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TURBULENT CHANNEL FLOW IN WEIGHTED SOBOLEV SPACES USING THE ANISOTROPIC LAGRANGIAN AVERAGED NAVIER-STOKES (LANS-α) EQUATIONS

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Abstract. Modeling the mean characteristics of turbulent channel flow has been one of the longstanding problems in fluid dynamics. While a great number of mathematical models have been proposed for isotropic turbulence, there are relatively few, if any, turbulence models in the anisotropic wall-bounded regime which hold throughout the *entire* channel. Recently, the anisotropic Lagrangian averaged Navier-Stokes equations (LANS-α) have been derived in [7]. This paper is devoted to the analysis of this coupled system of nonlinear PDE for the mean velocity and covariasnce tensor in the channel geometry. The vanishing of the covariance along the walls induces certain degenerate elliptic operators into the model, which require weighted Sobolev spaces to study. We prove that when the no-slip boundary conditions are prescribed for the mean velocity, the LANS-α equations possess unique global weak solutions which converge as time tends to infinity towards the unique stationary solutions. Qualitative properties of the stationary solutions are also established.

1. **Introduction.** Predicting the mean velocity profile of a turbulent incompressible fluid flow in a channel has been one of the fundamental testbeds for turbulence models. Recently, Chen et al. in [2] and [3] have proposed using the stationary solutions of the *isotropic* Lagrangian averaged Navier-Stokes (LANS-α) equations[1] as a closure approximation to the Reynolds equations whose solutions are believed to capture the statistical steady states of the original Navier-Stokes system, and hence the mean profiles throughout the channel. The isotropic theory, which subsumes isotropy in the covariance matrix, possesses inherent limitations as a model throughout the *entire* channel, and as such the results in [2] and [3] appear to be in good agreement with experimental data only away from the viscous sublayer.

Clearly, in the near-wall region, the fluctuations are highly anisotropic, and degenerate to zero along the wall. In this paper, we propose a turbulent channel flow theory founded upon the *anisotropic* Lagrangian averaged Navier-Stokes (LANS-α) equations developed by Marsden & Shkoller in [7] (see also [5] for a similar model). Unlike the isotropic theory which is not intended to model wall-bounded turbulence, the anisotropic theory, holds throughout the entire channel width.

The anisotropic theory has two major advantages over the isotropic theory proposed by Chen et al. First, the isotropic theory leaves the question of appropriate boundary conditions along the wall open, and as such creates a two-parameter

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[1]In [2] and [3], the LANS-α equations are referred to as the viscous Camassa-Holm or Navier-Stokes-α equations.

family of profiles; the parameters must then be chosen to fit experimental data. We show that the *degeneracy* of the covariance matrix in the anisotropic theory serves as the unknown boundary condition in the problem. Second, the isotropic theory assumes that the stationary solutions of the LANS-α equations are the statistical steady states of the Navier-Stokes equations. In our theory, we solve the time-dependent anisotropic LANS-α equations in the channel, and prove that our solutions asymptotically converges as time $t \to \infty$ towards the stationary solutions of the Lagrangian averaged Navier-Stokes equations.

The paper is structured as follows. In section 2, we introduce the anisotropic Lagrangian averaged Navier-Stokes equations on general domains. In section 3, we restrict our attention to the channel geometry, and write the LANS-α equations under the usual assumptions made for channel flow. Mathematically, we must work in weighted Sobolev spaces that compensate for the degeneracy of certain elliptic operators at the walls of the channel, and section 4 is devoted to the introduction of the weighted spaces that we shall use, as well as some fundamental inequalities. In section 5, we establish that the covariance tensor F must degenerate like $\rho = d\sqrt{|\log d|}$ in the viscous sublayer, where d denotes the normalized distance function to the wall. We show that any other choice of weight ρ produces physically unrealistic velocity profiles whose slope either vanishes or is infinite at the walls. Using this weight, in section 6, we proceed to establish global existence and uniqueness of weak solutions to the anisotropic LANS-α equations restricted to the channel. We then prove in section 7 the existence of unique stationary weak solutions, and show that these stationary solutions are the limits of the time-dependent solutions as time $t \to \infty$. Finally, in section 8, we prove that the stationary solution must be even, and moreover concave near the center of the channel.

2. **The anisotropic LANS-α model.**

The anisotropic Lagrangian averaged Navier-Stokes (LANS-α) equations on a domain $\Omega \subset \mathbb{R}^3$ are given by

$$(1-\alpha^2 C)\left(\frac{Du}{dt} - \nu \mathbf{P} C u\right) = -\operatorname{grad} p\,, \tag{1a}$$

$$\operatorname{div} \mathbf{u}(t,\mathbf{x}) = 0\,, \tag{1b}$$

$$\partial_t F + \nabla F \cdot \mathbf{u} = F \cdot \nabla \mathbf{u} + \nabla \mathbf{u}^T \cdot F\,, \tag{1c}$$

$$\mathbf{u}(0,\mathbf{x}) = \mathbf{u}_0(\mathbf{x}), F(0,\mathbf{x}) = F_0(\mathbf{x}) \geq 0, \tag{1d}$$

where $\mathbf{u}$ denotes the divergence-free velocity vector, F denotes the covariance tensor, a 3×3 matrix, and the notation $F_0 \geq 0$ means that the initial covariance matrix is assumed positive semi-definite. Furthermore, D/Dt denotes the total derivative $\partial_t + (u \cdot \nabla)$, and

$$C\mathbf{u} = \operatorname{div}[\nabla \mathbf{u} \cdot F], \tag{2}$$

or in components $(C\mathbf{u})^i = \partial_{x^k}(F^{jk}\partial_{x^j}\mathbf{u}^i)$, where the Einstein summation convention is used. The number $\alpha > 0$ represents a small spatial scale, below which, the fluid motion is averaged. The operator $\mathbf{P}$ denotes either the Leray (or Hodge) projector onto the space of L^2 divergence-free vector fields which are tangential to the boundary $\partial\Omega$, or the Stokes projector onto the space of H^1 divergence-free vector fields that vanish on $\partial\Omega$ (see Definition 2 in [7] and [8] for details on the Stokes projector). For the purpose of this article, either choice of the projector yields the same result. We refer the reader to Marsden & Shkoller [7] for a detailed derivation of this anisotropic model, as well as numerous references and history of the Lagrangian averaged Navier-Stokes equations.

Along the boundary $\partial\Omega$ of the fluid domain, we impose the no-slip boundary conditions given by

$$\mathbf{u} = 0 \text{ on } \partial\Omega. \tag{3}$$

We remark that since the commutator $[\partial_t + (u \cdot \nabla), C] = 0$, equation (1a) is equivalent to

$$\partial_t(1-\alpha^2 C)u + (u\cdot\nabla)(1-\alpha^2 C)u = -\operatorname{grad} p + \nu(1-\alpha^2 C)\mathbf{P}Cu. \tag{4}$$

The covariance tensor F is the ensemble average of the tensor product of the Lagrangian fluctuation vector ξ', and is given by

$$F(t,\mathbf{x}) = \langle \xi'(t,\mathbf{x}) \otimes \xi'(t,\mathbf{x})\rangle.$$

Because the fluctuations are necessarily zero along $\partial\Omega$,

$$F(t,\mathbf{x}) = 0 \text{ for } t \geq 0, \mathbf{x} \in \partial\Omega.$$

Thus, we must contend with an operator C whose ellipticity degenerates at the boundary Ω. In fact, not only does F degenerate at $\partial\Omega$, we also do not know *a priori* if $\|F(t,\cdot)\|_{L^\infty(\Omega)}$ remains bounded for all $t \geq 0$. In [4], we prove the existence of global weak solutions to (1), but those solutions are sufficiently weak to leave the question concerning the blowup of $\|F\|_{L^\infty(\Omega)}$ open.

We shall prove, however, that for the channel geometry with the standard assumptions on the velocity field in a channel, we are able to establish global existence and uniqueness in certain weighted Sobolev spaces that we shall describe below.

3. **Channel geometry and basic assumptions.**

The three-dimensional channel is given by $\Omega = \mathbb{R}^2 \times [-h,h]$ with coordinates $\mathbf{x} = (x,y,z)$. We shall assume that the velocity vector is of the form

$$\mathbf{u}(t,x,y,z) = (u(t,z),0,0), \tag{5}$$

and that the initial covariance matrix

$$F_0(t,x,y,z) = \rho(z)\,\mathrm{Id}\,. \tag{6}$$

We let η denote the Lagrangian flow of $\mathbf{u}$ satisfying $\partial_t\eta(t,\mathbf{x}) = \mathbf{u}(t,\eta(t,\mathbf{x}))$ with $\eta(0,\mathbf{x}) = \mathbf{x}$. Then (5) implies that

$$\eta(t,\mathbf{x}) = \left(x + \int_0^t u(s,z)ds, y, z\right).$$

Letting

$$u'(s,z) := \partial_z u(s,z),$$

we define

$$U(t,\mathbf{x}) = \int_0^t u'(s,z)ds\ .$$

Then

$$F(t,\mathbf{x}) = \rho(z)\begin{bmatrix} 1+U^2 & 0 & U \\ 0 & 1 & 0 \\ U & 0 & 1 \end{bmatrix}, \tag{7}$$

which together with (5) implies that

$$C\mathbf{u} = ((\rho u')', 0, 0)\,.$$

Thus, the nonlinear term in (4) vanishes and the LANS-α equations reduce to the following system:

$$\partial_t \left(u - \alpha^2(\rho u')'\right) - \nu \left((\rho u')' - \alpha^2(\rho(\rho u')'')'\right) = -\partial_x p\,, \tag{8a}$$

$$0 = -\partial_y p\,, \tag{8b}$$

$$0 = -\partial_z p\,, \tag{8c}$$

where $-\partial_x p$ is a constant that we shall denote by c.

4. **Weighted Sobolev Spaces and some inequalities.** We introduce certain weighted Sobolev spaces for functions on the interval $\omega = (-h, h)$.

Let σ be an almost everywhere (a.e.) positive, measurable function on $[-h, h]$. The space $H^m(\sigma)$ is defined as the set of all functions $u = u(z)$ which are defined a.e. on $(-h, h)$ and whose distributional derivatives $D^j u$ for $0 \leq j \leq m$ satisfy

$$\int_{-h}^{h} |D^j u(z)|^2 \sigma(z) dz < \infty,$$

where $D^j = d^j/dz^j$. $H^m(\sigma)$ is a Hilbert space (see [6]), if equipped with the norm

$$\|u\|^2_{H^m(\sigma)} = \sum_{j=0}^{m} \int_{-h}^{h} |D^j u(z)|^2 \sigma(z) dz.$$

When $\sigma = 1$, we shall denote $H^m(\sigma)$ simply by H^m.

Let d denote the normalized distance function to the boundary, so that for $z \in [0, h]$, $d(z) = \dfrac{h - z}{h}$, and for $z \in [-h, 0]$, $d(z) = \dfrac{h + z}{h}$. For $\epsilon < \delta << 1$, let $\rho \in C^\infty([-h, h])$ be a positive function such that

$$\rho(z) = \begin{cases} d\sqrt{|\log d|}, & 0 \leq d(z) \leq \epsilon, \\ 1, & d(z) \geq \delta \end{cases} .$$

We will also need another type of weighted Sobolev space which provides the appropriate functional framework for weak solutions to the anisotropic LANS-α model. The definitions and lemmas that follow are specific to the natural, but non-standard, weighted spaces inherent to our problem.

Definition 4.1. *The space $V^2(\rho)$ is defined as the set of all integrable functions $u = u(z)$ on $(-h, h)$ whose distributional derivatives $D^j u$ for $0 \leq j \leq 2$ satisfy*

$$\int_{-h}^{h} |u(z)|^2 + \rho(z)\ |u'(z)|^2 + |(\rho\ u')'(z)|^2 dz < \infty\ .$$

It is readily seen that $V^2(\rho)$ is a Hilbert space, if equipped with the norm

$$\|u\|^2_{V^2(\rho)} = \int_{-h}^{h} |u(z)|^2 + \rho(z)\ |u'(z)|^2 + |(\rho\ u')'(z)|^2 dz\ .$$

For the remainder of the paper we shall use the symbol $\hookrightarrow$ to denote continuous embedding. We will need the following lemmas.

Lemma 4.1. *For all $u \in V^2(\rho)$, $\rho u'(z) = 0$ at $z = h$ and $z = -h$.*

Proof. Since $\rho^2 \le c\,\rho$ for some constant c we see that we have

$$\int_{-h}^{h} |\rho(z)u'(z)|^2 + |(\rho\ u')'(z)|^2\ dz \le (1+c)\ \|u\|^2_{V^2(\rho)}\ ,$$

which proves that $\rho u' \in H^1 \hookrightarrow C^{0,\frac{1}{2}}$, this embedding allowing us to define the trace.

Now, suppose that $\rho u'(-h) = \beta \ne 0$. Then,

$$\rho(z)u'(z)^2 = \frac{\beta^2}{\rho(z)}\ (1+o(1)),$$

where $o(1) \to 0$ as $z \to -h$. (Note that the right hand side of the above equality simply indicates the continuity of $\rho u'(z)$ at $z=-h$.) Consequently,

$$\int_{-h}^{h} \rho(z)|u'(z)|^2\ dz = \infty\ ,$$

since $\rho = d\sqrt{|\log d|}$ which defines a non-convergent integral. This is impossible, however, since

$$\int_{-h}^{h} \rho(z)|u'(z)|^2\ dz \le \|u\|^2_{V^2(\rho)}\ .$$

Hence $\rho u'(-h) = 0$. The argument for $z=h$ is identical. ∎

LEMMA 4.2. *For any $p \in (1,2)$, $V^2(\rho)$ is continuously embedded in $W^{1,p} \hookrightarrow C^0(\overline{\omega})$.*

Proof. Let $u \in V^2(\rho)$. From the proof of Lemma 4.1, we know that $\rho u' \in H^1$ and that $\rho u'(-h) = 0$ and $\rho u'(h) = 0$. Thus for any $z \in (-h, 0]$ we have

$$\rho u'(z) = \int_{-h}^{z} (\rho u')'(x)\ dx\ ,$$

and thus

$$u'(z) = \frac{1}{\rho(z)}\ \int_{-h}^{z} (\rho u')'(x)\ dx\ .$$

From the Cauchy-Schwartz inequality, we infer that

$$|u'(z)| \le \frac{\sqrt{h\,d(z)}}{\rho(z)}\|(\rho u')'\|_{L^2}.$$

The same inequality also holds for $z \in [0,h)$. Let c be a constant such that for any $z \in (-h,h)$, we have $\rho(z) \ge cd(z)\ \sqrt{|\log(d(z))|}$. Then, for any $z \in (-h,h)$,

$$|u'(z)| \le \frac{\sqrt{h}\|u\|_{V^2(\rho)}}{c\sqrt{d(z)}\ \sqrt{|\log(d(z))|}}.$$

Since the function on the right hand side of this inequality belongs to L^p, we deduce that

$$\int_{-h}^{h} |u'(z)|^p\ dz \le C_p\sqrt{h}\ \|u\|^p_{V^2(\rho)}$$

where $C_p = \displaystyle\int_{-h}^{h} \frac{1}{\left(c\sqrt{d(z)}\ \sqrt{|\log(d(z))|}\right)^p}\ dz$, which defines a convergent integral for $p<2$; the condition $p>1$ ensures the continuous embedding of $W^{1,p}$ into $C^0(\overline{\omega})$. Since $\|u\|_{L^2} \le \|u\|_{V^2(\rho)}$ we deduce that $u \in W^{1,p}$, with a continuous embedding. ∎

Since for $p > 1$, $W^{1,p}$ is continuously embedded in $C^0([-h,h])$, the previous lemma allows us to define a linear and continuous trace operator $Tr : V^2(\rho) \to L^\infty(\{-h,h\})$ by $Tr(u)(-h) = \lim_{z\to -h} u(z)$ and $Tr(u)(h) = \lim_{z\to h} u(z)$.

We are now in a position to define the set

$$V = \{u \in V^2(\rho) \mid Tr(u) = 0\} ,$$

which is a closed subspace of $V^2(\rho)$ and is consequently a Hilbert space when endowed with the norm $\|\cdot\|_{V^2(\rho)}$.

Lemma 4.3 (Poincaré inequality). *For all $u \in V$, there exists $C > 1$ such that*

$$\|u\|^2_{V^2(\rho)} \le CJ(u) \le C\|u\|^2_{V^2(\rho)}, \tag{9}$$

where

$$J(u) := \int_{-h}^{h} \left[\rho u'^2 + (\rho u')'^2\right] dz.$$

Proof. Assume to the contrary; then there exists a sequence u_n in V such that

$$\begin{cases} \|u_n\|_{V^2(\rho)} = 1, \\ J(u_n) \to 0 \end{cases} .$$

Hence, there exists a subsequence u_{n_j} such that as $n_j \to \infty$,

$$\begin{aligned} u_{n_j} &\rightharpoonup u \ \text{ in } V \\ u_{n_j} &\rightharpoonup u \ \text{ in } W^{1,\frac{3}{2}} \\ u_{n_j} &\to u \ \text{ in } L^2. \end{aligned}$$

Then $J(u) \le \liminf_{n_j\to 0} J(u_{n_j}) = 0$, which implies that $u' = 0$, and thus that $u = 0$ (since $Tr(u) = 0$). Thus, $u_{n_j} \to 0$ in L^2. Since $\|u_{n_j}\|^2_{V^2(\rho)} = J(u_{n_j}) + \|u_{n_j}\|^2_{L^2}$, we deduce that $\|u_{n_j}\|_{V^2(\rho)} \to 0$, which contradicts our assumption that $\|u_{n_j}\|_{V^2(\rho)} = 1$, and thus establishes the lemma. ∎

Having proved that $J(u)$ provides an equivalent norm on $V^2(\rho)$, we next establish the following characterization of the Hilbert space V.

Lemma 4.4. *$V = V^2_0(\rho)$, where $V^2_0(\rho)$ is the closure of $\mathcal{D}(\omega)$ in $V^2(\rho)$.*

Proof. The inclusion $V^2_0(\rho) \subset V$ follows from the continuity of the trace operator. For the reverse inclusion, fix an element $u \in V$. From the proof of Lemma 4.2, we know that for any $z \in (-h,h)$ we have that

$$u'(z) = \frac{1}{\rho(z)} \int_{-h}^{z} (\rho u')'(x)\, dx ,$$

and also that

$$\int_{-h}^{h} (\rho u')'(x)\, dx = 0 , \tag{10}$$

since $\rho u' \in H^1_0$. From the continuity of u on $[-h,h]$, with $u(-h) = 0$, we also have for any $z \in [-h,h]$,

$$u(z) = \int_{-h}^{z} \frac{1}{\rho(z')} \int_{-h}^{z'} (\rho u')'(x)\, dx\, dz' ,$$

and from $u(h) = 0$ we infer the condition

$$\int_{-h}^{h} \frac{1}{\rho(z')} \int_{-h}^{z'} (\rho u')'(x)\ dx\ dz' = 0\ . \tag{11}$$

Now, let $(f_n)_{n\geq 1}$ be a sequence of elements of $\mathcal{D}(\omega)$ converging in L^2 towards $(\rho u')'$. Let $\psi \geq 0$ be a given element of $\mathcal{D}(\omega)$, non identically zero, and let $g_n = f_n - c_n\psi$ where

$$c_n = \frac{\int_{-h}^{h} f_n(x)\ dx}{\int_{-h}^{h} \psi(x)\ dx}\ .$$

We obviously have $g_n \in \mathcal{D}(\omega)$ with $\int_{-h}^{h} g_n(x)\ dx = 0$. From (10), we also get $c_n \to 0$ as $n \to \infty$, and thus $g_n \to (\rho u')'$ in L^2 as $n \to \infty$. Now, let $\xi \in \mathcal{D}(\omega)$ be such that $\displaystyle\int_{-h}^{h} \xi(x)\ dx = 0$ and $\displaystyle\int_{-h}^{h} \frac{\rho'(x)}{\rho^2(x)}\ \xi(x)\ dx \neq 0$ (which is realized if ξ is odd and non identically zero). Then a simple integration by parts shows that $\displaystyle\int_{-h}^{h} \frac{1}{\rho(z)} \int_{-h}^{z} \xi(x)\ dx\ dz = \int_{-h}^{h} \frac{\rho'(x)}{\rho^2(x)}\ \xi(x)\ dx \neq 0$. Now, let $h_n = g_n - d_n\xi$ where

$$d_n \int_{-h}^{h} \frac{1}{\rho(z)} \int_{-h}^{z} \xi(x)\ dx\ dz = \int_{-h}^{h} \frac{1}{\rho(z')} \int_{-h}^{z'} g_n(x)\ dx\ dz'\ . \tag{12}$$

Note that the integral on the right hand side is well defined at $-h$ since $\int_{-h}^{z'} g_n(x)\ dx = 0$ for z' close enough to $-h$ (since g_n has a compact support). From $\int_{-h}^{h} g_n(x)\ dx = 0$, we infer that $\int_{-h}^{z'} g_n(x)\ dx = -\int_{h}^{z'} g_n(x)\ dx$ and thus $\int_{-h}^{z'} g_n(x)\ dx = 0$ for z' close enough to h, which explains why the integral is also convergent at h. The integral on the left hand side of (12) converges by the same argument. We obviously have $h_n \in \mathcal{D}(\omega)$ with $\int_{-h}^{h} h_n(x)\ dx = 0$.

Now we prove that $d_n \to 0$ as $n \to \infty$. From (11), we get

$$\int_{-h}^{h} \frac{1}{\rho(z)} \int_{-h}^{z} g_n(z')\ dz'\ dz = \int_{-h}^{h} \frac{1}{\rho(z)} \int_{-h}^{z} g_n(z') - (\rho u')'(z')\ dz'\ dz\ .$$

Thanks to (10) and $\int_{-h}^{h} g_n(x)\ dx = 0$, we deduce from this equality

$$\begin{aligned}\int_{-h}^{h} \frac{1}{\rho(z)} \int_{-h}^{z} g_n(z')\ dz'\ dz &= \int_{-h}^{0} \frac{1}{\rho(z)} \int_{-h}^{z} g_n(z') - (\rho u')'(z')\ dz'\ dz \\ &\quad - \int_{0}^{h} \frac{1}{\rho(z)} \int_{z}^{h} g_n(z') - (\rho u')'(z')\ dz'\ dz\end{aligned}$$

With c a positive constant such that for any $z \in \omega$ we have $\rho(z) \geq c\, d(z)\ \sqrt{|\log(d(z)|}$, we deduce from the Cauchy-Schwartz inequality that

$$\begin{aligned}\left|\int_{-h}^{h} \frac{1}{\rho(z)} \int_{-h}^{z} g_n(z')\ dz'\ dz\right| &\leq \|g_n - (\rho u')'\|_{L^2} \int_{-h}^{0} \frac{\sqrt{h}}{c\sqrt{d(z)|\log(d(z))|}}\ dz \\ &\quad + \|g_n - (\rho u')'\|_{L^2} \int_{0}^{h} \frac{\sqrt{h}}{c\sqrt{d(z)|\log(d(z))|}}\ dz\ ,\end{aligned}$$

and thus

$$\left| \int_{-h}^{h} \frac{1}{\rho(z)} \int_{-h}^{z} g_n(z')\, dz'\, dz \right| \le \|g_n - (\rho u')'\|_{L^2} \int_{-h}^{h} \frac{\sqrt{h}}{c\sqrt{d(z)|\log(d(z))|}}\, dz\ ,$$

which, with (12), shows that $d_n \to 0$ as $n \to \infty$. As a consequence, $h_n \to (\rho u')'$ in L^2 as $n \to \infty$.

Now let us define for any $z \in [-h, h]$,

$$u_n(z) = \int_{-h}^{z} \frac{1}{\rho(z')} \int_{-h}^{z'} h_n(x)\, dx\, dz'\ . \tag{13}$$

This integral is well defined at $-h$ since $\int_{-h}^{z'} h_n(x)\, dx = 0$ for z' close enough to $-h$ (since h_n has a compact support), and thus $u_n(-h) = 0$. From the definition of d_n we also infer that

$$\int_{-h}^{h} \frac{1}{\rho(z')} \int_{-h}^{z'} h_n(x)\, dx\, dz' = 0\ ,$$

and then that $u_n(h) = 0$.

Moreover for any $z \in (-h, h)$,

$$\rho(z) u_n'(z) = \int_{-h}^{z} h_n(x)\, dx\ ,$$

which shows that $u_n \in C^\infty(\omega)$ and that $u_n' = 0$ in a neighborhood of $-h$ (since h_n has a compact support in ω). Since we also have

$$\rho(z) u_n'(z) = -\int_{h}^{z} h_n(x)\, dx\ ,$$

(from $\int_{-h}^{h} h_n(x)\, dx = 0$), we have in the same fashion $u_n' = 0$ in a neighborhood of h. Consequently, u_n is constant in a neighborhood of $-h$ and h, which shows that $u_n \in \mathcal{D}(\omega)$ (from $u_n(-h) = u_n(h) = 0$).

Now, we show that $u_n \to u$ in $V^2(\rho)$ as $n \to \infty$, which will prove the denseness lemma.

First, from $h_n \to (\rho u')'$ in L^2 as $n \to \infty$ and $h_n = (\rho u_n')'$, we infer that $(\rho u_n')' \to (\rho u')'$ in L^2 as $n \to \infty$.

On the other hand,

$$\|\sqrt{\rho}(u_n' - u')^2\|_{L^2}^2 = \int_{-h}^{h} \frac{1}{\rho(z)} \left[\int_{-h}^{z} h_n(z') - (\rho u')'(z')\, dz' \right]^2 dz\ ,$$

and from (10) and $\int_{-h}^{h} h_n(x)\, dx = 0$,

$$\begin{aligned} \|\sqrt{\rho}(u_n' - u')^2\|_{L^2}^2 = & \int_{-h}^{0} \frac{1}{\rho(z)} \left[\int_{-h}^{z} h_n(z') - (\rho u')'(z')\, dz' \right]^2 dz \\ & + \int_{0}^{h} \frac{1}{\rho(z)} \left[\int_{z}^{h} h_n(z') - (\rho u')'(z')\, dz' \right]^2 dz \end{aligned}$$

With c a positive constant such that for any $z \in \omega$ we have $\rho(z) \geq c\, d(z)\ \sqrt{|\log(d(z)|}$, we deduce from the Cauchy-Schwartz inequality that

$$\|\sqrt{\rho}(u_n' - u')^2\|_{L^2}^2 \leq \int_{-h}^{0} \frac{h}{c\sqrt{|\log(d(z))|}}\ \int_{-h}^{z} (h_n(z') - (\rho u')'(z'))^2\ dz'\ dz$$
$$+ \int_{0}^{h} \frac{h}{c\sqrt{|\log(d(z))|}}\ \int_{z}^{h} (h_n(z') - (\rho u')'(z'))^2\ dz'\ dz\ ,$$

and thus

$$\|\sqrt{\rho}(u_n' - u')^2\|_{L^2}^2 \leq \|h_n - (\rho u')'\|_{L^2}^2 \int_{-h}^{h} \frac{h}{c\sqrt{|\log(d(z))|}} dz\ ,$$

which shows that $\|\sqrt{\rho}(u_n' - u')^2\|_{L^2}^2 \to 0$ as $n \to \infty$. Thus we finally get $J(u_n - u) \to 0$ as $n \to \infty$, which establishes that $\|u_n - u\|_{V^2(\rho)} \to 0$ as $n \to \infty$ by the use of Lemma 4.3. ∎

Remark 1. *First, note that it is somewhat surprising that the closure of $\mathcal{D}(\omega)$ in the $V^2(\rho)$ norm has only the zero trace of the function defined, but not the zero trace of the derivative, as one has in the traditional case when the weight $\rho = 1$. Second, note that the regularity of the approximating sequence which we constructed relies on the C^∞ regularity of ρ on $(-h, h)$. In the case that ρ is only C^3, then u_n is C^4 by equation (13), and the subsequent analysis remains unchanged.*

Lemma 4.5. *$V^2(\rho)$ is separable.*

Proof. Just as for the case of standard Sobolev spaces (see [1]), we introduce the operator

$$T : V^2(\rho) \to (L^2)^3, \quad u \mapsto (u, \sqrt{\rho}u', (\rho u')')\,.$$

It is readily seen that $T(V^2(\rho))$ is closed in the separable space $(L^2)^3$; consequently, $T(V^2(\rho))$ is also separable (see [1]), and thus $V^2(\rho)$ is also separable. ∎

Lemma 4.6. *For all $f \in L^2$, there exists a unique $u \in V$ such that for all $v \in V$,*

$$\int_{-h}^{h} [\rho u'v' + (\rho u')'(\rho v')']\, dx = \int_{-h}^{h} f\, v dx.$$

Proof. From Lemma 4.3, $\int_{-h}^{h} [\rho u'v' + (\rho u')'(\rho v')']\, dx$ is an $V^2(\rho)$ equivalent inner-product, so the lemma follows from the Lax-Milgram theorem. ∎

5. **Justification of the weighting function ρ.**

In the following, we consider a general family of even weights, which near the boundary are of the type $\rho(z) = d(z)^\gamma\ |\log(d(z))|^\beta$, where $\gamma > 0$ (in order to have degeneracy on the boundary). We shall once again use the notation $V^2(\rho)$ as in the previous section, but for any choice of weight ρ. We also assume that a trace operator can be defined on $V^2(\rho)$, which is a natural requirement since the condition

$$u = 0 \text{ on the boundary} \tag{14}$$

is part of the formulation of the problem.

Next, we set

$$V = \{u \in V^2(\rho) \mid Tr(u) = 0\}.$$

We assume that the even weight $\rho \in C^2([-h,h])$ ($\rho > 0$ in $(-h,h)$ and $\rho(h) = \rho(-h) = 0$) is such that there exists an even solution to the following variational problem: Find $u \in V$ such that

$$\int_{-h}^{h} \rho u'v' \ + \ \alpha^2 (\rho u')'(\rho v')' \ dz \ - \ \int_{-h}^{h} cv \ dz = 0 \ , \tag{15}$$

for any $v \in V$, where c denotes a non zero constant.

We also assume that the weight ρ is given in such a way that for any $v \in V^2(\rho)$ we have:

$$\rho(z)u'(z) \to 0 \text{ as } z \to -h \text{ and as } z \to h \ . \tag{16}$$

We note that this condition is obviously natural if we want to define a problem where the solution has a finite slope on the boundary.

The previous requirement on the trace operator then leads us to the condition $\gamma < 2$, that we assume satisfied. We note that from Lemmas 4.1, 4.2, and 4.6 the assumptions on the weight ρ are satisfied if $\gamma = 1$ and $\beta = \frac{1}{2}$. In the last section of the paper, we shall prove that this solution is even for the choice of weight ρ considered above.

We prove in this section that among this family of weights, assuming the existence of an even solution to (15), there is only one possible choice such that the solution of problem (15) can have a finite and nonzero derivative on the boundary; namely, for $\gamma = 1$ and $\beta = \frac{1}{2}$, which justifies the framework in which is cast our study (both in the dynamical and static cases).

Since we have assumed that the solution is even, we restrict our study of the derivative of u to the point $-h$.

Henceforth, u denotes the solution to the problem (15). By interior regularity results for elliptic systems of order 4, we infer that $u \in C^4((-h,h))$ and that for any $z \in (-h,h)$,

$$(\rho u')'(z) - \alpha^2 \ (\rho(\rho u')'')'(z) = -c \ . \tag{17}$$

By integrating (17) from 0 to $z \in (-h,h)$, and using the fact that u is even, which implies that $\rho \ u'$ and $\rho \ (\rho \ u')''$ are odd and are hence null at 0, we first obtain:

$$u'(z) - \alpha^2 \ (\rho u')''(z) = -\frac{cz}{\rho(z)} \ ,$$

for any $z \in (-h,h)$.

Integrating this equation from 0 to $z \in (-h,h)$, we obtain

$$u(z) - \alpha^2 \ (\rho u')'(z) = -\int_0^z \frac{cz'}{\rho(z')} dz' + D \ , \tag{18}$$

where D denotes a constant, which by means of another integration from 0 to $z \in (-h,h)$ shows, after using the fact that $u'(0) = 0$, that

$$u'(z) = \frac{c}{\alpha^2 \rho(z)} \ \int_0^z \int_0^{z'} \frac{z''}{\rho(z'')} dz'' dz' + \frac{1}{\alpha^2 \rho(z)} \int_0^z u(z') dz' \ - \ \frac{Dz}{\alpha^2 \rho(z)} \ . \tag{19}$$

Since for any function in $V^2(\rho)$ we have the identity $\rho(z)u'(z) \to 0$ as $z \to -h$, we then infer from (19) that

$$D = -\frac{1}{h} \int_0^{-h} \int_0^{z'} \frac{cz''}{\rho(z'')} dz'' dz' - \frac{1}{h} \int_0^{-h} u(z') dz' \ ,$$

which by substitution in (19), yields

$$\begin{aligned} u'(z) = & \frac{zc}{\alpha^2\rho(z)} \left[\frac{1}{z} \int_0^z \int_0^{z'} \frac{z''}{\rho(z'')} dz'' dz' - \frac{1}{-h} \int_0^{-h} \int_0^{z'} \frac{z''}{\rho(z'')} dz'' dz' \right] \\ & + \frac{1}{\alpha^2\rho(z)} \int_0^z u(z') dz' + \frac{z}{h\alpha^2\rho(z)} \int_0^{-h} u(z') dz' . \end{aligned} \tag{20}$$

By defining the functions f and g in $C^2((-h,h)-\{0\}) \cap C^0([-h,h]-\{0\})$ by

$$f(z) = \frac{1}{z} \int_0^z \int_0^{z'} \frac{z''}{\rho(z'')} dz'' dz' ,$$

$$g(z) = \frac{1}{z} \int_0^z u(z')\, dz' ,$$

for any $z \in (-h,h) - \{0\}$ we can write (20) as

$$u'(z) = \frac{zc}{\alpha^2\rho(z)} [f(z) - f(-h)] + \frac{z}{\alpha^2\rho(z)} [g(z) - g(-h)]. \tag{21}$$

In the following, we assume that $z \in (-h, 0)$. From the mean value theorem, let $(\bar{z}, \tilde{z}) \in (-h, z)^2$ be such that

$$f(z) - f(-h) = (z+h)\, f'(\bar{z}), \tag{22}$$

$$g(z) - g(-h) = (z+h)\, g'(\tilde{z}). \tag{23}$$

A simple computation shows that

$$f'(z) = -\frac{1}{z^2} \int_0^z \int_0^{z'} \frac{z''}{\rho(z'')} dz'' dz' + \frac{1}{z} \int_0^z \frac{z'}{\rho(z')} dz' , \tag{24}$$

$$g'(z) = -\frac{1}{z^2} \int_0^z u(z')\, dz' + \frac{u(z)}{z} , \tag{25}$$

for any $z \in (-h, 0)$.

Hence, we see that for $0 < \gamma < 1$,

$$f'(\bar{z}) \to -\frac{1}{h^2} \int_0^{-h} \int_0^{z'} \frac{z''}{\rho(z'')} dz'' dz' - \frac{1}{h} \int_0^x \frac{z'}{\rho(z')} dz' \in \mathbb{R} \text{ as } \bar{z} \to -h ,$$

$$g'(\tilde{z}) \to -\frac{1}{h^2} \int_0^{-h} u(z')\, dz' \in \mathbb{R} \text{ as } \tilde{z} \to -h ,$$

which, with (22), (23) and (21), shows that $u'(z) \to 0$ as $z \to -h$.

In a similar fashion, if $\gamma = 1$ and $\beta > 1$,

$$f'(\bar{z}) \to -\frac{1}{h^2} \int_0^{-h} \int_0^{z'} \frac{z''}{\rho(z'')} dz'' dz' - \frac{1}{h} \int_0^x \frac{z'}{\rho(z')} dz' \in \mathbb{R} \text{ as } \bar{z} \to -h ,$$

$$g'(\bar{z}) \to -\frac{1}{h^2} \int_0^{-h} u(z')\, dz' \in \mathbb{R} \text{ as } \tilde{z} \to -h ,$$

which, with (22), (23) and (21), shows that $u'(z) \to 0$ as $z \to -h$.

On the other hand, if $1 < \gamma < 2$, let $\gamma' \in (1, \gamma)$. Since the first term of the right hand side of (24) is defined at $-h$, the simple minoration, on the second integral of the right hand side of this equation, of $d(z')^{-\gamma}|\log(d(z'))|^{-\beta}$ by $d(z')^{-\gamma'}$ if z' is close enough to $-h$ yields:

$$|f'(\bar{z})| \geq \frac{h}{(1-\gamma')}\ d(\bar{z})^{1-\gamma'}\ (1+o(1))\ ,$$

where $o(1) \to 0$ as $\bar{z} \to -h$.

Since

$$f(z) - f(-h) = (z+h)\ \int_0^1 f'(-h+t(z+h))\ dt\ ,$$

we deduce from the previous bound that:

$$|f(z)-f(-h)| \geq (1+o(1))\frac{(z+h)h}{(1-\gamma')}\ \int_0^1 t^{1-\gamma'}(d(z))^{1-\gamma'}\ dt\ ,$$

where $o(1) \to 0$ as $\bar{z} \to -h$.

This later bound enables us to assert from (21) and (23) that in a neighborhood of $-h$,

$$\begin{aligned}|u'(z)| \geq &- \frac{z|c|h^2}{\alpha^2}(-1+\gamma')^{-1}(2-\gamma')^{-1}\ \frac{d(z)^{2-\gamma-\gamma'}}{|\log(d(z))|^{\beta}} \\ &- \frac{1}{\alpha^2}\ \frac{d(z)^{1-\gamma}}{|\log(d(z))|^{\beta}}\ (\ |\int_0^{-h} u(z')dz'|\ +1)\ ,\end{aligned} .$$

Since $2-\gamma-\gamma' < 1-\gamma < 0$ the second term appearing on the right hand side of the previous inequality is negligible with respect to the first one, which establishes that $|u'(z)|$ tends to ∞ as z goes to $-h$.

Concerning the case $\gamma = 1$ and $\beta < 1$, we notice that from the expression for f',

$$f'(z) = -\frac{|\log(d(z))|^{1-\beta}}{1-\beta}\ h(1+o(1))\ , \tag{26}$$

where $o(1) \to 0$ as $z \to -h$, so that

$$f'(z) \to -\infty \quad \text{as } z \to -h\ .$$

Consequently, from (22), (23) and (21) we deduce that

$$\begin{aligned}u'(z) =&(1+o(1))\ \frac{h^3 c}{\alpha^2}\ \frac{|\log(d(\bar{z}))|^{1-\beta}}{1-\beta}\ |\log(d(z))|^{-\beta} \\ &- \frac{1}{\alpha^2}\ \frac{1}{|\log(d(z))|^{\beta}}\ (\ \int_0^{-h} u(z')dz'\ +o(1))\ ,\end{aligned} \tag{27}$$

where $o(1) \to 0$ as $z \to -h$. Since

$$|\log(d(\bar{z}))|^{1-\beta} \geq |\log(d(z))|^{1-\beta}\ ,$$

we infer from (27) that

$$\begin{aligned}|u'(z)| \geq&(1+o(1))\ \frac{h^3|c|}{\alpha^2}\ \frac{|\log(z+h)|^{1-2\beta}}{1-\beta} \\ &- \frac{1}{\alpha^2}\ \frac{1}{|\log(z+h)|^{\beta}}\ (\ \int_0^{-h} u(z')dz'\ +o(1))\ ,\end{aligned}$$

where $o(1) \to 0$ as $z \to -h$.

Consequently, if $\beta < \dfrac{1}{2}$, then $1-2\beta > 0$ and $1-2\beta > -\beta$, which implies that

$|u'(z)|$ tends to ∞ as z goes to $-h$.

In the following, we consider the remaining case $\gamma = 1$, and $\frac{1}{2} < \beta \leq 1$. In this case we have the following equivalent of $f(z) - f(-h)$ as $z \to -h$ (we point out here that f' is not defined at $-h$):

LEMMA 5.1. *If* $\frac{1}{2} < \beta < 1$, *then*

$$f(z) - f(-h) = -h^2 \; d(z) |\log(d(z))|^{1-\beta} \frac{(1+o(1))}{1-\beta},$$

where $o(1) \to 0$ *as* $z \to -h$. *If* $\beta = 1$, *then*

$$f(z) - f(-h) = -h^2 \; d(z) \, |\log|\log d(z)|| \, (1+o(1)),$$

where $o(1) \to 0$ *as* $z \to -h$.

Proof. We fix $z \in (-h, 0)$. Given $\epsilon \in (0,1)$, two simple integrations by parts show that

$$\begin{aligned} f(z) =& f(-h+\epsilon(z+h)) + (z+h)f'(z) - (z+h) \; \epsilon \; f'(-h+\epsilon(z+h)) \\ & - (z+h)^2 \int_\epsilon^1 t f''(-h+t(z+h)) \; dt \; . \end{aligned} \tag{28}$$

If $\beta < 1$, we infer from (26) that we have

$$f'(-h+\epsilon(z+h)) = -h \; \frac{|\log(\epsilon d(z))|^{1-\beta}}{1-\beta} \; (1+o(1)) \; ,$$

where $o(1) \to 0$ as $\epsilon \to 0$.

If $\beta = 1$, we have

$$f'(-h+\epsilon(z+h)) = -h \; |\log|\log(\epsilon d(z)|)| \; (1+o(1)) \; ,$$

where $o(1) \to 0$ as $\epsilon \to 0$.

Consequently, we find that for any $\beta \in [\frac{1}{2}, 1]$,

$$\epsilon \; f'(-h+\epsilon(z+h)) \to 0 \quad \text{as} \quad \epsilon \to 0 \; .$$

Since moreover f is continuous at $-h$, we infer from (28) and the previous relation that at the limit $\epsilon \to 0$ (z being fixed)

$$f(z) = f(-h) + (z+h)f'(z) - (z+h)^2 \int_0^1 t f''(-h+t(z+h)) \; dt \; , \tag{29}$$

for any $z \in (-h, 0)$. An elementary computation shows that

$$f''(x) = \frac{2}{x^3} \int_0^x \int_0^{z'} \frac{z''}{\rho(z'')} dz'' dz' \; - \frac{2}{x^2} \int_0^x \frac{z'}{\rho(z')} dz' \; + \; \frac{1}{\rho(x)},$$

for any $x \in (-h, 0)$, from which we deduce that

$$f''(x) = \frac{1}{(d(x))|\log(d(x))|^\beta}(1+o(1)), \tag{30}$$

where $o(1) \to 0$ as $x \to -h$.

Furthermore,

$$(z+h) \int_0^1 \frac{t}{td(z) \; |\log(td(z))|^\beta} dt = \frac{h}{d(z)} \int_0^{d(z)} \frac{1}{|\log(t')|^\beta} dt' \; ,$$

and thus,

$$(z+h)\int_0^1 \frac{t}{td(z)\ |\log(td(z))|^\beta}dt \le \frac{h}{|\log(d(z))|^\beta}\ . \tag{31}$$

From (29), (30) and (31) we infer that:

$$f(z) = f(-h) + (z+h)(f'(z) + o(1))\ , \tag{32}$$

where $o(1) \to 0$ as $z \to -h$.

If $\beta < 1$, (26) yields

$$f(z) - f(-h) = -(z+h)h\ |\log(d(z))|^{1-\beta}\frac{(1+o(1))}{1-\beta}\ , \tag{33}$$

where $o(1) \to 0$ as $z \to -h$.

If $\beta = 1$, using

$$f'(z) = -h|\log|\log(d(z))||\ (1+o(1))\ ,$$

where $o(1) \to 0$ as $z \to -h$, we deduce from (32) that

$$f(z) - f(-h) = -h(z+h)\ |\log|\log(d(z))||(1+o(1))\ , \tag{34}$$

where $o(1) \to 0$ as $z \to -h$. ■

For $\frac{1}{2} < \beta < 1$, by using (33) and (23) into (21), we deduce that in $(-h, 0)$:

$$u'(z) = -zch^2\ |\log(d(z))|^{1-2\beta}\frac{(1+o(1))}{\alpha^2(1-\beta)}\ + \frac{\int_0^{-h} u(z')dz' + o(1)}{\alpha^2|\log(d(z))|^\beta}\ ,$$

where $o(1) \to 0$ as $z \to -h$.

Consequently, we see that if

$$\beta = \frac{1}{2} \text{ then } u'(z) \to \frac{2h^3c}{\alpha^2} \neq 0 \text{ as } z \to -h\ , \tag{35}$$

and if $\beta > \frac{1}{2}$ then $u'(z) \to 0$ as $z \to -h$.

For $\beta = 1$, (34) and (21) yield in $(-h, 0)$:

$$u'(z) = -zch^2\ \frac{|\log|\log(d(z))||}{|\log(d(z))|}\frac{(1+o(1))}{\alpha^2}\ + \frac{\int_0^{-h} u(z')dz' + o(1)}{\alpha^2|\log(d(z))|}\ ,$$

where $o(1) \to 0$ as $z \to -h$, which shows that

$$u'(z) \to 0 \text{ as } z \to -h\ .$$

As a conclusion of this section, we have indeed proved that if there exists an even solution to (15) for a weight of the family considered, then the only possible weight for which the solution has a finite slope at the boundary is given by $\gamma = 1$ and $\beta = \frac{1}{2}$.

In the next sections, ρ denotes a C^∞, even weight which, near the boundary, is of the type $\rho(z) = d(z)\ \sqrt{|\log(d(z))|}$.

6. **Existence and uniqueness of a weak solution.** We first define the functional framework in which the initial data will be taken. In this and in the following sections, ω denotes the open interval $(-h,h)$.

Definition 6.1. *Let $H_0^1(\rho)$ denote the closure of $\mathcal{D}(\omega)$ in $H^1(\rho)$.*

$H_0^1(\rho)$ is of course a closed set of $H^1(\rho)$, and consequently a Hilbert space, endowed with the norm of $H^1(\rho)$. From [6], we emphasize that a trace operator cannot be defined on $H^1(\rho)$, and consequently $H_0^1(\rho)$ cannot be characterized as the subset of $H^1(\rho)$ with zero trace. However, such a characterization is by no means necessary. The key argument for the introduction of this framework is the denseness of V into $H_0^1(\rho)$, which is used in an essential way to get a solution taking into account the initial condition $u(0,\cdot)=u_0(\cdot)$.

In this section, the initial velocity u_0 is assumed to belong to $H_0^1(\rho)$. We first define in a classical way a weak solution to the anisotropic LANS-α equations:

Definition 6.2. *For any $T>0$, a function*

$$u\in C([0,T];H_0^1(\rho))\cap L^2(0,T;V)$$

with $\partial_t(u-\alpha^2(\rho u')')\in L^2(0,T;V')$ and $u(0)=u_0$ is said to be a weak solution to the LANS-α equations with initial data u_0, in the interval $[0,T]$ provided that

$$\begin{aligned}\int_0^T \left\langle \partial_t(u-\alpha^2(\rho\ u')'),v\right\rangle_V dt+\nu\int_0^T\int_\omega \rho u'v' \ +\alpha^2(\rho u')'(\rho v')'\ dzdt&\\ =c\int_0^T\int_\omega v\ dzdt\ ,&\end{aligned}\tag{36}$$

for every $v\in L^2(0,T;V)$, where $\langle\cdot,\cdot\rangle_V$ denotes the dual product between V' and V.

The existence of a weak solution to this problem will be established via a Galerkin type method after the introduction of an appropriate projector at each rank.

Definition 6.3. *Using Lemmas 4.4 and 4.5, we may choose (and fix) a sequence $(e_n)_{n\geq 1}$ in $\mathcal{D}(\omega)$ which forms a Hilbert basis of V. We denote by*

$$E_n=\mathrm{span}[e_i]_{1\leq n}$$

the subspace of $H^4\cap V$ and let

$$V_n=\mathrm{span}[e_i-\alpha^2(\rho\ e_i')']_{1\leq n}$$

denote the subspace of H^2.

We next introduce the following projection operator onto E_n.

Definition 6.4. *For any $n\geq 1$, let P_n denote the orthogonal projector of L^2 (endowed with its usual norm) onto E_n.*

Concerning the dimension of $P_n(V_n)$, we have the following

Lemma 6.1. *For any $n\geq 1$, $dim(P_n(V_n))=n$.*

Proof. Let $g_n \in E_n$ be such that $P_n(g_n - \alpha^2(\rho\, g_n')') = 0$. We multiply this relation by $g_n \in E_n$ and integrate over ω. Since $g_n = 0$ on $\partial\omega$, we obtain:

$$\int_\omega g_n^2 \ + \ \alpha^2 \rho \ g_n'^2 \ dz = 0 \ ,$$

which obviously yields $g_n = 0$ and the desired result. ∎

Concerning the approximation of the initial data, we have the following

LEMMA 6.2. *For any $u_0 \in H^1_0(\rho)$ there exists a sequence $(u_n^0)_{n\geq 1}$ such that for each n, $u_n^0 \in E_n$ and $u_n^0 \to u_0$ in $H^1_0(\rho)$ as $n \to \infty$.*

Proof. From the definition of $H^1_0(\rho)$, there exists a sequence $(v_n^0)_{n\geq 1}$ of elements of $\mathcal{D}(\omega)$ such that $\|v_n - u_0\|_{H^1(\rho)} \to 0$ as $n \to \infty$. Since each $v_n \in V$, and $(e_p)_{p\geq 1}$ is a Hilbert basis of V, let $u_{\sigma(n)} \in E_{\sigma(n)}$ (where σ is strictly increasing, with $\sigma(0) = 1$) be such that $\|u_{\sigma(n)} - v_n\|_{V^2(\rho)} \leq \frac{1}{n}$. From the continuity of the embedding of $V^2(\rho)$ into $H^1(\rho)$, we deduce that $\|u_{\sigma(n)} - v_n\|_{H^1(\rho)} \to 0$ as $n \to \infty$, and consequently $\|u_{\sigma(n)} - u_0\|_{H^1(\rho)} \to 0$ as $n \to \infty$. Now, if we define the sequence $(u_p^0)_{p\geq 1}$ by

$$u_p^0 = u_{\sigma(n)} \text{ for } \sigma(n-1) \leq p < \sigma(n) \ ,$$

we see that this sequence satisfies the statement of the lemma. ∎

In the following, $(u_n^0)_{n\geq 1}$ will denote a sequence satisfying the statement of the previous lemma. We are now in a position to define the Galerkin approximation of (36) at each rank $n \geq 1$.

DEFINITION 6.5. *For any $n \geq 1$, $u_n \in C^1([0,T]; E_n)$ is said to be the solution of the Galerkin approximation of (36) at rank n on [0,T] if*

$$P_n\left(\partial_t\left(u_n - \alpha^2(\rho u_n')'\right)\right) - \nu \ P_n\left((\rho u_n')' - \alpha^2(\rho(\rho u_n')'')'\right) - P_n(c) = 0 \ , \qquad (37)$$

in L^2 for any $t \in [0,T]$, and

$$u_n(0) = u_n^0 \in E_n \ . \qquad (38)$$

This problem admits an unique solution on $\mathbb{R}$ as shown by the following

LEMMA 6.3. *For any $n \geq 1$ there is an unique solution $u_n \in C^1([0,\infty); E_n)$ of problem (37).*

Proof. Let us denote by $(f_i)_{1\leq i\leq n}$ an orthonormal basis of E_n with respect to the L^2 inner product, and set $v_i = f_i - \alpha^2(\rho f_i')'$ for $1 \leq i \leq n$. Letting $u_n(t,z) = \Sigma_{i=1}^n \lambda_i(t) f_i(z)$ we see that (37) is equivalent to

$$\Sigma_{i=1}^n \lambda_i{}'(t) \ P_n(v_i) - \nu \ \Sigma_{i=1}^n \ \lambda_i(t) \ P_n\left((\rho f_i')' - \alpha^2(\rho(\rho f_i')'')'\right) - P_n(c) = 0 \ ,$$

By Lemma 6.1, $(P_n(v_i))_{1\leq i\leq n}$ is a basis of $P_n(V_n)$. Note well that since the f_i are elements of $\mathcal{D}(\omega)$ and hence have compact support, the term $(\rho f_i')' - \alpha^2(\rho(\rho f_i')'')'$ is in L^2, so that the projection P_n is well-defined[2]. Hence, by expressing the projected quantities $P_n\left((\rho f_i')' - \alpha^2(\rho(\rho f_i')'')'\right)$ and $P_n(c)$ in this basis, we see that the previous equation is equivalent to an equation of the type

$$\lambda'(t) + A_n \ \lambda(t) + B_n = 0 \ , \qquad (39)$$

[2]Since the weight ρ is given by $d\sqrt{|\log d|}$, we see that ρ' is in L^2 so that for smooth f, a term of the type $\rho' f'$ is also in L^2 even if f does have compact support.

where $\lambda(t) = (\lambda_1, ..., \lambda_n)^T$ and A_n is a matrix depending on n, and B_n a vector depending on n. On the other hand, the initial condition (38) may be written simply as

$$\lambda(0) = \lambda_0, \tag{40}$$

where $\lambda_0 = (\lambda_0^1, ..., \lambda_0^n)^T$, the $(\lambda_0^i)_{1\leq i\leq n}$ denoting the coordinates of the projection $P_n(u_n^0)$ in the basis f_i of E_n. Since the first order system of ODE with given initial value (39) and (40) admits a unique solution on $[0, \infty)$, then we infer that u_n is unique and defined on $[0, \infty)$. ■

We will also need the following result, whose proof follows from the same arguments as in [9]:

LEMMA 6.4. *If $v \in L^2(0, T; V)$ and*

$$\partial_t(v - \alpha^2(\rho\ v')') \in L^2(0, T; V')\ ,$$

then we have in the distributional sense on $(0, T)$:

$$\frac{d}{dt}\left((v, v)_{L^2} + \alpha^2(\rho\ v', v')_{L^2}\right) = 2\left(\partial_t(v - \alpha^2(\rho\ v')'), v\right)_{L^2}\ .$$

The next theorem asserts the existence of a unique global weak solution.

THEOREM 6.1. *For any initial data $u_0 \in H_0^1(\rho)$, there exists an unique $u \in L^2(0, T; V) \cap C([0, \infty); H_0^1(\rho))$ which is a solution of (36) for any $T > 0$.*

Proof. We first establish estimates independent on n, in various norms, of the solutions of the Galerkin approximations, which will yield at the limit a weak solution to the anisotropic LANS-α equations.

First, let us take $0 < T$. Multiplying (37) by $u_n \in\ E_n$ and integrating the obtained equality on ω yields

$$\int_\omega \partial_t\left(u_n - \alpha^2(\rho u_n')'\right)\ u_n\ dz\ - \int_\omega \left(\nu\left((\rho u_n')' - \alpha^2(\rho(\rho u_n')'')'\right) + c\right) u_n\ dz = 0\ , \tag{41}$$

for any $t \in [0, T]$.

By taking into account the condition $u_n = 0$ on $\partial\omega$, a simple integration by parts leads to

$$\int_\omega \partial_t\left(u_n - \alpha^2(\rho u_n')'\right)\ u_n\ dz = \int_\omega \partial_t u_n\ u_n + \alpha^2\rho\partial_t u_n'\ u_n'\ dz\ ,$$

and thus to

$$\int_\omega \partial_t\left(u_n - \alpha^2(\rho u_n')'\right)\ u_n\ dz = \frac{1}{2}\frac{d}{dt}\int_\omega u_n^2 + \alpha^2\rho u_n'^2\ dz\ . \tag{42}$$

We now identify an energy type law by means of some integrations by parts. Since u_n has compact support, we immediately notice that

$$\int_\omega (\rho u_n')'\ u_n\ dz = -\int_\omega (\rho u_n')\ u_n'\ dz\ . \tag{43}$$

Similarly, a first integration by parts shows that

$$\int_\omega (\rho(\rho u_n')'')'\ u_n\ dz = -\ \int_\omega \rho(\rho u_n')''\ u_n'\ dz\ ,$$

and a second integration by parts yields

$$\int_\omega (\rho(\rho u_n')'')' \; u_n \; dz = \int_\omega (\rho u_n')'^2 \; dz \; . \tag{44}$$

From (41), (42), (43) and (44) we deduce the following energy law:

$$\frac{1}{2}\frac{d}{dt}\int_\omega u_n^2 + \alpha^2 \rho u_n'^2 dz + \; \nu \int_\omega \rho u_n'^2 + \alpha^2 (\rho u_n')'^2 \; dz = c \int_\omega u_n \; dz \; . \tag{45}$$

From the Poincaré inequality (9) and the embedding of Lemma 4.2 we deduce the existence of $M > 0$ (independent of n) such that

$$M \int_\omega u_n^2 \; dz \leq \int_\omega \rho u_n'^2 + \alpha^2 (\rho u_n')'^2 \; dz \; .$$

Hence, we deduce from (45) that

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\int_\omega u_n^2 + \alpha^2 \rho u_n'^2 \; dz \\ &\quad + \; \frac{\nu}{2} \int_\omega \rho u_n'^2 + \alpha^2 (\rho u_n')'^2 \; dz \; + \frac{M\nu}{2} \int_\omega u_n^2 \; dz \leq c \int_\omega u_n \; dz \; , \end{aligned}$$

and an immediate application of the Cauchy-Schwartz and Young inequalities yields:

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\int_\omega u_n^2 + \alpha^2 \rho u_n'^2 \; dz \\ &\quad + \; \frac{\nu}{2} \int_\omega \rho u_n'^2 + \alpha^2 (\rho u_n')'^2 \; dz \; + \frac{M\nu}{4} \int_\omega u_n^2 \; dz \leq \frac{c^2}{M\nu} \; |\omega|^2 \; , \end{aligned}$$

for any $t \geq 0$. By defining $m = \min(\frac{1}{2}, \frac{M}{4})$, we see that the previous inequality leads us to

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\int_\omega u_n^2 + \alpha^2 \rho u_n'^2 \; dz \\ &\quad + \; m\nu \int_\omega u_n^2 + \rho u_n'^2 + \alpha^2 (\rho u_n')'^2 \; dz \; \leq \frac{c^2}{M\nu} \; |\omega|^2 \; . \end{aligned} \tag{46}$$

Let us denote

$$E(u_n)(t) = \frac{1}{2} \int_\omega u_n^2(t,z) + \alpha^2 \rho(z) u_n'^2(t,z) \; dz \; .$$

Integrating (46) from 0 to $T \geq 0$ yields:

$$\begin{aligned} E(u_n)(T) + \; m\nu \int_0^T \int_\omega u_n^2 + \rho u_n'^2 + \alpha^2 (\rho u_n')'^2 \; dz \; dt \leq & \frac{c^2}{M\nu} \; |\omega|^2 \; T \\ & + E(u_n)(0) \; . \end{aligned} \tag{47}$$

By simply noticing that

$$E(u_n)(0) = \int_\omega {u_n^0}^2 + \alpha^2 \rho {u_n^0}'^2 \;\; dz,$$

the initial condition (38) yields:

$$E(u_n)(0) \to \int_\omega u_0^2 + \alpha^2 \rho u_0'^2 \; dz, \text{ as } n \to \infty \; , \tag{48}$$

which shows that the sequence $(E(u_n)(0))_{n\geq 1}$ is bounded.

From (47) and (48), we see that $(u_n)_{n\geq 1}$ belongs to a bounded set of $L^2(0,T;V)$ (independently of n), which yields the existence of a subsequence, still noted $(u_n)_{n\geq 1}$, weakly convergent in the Hilbert space $L^2(0,T;V)$ towards an element $u \in L^2(0,T;V)$. Moreover, since $(u_n)_{n\geq 1}$ belongs to a bounded set of $L^\infty(0,T;H^1_0(\rho))$ (independently of n), then $u \in L^\infty(0,T;H^1_0(\rho))$.

Now let $\psi \in C^\infty([0,T])$ and $v_i \in E_p$ be given, with $\psi(T)=0$. For $n \geq p$, we have $v_i \in E_n$. Let us multiply (37) by ψv_i and integrate the obtained equality on $(0,T)\times\omega$. From the same kind of integration by parts as used in the proof of (45) we infer that:

$$\int_0^T \big(\partial_t(u_n - \alpha^2(\rho\, u_n')'), v_i\big)_{L^2}\ \psi\ dt$$
$$+\nu\int_0^T\int_\omega (\rho u_n' v_i'\ + \alpha^2(\rho u_n')'(\rho v_i')')\psi\ dzdt = c\int_0^T\int_\omega v_i\ \psi\ dzdt\ ,$$

Integrating by parts the previous relation with respect to the time variable yields:

$$-\int_0^T \big(u_n - \alpha^2(\rho\, u_n')', v_i\ \big)_{L^2}\ \frac{d\psi}{dt}\ dt - [\big(u_n^0, v_i\big)_{L^2} + \alpha^2\left(\rho\ u_n^{0\,\prime}, v_i'\right)_{L^2}]\psi(0)$$
$$+\nu\int_0^T\int_\omega (\rho u_n' v_i'\ + \alpha^2(\rho u_n')'(\rho v_i')')\psi\ dzdt = c\int_0^T\int_\omega v_i\ \psi\ dzdt\ ,$$

By letting $n\to\infty$ we get by weak convergence on the time integrals and strong convergence on the initial data that

$$-\int_0^T \big(u - \alpha^2(\rho\, u')', v_i\ \big)_{L^2}\ \frac{d\psi}{dt}\ dt - [(u_0, v_i)_{L^2} + \alpha^2\,(\rho\ u_0{}', v_i')_{L^2}]\psi(0)$$
$$+\nu\int_0^T\int_\omega (\rho u' v_i'\ + \alpha^2(\rho u')'(\rho v_i')')\psi\ dzdt = c\int_0^T\int_\omega v_i\ \psi\ dzdt\ .$$

The previous relation being true for any $v_i \in E_p$, for any $p \geq 1$, since $(e_i)_{i\geq 1}$ is a Hilbert basis of V, we may infer that for any $v \in V$,

$$\begin{aligned}&-\int_0^T \big(u - \alpha^2(\rho\, u')', v\ \big)_{L^2}\ \frac{d\psi}{dt}\ dt - [(u_0, v)_{L^2} + \alpha^2\,(\rho\ u_0{}', v')_{L^2}]\psi(0)\\ &\quad+\nu\int_0^T\int_\omega (\rho u' v'\ + \alpha^2(\rho u')'(\rho v')')\psi\ dzdt = c\int_0^T\int_\omega v\ \psi\ dzdt\ .\end{aligned} \tag{49}$$

Hence, in the distributional sense on $(0,T)$,

$$\frac{d}{dt}\big(u - \alpha^2(\rho\, u')', v\ \big)_{L^2}\ +\ \nu\,(\rho u', v')_{L^2}\ + \nu\alpha^2\,((\rho u')', (\rho v')')_{L^2} = (c, v)_{L^2}\ . \tag{50}$$

From classical results (see [9]) we then infer that

$$u - \alpha^2(\rho\, u')' \in C([0,T];V'), \tag{51}$$

$$\partial_t(u - \alpha^2(\rho\, u')') \in L^2(0,T;V')\ . \tag{52}$$

and that in $L^2(0,T;V')$,

$$\partial_t(u - \alpha^2(\rho\, u')')\ - \nu\left((\rho\, u')' - \alpha^2\ (\rho(\rho\, u')'')'\right) = c\ . \tag{53}$$

As a consequence of Lemma 6.4 and equation (50), we have that in the distributional sense on $(0,T)$

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\left((u,u)_{L^2}+\alpha^2(\rho\ u',u')_{L^2}\right) \\ &+\ \nu\left((\rho u',u')_{L^2}\ +\alpha^2\left((\rho\ u')',(\rho\ u')'\right)_{L^2}\right)=(c,u)_{L^2}\ . \end{aligned} \tag{54}$$

The previous statement leads in the same fashion as in [9] to the fact that

$$u\in C([0,T];H^1_0(\rho)). \tag{55}$$

To obtain the initial condition, let us rewrite (50) as

$$\frac{1}{2}\frac{d}{dt}\left((u,v)_{L^2}+\alpha^2(\rho\ u',v')_{L^2}\right)+\ \nu\,(\rho u',v')_{L^2}\ +\nu\alpha^2\left((\rho\ u')',(\rho\ v')'\right)_{L^2}=(c,v)_{L^2}\ ,$$

Multiplying the previous relation by ψ and integrating from 0 to T yields

$$\begin{aligned} -\int_0^T\left(u-\alpha^2(\rho\ u')',v\ \right)_{L^2}\ \frac{d\psi}{dt}\ dt-[(u(0),v)_{L^2}+\alpha^2\left(\rho\ u(0)',v'\right)_{L^2}]\psi(0) \\ +\nu\int_0^T\int_\omega(\rho u'v'\ +\alpha^2(\rho u')'(\rho v')')\psi\ dzdt=c\int_0^T\int_\omega v\ \psi\ dzdt\ . \end{aligned} \tag{56}$$

By comparing (49) and (56) we then find:

$$[(u(0)-u_0,v)_{L^2}+\alpha^2\left(\rho\ (u(0)'-u_0{}'),v'\right)_{L^2}]\psi(0)=0\ .$$

Since $\psi(0)\neq 0$ we have

$$(u(0)-u_0,v)_{L^2}+\alpha^2\left(\rho\ (u(0)'-u_0{}'),v'\right)_{L^2}=0\ , \tag{57}$$

this holding for any $v\in V$, and hence for any $v\in\mathcal{D}(\omega)$. By the definition of $H^1_0(\rho)$, the previous equality also holds true for any $v\in H^1_0(\rho)$, and hence in particular for $v=u(0)'-u_0{}'$, which implies

$$u(0)=u_0\ . \tag{58}$$

From (52), (53), (54), (56), (58), we infer that u is a weak solution of the problem on [0,T].

Now, to prove uniqueness of such a solution on [0,T], we simply notice that if we let $\bar u$ denote another solution, then the difference $w=u-\bar u$ satisfies:

$$\partial_t(w-\alpha^2(\rho\ w')')\in L^2(0,T;V')\ ,$$

$$w-\alpha^2(\rho\ w')'\in C([0,T];V')\ ,$$

$$w\in L^2(0,T;V)\ ,$$

and

$$\partial_t(w-\alpha^2(\rho\ w')')\ -\nu\ \left((\rho\ w')'-\alpha^2\ (\rho(\rho\ w')'')'\right)=0$$

in $L^2(0,T;V')$. From Lemma 6.4, we also have in the distributional sense on $(0,T)$,

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\left((w,w)_{L^2}+\alpha^2(\rho\ w',w')_{L^2}\right) \\ &+\ \nu\left((\rho w',w')_{L^2}\ +\alpha^2\left((\rho\ w')',(\rho\ w')'\right)_{L^2}\right)=0\ . \end{aligned}$$

Since $w(0) = 0$ and w is continuous from $[0,T]$ into $H_0^1(\rho)$, the integration of the previous relation from 0 to any $t \in [0,T]$ yields

$$\frac{1}{2}(w(t,\cdot), w(t,\cdot))_{L^2} + \frac{\alpha^2}{2}(\rho\ w'(t,\cdot), w'(t,\cdot))_{L^2}$$
$$+\,\nu \int_0^t (\rho w', w')_{L^2}\ + \alpha^2\,((\rho\ w')', (\rho\ w')')_{L^2}\ dt = 0\ ,$$

which obviously leads to $w = 0$, which proves the uniqueness of the weak solution. ■

7. The stationary solution in the limit $t \to \infty$.

Definition 7.1. *We say that an element $u_s \in V$ is a stationary solution to the channel problem if the variational problem*

$$\nu \int_\omega (\rho u_s' v'\ + \alpha^2 (\rho u_s')'(\rho v')')\ dz = c \int_0^T \int_\omega v\ dz dt\ ,$$

is satisfied for any $v \in V$.

We have established in Lemma 4.6 that there exists a unique solution to this variational problem, which we will denote u_s. With the same assumptions on the initial data u_0 as in the previous section, let u denote the (dynamic) weak solution to problem (37) (guaranteed to exist by the previous theorem). Then, regardless of the choice of the initial velocity, we have the following asymptotic behavior:

Theorem 7.1. *As $t \to \infty$, $u(t,\cdot) \to u_s(\cdot)$ in $H_0^1(\rho)$.*

Proof. If we denote $w = u - u_s$ then w satisfies

$$\partial_t (w - \alpha^2 (\rho\ w')') \in L^2(0,T;V')\ ,$$
$$w - \alpha^2(\rho\ w')' \in C([0,T];V')\ ,$$
$$w \in L^2(0,T;V)\ ,$$
$$w\ \text{ is continuous from } [0,T] \text{ into } H_0^1(\rho)\ ,$$

and in $L^2(0,T;V')$,

$$\partial_t (w - \alpha^2 (\rho\ w')')\ - \nu\ ((\rho\ w')' - \alpha^2\ (\rho (\rho\ w')'')') = 0\ .$$

From Lemma 6.4, we have in the distributional sense on $(0,T)$,

$$\frac{1}{2}\frac{d}{dt}\left((w,w)_{L^2} + \alpha^2 (\rho\ w', w')_{L^2}\right)$$
$$+\ \nu\left((\rho w', w')_{L^2}\ + \alpha^2\,((\rho\ w')', (\rho\ w')')_{L^2}\right) = 0\ .$$

Since $w(0) = u_0 - u_s$ and w is continuous from $[0,T]$ into $H_0^1(\rho)$, the integration of the previous relation from 0 to any $t \in [0,T]$ yields

$$E(t) + \nu \int_0^t (\rho w', w')_{L^2}\ + \alpha^2\,((\rho\ w')', (\rho\ w')')_{L^2}\ dt = E(0)\ , \tag{59}$$

with

$$E(t) = \frac{1}{2}(w(t,\cdot), w(t,\cdot))_{L^2} + \frac{\alpha^2}{2}(\rho\ w'(t,\cdot), w'(t,\cdot))_{L^2}\ .$$

From (59), we infer that

$$\int_0^\infty (\rho w', w')_{L^2} \;+ \alpha^2 \left((\rho\, w')', (\rho\, w')'\right)_{L^2} \; dt \in \mathbb{R} \ ,$$

which provides the existence of a sequence $t_n \to \infty$ such that

$$(\rho w'(t_n,\cdot), w'(t_n,\cdot))_{L^2} \;+ \alpha^2 \left((\rho\, w')'(t_n,\cdot), (\rho\, w')'(t_n,\cdot)\right)_{L^2} \to 0 \text{ as } n \to \infty \ .$$

From the continuity of the embedding of V into $H^1(\rho)$, the previous relation yields

$$(\rho w(t_n,\cdot), w(t_n,\cdot))_{L^2} \;+ \alpha^2 \left(\rho\, w'(t_n,\cdot), w'(t_n,\cdot)\right)_{L^2} \to 0 \text{ as } n \to \infty \ . \tag{60}$$

Since the energy E is a decreasing function of the time (as we can see in (59), we then infer from (60) that

$$E(t) \to 0 \text{ as } t \to \infty \ ,$$

which precisely proves the asymptotic result.

■

8. **Qualitative behavior for the asymptotic mean velocity profile.** We establish hereafter some properties satisfied by the solution to the stationary channel problem of Definition 7.1.

In what follows, c is assumed to be a strictly positive constant.

We first have the required property of symmetry for a Poiseuille flow:

THEOREM 8.1. *The solution u_s is even.*

Proof. We know from the proof of Lemma 4.6 that u_s is the unique minimizer on V of the following functional:

$$J(u) = \frac{\nu}{2} \int_\omega \rho u'^2 \;+ \alpha^2 (\rho u')'^2 \; dz \;\; - c \int_\omega u dz \ .$$

By letting $u_s^- \in V$ be defined on $[-h, h]$ by $u_s^-(z) = u(-z)$, we see from the expression of J that since ρ is even, we have:

$$J(u_s^-) = J(u_s) \ ,$$

which implies that u_s^- also minimizes J on V. From the uniqueness of this minimizer this yields

$$u_s = u_s^- \ ,$$

which proves the theorem. ■

THEOREM 8.2. *The solution u_s belongs to $C^1(\bar\omega) \cap C^2(\omega)$.*

Proof. From formula (19) with c/ν replacing the constant c, we obviously have $u \in C^2(\omega)$. From equation (35), we also have (since $\rho(z) = d(z)\ \sqrt{\log(|d(z)|)}$ close to $\partial\omega$),

$$u_s'(z) \to \frac{2h^3 c}{\nu\alpha^2} > 0 \text{ as } z \to -h \ ,$$

which establishes that $u_s \in C^1(\bar\omega)$. ■

Concerning the shape of the profile, we have the following:

THEOREM 8.3. *$u_s' > 0$ on $[-h, 0)$ (and $u_s' < 0$ on $(0, h]$ by evenness), and u_s is concave at the center of the channel.*

Proof. From formula (18), we have

$$u_s(z) - \alpha^2\ (\rho u_s')'(z) = -\int_0^z \frac{cz'}{\nu\rho(z')}dz' + D\ ,$$

where D is in $\mathbb{R}$. Consequently, we obtain the following asymptotic behavior close to $-h$:

$$(\rho u_s')'(z) = (1+o(1))\frac{ch^2}{\nu\alpha^2}\sqrt{\log(|d(z)|)},$$

■

where $o(1) \to 0$ as $z \to -h$. This shows that $(\rho u_s')'$ is positive in a neighborhood of $-h$ and goes to ∞ as $z \to -h$. Now, if $(\rho u_s')' > 0$ on $(-h, 0)$, $\rho u_s'$ would be increasing (strictly) from $\rho u_s'(-d) = 0$ to $\rho u_s'(0) = 0$ (since u_s is even), which obviously leads to a contradiction. Hence, let $h_0 \in [0, h]$ be the supremum of all values of $\tilde{h}$ such that $(\rho u_s')'(-\tilde{h}) = 0$. Then $h_0 \in (0, h)$ since $(\rho u')$ is infinite at $-h$, and $(\rho u_s')'(-h_0) = 0$, which implies $(\rho u_s')'(h_0) = 0$ since $(\rho u_s')'$ is even.

Now, if we denote $v = (\rho u_s')'$, v is a solution on $[-h_0, h_0]$ of

$$v - \alpha^2\ (\rho v')' = -\frac{c}{\nu} < 0\ ,$$

with the boundary condition $v(h_0) = 0 = v(-h_0)$. The maximum principle shows that $v \le 0$ on $[-h_0, h_0]$.

Consequently, $\rho u_s'$ is decreasing on $[-h_0, 0]$, which shows that $\rho u_s' \ge 0$ on $[-h_0, 0]$ (since $u_s'(0) = 0$).

Now, if for some $z \in [-h_0, 0)$, we have $\rho u_s'(z) = 0$, the monotonicity previously established implies that $\rho u_s' = 0$ on $[z, 0]$ and hence that u_s is constant on $[-h, 0]$. From (18), this implies that

$$\int_0^z \frac{cz'}{\nu\rho(z')}dz'$$

is a constant on $[-h, 0]$ (equal to 0, its value at $z = 0$), which is obviously not the case. Hence, $u' > 0$ on $[-h_0, 0)$.

We also know that on $[-h, -h_0)$, $(\rho u_s')' > 0$, which implies that $\rho u_s'$ is increasing strictly from its value 0 at $-h$ and that $u_s' > 0$ on $[-h, -h_0]$.

We then have established that $u_s' > 0$ on $[-h, 0)$, which is the first part of the assertion of the theorem.

Now, concerning the concavity, from the fact that $v \le 0$ on $[-h_0, h_0]$, we simply have $\rho u_s'' + \rho' u_s' \le 0$ on $[-h_0, 0)$, and since $\rho' \ge 0$ and $u_s' > 0$ on $[-h_0, 0]$, we deduce that $u_s'' < 0$ on $[-h_0, 0]$ and on $[-h_0, h_0]$ as well since u_s'' is even, which proves the concavity of u_s on $[-h_0, h_0]$.

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