T}(e_2(t) - \mathrm{e}_2) = 0\,. \tag{7.36}$$

Therefore

$$\boldsymbol{\delta}\eta(t) = P(t)^{-1}\begin{pmatrix} e^{\lambda_1(t)} & 0 \\ 0 & e^{\lambda_2(t)} \end{pmatrix} P(t)\ \boldsymbol{\delta}\eta(T_0)\,, \tag{7.37}$$

with $P(t)$ denoting a matrix converging to the identity matrix as $t \to T$ thanks to (7.36). Now, let us assume that

$$\boldsymbol{\delta}\eta_1(T_0) \neq 0\,. \tag{7.38}$$

From (7.35) and (7.34), we infer that

$$\lim_{t\to T} e^{\lambda_1(t)} = 1\,, \quad \lim_{t\to T} e^{\lambda_2(t)} = 0\,, \tag{7.39}$$

which with (7.37) and the assumption (7.38) provides:

$$\lim_{t\to T}(\boldsymbol{\delta}\eta_1(t) - \boldsymbol{\delta}\eta_1(T_0)) = 0\,,$$

which is a contradiction with $\boldsymbol{\delta}\eta(T) = 0$. Therefore the assumption (7.38) was wrong, which establishes that

$$\boldsymbol{\delta}\eta_1(T_0) = 0\,.$$

This property could have been established from any choice of origin between T_0 and T. Thus

$$\forall t \in [T_0, T)\,, \quad \boldsymbol{\delta}\eta_1(t) = 0\,, \tag{7.40}$$

which shows that the splat contact would occur with a relative motion along the normal to the surface at the point of contact. Thus,

$$\boldsymbol{\delta}\eta_2(T_0) \neq 0\,, \tag{7.41}$$

which then in turn implies

$$\lim_{t\to T}\frac{\boldsymbol{\delta}\eta_2(t)}{e^{\lambda_2(t)}} = \boldsymbol{\delta}\eta_2(T_0)\,,$$

i.e.

$$\lim_{t\to T}\left(\boldsymbol{\delta}\eta_2(t)\left(\frac{T-t}{T-T_0}\right)^{\alpha_{22}(t_2(t))}\right) = \boldsymbol{\delta}\eta_2(T_0) \neq 0\,, \tag{7.42}$$

with $\alpha_{22}(t) \in [-1-\epsilon, 1+\epsilon]$. This relation was established for any $\epsilon > 0$. Therefore, $\alpha_{22} \in [-1,1]$, and we can write $\alpha_{22}(t_2(t)) = -1 + \sigma(t)$, with $\sigma \geq 0$. Thus

$$\lim_{t\to T}\left(\frac{\boldsymbol{\delta}\eta_2(t)}{T-t}\left(\frac{T-t}{T-T_0}\right)^{(\sigma(t))}\right) = \frac{\boldsymbol{\delta}\eta_2(T_0)}{T-T_0} \neq 0\,. \tag{7.43}$$

Now since

$$\boldsymbol{\delta}\eta_2(t) = \boldsymbol{\delta}\eta_2(T) + \int_T^t \boldsymbol{\delta}u_2^-(s)\ ds = \int_T^t \boldsymbol{\delta}u_2^-(s)\,,$$

where $\boldsymbol{\delta} u_2^-(s) = u_2^-(\eta(x_0,s),s) - u_2^-(\eta(x_1,s),s)$, we obtain with (7.43) that

$$-\lim_{t\to T}\left(\boldsymbol{\delta} u_2^-(T)\left(\frac{T-t}{T-T_0}\right)^{(\sigma(t))}\right) = \frac{\boldsymbol{\delta}\eta_2(T_0)}{T-T_0} \neq 0\,.$$

Since $\left(\frac{T-t}{T-T_0}\right)^{(\sigma(t))} \leq 1$, this then provides us with

$$|\delta u_2(T)| \geq \frac{|\boldsymbol{\delta}\eta_2(T_0)|}{T-T_0} > 0\,, \tag{7.44}$$

which shows that contact occurs with a non zero relative normal velocity. Furthermore, from time differentiating (7.40), we also have that

$$\forall t \in [T_0, T)\,, \quad \boldsymbol{\delta} u_1^-(t) = 0\,, \tag{7.45}$$

which shows that contact occurs in the normal direction. The same occurs for any point of γ_0 and γ_1. Since u^- and τ are continuous along $\eta(\gamma_0,[T_0,T])$ and $\eta(\gamma_1,[T_0,T])$ (the part on u^- resulting from $\gamma_0 \subset \mathcal{B}_0$ and $\gamma_1 \subset \mathcal{B}_1$), by restricting if necessary our neighborhoods we can assume that the tangential vector τ is always at a distance less than ϵ from e_1, and that the difference of velocities field δu is at a distance less than ϵ from some $C_0\mathrm{e}_2$ ($C_0 \neq 0$).

We now call $\omega(t)$ the rectangle-like subset of $\Omega^-(t)$ whose boundary is made with $\eta(\gamma_0,t)$, $\eta(\gamma_1,t)$ and the two vertical-like curves joining the extremities of $\eta(\gamma_0,t)$ and $\eta(\gamma_1,t)$.

We note that the horizontal like sides of this rectangle like domain are with a length greater than some $L > 0$ (independent of t close to T), whereas the vertical ones are of length less than $2C_0\delta t$.

We now notice that

$$\int_{\omega(t)} \frac{\partial u_2^-}{\partial x_2}dx = \int_{\partial\omega(t)} u_2^- n_2 dl(t) = O(C_0 L) \text{ as } t \to T\,. \tag{7.46}$$

Similarly,

$$\left|\int_{\omega(t)} \frac{\partial u_1^-}{\partial x_1}dx\right| = \left|\int_{\partial\omega(t)} u_1^- n_1 dl(t)\right| \leq CL\epsilon + 4|C_0|\delta t\|u^-\|_{L^\infty}\,. \tag{7.47}$$

Comparing (7.46) and (7.47) (due to the divergence free condition), we find that

$$|C_0|L \leq CL\epsilon + 4|C_0|\delta t\|u^-\|_{L^\infty}$$

which is a contradiction for $\epsilon < \frac{C}{|C_0|}$ and δt small enough. This establishes the impossibility of a splat contact at time T.

Furthermore, we see the analysis was done only in a subset of an eventual splat: This means any combination of splats and splashes at time T is excluded (since the above study can be done on an individual splat without change). This finishes the proof of our exclusion of splat or splash singularities in finite time.

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