Arch. Rational Mech. Anal. (2005) 25–102
Digital Object Identifier (DOI) 10.1007/s00205-004-0340-7

Motion of an Elastic Solid inside an Incompressible Viscous Fluid

DANIEL COUTAND & STEVE SHKOLLER

Communicated by V. ŠVERÁK

Abstract

The motion of an elastic solid inside an incompressible viscous fluid is ubiquitous in nature. Mathematically, such motion is described by a PDE system that couples the parabolic and hyperbolic phases, the latter inducing a loss of regularity which has left the basic question of existence open until now.

In this paper, we prove the existence and uniqueness of such motions (locally in time), when the elastic solid is the linear Kirchhoff elastic material. The solution is found using a topological fixed-point theorem that requires the analysis of a linear problem consisting of the coupling between the time-dependent Navier-Stokes equations set in Lagrangian variables and the linear equations of elastodynamics, for which we prove the existence of a unique weak solution. We then establish the regularity of the weak solution; this regularity is obtained in function spaces that scale in a hyperbolic fashion in both the fluid and solid phases. Our functional framework is optimal, and provides the *a priori* estimates necessary for us to employ our fixed-point procedure.

1. Introduction

We are concerned with establishing the existence (and uniqueness) of solutions for the equations of motion of linearly elastic solids moving and interacting with an incompressible viscous fluid, with the natural conditions of continuity of the velocity fields and normal components of the stress tensors along the moving interface between the two materials.

The analysis of interacting fluid-structure problems has been the subject of active research since the late nineties. As of now, only the question of the possible motion of a solid inside a viscous flow, in which the solid is either *rigid* or consists of a *finite number of modes*, has been settled. In [12], existence and uniqueness (locally in time) of smooth solutions has been obtained using a Lagrangian framework, for the rigid body case, provided that the rigid disk is sufficiently heavy. In [6], for the same problem, but with an arbitrary number of rigid solid bodies, existence of at least one weak solution has been established in an Eulerian formulation by a global variational approach; their result holds for all time in two space dimensions as long as no collisions occur between solids or with the boundary, and is local in time for the three-dimensional case. In [7], by generalizing the methods of [6], the case of an elastic body following the linear Kirchhoff law, with the important restrictions of allowing only a *finite number of modes*, and a relaxation of the continuity of the normal stress along the boundary of the solids, has been considered. The above list of references for contributions to this area is by no means exhaustive; see for instance [4, 10, 13]. Note also that the related problem of the free fall of a rigid body in a Stokes flow in the full space has been considered in [17], for the stationary case, and in [14] for the stationary as well as the time-dependent case.

More recently, the interaction of a viscous incompressible flow with an elastic plate (without the restriction of a finite number of modes), whose constitutive law comprises a *parabolic* hyperviscosity term in the plate, has been studied in [2]. We remark that this additional hyperviscosity term is of crucial importance in that study. (Note also that two-dimensional plate models that approximate thin three-dimensional structures usually contain fourth-order operators arising from bending stresses, whereas models of elastic solids have only second-order operators; as such, plate models can provide better *a priori* control for the motion of the material interface.)

In the *steady-state* situation in which both phases are governed by elliptic operators, [11] has obtained an existence result (for the case where the solid follows the nonlinear Saint Venant-Kirchhoff law) by the use of a fixed-point method that iterates between fluid and solid phases. This approach is indeed natural for the steady-state problem since the analysis can make use of elliptic-regularity theory. For the dynamic problem, however, such an iteration procedure appears to *fail* because of a consequent loss of regularity induced by either a fluid-solid-fluid iteration or a solid-fluid-solid iteration. This loss of regularity is due to the fact that hyperbolic and parabolic systems do not have the same regularity requirements and properties, which is in fact the heart of the difficulty in the coupling of the two phases.

Whereas the coupling between the Navier-Stokes equations and the linear Kirchhoff law is perhaps the most fundamental problem to consider in regards to the motion and interaction of an elastic body in a viscous incompressible fluid, none of the methods that have been developed to date can handle this system, mostly because of the differences between parabolic and hyperbolic regularity, i.e., in both the requirements on the function spaces for the prescribed data, as well as the functional framework of the solution space.

We now come to the formulation of the problem. The motion of the fluid is described by the time-dependent incompressible Navier-Stokes equations, while the deformation of the solid body is governed by the linear Kirchhoff equations. The two models are coupled along the moving material interface by imposing the continuity of the normal component of the stress tensors as well as the particle displacement fields. From the point-of-view of mathematical analysis, the

Navier-Stokes equations are traditionally studied in the Eulerian (or spatial) description, while the elastic body is studied in the Lagrangian (or material) frame. Because the material interface is fixed in the Lagrangian representation, we shall study this problem entirely in material coordinates. This Lagrangian framework also has the advantage of keeping the hyperbolic problem (where the loss of regularity occurs) linear, which is of paramount importance here. Note, however, that a semi-linear elastic system, as for some plate or shell models (see for instance [3]), can be handled without any difficulty by our methodology. The question of existence for the case of a quasilinear elasticity law can also be obtained (and shall be addressed in later work), requiring a smoother functional framework leading to more compatibility conditions at the origin.

Let us now set the equations. Let $\Omega \subset \mathbb{R}^3$ denote an open, bounded, connected and smooth domain with smooth boundary $\partial\Omega$ which represents the fluid container in which both the solid and fluid move. Let $\overline{\Omega^s(t)} \subset \Omega$ denote the closure of an open and bounded subset representing the solid body at each instant of time $t \in [0, T]$ with $\Omega^f(t) := \Omega/\overline{\Omega^s(t)}$ denoting the fluid domain at each $t \in [0, T]$. Note that in our analysis $\Omega^s(t)$ is not necessarily connected, which allows us to handle the case of several elastic bodies moving in the fluid.

Remark 1. If a function u is defined on all of Ω, we will define $u^f = u\ 1_{\overline{\Omega^f_0}}$ and $u^s = u\ 1_{\overline{\Omega^s_0}}$. This allows us to indicate from which phase the traces on

$$\Gamma(0) := \overline{\Omega^f(0)} \cap \overline{\Omega^s(0)}$$

of various discontinuous terms arise, and also to specify functions that are associated with the fluid and solid phases.

For each $t \in (0, T]$, we wish to find the location of these domains inside Ω, the divergence-free velocity field $u^f(t, \cdot)$ of the fluid, the fluid pressure function $p(t, \cdot)$ on $\Omega^f(t)$, the fluid Lagrangian volume-preserving configuration $\eta^f(t, \cdot) : \Omega^f(0) = \Omega^f_0 \to \Omega^f(t)$, and the elastic Lagrangian configuration field $\eta^s(t, \cdot) : \Omega^s(0) = \Omega^s_0 \to \Omega^s(t)$ such that

$$\Omega = \eta^s(t, \overline{\Omega^s_0}) \cup \eta^f(t, \Omega^f_0)\,, \tag{1a}$$

where

$$\eta^f_t(t, x) = u^f(t, \eta^f(t, x))\,, \tag{1b}$$

and u^f solves the Navier-Stokes equations in $\Omega^f(t)$:

$$u^f_t + (u^f \cdot \nabla)u^f = \operatorname{div} T^f + f_f\,, \tag{1c}$$

$$\operatorname{div} u^f = 0\,, \tag{1d}$$

with

$$T^f = \nu\ \operatorname{Def} u^f - p\,\mathrm{I}\,. \tag{1e}$$

The function η^s solves the elasticity equations on $\Omega^s(0)$:

$$\ddot{\eta}^s = \operatorname{div} T^s + f_s \,, \tag{1f}$$

with

$$T^s = \lambda \operatorname{Trace}(\nabla\eta^s - I)I \;+ \mu \,(\nabla\eta^s + \nabla\eta^{sT} - 2\,I)\,, \tag{1g}$$

and where the equations are coupled together by the continuity of the normal component of stress along the material interface $\Gamma(t) := \overline{\Omega^s}(t) \cap \overline{\Omega^f}(t)$ expressed in the Lagrangian representation on

$$\Gamma_0 := \Gamma(0)$$

as

$$T^s\; N = [T^f \circ \eta^f]\,[(\nabla\eta^f)^{-1}\; N]\,, \tag{1h}$$

and the continuity of particle displacement fields along Γ_0

$$\eta^f = \eta^s\,, \tag{1i}$$

together with the initial conditions

$$u(0, x) = u_0(x)\,, \tag{1j}$$

$$\eta(0, x) = x\,, \tag{1k}$$

and the Dirichlet (no-slip) condition on the boundary $\partial\Omega$ of the container

$$u^f = 0\,, \tag{1l}$$

where $\nu > 0$ is the kinematic viscosity of the fluid, $\lambda > 0$ and $\mu > 0$ denote the Lamé constants of the elastic material, N is the outward unit normal to Γ_0 and Def u is twice the rate of deformation tensor of u, given in coordinates by $u^i,_j + u^j,_i$. All Latin indices run through 1, 2, 3, the Einstein summation convention is employed, and indices after commas denote partial derivatives.

We now briefly outline the proof. As the solid and fluid phases are naturally expressed in the Lagrangian and Eulerian framework, respectively, we begin by transforming the fluid phase into Lagrangian coordinates, leading us to the system of equations (4) of Section 3. This system of partial differential equations is both parabolic (in the fluid) and hyperbolic (in the solid) in character; hence, one of the fundamental difficulties that must be overcome is an appropriate functional framework accommodating both features. Sections 4 and 6 are devoted to the setting of our functional framework, which appears to be of hyperbolic type in both solid and fluid phases, and is necessitated by the estimate of the elastic energy. This hyperbolic scaling in turn requires the initial data to possess more regularity, and thus produces more compatibility conditions in the fluid phase than if a parabolic scaling were used (as seen in the statement of the existence theorem in Section 5). Whereas the choice of working in Eulerian or Lagrangian variables may seem arbitrary, at the level of the functional framework, it appears that the problem truly requires this hyperbolic functional framework for both phases, regardless of the choice of spatial or material coordinates.

In order to solve (4), we use a fixed-point approach, where we solve the linear system (20) for the Lagrangian velocity w, the coefficients $a^i_j(\eta)$ coming from the flow map η of a *given* velocity v. The study of the regularity of the solutions to this problem, which constitutes the main part of this paper, is given in Sections 9 and 10. It appears that the regularity theory for (20) cannot be obtained directly by solving the problem with the actual coefficients $a^j_i(\eta)$. In Section 8 we explain the smoothing process for the problem: we introduce smoothed velocity fields v_n which provide us with smoothed coefficients $a^f_i(\eta_n)$ (which we denote generically by $\tilde{v}$ and $\tilde{a}$). We also present two versions of what we term the *Lagrange multiplier lemma* (which associates a pressure function with the weak solution) that will be of basic use throughout this paper.

We study in Section 9 the existence of weak solutions $\tilde{w}$ to (20) (with regularized coefficients), as the limit of penalized problems. Whereas these penalized problems are not necessary merely to obtain existence of weak solutions, they are of *paramount* importance in getting the appropriate regularity results for $\tilde{w}_t$ and $\tilde{w}_{tt}$, the primary reason being that the pressure associated with (20) with the Dirichlet boundary condition cannot be obtained simply from the variational form of the problem, and requires the study of the time-differentiated problem in order to get more information on $\tilde{w}_t$ (which would need to be in $L^2(0, T; H^{-1}(\Omega; \mathbb{R}^3))$ for the Lagrange multiplier lemma). Unfortunately, this time-differentiated problem contains $p \circ \eta$ in its formulation, which leads to a circular argument, and thus explains the need for the penalized problem. We then obtain the regularity for the problem by the energy inequality for $\tilde{w}_{tt}$ and some difference-quotient inequalities for $\tilde{w}_t$ and $\tilde{w}$ carried out in Lagrangian variables in a neighborhood of the interface Γ_0. This, in turn, provides us with an estimate for the *trace* of $\tilde{w}$ and $\tilde{w}_t$ on Γ_0, which after a return to the Eulerian variables for the fluid phase, immediately provides the regularity in the fluid domain. The regularity in the solid phase is then obtained in a straightforward manner from elliptic regularity and the already-obtained trace estimate. We note that the estimates proved at this stage blow up as the regularized coefficients tend to the true coefficients, i.e., as the regularization parameter tends to zero.

For this reason, in Section 10, we obtain a different set of estimates (founded upon interpolation inequalities) for the solutions of the regularized problems, and conclude that the norms of the regularized solutions are actually uniformly bounded in the appropriate spaces, which thus provides, by weak convergence, a solution to (20) with the appropriate *a priori* estimates.

Finally, we conclude the proof of the existence theorem in Sections 11 and 12 by means of the Tychonoff fixed-point theorem. Although it might be possible to employ the Schauder theorem instead, it appears that the strong convergence requirements of the Schauder theorem are not very convenient to write and are, in particular, unnecessary for the use of the Tychonoff theorem.

Uniqueness is proved in Section 13 with further regularity requirements on the data.

2. Notational simplification

Although a fluid with a Neumann (free-slip) boundary condition indeed obeys the constitutive law (1e), it turns out that the notation is substantially simplified (particularly in Section 9 wherein we analyze the twice differentiated-in-time problem in Lagrangian coordinates) if we replace (1e) with

$$T^f = \nu \nabla u^f - p\mathrm{I}; \tag{2}$$

this amounts to replacing the energy $\int_{\Omega_0^f} \mathrm{Def}\, u^f : \mathrm{Def}\, v$ by $\int_{\Omega_0^f} \nabla u^f : \nabla v$, which is an equivalent form when $u^f = 0$ on $\partial\Omega$ due to the well-known Korn inequality. Henceforth, we shall take (2) as the fluid constitutive law.

3. Lagrangian formulation of the problem

In regards to the forcing functions, we shall use the convention of denoting both the fluid forcing f_f and the solid forcing f_s by the same letter f. Since f_f has to be defined in Ω (because of the composition with η), and f_s must be defined in Ω_0^s, we will assume that the forcing f is defined over the entire domain Ω.

Let

$$a(x) = [\nabla \eta^f(x)]^{-1}, \tag{3}$$

where $(\nabla \eta^f(x))^i_j = \partial(\eta^f)^i / \partial x^j(x)$ denotes the matrix of partial derivatives of η^f. Clearly, the matrix a depends on η and we shall sometimes use the notation $a^i_j(\eta)$ to denote the formula (3).

Let $v = u \circ \eta$ denote the Lagrangian or material velocity field, $q = p \circ \eta$ is the Lagrangian pressure function (in the fluid), and $F = f^f \circ \eta^f$ is the fluid forcing function in the material frame. Then, as long as no collisions occur between the solids (if there are initially more than one) or between a solid and $\partial\Omega$, the system (1) can be reformulated as

$$\eta_t = v \qquad \text{in } (0,T) \times \Omega, \tag{4a}$$

$$v^i_t - \nu(a^j_l a^k_l v^i{,}_k){,}_j + (a^k_i q){,}_k = F^i \qquad \text{in } (0,T) \times \Omega_0^f, \tag{4b}$$

$$a^k_i v^i{,}_k = 0 \qquad \text{in } (0,T) \times \Omega_0^f, \tag{4c}$$

$$v^i_t - \left[c^{ijkl} \int_0^t v^k{,}_l \right]{,}_j = f^i \qquad \text{in } (0,T) \times \Omega_0^s, \tag{4d}$$

$$\nu\, v^i{,}_k a^k_l a^j_l N_j - q a^j_i N_j = c^{ijkl} \int_0^t v^k{,}_l \; N_j \quad \text{on } (0,T) \times \Gamma_0, \tag{4e}$$

$$v(t,\cdot) \in H^1_0(\Omega; \mathbb{R}^3) \qquad \text{a.e. in } (0,T), \tag{4f}$$

$$v = u_0 \qquad \text{on } \Omega_0 \times \{t=0\}, \tag{4g}$$

$$\eta = \mathrm{Id} \qquad \text{on } \Omega_0 \times \{t=0\}, \tag{4h}$$

where N denotes the outward-pointing unit normal to Γ_0 (pointing into the solid phase), and

$$c^{ijkl} = \lambda \delta^{ij} \delta^{kl} + \mu(\delta^{ik} \delta^{jl} + \delta^{il} \delta^{jk}) .$$

Throughout the paper, all Greek indices run through 1, 2 and all Latin indices run through 1, 2, 3. Note that the continuity of the velocity (1i) along the interface is satisfied in the sense of traces on Γ_0 by condition (4f), whereas the continuity of the normal stress along the interface is represented by (4e).

Remark 2. The case in which the viscosity or Lamé coefficients are variable functions depending on $x \in \Omega$ and satisfying the usual assumptions, can be handled by our methodology without any supplementary mathematical difficulties.

4. Notation and conventions

We begin by specifying our notation for certain vector and matrix operations.

We write the Euclidean inner-product between two vectors x and y as $x \cdot y$, so that $x \cdot y = x^i \, y^i$.

The transpose of a matrix A will be denoted by A^T, i.e., $(A^T)^i_j = A^j_i$.

We write the product of a matrix A and a vector b as $A\, b$, i.e., $(A\, b)^i = A^i_j b^j$.

The product of two matrices A and S will be denoted by $A \cdot S$, i.e., $(A \cdot S)^i_j = A^i_k \, S^k_j$.

The trace of the product of two matrices A and S will be denoted by $A : S$, i.e., $A : S = \text{Trace}(A \cdot S) = A^i_j \, S^j_i$.

For $s \geqq 0$ and a Hilbert space $(X, \| \cdot \|_X)$, $H^s(\Omega; \mathbb{R}^3)$ denotes the Sobolev space of $\mathbb{R}^3$-valued functions with s distributional derivatives in $L^2(\Omega; \mathbb{R}^3)$, while $L^2(0, T; X)$ denotes the equivalence class of functions which are measurable and have finite $\| \cdot \|_{L^2}$ norm, where $\|f\|^2_{L^2(0,T;X)} = \int_0^T \|f(t)\|^2_X dt$.

We also set $H^1_{\partial\Omega}(\Omega^f_0; \mathbb{R}^3) = \{u \in H^1(\Omega^f_0; \mathbb{R}^3) |\, u = 0 \text{ on } \partial\Omega\}$.

For $T > 0$, we set

$$\begin{aligned}
V^2_f(T) &= \{w \in L^2(0, T; H^2(\Omega^f_0; \mathbb{R}^3)) \mid w_t \in L^2(0, T; H^1(\Omega^f_0; \mathbb{R}^3)) \\
&\qquad w_{tt} \in L^2(0, T; L^2(\Omega^f_0; \mathbb{R}^3))\}, \\
V^3_f(T) &= \{w \in L^2(0, T; H^3(\Omega^f_0; \mathbb{R}^3)) \mid w_t \in L^2(0, T; H^2(\Omega^f_0; \mathbb{R}^3)) \mid \\
&\qquad w_{tt} \in L^2(0, T; H^1(\Omega^f_0; \mathbb{R}^3))\}, \\
V^2_s(T) &= \{w \in L^2(0, T; H^2(\Omega^s_0; \mathbb{R}^3)) \mid w_t \in L^2(0, T; H^1(\Omega^s_0; \mathbb{R}^3)) \mid \\
&\qquad w_{tt} \in L^2(0, T; L^2(\Omega^s_0; \mathbb{R}^3))\} , \\
V^3_s(T) &= \{w \in L^2(0, T; H^3(\Omega^s_0; \mathbb{R}^3)) \mid w_t \in L^2(0, T; H^2(\Omega^s_0; \mathbb{R}^3))| \\
&\qquad w_{tt} \in L^2(0, T; H^1(\Omega^s_0; \mathbb{R}^3))\}.
\end{aligned}$$

We will solve (4) by a fixed-point method, set in an appropriate subset of $V^3_f(T) \times V^3_s(T)$. We assume in what follows that $v \in V^3_f(T)$ is given in such a way that the matrix $a_i^j(\eta)$ associated with the flow η of this velocity field v is well defined.

We then introduce the space (of weak solutions)

$$\mathcal{V}_v([0,T]) = \Big\{ w \in L^2(0,T;L^2(\Omega;\mathbb{R}^3)) \mid \int_0^\cdot w \in L^2(0,T;H^1(\Omega;\mathbb{R}^3)),$$
$$w \in L^2(0,T;H^1(\Omega_0^f;\mathbb{R}^3)), a_i^j w^i,_j = 0 \text{ in } [0,T]\times\Omega_0^f,$$
$$w = 0 \text{ on } \partial\Omega \Big\}.$$

Note that we impose the condition $\int_0^\cdot w \in L^2(0,T;H^1(\Omega;\mathbb{R}^3))$ to ensure continuity of the displacement field, in the sense of traces, between the solid and fluid phases along the interface Γ_0. We will also define for $t \in [0,T]$,

$$\mathcal{V}_v(t) = \{\psi \in H_0^1(\Omega;\mathbb{R}^3) \mid a_i^j(t)\psi^i,_j = 0 \text{ in } \Omega_0^f\}.$$

Furthermore, we will need the space

$$\mathcal{W}([0,T]) = \Big\{ w \in L^2(0,T;L^2(\Omega;\mathbb{R}^3)) \mid \int_0^\cdot w \in L^2(0,T;H^1(\Omega;\mathbb{R}^3)),$$
$$w \in L^2(0,T;H^1(\Omega_0^f;\mathbb{R}^3)) \ \ w = 0 \text{ on } \partial\Omega \Big\},$$

with the "divergence-free" constraint removed.

In order to specify the initial data for the weak formulation, we introduce the space

$$L^2_{div,f} = \{\psi \in L^2(\Omega;\mathbb{R}^3) \mid \operatorname{div}\psi = 0 \text{ in } \Omega_0^f, \ \ \psi\cdot N = 0 \text{ on } \partial\Omega\},$$

which is endowed with the $L^2(\Omega;\mathbb{R}^3)$ scalar product.

The space of velocities, X_T, is defined as the following separable Hilbert space:

$$X_T = \Big\{ u \in L^2(0,T;H_0^1(\Omega;\mathbb{R}^3)) \mid \Big(u^f, \int_0^\cdot u^s\Big) \in V_f^3(T)\times V_s^3(T) \Big\}, \quad (5)$$

endowed with its natural Hilbert norm

$$\|u\|^2_{X_T} = \|u\|^2_{L^2(0,T;H^1(\Omega;\mathbb{R}^3))} + \|u\|^2_{L^2(0,T;H^3(\Omega_0^f;\mathbb{R}^3))} + \|u_t\|^2_{L^2(0,T;H^2(\Omega_0^f;\mathbb{R}^3))}$$
$$+ \Big\|\int_0^\cdot u\Big\|^2_{L^2(0,T;H^3(\Omega_0^s;\mathbb{R}^3))} + \|u\|^2_{L^2(0,T;H^2(\Omega_0^s;\mathbb{R}^3))}$$
$$+ \|u_t\|^2_{L^2(0,T;H^1(\Omega_0^s;\mathbb{R}^3))} + \|u_{tt}\|^2_{L^2(0,T;H^1(\Omega_0^f;\mathbb{R}^3))}.$$

The existence of solutions to (4) will be obtained in the separable Hilbert space

$$Y_T = \{(u,p) \in X_T \times L^2(0,T;H^2(\Omega_0^f;\mathbb{R})) \mid p_t \in L^2(0,T;H^1(\Omega_0^f;\mathbb{R}))\},$$

endowed with its natural Hilbert norm

$$\|(u,p)\|^2_{Y_T} = \|u\|^2_{X_T} + \|p\|^2_{L^2(0,T;H^2(\Omega_0^f;\mathbb{R}))} + \|p_t\|^2_{L^2(0,T;H^1(\Omega_0^f;\mathbb{R}))}.$$

Remark 3. Note well that our method does not require any *a priori* knowledge of the regularity of the second time derivative of the pressure function p_{tt}; this is due to the Dirichlet boundary condition on $\partial\Omega$ as well as the Lagrangian representation of the problem that we employ.

We shall also need L^∞-in-time control of certain norms of the velocity, which necessitates the use of the following closed subspace of X_T:

$$W_T = \Big\{ u \in X_T |\ u_{tt} \in L^\infty(0,T; L^2(\Omega;\mathbb{R}^3)),\ u_t \in L^\infty(0,T; H^1(\Omega_0^s;\mathbb{R}^3)),$$
$$u \in L^\infty(0,T; H^2(\Omega_0^s;\mathbb{R}^3)),\ \int_0^\cdot u \in L^\infty(0,T; H^3(\Omega_0^s;\mathbb{R}^3)) \Big\},$$

endowed with the norm

$$\|u\|^2_{W_T} = \|u\|^2_{X_T} + \|u_{tt}\|^2_{L^\infty(0,T;L^2(\Omega;\mathbb{R}^3))} + \left\| \int_0^\cdot u \right\|^2_{L^\infty(0,T;H^3(\Omega_0^s;\mathbb{R}^3))}$$
$$+ \|u\|^2_{L^\infty(0,T;H^2(\Omega_0^s;\mathbb{R}^3))} + \|u_t\|^2_{L^\infty(0,T;H^1(\Omega_0^s;\mathbb{R}^3))}.$$

For some of our estimates, we will also make use of the space

$$Z_T = \{(u,p) \in W_T \times L^2(0,T; H^2(\Omega_0^f;\mathbb{R}))|\ p_t \in L^2(0,T; H^1(\Omega_0^f;\mathbb{R}))\},$$

endowed with its natural norm

$$\|(u,p)\|^2_{Z_T} = \|u\|^2_{W_T} + \|p\|^2_{L^2(0,T;H^2(\Omega_0^f;\mathbb{R}))} + \|p_t\|^2_{L^2(0,T;H^1(\Omega_0^f;\mathbb{R}))}.$$

Throughout the paper, we shall use C to denote a generic constant, which may possibly depend on the coefficients ν, λ, μ, or on the initial geometry given by Ω and Ω_0^f (such as a Sobolev constant or an elliptic constant). Similarly, we will denote by $C(M)$ a generic constant which depends on the same variables as C as well as on M (which is a variable defined in the next section) and $\|u_0\|_{H^5(\Omega_0^f;\mathbb{R}^3)}$, $\|f(0)\|_{H^3(\Omega;\mathbb{R}^3)}$ and the fixed time $\bar{T}$ for which the forcing functions are defined. We note that these constants do not blow up whenever the quantities they depend upon remain finite.

For the sake of notational convenience, we will also write $u(t)$ for $u(t,\cdot)$.

5. The main theorem

Theorem 1. *Let $\Omega \subset \mathbb{R}^3$ be a bounded domain of class H^3, and let Ω_0^s be an open set (with a finite number $\geqq 1$ of connected components) of class H^4 such that $\overline{\Omega_0^s} \subset \Omega$. Define $\Omega_0^f = \Omega \cap (\overline{\Omega_0^s})^c$. Let $\nu > 0$, $\lambda > 0$, $\mu > 0$ be given. Let*

$$f \in L^2(0,\bar{T}; H^2(\Omega;\mathbb{R}^3)),\ f_t \in L^2(0,\bar{T}; H^1(\Omega;\mathbb{R}^3)),$$
$$f_{tt} \in L^2(0,\bar{T}; L^2(\Omega;\mathbb{R}^3)), \tag{6a}$$
$$f(0) \in H^3(\Omega;\mathbb{R}^3). \tag{6b}$$

Assume that the initial data satisfies

$$u_0 \in H^5(\Omega_0^f;\mathbb{R}^3) \cap H^2(\Omega_0^s;\mathbb{R}^3) \cap H_0^1(\Omega;\mathbb{R}^3) \cap L^2_{\rm div,f}$$

as well as the compatibility conditions

$$[\nabla u_0^f\, N]_{\tan} = 0 \ \ on \ \ \Gamma_0, \quad w_1 = 0 \ on \ \partial\Omega, \quad \nu\triangle u_0^f - \nabla q_0 = 0 \ on \ \Gamma_0, \tag{7a}$$

$$[(\nu[\nabla w_1^f\, N]^i + \nu[\, u_0^f{,}_k^i\, (a_l^k a_l^j)_t(0)\,]\, N_j)_{i=1}^3]_{\tan} = [(c^{ijkl} u_0^f{,}_l^k\, N_j)_{i=1}^3]_{\tan} \ on \ \Gamma_0, \tag{7b}$$

with $q_0 \in H^4(\Omega_0^f;\mathbb{R})$ *defined by*

$$\triangle q_0 = {\rm div}\, f(0) + (a_i^j)_t(0) u_0^i{,}_j \ in \ \Omega_0^f, \tag{8a}$$

$$q_0 = \nu[\nabla u_0^f\, N]\cdot N \ on \ \Gamma_0, \tag{8b}$$

$$\frac{\partial q_0}{\partial N} = f(0)\cdot N + \nu\triangle u_0 \cdot N \ on \ \partial\Omega, \tag{8c}$$

and $w_1 \in H_0^1(\Omega;\mathbb{R}^3) \cap H^3(\Omega_0^s;\mathbb{R}^3) \cap H^3(\Omega_0^f;\mathbb{R}^3)$ *defined by*

$$w_1 = \nu\triangle u_0 - \nabla q_0 + f(0) \ \ in \ \ \Omega_0^f, \tag{9a}$$

$$w_1 = f(0) \ in \ \Omega_0^s\, . \tag{9b}$$

(Note that $(a_i^j)_t|_{t=0}$ *depends only on* u_0 *and not on the values taken by* u *at times* $t > 0$.*)*

Then there exists $T \in (0, \bar{T})$ *depending on* u_0, f, *and* Ω_0^f, *such that there exists a solution* $(v, q) \in Z_T$ *of the problem* (4). *Furthermore,* $\eta \in C^0([0,T]; H^3(\Omega_0^f;\mathbb{R}^3) \cap H^3(\Omega_0^s;\mathbb{R}^3) \cap H^1(\Omega;\mathbb{R}^3))$.

Remark 4. In Theorem 6, assumptions ensuring uniqueness of the solutions are also given.

Remark 5. If we had not made the notational simplification of Section 2, we would have to modify (7) by

$$[{\rm Def}\, u_0^f\, N]_{\tan} = 0 \ \text{ on } \ \Gamma_0, \quad w_1 = 0 \text{ on } \partial\Omega, \quad \nu\triangle u_0^f - \nabla q_0 = 0 \text{ on } \Gamma_0,$$

$$[(\nu[{\rm Def}\, w_1^f\, N]^i + \nu[\, u_0^f{,}_k^i\, (a_l^k a_l^j)_t(0) + u_0^f{,}_k^l\, (a_i^k a_l^j)_t(0)\,]\, N_j)_{i=1}^3]_{\tan} = [(c^{ijkl} u_0^f{,}_l^k\, N_j)_{i=1}^3]_{\tan} \text{ on } \Gamma_0,$$

and (8b) would be replaced by $q_0 = \nu\, [{\rm Def}\, u_0^f\, N]\cdot N$ on Γ_0.

Remark 6. The regularity of our solution $v \in W_T$ implies that for each $t \in (0, T]$ the solid domain $\Omega_s(t)$ is of class H^3. Also, although we have stated our results for three-dimensional motion, all of our results hold in the two-dimensional case as well.

Remark 7. We have stated our results using the convention of Section 3, wherein the forcing function f is taken to be defined over the entire domain Ω. It is certainly possible to define separate forcing functions for the solid and fluid phase, in which case we would need the following regularity:

$$f_f \in L^2(0,\bar{T};H^1(\Omega;\mathbb{R}^3)),\ \ f_{f_t} \in L^2(0,\bar{T};L^2(\Omega;\mathbb{R}^3)),$$
$$f_{f_{tt}} \in L^2(0,\bar{T};H^1_{\partial\Omega}(\Omega;\mathbb{R}^3)'),\ \ f_s \in L^2(0,\bar{T};H^2(\Omega_0^s;\mathbb{R}^3)),$$
$$f_{s_t} \in L^2(0,\bar{T};H^1(\Omega_0^s;\mathbb{R}^3)),\ f_{s_{tt}} \in L^2(0,\bar{T};L^2(\Omega_0^s;\mathbb{R}^3)),\ f_f(0) \in H^3(\Omega_0^f;\mathbb{R}^3).$$

The compatibility condition $\nu\triangle u_0^f - \nabla q_0 = 0$ on Γ_0 in (7) would be replaced by $\nu\triangle u_0^f - \nabla q_0 + f_f(0) = f_s(0)$ on Γ_0. In the definition of q_0, $f_f(0)$ replaces $f(0)$ and in the definition of w_1, $f(0)$ is replaced by $f_f(0)$ and $f_s(0)$ respectively in Ω_0^f and Ω_0^s.

Remark 8. Note that the supplementary regularity condition for $u(0)$ and $f(0)$ is due to the hyperbolic scaling of the velocity and forcing in the fluid. A parabolic scaling in the fluid, which may appear to be more appropriate, would not, however, lead to the necessary estimates, except for the case in which the initial solid-fluid interface is flat (which is not the case considered herein). This is due to an elastic energy integral (which we shall shortly identify) that requires the hyperbolic scaling in order to be estimated.

Remark 9. Note also the presence of two compatibility conditions for the stresses on Γ_0, which is also a consequence of the hyperbolic scaling. A fluid-fluid interface problem would require only one compatibility condition.

Remark 10. Note that the proof of *existence* of solutions requires only the "minimal" regularity assumptions (6) on the forcing function f; this is due to our method of proof which employs the Tychonoff fixed-point theorem instead of a Banach-type contraction mapping. Note also that unlike the case of a free-surface fluid problem, a Banach contraction method does not work for the problem that we study herein. We will see later that some additional Lipschitz assumptions (124) are necessary for uniqueness.

Remark 11. We also remark that our technique is restricted to the case where the elastic constitutive law in the solid is either linear or semi-linear. Whereas the paper is written with a linear elasticity law, we can handle in the same fashion and with the exact same methods, the case where an extra contribution of the type $F(\nabla\eta,\eta)$ is added, with F satisfying the usual regularity and growth assumptions. In that case the linear problem (20), defined hereafter, which is used in the fixed-point approach would be replaced by a similar problem, with (20c) replaced by

$$w_t^i - \left[c^{ijkl}\int_0^t w^k,_l\right],_j + F\left(\nabla\int_0^t w, \int_0^t w\right) = f^i \text{ in } (0,T)\times\Omega_0^s\,,$$

which does not create any additional difficulties with respect to the analysis of the linear case.

The consideration of a quasilinear elastic law such as the nonlinear Saint-Venant Kirchhoff material, involves a smoother functional framework and will be developed in a future article.

6. A bounded convex closed set of W_T

Definition 1. Let $M > 0$ be given. We let $C_T(M)$ denote the subset of W_T consisting of elements $u \in W_T$ such that

$$\|u\|^2_{W_T} \leqq M, \tag{10}$$

and such that

$$u(0)|_{\Omega_0^f} = u_0|_{\Omega_0^f}, \quad \text{and } u_t(0)|_{\Omega_0^f} = w_1|_{\Omega_0^f}, \tag{11}$$

with w_1 defined in Theorem 1, and where we continue to assume that the conditions stated in Theorem 1 for the forcing function f and the initial data u_0 are satisfied.

Lemma 1. *There exists $M_0 > 0$ such that $C_T(M)$ is non-empty for $M > M_0$. Furthermore, $C_T(M)$ is a convex, bounded and closed subset of X_T.*

Proof. We note that if $\check{v}(t) = u_0 + tw_1$, then $\check{v} \in C_M(T)$ for $M \geqq M_0 = \|\check{v}\|^2_{W_T}$. The fact that $C_T(M)$ is closed follows from Mazur's lemma. □

Remark 12. Note also that if $0 < T' \leqq T$, then $C_{T'}(M)$ is non-empty. Henceforth, M is assumed to be larger that M_0.

In the remainder of the paper, we will assume that

$$0 < T < \bar{T},$$

where the forcing f is defined on the time interval $[0, \bar{T}]$; we will have to choose T sufficiently small to ensure existence of solutions to our problem.

We will need the following series of simple lemmas on the set $C_T(M)$.

Lemma 2. *There exists $T_0 \in (0, \bar{T})$ such that for all $T \in (0, T_0)$ and for all $v \in C_T(M)$, the matrix a is well defined, and satisfies the estimate (which is independent of $v \in C_T(M)$)*

$$\begin{aligned}&\|a\|_{L^\infty(0,T;H^2(\Omega_0^f;\mathbb{R}^9))} + \|a_t\|_{L^\infty(0,T;H^1(\Omega_0^f;\mathbb{R}^9))} + \|a_{tt}\|_{L^\infty(0,T;L^2(\Omega_0^f;\mathbb{R}^9))}\\ &\quad+\|a_t\|_{L^2(0,T;H^2(\Omega_0^f;\mathbb{R}^9))} + \|a_{tt}\|_{L^2(0,T;H^1(\Omega_0^f;\mathbb{R}^9))}\\ &\quad+\|a_{ttt}\|_{L^2(0,T;L^2(\Omega_0^f;\mathbb{R}^9))} \leqq C(M).\end{aligned} \tag{12}$$

Proof. Notice that in the separable Hilbert space $H^3(\Omega_0^f;\mathbb{R}^3)$ (for which the Bochner integral is well defined),

$$\eta(t) = \text{Id} + \int_0^t v(s)ds;$$

this together with the Jensen and Cauchy-Schwarz inequalities shows that

$$\|\eta - \mathrm{Id}\|_{L^\infty(0,T;H^3(\Omega_0^f;\mathbb{R}^3))} \leqq C\,\sqrt{T}\,\|v\|_{L^2(0,T;H^3(\Omega_0^f;\mathbb{R}^3))},$$

and thus

$$\|\nabla\eta - \mathrm{I}\|_{L^\infty(0,T;H^2(\Omega_0^f;\mathbb{R}^9))} \leqq C\,\sqrt{T}\,\|v\|_{X_T} \leqq C\,\sqrt{T}\,\sqrt{M}. \tag{13}$$

Next, choose $R > 0$ to be such that for any 3×3 matrix b satisfying $\|b-\mathrm{I}\|_{\mathbb{R}^9} \leqq R$, we have $\det b \geqq \frac{1}{2}$.

We then see from (13) and the Sobolev inequalities that, for $T \leqq T_0 = \frac{CR^2}{M}$, $\nabla\eta(t)$ is invertible for $t \in [0, T]$ in Ω_0^f for any $v \in C_T(M)$. From now on, T is assumed to be in $(0, T_0)$. Since

$$a(t) = \frac{1}{\det\nabla\eta(t)}\,\mathrm{Cof}\nabla\eta(t) \text{ in } \Omega_0^f,$$

we then see from (13) that

$$\|a\|_{L^\infty(0,T;H^2(\Omega_0^f;\mathbb{R}^9))} \leqq C(1+\sqrt{TM})^5.$$

Similarly,

$$\|v - u_0\|_{L^\infty(0,T;H^2(\Omega_0^f;\mathbb{R}^3))} \leqq C\,\sqrt{TM}, \tag{14}$$

providing

$$\|a_t\|_{L^\infty(0,T;H^1(\Omega_0^f;\mathbb{R}^9))} \leqq C(1+\|u_0\|_{H^3(\Omega_0^f;\mathbb{R}^3)}+\sqrt{TM})^5\ .$$

In the same fashion,

$$\|v_t - w_1\|_{L^\infty(0,T;H^1(\Omega_0^f;\mathbb{R}^3))} \leqq C\,\sqrt{T}\,M, \tag{15}$$

providing

$$\|a_{tt}\|_{L^\infty(0,T;H^1(\Omega_0^f;\mathbb{R}^9))} \leqq C(1+\|w_1\|_{H^1(\Omega_0^f;\mathbb{R}^3)}+\sqrt{T}M)^5\ .$$

The L^2-in-time estimates are established in a straightforward manner from the definition of $C_T(M)$, which concludes the proof of the lemma. □

Remark 13. Note that T_0 also depends on M.

In the following, T is taken in $(0, T_0)$ (and M is still taken in $(0, M_0)$). By the same arguments as above, we can easily prove the following results:

Lemma 3. *For all $v \in C_M(T)$,*

$$\|a-a(0)\|^2_{L^\infty(0,T;H^2(\Omega_0^f;\mathbb{R}^9))} + \|a_t - a_t(0)\|^2_{L^\infty(0,T;H^1(\Omega_0^f;\mathbb{R}^9))} \leqq C(M)\,T\ . \tag{16}$$

Lemma 4. *There exists* $T_1 \in (0, T_0)$ *which depends on* M, *and a constant* $C > 0$ *which depends on* u_0 *but does not depend on* M, *such that for all* $v \in C_M(T)$,

$$\|\eta\|^2_{L^\infty(0,T;H^3(\Omega_0^f;\mathbb{R}^3))} + \|v\|^2_{L^\infty(0,T;H^2(\Omega_0^f;\mathbb{R}^3))} + \|v_t\|^2_{L^\infty(0,T;H^1(\Omega_0^f;\mathbb{R}^3))} \leqq C. \tag{17}$$

The next result concerns potential solid-solid or solid-container collisions for a short time.

Lemma 5. *Let* $d > 0$ *denote the infimum of the distances between two distinct connected components of* Ω_0^s (*if we have more than one solid in the problem*) *and of the distance between* Ω_0^s *and* $\partial\Omega$. *Then, there exists* $T_2 \in (0, T_0)$ *such that for all* $v \in C_M(T_2)$,

$$\int_0^{T_2} \|v\|_{L^\infty(\Omega_0^f;\mathbb{R}^3)} \leqq \frac{d}{2}. \tag{18}$$

Proof. The inequality $\int_0^T \|v\|_{L^\infty(\Omega_0^f;\mathbb{R}^3)} \leqq C\sqrt{T}\,[\int_0^T \|v\|^2_{H^2(\Omega_0^f;\mathbb{R}^3)}]^{\frac{1}{2}}$ proves the result. □

Henceforth, we shall require

$$T \in (0, T_M), \quad T_M = \min(T_1, T_2).$$

The series of estimates in Sections 9 and 10 will show that M must first be chosen sufficiently large, and then T must be chosen sufficiently small.

The next result is crucial for the derivation of appropriate estimates; while it appears that we should require an estimate of q_{tt} in $L^2(0, T; L^2(\Omega_0^f; \mathbb{R}))$, we are not able to obtain such an estimate, and effectively replace it with an estimate of q_t in $L^\infty(0, T; L^2(\Omega_0^f; \mathbb{R}))$.

Lemma 6. *For all* $v \in C_M(T)$,

$$\|a_{tt}(t)\|_{L^\infty(0,T;L^3(\Omega_0^f;\mathbb{R}^9))} \leqq C(M)\,. \tag{19}$$

Proof. Let $\psi(t) = \int_{\Omega_0^f} |a_{tt}(t)|^3 + 1 \geqq 1$. We then have in the distributional sense

$$\psi'(t) = 3\int_{\Omega_0^f} |a_{tt}(t)|^2 a_{ttt}(t).$$

Thus,

$$\psi'(t) \leqq C\,\|a_{tt}(t)\|^2_{L^4(\Omega_0^f;\mathbb{R}^9)}\|a_{ttt}(t)\|_{L^2(\Omega_0^f;\mathbb{R}^9)},$$

which by interpolation yields

$$\psi'(t) \leqq C\,\|a_{tt}(t)\|_{L^3(\Omega_0^f;\mathbb{R}^9)}\|a_{tt}(t)\|_{H^1(\Omega_0^f;\mathbb{R}^9)}\|a_{ttt}(t)\|_{L^2(\Omega_0^f;\mathbb{R}^9)},$$

i.e.,

$$\psi'(t) \leqq C\,[\psi(t)]^{\frac{1}{3}}\,\|a_{tt}(t)\|_{H^1(\Omega_0^f;\mathbb{R}^9)}\|a_{ttt}(t)\|_{L^2(\Omega_0^f;\mathbb{R}^9)}\,.$$

Thus, since $\psi(t) \geqq 1$,

$$\psi(t) \leqq \left[\psi(0)^{\frac{2}{3}} + C \int_0^t \|a_{tt}(t)\|_{H^1(\Omega_0^f;\mathbb{R}^9)} \|a_{ttt}(t)\|_{L^2(\Omega_0^f;\mathbb{R}^9)}\right]^{\frac{3}{2}},$$

which by (12) provides

$$\psi(t) \leqq [\psi(0)^{\frac{2}{3}} + C(M)]^{\frac{3}{2}},$$

which establishes (19). □

Remark 14. Note that in the above L^3 estimate, the exponent 3 is the limiting case for this lemma.

Remark 15. Had we not made the notational (constitutive) simplification of Section 2, we would require the following Korn-type lemma in the Lagrangian setting (this is the only mathematical issue that the actual constitutive law (1e) requires):

Lemma 7. *There exists $T_3 \in (0, T)$ such that for any $T \in (0, T_3)$ and $v \in C_T(M)$, for all $\phi \in H_0^1(\Omega_0^f; \mathbb{R}^3)$ and $t \in [0, T]$,*

$$\int_{\Omega_0^f} (a_j^k(t)\phi,_k^i + a_i^k(t)\phi,_k^j)(a_j^k(t)\phi,_k^i + a_i^k(t)\phi,_k^j) \geqq C\, \|\phi\|^2_{H_0^1(\Omega_0^f;\mathbb{R}^3)}.$$

Proof. To prove this result, we let $a(t) = I + [a(t) - a(0)]$ and apply (16) followed by the Korn inequality. □

7. The basic linear problem

Suppose that $M \geqq M_0$, $T \in (0, T_M)$ and $v \in C_T(M)$ are given. Let $\eta = \mathrm{Id} + \int_0^\cdot v$ and let a_i^j be the quantity associated with η through (3).

We are concerned with the following time-dependent linear problem, whose fixed-point $w = v$ provides a solution to (4):

$$w_t^i - \nu(a_l^j a_l^k w^i,_k),_j + (a_i^k q),_k = f \circ \eta \quad \text{in } (0, T) \times \Omega_0^f, \tag{20a}$$

$$a_i^k w^i,_k = 0 \quad \text{in } (0, T) \times \Omega_0^f, \tag{20b}$$

$$w_t^i - \left[c^{ijkl} \int_0^t w^k,_l\right],_j = f^i \quad \text{in } (0, T) \times \Omega_0^s, \tag{20c}$$

$$\nu w^i,_k a_l^k a_l^j N_j - q a_i^j N_j = c^{ijkl} \int_0^t w^k,_l \; N_j \text{on } (0, T) \times \Gamma_0, \tag{20d}$$

$$w(t, \cdot) \in H_0^1(\Omega; \mathbb{R}^3) \quad \text{a.e. in } (0, T), \tag{20e}$$

$$w = u_0 \quad \text{on } \Omega_0 \times \{t = 0\}, \tag{20f}$$

$$\eta = \mathrm{Id} \quad \text{on } \Omega_0 \times \{t = 0\}, \tag{20g}$$

The following regularity result will be of paramount importance in our analysis:

Theorem 2. *Given f and u_0 satisfying the assumptions of Theorem 1, there exists $M > 0$, $T > 0$, such that for any $v \in C_T(M)$, there exists a unique solution $(w, p) \in Z_T$ of (20). Furthermore, $w \in C_T(M)$.*

Sections 9 and 10 are devoted to the proof of this theorem. In the following, we set

$$
\begin{aligned}
N(u_0, f)^2 = (1 + \; & \|u_0\|^2_{H^5(\Omega_0^f;\mathbb{R}^3)} + \|u_0\|^2_{H^2(\Omega_0^s;\mathbb{R}^3)} + \|f(0)\|^2_{H^3(\Omega;\mathbb{R}^3)} \\
& + \|f\|^2_{L^2(0,T;H^2(\Omega;\mathbb{R}^3))} + \|f_t\|^2_{L^2(0,T;H^1(\Omega;\mathbb{R}^3))} \\
& + \|f_{tt}\|^2_{L^2(0,T;L^2(\Omega;\mathbb{R}^3))})^4 \, . \qquad (21)
\end{aligned}
$$

8. Preliminary results

8.1. Divergence, extension and regularization-type results

We first state the following result, whose proof follows the same argument as for the case of a smooth boundary, with the exception that the regularity results for elliptic systems of [8] are used instead of the more classical results wherein the boundary is smooth.

Lemma 8. *Let Ω' be a domain of class H^k ($k \geqq 3$). Then, for $0 \leqq m \leqq k-2$, there exists a continuous linear operator*

$$
\begin{aligned}
L(\Omega') : & \left\{ (d, r) \in H^m(\Omega'; \mathbb{R}) \times H^{m+0.5}(\partial\Omega'; \mathbb{R}^3) | \int_\Omega d = \int_{\partial\Omega} r \cdot n \right\} \\
& \to H^{m+1}(\Omega'; \mathbb{R}^3)
\end{aligned}
$$

such that $u = L(\Omega')(d, r)$ satisfies

$$
\begin{aligned}
\operatorname{div} u &= d \text{ in } \Omega', \\
u &= r \text{ on } \partial\Omega'.
\end{aligned}
$$

Furthermore, the operator norm of $L(\Omega')$ remains bounded as the norm of the charts defining Ω' stays in a bounded set of H^k.

We will need the following extension:

Lemma 9. *Recalling that Ω_0^f is of class H^3, for each $1 \leqq m \leqq 3$, there exists a continuous linear operator*

$$
E : H^m(\Omega_0^f; \mathbb{R}^3) \cap H^1_{\partial\Omega}(\Omega_0^f; \mathbb{R}^3) \to H^m(\Omega; \mathbb{R}^3) \cap H^1_0(\Omega; \mathbb{R}^3)
$$

such that $E(u) = u$ in Ω_0^f.

Proof. The result is well known in the case where $\Omega_0^f = \mathbb{R}^3_+$; let $\Pi : \mathbb{R}^3_+ \to \mathbb{R}^3$ denote this extension operator, and let $\{\Psi_i\}_{i=1}^N$ denote a collection of charts in a neighborhood of Γ_0 (each Ψ_i is a map of class H^3 from the unit ball in $\mathbb{R}^3$ into an open set containing a coordinate patch of Γ_0), and let $(\theta_i)_{i=1}^N$ denote the associated partition of the unity. We see that

$$F(u) = \sum_{i=1}^{N} \Pi[(\theta_i u) \circ \Psi_i] \circ \Psi_i^{-1}$$

is an extension of u into a neighborhood of Γ_0. By introducing a smooth cut-off function ξ, equal to 1 in Ω_0^f and equal to 0 in the complementary part of this neighborhood included in Ω_0^s, we see that $E(u) = \xi\ F(u)$ satisfies the statement of the lemma. □

In a similar fashion, we can also extend from Ω to $\mathbb{R}^3$, with the same arguments.

Lemma 10. *There exists a linear and continuous operator E_g from $H^m(\Omega; \mathbb{R}^3)$ into $H^m(\mathbb{R}^3; \mathbb{R}^3)$ (for each $1 \leqq m \leqq 3$) such that $E_g(u) = u$ in Ω.*

We also need a regularization lemma for the coefficients a_i^j, which we shall use to obtain estimates for the solutions of the regularized problems (whose coefficients by definition use these regularized coefficients); we will then pass to the limit as the regularization parameter tends to zero.

Lemma 11. *Let $v \in C_T(M)$ and $\eta = \mathrm{Id} + \int_0^\cdot v$. Then, there exists a sequence $v_n \in V_f^{\mathrm{reg}}(T) = \{u \in L^2(0, T; H^3(\Omega_0^f; \mathbb{R}^3))|\ u_t \in L^2(0, T; H^3(\Omega_0^f; \mathbb{R}^3)),\ u_{tt} \in L^2(0, T; H^3(\Omega_0^f; \mathbb{R}^3))\}$, such that $v_n(0)|_{\Omega_0^f} = u_0|_{\Omega_0^f}$, $v_{nt}(0)|_{\Omega_0^f} = w_1|_{\Omega_0^f}$, and*

$$\|v_n - v\|_{V_f^3(T)} + \|(v_n - v)_{tt}\|_{L^\infty(0,T;L^2(\Omega_0^f;\mathbb{R}^3))} \to 0.$$

Proof. Let $\rho \in \mathcal{D}(B(0, 1))$ be such that $\rho \geqq 0$ and $\int_{B(0,1)} \rho = 1$, and let $\rho_n(x) = n^3 \rho(xn)$ denote the usual mollifier.

From Lemma 9, for any $t \in [0, T]$, let $\bar{v}(t) = E(v(t))$, so that $\bar{v} \in V_3^f(T) \cap V_3^s(T)$ with $\|\bar{v}\|_{V_f^3(T)} + \|\bar{v}\|_{V_s^3(T)} \leqq C\ \|v\|_{V_f^3(T)}$. We extend to $\mathbb{R}^3$ by setting $v' = E_g(\bar{v})$.

Then let $\tilde{v}_n$ be defined for any $t \in [0, T]$ by

$$\tilde{v}_n(t) = \rho_n \star v'(t).$$

From the properties of the space convolution, we know that $v_n \in V_f^{\mathrm{reg}}(T) \cap V_s^3(T)$ and that

$$\|\tilde{v}_n - v'\|_{V_f^3(T)} + \|\tilde{v}_n - v'\|_{V_s^3(T)} + \|(v_n - v')_{tt}\|_{L^\infty(0,T;L^2(\Omega_0^f;\mathbb{R}^3))} \to 0$$

as $n \to \infty$. This in turn implies that

$$\|\tilde{v}_n - v\|_{V_f^3(T)} + \|(v_n - v)_{tt}\|_{L^\infty(0,T;L^2(\Omega_0^f;\mathbb{R}^3))} \to 0$$

as $n \to \infty$. Now, for the initial conditions, let us define

$$v_n(t) = u_0 + t w_1 - \int_0^t (t' - t)(\tilde{v}_n)_{tt}\, dt'$$

(the Bochner integral being well defined in the Hilbert space $H^3(\Omega_0^f; \mathbb{R}^3)$). We then have $v_n(0) = u_0$, $v_{nt}(0) = w_1$.

Moreover,

$$v_n = \tilde{v}_n + E_g(E(u_0)) - \rho_n \star E_g(E(u_0)) + t\,[E_g(E(w_1)) - \rho_n \star E_g(E(w_1))],$$

which yields $\|v_n - v\|_{V_f^3(T)} + \|(v_n - v)_{tt}\|_{L^\infty(0,T;L^2(\Omega_0^f;\mathbb{R}^3))} \to 0$, as $n \to \infty$. □

Remark 16. Our construction does not necessarily yield $v_n = 0$ on $\partial\Omega$. Consequently, with $\eta_n = \mathrm{Id} + \int_0^\cdot v_n$ we do not have $\eta_n(\Omega) = \Omega$. It, nevertheless, does not matter for the purpose of our analysis.

Remark 17. In the following, we will solve (20) as the limit as $n \to \infty$ of the solutions w_n to the problems (20) associated with these regularized v_n. The interest of this regularizing process is that for a given n, $a(\eta_n)$ and its first and second time derivatives are in $L^\infty(0, T; H^2(\Omega_0^f; \mathbb{R}^9)) \subset L^\infty(0, T; L^\infty(\Omega_0^f; \mathbb{R}^9))$ and its third time derivative is in $L^2(0, T; H^2(\Omega_0^f; \mathbb{R}^9)) \subset L^2(0, T; L^\infty(\Omega_0^f; \mathbb{R}^9))$ which is necessary in order to get the existence of regular solutions to (20). These bounds in those spaces of course blow up as $n \to \infty$ (except for the estimate for $a(\eta_n)$).

Nevertheless, using the fact that $\|v_n - v\|_{V_f^3(T)} \to 0$ as $n \to \infty$, $a(\eta_n)$, and its first, second and third time derivatives satisfy the same type of estimates as (12), (19) and (16), respectively, with a constant $C(M)$ which does not depend on n. This fact will be used, together with interpolation inequalities (that hold since the solutions w_n are regular) in order to get estimates in Y_T for w_n which are independent of n. By weak convergence, this will provide our smooth solution to (20).

We will also use the convention of denoting the regularized velocity fields v_n by $\tilde{v}$, and the corresponding regularized matrix $a(\eta_n)$ by $\tilde{a}$.

Remark 18. Since the fluid forcing in (20) is given $f \circ \eta$, we need to extend f to $\mathbb{R}^3$. Hence, when we solve this problem with the regularized coefficients arising from $\tilde{v}$, we in fact implicitly use the extension $E_g(f)$. This extension has the same regularity as f with $\mathbb{R}^3$ replacing Ω; this follows from the fact that E_g commutes with the time derivative. For notational convenience, we shall continue to denote the extended forcing function by the same letter f.

8.2. Pressure as a Lagrange multiplier

Lemma 12. *For all $p \in L^2(\Omega_0^f; \mathbb{R})$, $t \in [0, T]$, there exists a constant $C > 0$ and $\phi \in H_0^1(\Omega, \mathbb{R}^3)$ such that $a_i^j(t)\phi^i{}_{,j} = p$ in Ω_0^f and*

$$\|\phi\|^2_{H_0^1(\Omega;\mathbb{R}^3)} \leqq C\|p\|^2_{L^2(\Omega_0^f;\mathbb{R})}. \tag{22}$$

Proof. Let $p_1 \in L^2(\Omega_0^s; \mathbb{R})$ be such that $p_1 > 0$ in Ω_0^s. Let $\bar{p}$ be defined by $\bar{p} = p$ in Ω_0^f and

$$\bar{p} = -\frac{\int_{\Omega_0^f} p \ \det\nabla\eta(t)}{\int_{\Omega_0^s} p_1 \ \det\nabla\eta(t)} p_1$$

in Ω_0^s. Since

$$\int_{\eta(t,\Omega)} \bar{p} \circ \eta(t)^{-1} \, dx = \int_\Omega \bar{p}(x) \ \det\nabla\eta(t) dx = 0,$$

we then see that $\phi = L(\eta(t,\Omega))(\bar{p} \circ \eta(t)^{-1}, 0) \circ \eta(t) \in H_0^1(\Omega; \mathbb{R}^3)$ satisfies

$$\operatorname{div}(\phi \circ \eta(t)^{-1}) = \bar{p} \circ \eta(t)^{-1} \text{ in } \eta(t,\Omega),$$

and thus

$$a_i^j(t)\phi^i,_j = \operatorname{div}(\phi \circ \eta(t)^{-1}) \circ \eta(t) = \bar{p} = p \text{ in } \Omega_0^f.$$

The inequality (22) is then a simple consequence of the properties of L and of the condition $v \in C_T(M)$. □

We can now follow [16]. We define the linear functional on $H_0^1(\Omega; \mathbb{R}^3)$ by $(p, a_i^j(t)\varphi^i,_j)_{L^2(\Omega_0^f;\mathbb{R})}$, where $\varphi \in H_0^1(\Omega; \mathbb{R}^3)$. By the Riesz representation theorem, there is a bounded linear operator $Q(t) : L^2(\Omega_0^f; \mathbb{R}^3) \to H_0^1(\Omega; \mathbb{R}^3)$ such that

$$\forall\varphi \in H_0^1(\Omega; \mathbb{R}^3), \ (p, \ a_i^j(t)\varphi^i,_j)_{L^2(\Omega_0^f;\mathbb{R})} = (Q(t)p, \ \varphi)_{H_0^1(\Omega;\mathbb{R}^3)}.$$

Letting $\varphi = Q(t)p$ shows that

$$\|Q(t)p\|_{H_0^1(\Omega;\mathbb{R}^3)} \leqq C\|p\|_{L^2(\Omega_0^f;\mathbb{R})} \tag{23}$$

for some constant $C > 0$. Using Lemma 12, we have the estimate

$$\begin{aligned} \|p\|^2_{L^2(\Omega_0^f;\mathbb{R})} &\leqq C\|Q(t)p\|_{H_0^1(\Omega;\mathbb{R}^3)}\|\phi\|_{H_0^1(\Omega;\mathbb{R}^3)} \\ &\leqq C\|Q(t)p\|_{H_0^1(\Omega;\mathbb{R}^3)}\|p\|_{L^2(\Omega_0^f;\mathbb{R})}, \end{aligned} \tag{24}$$

which shows that $R(Q(t))$ is closed in $H_0^1(\Omega; \mathbb{R}^3)$. Since $\mathcal{V}_v(t) \subset R(Q(t))^\perp$ and $R(Q(t))^\perp \subset \mathcal{V}_v(t)$, it follows that

$$H_0^1(\Omega; \mathbb{R}^3) = R(Q(t)) \oplus_{H_0^1(\Omega;\mathbb{R}^3)} \mathcal{V}_v(t). \tag{25}$$

We can now introduce our Lagrange multiplier

Lemma 13. *Let $\mathfrak{L}(t) \in H^{-1}(\Omega; \mathbb{R}^3)$ be such that $\mathfrak{L}(t)\varphi = 0$ for any $\varphi \in \mathcal{V}_v(t)$. Then there exists a unique $q(t) \in L^2(\Omega_0^f; \mathbb{R})$, which is termed the pressure function, satisfying*

$$\forall \varphi \in H_0^1(\Omega; \mathbb{R}^3), \quad \mathfrak{L}(t)(\varphi) = (q(t),\ a_i^j \varphi^i,_j)_{L^2(\Omega_0^f;\mathbb{R})}.$$

Moreover, there is a $C > 0$ (which does not depend on $t \in [0, T]$ and on the choice of $v \in C_M(T)$) such that

$$\|q(t)\|_{L^2(\Omega_0^f;\mathbb{R})} \leqq C\ \|\mathfrak{L}(t)\|_{H^{-1}(\Omega;\mathbb{R}^3)}.$$

Proof. By the decomposition (25), for $v \in H_0^1(\Omega, \mathbb{R}^3)$, we let $\varphi = v_1 + v_2$, where $v_1 \in \mathcal{V}_v(t)$ and $v_2 \in R(Q(t))$. From our assumption, it follows that

$$\mathfrak{L}(t)(\varphi) = \mathfrak{L}(t)(v_2) = (\psi(t), v_2)_{H_0^1(\Omega,\mathbb{R}^3)} = (\psi(t), \varphi)_{H_0^1(\Omega,\mathbb{R}^3)}$$

for a unique $\psi(t) \in R(Q(t))$.

From the definition of $Q(t)$ we then get the existence of a unique $q(t) \in L^2(\Omega_0; \mathbb{R})$ such that

$$\forall \varphi \in H_0^1(\Omega; \mathbb{R}^3), \quad \mathfrak{L}(t)(\varphi) = (q(t),\ a_i^j \varphi^i,_j)_{L^2(\Omega_0^f;\mathbb{R})}.$$

The estimate stated in the lemma is then a simple consequence of (24). □

We will also need a version of the Lagrange multiplier lemma for the case where $\mathfrak{L}(t) \in H^{-1}(\Omega_0^f; \mathbb{R}^3)$, which implies an estimate on the pressure, modulo a constant. We first have

Lemma 14. *For all $p \in L^2(\Omega_0^f; \mathbb{R})$ such that $\int_{\Omega_0^f} p\ det\nabla\eta = 0$, $t \in [0, T]$, there exists a constant $C > 0$ and $\phi \in H_0^1(\Omega_0^f; \mathbb{R}^3)$ such that $a_i^j(t)\phi^i,_j = p$ in Ω_0^f and*

$$\|\phi\|^2_{H_0^1(\Omega_0^f;\mathbb{R}^3)} \leqq C\|p\|^2_{L^2(\Omega_0^f;\mathbb{R})}. \tag{26}$$

Proof. Since

$$\int_{\eta(t,\Omega_0^f)} \bar{p} \circ \eta(t)^{-1}\ dx = \int_{\Omega_0^f} \bar{p}(x)\ \ \det\nabla\eta(t) dx = 0,$$

we then define $\phi = L(\eta(t, \Omega_0^f))(\bar{p} \circ \eta(t)^{-1}, 0) \circ \eta(t) \in H_0^1(\Omega_0^f; \mathbb{R}^3)$.

The inequality (22) is then a simple consequence of the properties of L and of the condition $v \in C_T(M)$. □

In a similar fashion to the proof of Lemma 13, we can now establish our second Lagrange multiplier

Lemma 15. *Let* $\mathfrak{L}(t) \in H^{-1}(\Omega_0^f;\mathbb{R}^3)$ *be such that* $\mathfrak{L}(t)\varphi = 0$ *for any* $\varphi \in \mathcal{V}_v(t) \cap H_0^1(\Omega_0^f;\mathbb{R}^3)$. *Then there exists a unique* $q(t) \in L^2(\Omega_0^f;\mathbb{R})$, *satisfying* $\int_{\Omega_0^f} q(t)\det\nabla\eta = 0$, *which is termed the pressure function, satisfying*

$$\forall\varphi \in H_0^1(\Omega_0^f;\mathbb{R}^3),\ \ \mathfrak{L}(t)(\varphi) = (q(t),\ a_i^j \varphi^i,_j)_{L^2(\Omega_0^f;\mathbb{R})}.$$

Moreover, there is a $C > 0$ (*which does not depend on* $t \in [0,T]$ *and on the choice of* $v \in C_M(T)$) *such that*

$$\|q(t)\|_{L^2(\Omega_0^f;\mathbb{R})} \leqq C\ \|\mathfrak{L}(t)\|_{H^{-1}(\Omega_0^f;\mathbb{R}^3)}.$$

Remark 19. The four previous lemmas do not rely on the fact that $v = 0$ on $\partial\Omega$. Therefore, they are also true for the case where the coefficients $\tilde{a}$ are associated with $\tilde{v}$. The important point is that the estimates (12), (19) and (16) are also satisfied by the regularized matrix $\tilde{a}$ and velocity $\tilde{v}$.

9. Estimates for (20): the case of the regularized coefficients

9.1. Weak solutions

Definition 2. A vector $w \in \mathcal{V}_v([0,T])$ with $w_t \in \mathcal{V}_v(t)'$ for a.e. $t \in (0,T)$ is a weak solution of (20) provided that

$$\text{(i)}\ \langle w_t,\phi\rangle + \nu(a_k^r w^i,_r\,, a_k^s\phi^i,_s)_{L^2(\Omega_0^f;\mathbb{R}^9)} + \left(c^{ijkl}\int_0^t w^k,_l\,,\phi^i,_j\right)_{L^2(\Omega_0^s;\mathbb{R})}$$

$$= (F,\phi)_{L^2(\Omega_0^f;\mathbb{R}^3)} + (f,\phi)_{L^2(\Omega_0^s;\mathbb{R}^3)},\ \forall\phi \in \mathcal{V}_v(t)\,,\ \text{and} \tag{27a}$$

$$\text{(ii)}\ w(0,\cdot) = u_0, \tag{27b}$$

for a.e. $0 \leqq t \leqq T$, where $\langle\cdot,\cdot\rangle$ denotes the duality product between $\mathcal{V}_v(t)$ and its dual $\mathcal{V}_v(t)'$.

9.2. Penalized problems

Whereas the existence of a weak solution can be proved directly in the space $\mathcal{V}_v([0,T])$, with $w_t \in \mathcal{V}_v(t)'$, this framework is not suitable for finding the pressure estimate required by our analysis. (Even for the well-studied problem of the Navier-Stokes equations on a fixed and smooth bounded domain, the weak solution only provides a pressure estimate of the form $\int_0^\cdot p \in L^2(0,T;L^2(\Omega_0^f;\mathbb{R}))$.) A penalized form of the problem, however, together with the penalized form for the time-differentiated problem, provides the correct pressure estimate in the limit as the penalization parameter tends to zero.

As we noted following Lemma 11, we will work with a regularized sequence of velocities v_n, and we shall generically denote elements of this sequence simply as $\tilde{v}$, and the associated regularized matrices $a(\eta_n)$ as $\tilde{a}$.

Given the regularity assumptions in (6), $f_t \in C([0,T]; L^2(\Omega;\mathbb{R}^3))$, so that $f_t(0) \in L^2(\Omega_0^f;\mathbb{R}^3))$.

Then, let $w_2 \in L^2(\Omega;\mathbb{R}^3)$ be defined by

$$w_2^i = \nu \triangle w_1^i + \nu((a_l^j a_l^k)_t(0) u_0^i,_k),_j + F_t(0) - ((a_i^j)_t(0) q_0),_j - q_1,_i \text{ in } \Omega_0^f, \tag{28a}$$

$$w_2^i = f_t^i(0) + [c^{ijkl} u_0^k,_l],_j \text{ in } \Omega_0^s, \tag{28b}$$

where $q_1 \in H^1(\Omega_0^f;\mathbb{R})$ is defined by

$$\begin{aligned} \triangle q_1 &= \frac{\partial}{\partial x^i}[\nu \triangle (w^i)_1 + (F^i)_t(0) + \nu((a_l^j a_l^k)_t(0) u_0^i,_k),_j - ((a_i^j)_t(0) q_0),_j] \\ &\quad + 2(a_i^j)_t(0) w_1^i,_j + (a_i^j)_{tt}(0) u_0^i,_j \text{ in } \Omega_0^f, \end{aligned} \tag{29a}$$

$$\begin{aligned} q_1 &= \nu[\nabla w_1 N \cdot N + (a_l^k a_l^j)_t(0) u_0^i,_k N_j N_i] - c^{ijkl} u_0^k,_l N_j N_i \text{ on } \Gamma_0 \\ &\quad + q_0 (a_i^j)_t(0) N_j N_i \end{aligned} \tag{29b}$$

$$\begin{aligned} \frac{\partial q_1}{\partial N} &= F_t(0) \cdot N - [(a_i^j)_t(0) q_0],_j + \nu \triangle w_1 \cdot N \\ &\quad + \nu((a_l^j a_l^k)_t(0) u_0^i,_k),_j N_i \text{ on } \partial\Omega. \end{aligned} \tag{29c}$$

Once again, we remind the reader that $(a_i^j)_t(0)$ and $(a_i^j)_{tt}(0)$ depend only on u_0 and w_1, and we note that they are equal to $(\tilde{a}_i^j)_t(0)$ and $(\tilde{a}_i^j)_{tt}(0)$, respectively.

Letting $\varepsilon > 0$ denote the penalization parameter, we define $w_\varepsilon \in \mathcal{W}([0,T])$ to be the unique weak solution of the problem (whose existence can be obtained via a standard Galerkin method in a basis of $H_0^1(\Omega;\mathbb{R}^3)$):

$$\begin{aligned} \text{(i) } & \langle w_{\varepsilon t}, \phi \rangle + \nu(\tilde{a}_k^r w_\varepsilon^i,_r , \tilde{a}_k^s \phi^i,_s)_{L^2(\Omega_0^f;\mathbb{R})} + \left(c^{ijkl} \int_0^t w_\varepsilon^k,_l , \phi^i,_j \right)_{L^2(\Omega_0^s;\mathbb{R})} \\ & + \left(\frac{1}{\varepsilon} \tilde{a}_j^i w_\varepsilon^j,_i - q_0 - t q_1, \tilde{a}_k^l \phi^k,_l \right)_{L^2(\Omega_0^f;\mathbb{R})} \\ & = (F, \phi)_{L^2(\Omega_0^f;\mathbb{R}^3)} + (f, \phi)_{L^2(\Omega_0^s;\mathbb{R}^3)} \end{aligned} \tag{30a}$$

$$\forall \phi \in H_0^1(\Omega;\mathbb{R}^3), \text{ and}$$

$$\text{(ii) } w(0,\cdot) = u_0, \tag{30b}$$

where $\langle \cdot, \cdot \rangle$ denotes the duality product between $H_0^1(\Omega;\mathbb{R}^3)$ and its dual.

9.3. Weak solutions for the penalized problem

The aim of this section is to establish the existence of w_ε, as well as the energy equalities satisfied by w_ε and $w_{\varepsilon t}$, and the energy inequality satisfied by $w_{\varepsilon tt}$. It turns out that the exposition is simplified if we first study the twice differentiated-in-time problem, that we introduce now.

Step 1. Galerkin sequence. By introducing a basis $(e_l)_{l=1}^{\infty}$ of $H_0^1(\Omega;\mathbb{R}^3)$ and $L^2(\Omega;\mathbb{R}^3)$, and taking the approximation at rank $l \geqq 2$ under the form

$$w_l(t,x) = \sum_{k=1}^{l} y_k(t)\, e_k(x),$$

and satisfying on $[0, T]$

$$\begin{aligned}
\text{(i)}\ \ & (w_{l\,ttt}, \phi)_{L^2(\Omega;\mathbb{R}^3)} + \nu(\tilde a_k^r w_{l\,tt},_r , \tilde a_k^s \phi,_s)_{L^2(\Omega_0^f;\mathbb{R}^3)} \\
&\quad + (c^{ijkl} w_{l\,t}^k,_l , \phi^i,_j)_{L^2(\Omega_0^s;\mathbb{R})} + \nu((\tilde a_k^r \tilde a_k^s)_{tt} w_l,_r ,\ \phi,_s)_{L^2(\Omega_0^f;\mathbb{R}^3)} \\
&\quad + 2\nu((\tilde a_k^r \tilde a_k^s)_t w_{l\,t},_r ,\ \phi,_s)_{L^2(\Omega_0^f;\mathbb{R}^3)} - ((\tilde a_i^j q_l)_{tt},\ \phi^i,_j)_{L^2(\Omega_0^f;\mathbb{R})} \\
&= (F_{tt}, \phi)_{L^2(\Omega_0^f;\mathbb{R}^3)} + (f_{tt}, \phi)_{L^2(\Omega_0^s;\mathbb{R}^3)},\ \forall \phi \in \operatorname{span}(e_1, ..., e_l), \\
\text{(ii)}\ \ & w_{l\,tt}(0) = (w_2)_l,\ w_{l\,t}(0) = (w_1)_l,\ w_l(0) = (u_0)_l,\ \text{in } \Omega,
\end{aligned}$$

where $q_l = q_0 + t\, q_1 - (1/\varepsilon)\tilde a_i^j w_l^i,_j$ and $(w_2)_l$ denotes the $L^2(\Omega;\mathbb{R}^3)$ projection of w_2 onto $\operatorname{span}(e_1, ..., e_l)$, and $(w_1)_l$ and $(u_0)_l$ denote the respective $H_0^1(\Omega;\mathbb{R}^3)$ projections of w_1 and u_0 on $\operatorname{span}(e_1, ..., e_l)$, we see that the Cauchy-Lipschitz theorem gives us the local well-posedness for w_l. The use of the test function $(w_l)_{tt}$ in this system of ordinary differential equations (which is allowed as it belongs to $\operatorname{span}(e_1, ..., e_l)$) gives us in turn the energy law

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|w_{l\,tt}\|^2_{L^2(\Omega;\mathbb{R}^3)} + \nu(\tilde a_k^r w_{l\,tt},_r ,\ \tilde a_k^s w_{l\,tt},_s)_{L^2(\Omega_0^f;\mathbb{R}^3)} \\
&\quad + \frac{1}{2}\frac{d}{dt}(c^{ijkl} w_{l\,t}^k,_l , w_{l\,t}^i,_j)_{L^2(\Omega_0^s;\mathbb{R})} + \nu((\tilde a_k^r \tilde a_k^s)_{tt} w_l,_r ,\ w_{l\,tt},_s)_{L^2(\Omega_0^f;\mathbb{R}^3)} \\
&\quad + 2\nu((\tilde a_k^r \tilde a_k^s)_t w_{l\,t},_r ,\ w_{l\,tt},_s)_{L^2(\Omega_0^f;\mathbb{R}^3)} - ((\tilde a_i^j q_l)_{tt},\ w_{l\,tt}^i,_j)_{L^2(\Omega_0^f;\mathbb{R})} \\
&= (F_{tt}, w_{l\,tt})_{L^2(\Omega_0^f;\mathbb{R}^3)} + (f_{tt}, w_{l\,tt})_{L^2(\Omega_0^s;\mathbb{R}^3)}.
\end{aligned}$$

After transforming the term with $(q_l)_{tt}$ (since it involves $\nabla w_{l\,tt}$) and integrating this relation from 0 to $t \in (0, T)$, we get

$$\begin{aligned}
&\frac{1}{2}\|w_{l\,tt}\|^2_{L^2(\Omega;\mathbb{R}^3)} + \nu\int_0^t (\tilde a_k^r w_{l\,tt},_r ,\ \tilde a_k^s w_{l\,tt},_s)_{L^2(\Omega_0^f;\mathbb{R}^3)} + \varepsilon\int_0^t\int_{\Omega_0^f} {q_l}_{tt}^2 \\
&\quad + \frac{1}{2}(c^{ijkl} w_{l\,t}^k,_l , w_{l\,t}^i,_j)_{L^2(\Omega_0^s;\mathbb{R})} + \int_0^t\int_{\Omega_0^f} q_{l\,tt}\, [2(\tilde a_i^j)_t w_{l\,t}^i,_j \\
&\quad + (\tilde a_i^j)_{tt} w_l{}^i,_j] - 2\int_0^t\int_{\Omega_0^f} (\tilde a_i^j)_t q_{l\,t} w_{l\,tt}^i,_j - \int_0^t\int_{\Omega_0^f} (\tilde a_i^j)_{tt} q_l w_{l\,tt}^i,_j \\
&\quad + \nu\int_0^t ((\tilde a_k^r \tilde a_k^s)_{tt} w_l,_r ,\ w_{l\,tt},_s)_{L^2(\Omega_0^f;\mathbb{R}^3)} \\
&\quad + 2\nu\int_0^t ((\tilde a_k^r \tilde a_k^s)_t w_{l\,t},_r , w_{l\,tt},_s)_{L^2(\Omega_0^f;\mathbb{R}^3)} \\
&\leqq C\, N(u_0, f)^2 + \int_0^t (F_{tt}, w_{l\,tt})_{L^2(\Omega_0^f;\mathbb{R}^3)} + \int_0^t (f_{tt}, w_{l\,tt})_{L^2(\Omega_0^s;\mathbb{R}^3)}. \qquad (31)
\end{aligned}$$

By noticing that, ε being fixed, the third term of the left-hand side of this inequality involving the square of $(q_l)_{tt}$ acts as a viscous energy term, and taking into account the $L^\infty(0,T;L^\infty(\Omega_0^f;\mathbb{R}))$ bound of each one of the regularized coefficients $\tilde{a}_i^j$ and their first and second time derivatives, we then get

$$\frac{1}{2}\|w_{l_{tt}}(t)\|^2_{L^2(\Omega;\mathbb{R}^3)} + \frac{\nu}{2}\int_0^t \|\nabla w_{l_{tt}}\|^2_{L^2(\Omega_0^f;\mathbb{R}^9)} + \frac{\varepsilon}{4}\int_0^t\int_{\Omega_0^f} q_{l_{tt}}^2$$
$$-\tilde{C}_\varepsilon\left[\int_0^t\int_0^{t'} \|\nabla w_{l_{tt}}\|^2_{L^2(\Omega_0^f;\mathbb{R}^9)} + N(u_0,f)^2 + \int_0^t\int_0^{t'} \|q_{l_{tt}}\|^2_{L^2(\Omega_0^f;\mathbb{R})}\right]$$
$$+\frac{1}{2}(c^{ijkl}w_{l_t}^k,_l(t),\, w_{l_t}^i,_j(t))_{L^2(\Omega_0^s;\mathbb{R})}$$
$$\leqq C\,N(u_0,f)^2 + \int_0^t (F_{tt},w_{l_{tt}})_{L^2(\Omega_0^f;\mathbb{R}^3)} + \int_0^t (f_{tt},w_{l_{tt}})_{L^2(\Omega_0^s;\mathbb{R}^3)},$$

where $\tilde{C}_\varepsilon$ depends on the regularizing parameter of $\tilde{a}$ and on ε, but not on l. By Gronwall's inequality, we then get an estimate on each of the integral terms multiplying $\tilde{C}_\varepsilon$, which in turn implies

$$\|w_{l_{tt}}(t)\|^2_{L^2(\Omega;\mathbb{R}^3)} + \int_0^t \|\nabla w_{l_{tt}}\|^2_{L^2(\Omega_0^f;\mathbb{R}^9)}$$
$$+\frac{1}{2}(c^{ijkl}w_{l_t}^k,_l(t),\, w_{l_t}^i,_j(t))_{L^2(\Omega_0^s;\mathbb{R})} + \varepsilon\int_0^t\int_{\Omega_0^f} q_{l_{tt}}^2 \leqq \tilde{C}_\varepsilon\,N(u_0,f)^2.$$

Step 2. Weak solution w_ε of the penalized problem and its time differentiated problem. We can then infer that w_l is defined on $[0,T]$, and that there is a subsequence, still denoted with the subscript l, satisfying

$$w_l \rightharpoonup w_\varepsilon \ \text{ in } L^2(0,T;H_0^1(\Omega;\mathbb{R}^3)), \tag{32a}$$
$$w_{l_t} \rightharpoonup w_{\varepsilon t} \ \text{ in } L^2(0,T;H_0^1(\Omega;\mathbb{R}^3)), \tag{32b}$$
$$w_{l_{tt}} \rightharpoonup w_{\varepsilon tt} \ \text{ in } L^2(0,T;L^2(\Omega;\mathbb{R}^3)) \text{ and in } L^2(0,T;H^1(\Omega_0^f;\mathbb{R}^3)), \tag{32c}$$
$$q_{l_{tt}} \rightharpoonup q_{\varepsilon tt} \ \text{ in } L^2(0,T;L^2(\Omega_0^f;\mathbb{R}^3)), \tag{32d}$$

where

$$q_\varepsilon = q_0 + tq_1 - \frac{1}{\varepsilon}\tilde{a}_i^j w_\varepsilon{}^i,_j\ . \tag{33}$$

From the standard procedure for weak solutions, we can now infer from these weak convergences and the definition of w_l that $w_{\varepsilon ttt} \in L^2(0,T;H^{-1}(\Omega;\mathbb{R}^3))$. In turn, $w_{\varepsilon tt} \in C^0([0,T];H^{-1}(\Omega;\mathbb{R}^3))$, $w_{\varepsilon t} \in C^0([0,T];L^2(\Omega;\mathbb{R}^3))$, $w_\varepsilon \in C^0([0,T];H_0^1(\Omega;\mathbb{R}^3))$, with $w_\varepsilon(0)=u_0$, $w_{\varepsilon t}(0)=w_1$, $w_{\varepsilon tt}(0)=w_2$.

We moreover have, for w_{l_t},

(i) $(w_{l_{tt}},\phi)_{L^2(\Omega;\mathbb{R}^3)} + \nu(\tilde{a}_k^r w_{l_t},_r\,,\ \tilde{a}_k^s\phi,_s)_{L^2(\Omega_0^f;\mathbb{R}^3)} + (c^{ijkl}w_l{}^k,_l\,,\phi^i,_j)_{L^2(\Omega_0^s;\mathbb{R})}$

$+\nu((\tilde{a}_k^r\tilde{a}_k^s)_t w_l,_r\,,\ \phi,_s)_{L^2(\Omega_0^f;\mathbb{R}^3)} - ((\tilde{a}_i^j q_l)_t,\ \phi^i,_j)_{L^2(\Omega_0^f;\mathbb{R})}$

$= (F_t,\phi)_{L^2(\Omega_0^f;\mathbb{R}^3)} + (f_t,\phi)_{L^2(\Omega_0^s;\mathbb{R}^3)} + c_l(\phi)\ \forall\phi \in \text{span}(e_1,\ldots,e_l)$,

(ii) $w_{l_t}(0) = (w_1)_l,\ w_l(0) = (u_0)_l$ in Ω , where $c_l(\phi) \in \mathbb{R}$ is given by

$$
\begin{aligned}
c_l(\phi) = \; & ((w_2)_l, \phi)_{L^2(\Omega;\mathbb{R}^3)} + \nu(\tilde{a}_k^r(0) w_{1l},_r\,,\ \tilde{a}_k^s(0)\phi,_s\,)_{L^2(\Omega_0^f;\mathbb{R}^3)} \\
& + (c^{ijkl}(w_1)_l^k,_l\,, \phi^i,_j\,)_{L^2(\Omega_0^s;\mathbb{R})} + \nu((\tilde{a}_k^r \tilde{a}_k^s)_t(0)(w_1)_l,_r\,,\ \phi,_s\,)_{L^2(\Omega_0^f;\mathbb{R}^3)} \\
& - ((\tilde{a}_i^j q_l)_t(0),\ \phi^i,_j\,)_{L^2(\Omega_0^f;\mathbb{R})} - (F_t(0), \phi)_{L^2(\Omega_0^f;\mathbb{R}^3)} - (f_t(0), \phi)_{L^2(\Omega_0^s;\mathbb{R}^3)}.
\end{aligned}
$$

Thus, $c_l(\phi)$ converges to the same expression, where the approximate initial data $(w_i)_l$ are replaced by the actual initial data w_i $(i = 0, 1, 2)$. From our compatibility conditions (7) together with (28), this leads us to

$$
\|c_l\|_{H^{-1}(\Omega;\mathbb{R}^3)} \to 0, \text{ as } l \to \infty. \tag{34}
$$

Similarly, for w_l,

$$
\begin{aligned}
\text{(i) } & (w_{lt}, \phi)_{L^2(\Omega;\mathbb{R}^3)} + \nu(\tilde{a}_k^r w_l,_r\,,\ \tilde{a}_k^s \phi,_s\,)_{L^2(\Omega_0^f;\mathbb{R}^3)} \\
& + \left(c^{ijkl} \int_0^\cdot w_l{}^k,_l\,, \phi^i,_j \right)_{L^2(\Omega_0^s;\mathbb{R})} - (\tilde{a}_i^j q_l,\ \phi^i,_j\,)_{L^2(\Omega_0^f;\mathbb{R}^3)} \\
& = (F, \phi)_{L^2(\Omega_0^f;\mathbb{R}^3)} + (f, \phi)_{L^2(\Omega_0^s;\mathbb{R}^3)} + c_l(\phi)t \\
& \quad + d_l(\phi) \forall \phi \in \text{ span}(e_1, ..., e_l),
\end{aligned} \tag{35a}
$$

$$
\text{(ii) } w_l(0) = (u_0)_l \text{ in } \Omega\ , \tag{35b}
$$

where $d_l(\phi) \in \mathbb{R}$ is given by

$$
\begin{aligned}
d_l(\phi) = \; & ((w_1)_l, \phi)_{L^2(\Omega;\mathbb{R}^3)} + \nu(\tilde{a}_k^r(0) u_{0l},_r\,,\ \tilde{a}_k^s(0)\phi,_s\,)_{L^2(\Omega_0^f;\mathbb{R}^3)} \\
& - ((\tilde{a}_i^j q_l)(0), \phi^i,_j\,)_{L^2(\Omega_0^f;\mathbb{R})} - (F(0), \phi)_{L^2(\Omega_0^f;\mathbb{R}^3)} - (f(0), \phi)_{L^2(\Omega_0^s;\mathbb{R}^3)}.
\end{aligned}
$$

Similarly as for $c_l(\phi)$, from our compatibility conditions (7),

$$
\|d_l\|_{H^{-1}(\Omega;\mathbb{R}^3)} \to 0, \quad \text{as } l \to \infty. \tag{36}
$$

We can thus infer now that at the limit w_ε satisfies, for all $\phi \in L^2(0, T; H_0^1(\Omega; \mathbb{R}^3))$,

$$
\begin{aligned}
& \int_0^T (w_{\varepsilon t}, \phi)_{L^2(\Omega;\mathbb{R}^3)}\, dt + \nu \int_0^T (\tilde{a}_k^r w_\varepsilon,_r\,,\ \tilde{a}_k^s \phi,_s\,)_{L^2(\Omega_0^f;\mathbb{R}^3)}\, dt \\
& \quad + \int_0^T \left(c^{ijkl} \int_0^t w_\varepsilon{}^k,_l\,, \phi^i,_j \right)_{L^2(\Omega_0^s;\mathbb{R})} dt - \int_0^T (q_\varepsilon,\ \tilde{a}_k^l \phi^k,_l\,)_{L^2(\Omega_0^f;\mathbb{R})} dt \\
& = \int_0^T (F, \phi)_{L^2(\Omega_0^f;\mathbb{R}^3)} + (f, \phi)_{L^2(\Omega_0^s;\mathbb{R}^3)}\, dt,
\end{aligned} \tag{37}
$$

which, combined with $w_\varepsilon(0) = u_0$, shows us that w_ε is a weak solution of (30).

Moreover, $w_{\varepsilon t}$ satisfies for all $\phi \in L^2(0,T;H^1_0(\Omega;\mathbb{R}^3))$,

$$\int_0^T (w_{\varepsilon tt},\phi)_{L^2(\Omega;\mathbb{R}^3)}\,dt + \nu\int_0^T ((\tilde{a}^s_k\tilde{a}^r_k w_\varepsilon,_r)_t,\ \phi,_s)_{L^2(\Omega^f_0;\mathbb{R}^3)}\,dt$$
$$+\int_0^T (c^{ijkl}w_\varepsilon{}^k,_l,\phi^i,_j)_{L^2(\Omega^s_0;\mathbb{R})}\,dt - \int_0^T ((\tilde{a}^l_k q_\varepsilon)_t,\ \phi^k,_l)_{L^2(\Omega^f_0;\mathbb{R})}dt$$
$$= \int_0^T (F_t,\phi)_{L^2(\Omega^f_0;\mathbb{R}^3)} + (f_t,\phi)_{L^2(\Omega^s_0;\mathbb{R}^3)}\,dt. \tag{38}$$

Step 3. Strong convergence for the Galerkin approximation. Since $w_\varepsilon \in L^2(0,T;H^1_0(\Omega;\mathbb{R}^3))$, we can use it as a test function in (37), which provides us on $(0,T)$ with the equality

$$\frac{1}{2}\|w_\varepsilon(t)\|^2_{L^2(\Omega;\mathbb{R}^3)} + \nu\int_0^t (\tilde{a}^r_k w_\varepsilon,_r,\ \tilde{a}^s_k w_\varepsilon,_s)_{L^2(\Omega^f_0;\mathbb{R}^3)}\,dt$$
$$+\frac{1}{2}\left(c^{ijkl}\int_0^t w_\varepsilon{}^k,_l,\int_0^t w_\varepsilon{}^i,_j\right)_{L^2(\Omega^s_0;\mathbb{R})}$$
$$+\int_0^t \varepsilon\|q_\varepsilon\|^2_{L^2(\Omega^f_0;\mathbb{R})} - \varepsilon(q_0+tq_1,\ q_\varepsilon)_{L^2(\Omega^f_0;\mathbb{R})}dt$$
$$= \frac{1}{2}\|u_0\|^2_{L^2(\Omega;\mathbb{R}^3)} + \int_0^t (F,w_\varepsilon)_{L^2(\Omega^f_0;\mathbb{R}^3)} + (f,w_\varepsilon)_{L^2(\Omega^s_0;\mathbb{R}^3)}\,dt. \tag{39}$$

Similarly since $w_l(t) \in \operatorname{span}(e_1,\dots,e_l)$ for all $t\in[0,T]$, we can use it as a test function in (35), which gives us

$$\frac{1}{2}\|w_l(t)\|^2_{L^2(\Omega;\mathbb{R}^3)} + \nu\int_0^t (\tilde{a}^r_k w_l,_r,\ \tilde{a}^s_k w_l,_s)_{L^2(\Omega^f_0;\mathbb{R}^3)}\,dt$$
$$+\frac{1}{2}\left(c^{ijkl}\int_0^t w_l{}^k,_l,\int_0^t w_\varepsilon{}^i,_j\right)_{L^2(\Omega^s_0;\mathbb{R})}$$
$$+\int_0^t \varepsilon\|q_l\|^2_{L^2(\Omega^f_0;\mathbb{R})} - \varepsilon(q_0+tq_1,\ q_l)_{L^2(\Omega^f_0;\mathbb{R})}dt$$
$$= \frac{1}{2}\|(u_0)_l\|^2_{L^2(\Omega;\mathbb{R}^3)} + \int_0^t (F,w_\varepsilon)_{L^2(\Omega^f_0;\mathbb{R}^3)} + (f,w_\varepsilon)_{L^2(\Omega^s_0;\mathbb{R}^3)}\,dt$$
$$+\int_0^t t d_l(w_l) + c_l(w_l)\,dt. \tag{40}$$

By (32), (34) and (36), we then infer by comparing (40) and (39), that as $l\to\infty$, for all $t\in[0,T]$,

$$\frac{1}{2}\|w_l(t)\|^2_{L^2(\Omega;\mathbb{R}^3)} + \nu\int_0^t (\tilde{a}^r_k w_l,_r,\ \tilde{a}^s_k w_l,_s)_{L^2(\Omega^f_0;\mathbb{R}^3)}\,dt$$
$$+\frac{1}{2}\left(c^{ijkl}\int_0^t w_l{}^k,_l,\int_0^t w_\varepsilon{}^i,_j\right)_{L^2(\Omega^s_0;\mathbb{R})}\,dt + \varepsilon\int_0^t \|q_l\|^2_{L^2(\Omega^f_0;\mathbb{R})}dt$$

$$\to \frac{1}{2}\|w_\varepsilon(t)\|^2_{L^2(\Omega;\mathbb{R}^3)} + \nu \int_0^t (\tilde{a}^r_k w_{\varepsilon},_r\,,\ \tilde{a}^s_k w_{\varepsilon},_s)_{L^2(\Omega_0^f;\mathbb{R}^3)}\,dt$$
$$+\frac{1}{2}\left(c^{ijkl}\int_0^t {w_\varepsilon}^k,_l\,,\int_0^t {w_\varepsilon}^i,_j\right)_{L^2(\Omega_0^s;\mathbb{R})} dt + \varepsilon\int_0^t \|q_\varepsilon\|^2_{L^2(\Omega_0^f;\mathbb{R})}dt,$$

which gives in turn the strong convergences

$$w_l \to w_\varepsilon \ \text{ in } L^2(0,T;H^1(\Omega_0^f;\mathbb{R}^3)), \tag{41a}$$
$$w_l \to w_\varepsilon \ \text{ in } L^2(0,T;L^2(\Omega;\mathbb{R}^3)), \tag{41b}$$
$$\int_0^\cdot w_l \to \int_0^\cdot w_\varepsilon \ \text{ in } L^2(0,T;H^1(\Omega_0^s;\mathbb{R}^3)), \tag{41c}$$
$$q_l \to q_\varepsilon \ \text{ in } L^2(0,T;L^2(\Omega_0^f;\mathbb{R})). \tag{41d}$$

Since $w_{\varepsilon t} \in L^2(0,T;H^1_0(\Omega;\mathbb{R}^3))$, we can prove in a similar fashion the strong convergences

$$w_{lt} \to w_{\varepsilon t} \ \text{ in } L^2(0,T;H^1(\Omega_0^f;\mathbb{R}^3)), \tag{42a}$$
$$w_{lt} \to w_{\varepsilon t} \ \text{ in } L^2(0,T;L^2(\Omega;\mathbb{R}^3)), \tag{42b}$$
$$w_l \to w_\varepsilon \ \text{ in } L^2(0,T;H^1(\Omega_0^s;\mathbb{R}^3)), \tag{42c}$$
$$q_{lt} \to q_{\varepsilon t} \ \text{ in } L^2(0,T;L^2(\Omega_0^f;\mathbb{R})). \tag{42d}$$

Step 4. Energy inequality for $w_{\varepsilon tt}$. By using the relation

$$\frac{1}{2}\|w_{ltt}\|^2_{L^2(\Omega;\mathbb{R}^3)} + \nu\int_0^t (\tilde{a}^r_k w_{ltt},_r\,,\ \tilde{a}^s_k w_{ltt},_s)_{L^2(\Omega_0^f;\mathbb{R}^3)}$$
$$+\frac{1}{2}(c^{ijkl}w^k_{lt},_l\,, w^i_{lt},_j)_{L^2(\Omega_0^s;\mathbb{R})} - 2\int_0^t\int_{\Omega_0^f}(\tilde{a}^j_i)_t q_{lt} w^i_{ltt},_j$$
$$+\varepsilon\int_0^t\int_{\Omega_0^f} q^2_{ltt} + \int_0^t\int_{\Omega_0^f} q_{ltt}\,[2(\tilde{a}^j_i)_t w^i_{lt},_j + (\tilde{a}^j_i)_{tt}{w_l}^i,_j]$$
$$-\int_0^t\int_{\Omega_0^f}(\tilde{a}^j_i)_{tt}q_l w^i_{ltt},_j + \nu\int_0^t((\tilde{a}^r_k\tilde{a}^s_k)_{tt}w_l,_r\,,\ w_{ltt},_s)_{L^2(\Omega_0^f;\mathbb{R}^3)}$$
$$+2\,\nu\int_0^t((\tilde{a}^r_k\tilde{a}^s_k)_t w_{lt},_r\,,\ w_{\varepsilon tt},_s)_{L^2(\Omega_0^f;\mathbb{R}^3)}$$
$$\leqq C\,N(u_0,f)^2 + \int_0^t(F_{tt},w_{ltt})_{L^2(\Omega_0^f;\mathbb{R}^3)} + \int_0^t(f_{tt},w_{ltt})_{L^2(\Omega_0^s;\mathbb{R}^3)},$$

and the weak convergences (32), along with the strong convergences (41) and (42), we then get

$$\frac{1}{2}\|w_{\varepsilon tt}\|^2_{L^2(\Omega;\mathbb{R}^3)} + \nu\int_0^t (\tilde{a}^r_k w_{\varepsilon tt},_r\,,\ \tilde{a}^s_k w_{\varepsilon tt},_s)_{L^2(\Omega_0^f;\mathbb{R}^3)}$$
$$+\frac{1}{2}(c^{ijkl}w^k_{\varepsilon t},_l\,, w^i_{\varepsilon t},_j)_{L^2(\Omega_0^s;\mathbb{R})} - 2\int_0^t\int_{\Omega_0^f}(\tilde{a}^j_i)_t q_{\varepsilon t} w^i_{\varepsilon tt},_j$$
$$+\varepsilon\int_0^t\int_{\Omega_0^f} q^2_{\varepsilon tt} + \int_0^t\int_{\Omega_0^f} q_{\varepsilon tt}\,[2(\tilde{a}^j_i)_t w^i_{\varepsilon t},_j + (\tilde{a}^j_i)_{tt}{w_\varepsilon}^i,_j]$$

$$
\begin{aligned}
&- \int_0^t \int_{\Omega_0^f} (\tilde{a}_i^j)_{tt} q_\varepsilon w_{\varepsilon tt}^i,_j + \nu \int_0^t ((\tilde{a}_k^r \tilde{a}_k^s)_{tt} w_\varepsilon,_r \,,\ w_{\varepsilon tt},_s)_{L^2(\Omega_0^f;\mathbb{R}^3)} \\
&+ 2\,\nu \int_0^t ((\tilde{a}_k^r \tilde{a}_k^s)_t w_{\varepsilon t},_r \,,\ w_{\varepsilon tt},_s)_{L^2(\Omega_0^f;\mathbb{R}^3)} \\
&\leqq C\, N(u_0, f)^2 + \int_0^t (F_{tt}, w_{\varepsilon tt})_{L^2(\Omega_0^f;\mathbb{R}^3)} + \int_0^t (f_{tt}, w_{\varepsilon tt})_{L^2(\Omega_0^s;\mathbb{R}^3)}.
\end{aligned} \tag{43}
$$

9.4. Existence of $\tilde{w}$, $\tilde{w}_t$, $\tilde{w}_{tt}$, uniqueness

In this section, we establish the existence of $\tilde{w}$, and its first and second time derivatives by taking the limit $\varepsilon \to 0$. The inequality (79) proved at the end of this section, holds for any regularized velocity field $\tilde{v} = v_n$, independently of n, and requires, in its proof, strong convergence results from their penalized counterparts since the regularity that we take on the data does not allow us to view $\tilde{w}_{tt}$ as a weak solution of a variational problem.

Theorem 3. *Suppose that u_0 and f satisfy the conditions stated in Theorem 1. Then, there exists a weak solution $\tilde{w}$ to the problem* (20) *with the mollified coefficients replacing the actual coefficients. Moreover, $\tilde{w}$ is in $L^2(0, T; \mathcal{V}_{\tilde{v}}(\cdot))$ and is unique, and $\tilde{w}_t \in \mathcal{W}([0, T])$.*

Proof. *Step 1. The limit as $\varepsilon \to 0$.* Let $\varepsilon = \frac{1}{m}$; we first pass to the weak limit as $m \to \infty$. The energy law (39) shows that there exists a subsequence $\{w_{\frac{1}{m_l}}\}$ such that

$$
w_{\frac{1}{m_l}} \rightharpoonup \tilde{w} \quad \text{in} \quad \mathcal{W}([0, T]). \tag{44}
$$

Moreover, since (39) also shows that $\|\tilde{a}_i^j w_{\frac{1}{m}}^i,_j\|_{L^2(0,T;L^2(\Omega_0^f;\mathbb{R}))} \to 0$ as $m \to \infty$, we then have $\|\tilde{a}_i^j \tilde{w}^i,_j\|_{L^2(0,T;L^2(\Omega_0^f;\mathbb{R}))} = 0$, i.e.,

$$
\tilde{w} \in \mathcal{V}_{\tilde{v}}([0, T])\,. \tag{45}
$$

Step 2. The penalized time differentiated problems and estimates independent of ε. Thanks to (42) and (43), we have

$$
w_{\varepsilon t} \in L^2(0, T; H_0^1(\Omega; \mathbb{R}^3)) \cap C^0([0, T]; L^2(\Omega; \mathbb{R}^3)).
$$

We can thus use it as a test function in (38), which gives us, for a.e. $t \in (0, T)$,

$$
\begin{aligned}
&\frac{1}{2}\|w_{\varepsilon t}\|^2_{L^2(\Omega;\mathbb{R}^3)} + \nu \int_0^t (\tilde{a}_k^r w_{\varepsilon t},_r \,,\ \tilde{a}_k^s w_{\varepsilon t},_s)_{L^2(\Omega_0^f;\mathbb{R}^3)} \\
&\quad + \frac{1}{2}(c^{ijkl} w_\varepsilon^k,_l \,,\, w_\varepsilon^i,_j)_{L^2(\Omega_0^s;\mathbb{R})} + \varepsilon \int_0^t \int_{\Omega_0^f} q_{\varepsilon t}^2 - \varepsilon \int_0^t \int_{\Omega_0^f} q_{\varepsilon t} q_1
\end{aligned}
$$

$$+\int_0^t \int_{\Omega_0^f} q_{\varepsilon t}(\tilde{a}_i^j)_t w_\varepsilon{}^i{}_{,j} - \int_0^t \int_{\Omega_0^f} (\tilde{a}_i^j)_t q_\varepsilon w_{\varepsilon t}{}^i{}_{,j}$$

$$+\nu \int_0^t ((\tilde{a}_k^r \tilde{a}_k^s)_t w_{\varepsilon,r}\, , \, w_{\varepsilon t,s})_{L^2(\Omega_0^f;\mathbb{R}^3)}$$

$$\leqq C\|w_1\|^2_{L^2(\Omega;\mathbb{R}^3)} + \int_0^t (F_t, w_{\varepsilon t})_{L^2(\Omega_0^f;\mathbb{R}^3)} + \int_0^t (f_t, w_{\varepsilon t})_{L^2(\Omega_0^s;\mathbb{R}^3)}.$$

At this stage, we remove the time derivative from the $q_{\varepsilon t}$ term in this inequality by integrating by parts:

$$\int_0^t q_{\varepsilon t}(\tilde{a}_i^j)_t w_\varepsilon{}^i{}_{,j} = -\int_0^t q_\varepsilon((\tilde{a}_i^j)_t w_\varepsilon{}^i{}_{,j})_t + (q_\varepsilon(\tilde{a}_i^j)_t w_\varepsilon{}^i{}_{,j})(t) - q_0(\tilde{a}_i^j)_t(0) u_0^i{}_{,j}$$

($q_\varepsilon(0) = q_0$ by $\operatorname{div} u_0 = 0$); we then infer by the regularity of $\tilde{a}$ and of w_ε that

$$\frac{1}{2}\|w_{\varepsilon t}\|^2_{L^2(\Omega;\mathbb{R}^3)} + \frac{\nu}{2}\int_0^t (\tilde{a}_k^r w_{\varepsilon t,r}\, , \, \tilde{a}_k^s w_{\varepsilon t,s})_{L^2(\Omega_0^f;\mathbb{R}^3)}$$

$$+\frac{1}{2}(c^{ijkl} w_\varepsilon^k{}_{,l}\, , \, w_\varepsilon^i{}_{,j})_{L^2(\Omega_0^s;\mathbb{R})}$$

$$\leqq \tilde{C} \int_0^t \|q_\varepsilon\|^2_{L^2(\Omega_0^f;\mathbb{R})} + \delta\, \|q_\varepsilon(t)\|^2_{L^2(\Omega_0^f;\mathbb{R})} + \tilde{C}\, C_\delta\, \|\nabla w_\varepsilon(t)\|^2_{L^2(\Omega_0^f;\mathbb{R}^9)}$$

$$+\tilde{C}\, N(u_0, f)^2 + C\int_0^t \|w_{\varepsilon t}\|^2_{L^2(\Omega;\mathbb{R}^3)}\,,$$

where $\delta > 0$ is arbitrary, and $\tilde{C}$ denotes a generic constant depending on the smoothing parameter n implicit in $\tilde{a}$.

Note that it is the presence of $\tilde{a}_{tt}$ which requires the use of the regularized coefficient matrix $\tilde{a}$; this is due to the fact that $a_{tt}(t)$ is not in Ł$^\infty$ as the presence of ∇w_ε and q_ε (both taken in L^2) would require. In order for us to be able to obtain consistent estimates later on which are independent of the regularization process, we must require the pressure of the penalized problem to be in H^1 (for a.e. $t \in (0, T)$); this requires difference-quotient methods. In order to achieve this, we first define this pressure function to be in L^2 (a.e. $t \in (0, T)$), and then find estimates which are independent of the regularization of a. Thus, in $(0, T)$,

$$\|w_{\varepsilon t}(t)\|^2_{L^2(\Omega;\mathbb{R}^3)} + \|w_\varepsilon(t)\|^2_{H^1(\Omega_0^s;\mathbb{R}^3)} + \int_0^t \|w_{\varepsilon t}\|^2_{H^1(\Omega_0^f;\mathbb{R}^3)}$$

$$\leqq \tilde{C} \int_0^t \|q_\varepsilon\|^2_{L^2(\Omega_0^f;\mathbb{R})} + \delta\|q_\varepsilon(t)\|^2_{L^2(\Omega_0^f;\mathbb{R})}$$

$$+\tilde{C}\, C_\delta\|\nabla w_\varepsilon(t)\|^2_{L^2(\Omega_0^f;\mathbb{R}^9)} + \tilde{C}\, N(u_0, f)^2 + C\int_0^t \|w_{\varepsilon t}\|^2_{L^2(\Omega;\mathbb{R}^3)}. \tag{46}$$

By the Lagrange multiplier Lemma 13, we also have

$$\|q_\varepsilon(t)\|^2_{L^2(\Omega_0^f;\mathbb{R})} \leqq C\left[\|w_{\varepsilon t}(t)\|^2_{L^2(\Omega;\mathbb{R}^3)} + \|\nabla w_\varepsilon(t)\|^2_{L^2(\Omega_0^f;\mathbb{R}^9)} + \left\|\nabla \int_0^t w_\varepsilon\right\|^2_{L^2(\Omega_0^s;\mathbb{R}^9)} + N(u_0, f)^2\right],$$

which coupled with (46) and (39), gives for a choice of $\delta > 0$ small enough,

$$\|q_\varepsilon(t)\|^2_{L^2(\Omega_0^f;\mathbb{R})} \leqq \tilde{C}\left[\int_0^t \|q_\varepsilon\|^2_{L^2(\Omega_0^f;\mathbb{R})} + \|\nabla w_\varepsilon(t)\|^2_{L^2(\Omega_0^f;\mathbb{R}^9)} + N(u_0, f)^2\right] + C\int_0^t \|w_{\varepsilon t}\|^2_{L^2(\Omega;\mathbb{R}^3)}.$$

Since $\int_0^t \|\nabla w_\varepsilon(t)\|^2_{L^2(\Omega_0^f;\mathbb{R}^9)} \leqq C\, N(u_0, f)^2$, we get by Gronwall's inequality an estimate on $\int_0^t \|q_\varepsilon\|^2_{L^2(\Omega_0^f;\mathbb{R})}$ which in turn provides

$$\|q_\varepsilon(t)\|^2_{L^2(\Omega_0^f;\mathbb{R})} \leqq \tilde{C}\,[\,\|\nabla w_\varepsilon(t)\|^2_{L^2(\Omega_0^f;\mathbb{R}^9)} + N(u_0, f)^2\,] + \tilde{C}\int_0^t \|w_{\varepsilon t}\|^2_{L^2(\Omega;\mathbb{R}^3)}. \tag{47}$$

Combined with (46), still for $\delta > 0$ small enough, this also gives

$$\begin{aligned}&\|w_{\varepsilon t}(t)\|^2_{L^2(\Omega;\mathbb{R}^3)} + \|w_\varepsilon(t)\|^2_{H^1(\Omega_0^s;\mathbb{R}^3)} + \int_0^t \|w_{\varepsilon t}\|^2_{H^1(\Omega_0^f;\mathbb{R}^3)} \\ &\quad\leqq\ \tilde{C}\,N(u_0, f)^2 + \tilde{C}\,\|w_\varepsilon(t)\|^2_{H^1(\Omega_0^f;\mathbb{R}^3)} + \tilde{C}\int_0^t \|w_{\varepsilon t}\|^2_{L^2(\Omega;\mathbb{R}^3)}.\end{aligned}$$

By Gronwall and (39), we first deduce a bound on $\int_0^t \|w_{\varepsilon t}\|^2_{L^2(\Omega;\mathbb{R}^3)}$ which in turn provides us with

$$\begin{aligned}&\|w_{\varepsilon t}(t)\|^2_{L^2(\Omega;\mathbb{R}^3)} + \|w_\varepsilon(t)\|^2_{H^1(\Omega_0^s;\mathbb{R}^3)} + \int_0^t \|w_{\varepsilon t}\|^2_{H^1(\Omega_0^f;\mathbb{R}^3)} \\ &\quad\leqq\ \tilde{C}\,N(u_0, f)^2 + \tilde{C}\,\|w_\varepsilon t\|^2_{H^1(\Omega_0^f;\mathbb{R}^3)}.\end{aligned} \tag{48}$$

Step 3. An estimate of $w_{\varepsilon t}$ on $[0, T]$ which is independent of ε. By using $w_\varepsilon(t) = u_0 + \int_0^t w_{\varepsilon t}$, we see that

$$\begin{aligned}&\|w_{\varepsilon t}(t)\|^2_{L^2(\Omega;\mathbb{R}^3)} + \|w_\varepsilon\|^2_{H^1(\Omega_0^s;\mathbb{R}^3)} + \int_0^t \|w_{\varepsilon t}\|^2_{H^1(\Omega_0^f;\mathbb{R}^3)} \\ &\quad\leqq \tilde{C}N(u_0, f)^2 + \tilde{C}_1 t\int_0^t \|w_{\varepsilon t}\|^2_{H^1(\Omega_0^f;\mathbb{R}^3)} + \tilde{C}\,\|u_0\|^2_{H^1(\Omega_0^f;\mathbb{R}^3)} \\ &\quad\leqq \tilde{C}N(u_0, f)^2 + \tilde{C}_1 t\int_0^t \|w_{\varepsilon t}\|^2_{H^1(\Omega_0^f;\mathbb{R}^3)},\end{aligned}$$

where we denote by $\tilde{C}_1$ a constant, dependent on the smoothing parameter of $\tilde{a}$ (but not on ε), which will remain unchanged in the following estimates.

Now, we see that for any $0 \leqq t \leqq t_1 = \mathrm{Min}(T_M, \frac{1}{2\tilde{C}_1})$, we have

$$\|w_{\varepsilon t}(t)\|^2_{L^2(\Omega;\mathbb{R}^3)} + \|w_\varepsilon\|^2_{H^1(\Omega_0^s;\mathbb{R}^3)} + \frac{1}{2}\int_0^t \|w_{\varepsilon t}\|^2_{H^1(\Omega_0^f;\mathbb{R}^3)} \leqq \tilde{C}\,N(u_0, f)^2,$$

which with $w_\varepsilon(t_1) = u_0 + \int_0^{t_1} w_{\varepsilon t}$ gives

$$\|w_\varepsilon(t_1)\|^2_{H^1(\Omega_0^f;\mathbb{R}^3)} \leqq \tilde{C}\, N(u_0, f)^2. \tag{49}$$

Next, we take $t \geqq t_1$ and let $w_\varepsilon(t) = w_\varepsilon(t_1) + \int_{t_1}^{t} w_{\varepsilon t}$; we know from (48) and (49) that

$$\begin{aligned} \|w_{\varepsilon t}(t)\|^2_{L^2(\Omega;\mathbb{R}^3)} &+ \|w_\varepsilon\|^2_{H^1(\Omega_0^s;\mathbb{R}^3)} + \int_0^t \|w_{\varepsilon t}\|^2_{H^1(\Omega_0^f;\mathbb{R}^3)} \\ &\leqq \tilde{C}\, N(u_0, f)^2 + \tilde{C}_1\, (t - t_1) \int_{t_1}^t \|w_{\varepsilon t}\|^2_{H^1(\Omega_0^f;\mathbb{R}^3)} \\ &\quad + \tilde{C}\, \|w_\varepsilon(t_1)\|^2_{H^1(\Omega_0^f;\mathbb{R}^3)} \\ &\leqq \tilde{C}\, N(u_0, f)^2 + \tilde{C}_1\, (t - t_1) \int_0^t \|w_{\varepsilon t}\|^2_{H^1(\Omega_0^f;\mathbb{R}^3)}. \end{aligned}$$

Now, we see that for any $t_1 \leqq t \leqq 2\, t_1$, we have

$$\|w_{\varepsilon t}(t)\|^2_{L^2(\Omega;\mathbb{R}^3)} + \|w_\varepsilon\|^2_{H^1(\Omega_0^s;\mathbb{R}^3)} + \frac{1}{2} \int_0^t \|w_{\varepsilon t}\|^2_{H^1(\Omega_0^f;\mathbb{R}^3)} \leqq \tilde{C}\, N(u_0, f)^2,$$

which with $w_\varepsilon(2t_1) = u_0 + \int_0^{2t_1} w_{\varepsilon t}$ gives

$$\|w_\varepsilon(2t_1)\|^2_{H^1(\Omega_0^f;\mathbb{R}^3)} \leqq \tilde{C}\, N(u_0, f)^2.$$

We then see by an easy induction argument that for any $t \in (0, T)$,

$$\|w_{\varepsilon t}(t)\|^2_{L^2(\Omega;\mathbb{R}^3)} + \|w_\varepsilon\|^2_{H^1(\Omega_0^s;\mathbb{R}^3)} + \frac{1}{2} \int_0^t \|w_{\varepsilon t}\|^2_{H^1(\Omega_0^f;\mathbb{R}^3)} \leqq \tilde{C}\, N(u_0, f)^2. \tag{50}$$

(As is evident in the proof, the constant $\tilde{C}$ grows as T increases and thus depends on T.) Thus with (47), for all $t \in [0, T]$,

$$\|q_\varepsilon(t)\|^2_{L^2(\Omega_0^f;\mathbb{R})} \leqq \tilde{C}\, N(u_0, f)^2. \tag{51}$$

Step 4. Weak convergence and limit problem. Since w_ε also satisfies (39), we thus deduce that for the choice $\varepsilon = \frac{1}{m_l}$ there is a subsequence, still noted $w_{\frac{1}{m_l}}$, such that

$$w_{\frac{1}{m_l}} \rightharpoonup \tilde{w} \text{ in } L^2(0, T; H^1(\Omega; \mathbb{R}^3)), \tag{52a}$$

$$w_{\frac{1}{m_l}\, t} \rightharpoonup \tilde{w}_t \text{ in } L^2(0, T; L^2(\Omega; \mathbb{R}^3)), \tag{52b}$$

$$q_{\frac{1}{m_l}} \rightharpoonup \tilde{q} \text{ in } L^2(0, T; L^2(\Omega_0^f; \mathbb{R}))\, . \tag{52c}$$

By the weak convergences (52), we infer from (37) that at the limit, for each $\phi \in L^2(0,T;H^1_0(\Omega;\mathbb{R}^3))$,

$$\int_0^T (\tilde{w}_t,\phi)_{L^2(\Omega;\mathbb{R}^3)}\,dt + \nu\int_0^T (\tilde{a}^r_k\tilde{w},_r\,,\ \tilde{a}^s_k\phi,_s)_{L^2(\Omega^f_0;\mathbb{R}^3)}\,dt$$
$$+\int_0^T\left(c^{ijkl}\int_0^t \tilde{w}^k,_l\,,\phi^i,_j\right)_{L^2(\Omega^s_0;\mathbb{R})}dt - \int_0^T(\tilde{q},\ \tilde{a}^l_k\phi^k,_l)_{L^2(\Omega^f_0;\mathbb{R})}dt$$
$$=\int_0^T (F,\phi)_{L^2(\Omega^f_0;\mathbb{R}^3)} + (f,\phi)_{L^2(\Omega^s_0;\mathbb{R}^3)}\,dt. \tag{53}$$

Now for the initial condition, we notice that $\tilde{w}\in C^0([0,T];L^2(\Omega;\mathbb{R}^3))$. From the following identities which hold in $L^2(\Omega;\mathbb{R}^3)$,

$$\tilde{w}(t) = w(0) + \int_0^t \tilde{w}_t,\quad w_\varepsilon(t) = u_0 + \int_0^t w_{\varepsilon t},$$

we deduce from the weak convergence of $\int_0^\cdot w_{\varepsilon t}$ to $\int_0^\cdot \tilde{w}_t$ in $L^2(0,T;L^2(\Omega;\mathbb{R}^3))$ that $w(0)=u_0$ in $L^2(\Omega;\mathbb{R}^3)$. Combined with (53), this shows that $\tilde{w}$ is a weak solution of (27) associated with $\tilde{v}$.

Now, let us prove that the sequences in (52) in fact converge strongly.

Step 5. Strong convergence. Since $\tilde{w}\in L^2(0,T;H^1_0(\Omega;\mathbb{R}^3))$ we can use $\tilde{w}$ as a test function in (53), which provides an energy law that we can compare to (39). By using the weak convergence in (52), and the fact that $\|\tilde{a}^l_k w_\varepsilon{}^k,_l\|_{L^2(0,T;L^2(\Omega^f_0;\mathbb{R}^3))}\to 0$ as $\varepsilon\to 0$ from (39), we deduce from this comparison that for any $t\in[0,T]$, as $\varepsilon\to 0$,

$$\frac{1}{2}\|w_\varepsilon(t)\|^2_{L^2(\Omega;\mathbb{R}^3)} + \nu\int_0^t\left(\tilde{a}^r_k w_\varepsilon,_r\,,\ \tilde{a}^s_k w_\varepsilon,_s\right)_{L^2(\Omega^f_0;\mathbb{R}^3)}dt$$
$$+\frac{1}{2}\left(c^{ijkl}\int_0^t w_\varepsilon{}^k,_l\,,\int_0^t w_\varepsilon{}^i,_j\right)_{L^2(\Omega^s_0;\mathbb{R})}$$
$$\to \frac{1}{2}\|\tilde{w}(t)\|^2_{L^2(\Omega;\mathbb{R}^3)} + \nu\int_0^t(\tilde{a}^r_k\tilde{w},_r\,,\ \tilde{a}^s_k\tilde{w},_s)_{L^2(\Omega^f_0;\mathbb{R}^3)}\,dt$$
$$+\frac{1}{2}\left(c^{ijkl}\int_0^t \tilde{w}^k,_l\,,\int_0^t\tilde{w}^i,_j\right)_{L^2(\Omega^s_0;\mathbb{R})},$$

which with (52) precisely gives the strong convergence

$$w_{\frac{1}{m_l}} \to \tilde{w}\ \text{ in } L^2(0,T;H^1(\Omega^f_0;\mathbb{R}^3)), \tag{54a}$$
$$w_{\frac{1}{m_l}}(t) \to \tilde{w}(t)\ \text{ in } L^2(\Omega;\mathbb{R}^3) \text{ for any } t\in[0,T]\,, \tag{54b}$$
$$\int_0^t w_{\frac{1}{m_l}} \to \int_0^t \tilde{w}\ \text{ in } H^1(\Omega^s_0;\mathbb{R}^3) \text{ for any } t\in[0,T]\,. \tag{54c}$$

Step 6. Uniqueness. Now, to prove uniqueness, let us assume that there exists another solution w' to (27), such that $w' \in L^2(0,T;\mathcal{V}_{\tilde{v}}(\cdot))$, $w'_t \in L^2(0,T;L^2(\Omega;\mathbb{R}^3))$. By defining $\delta w = \tilde{w} - w'$, we see that $\delta w \in L^2(0,T;\mathcal{V}_v)$ is a solution of

$$\text{(i) } (\delta w_t, \phi)_{L^2(\Omega;\mathbb{R}^3)} + \nu(\tilde{a}_k^r \delta w,_r\,,\ \tilde{a}_k^s \phi,_s)_{L^2(\Omega_0^f\ \mathbb{R}^3)} + \left(c^{ijkl}\int_0^t \delta w^k,_l\,, \phi^i,_j\right)_{L^2(\Omega_0^s;\mathbb{R})} = 0\ \forall \phi \in \mathcal{V}_v(t),$$

(ii) $\delta w(0) = 0$ in Ω .

Since $\delta w(t\cdot) \in L^2(0,T;\mathcal{V}_{\tilde{v}}(\cdot))$, we can use δw as a test function in (i), which gives a.e. in $(0,T)$,

$$\frac{1}{2}\frac{d}{dt}\left[\|\delta w\|^2_{L^2(\Omega;\mathbb{R}^3)} + \left(c^{ijkl}\int_0^t \delta w^k,_l\,,\ \delta w^i,_j\right)_{L^2(\Omega_0^s;\mathbb{R})}\right] + \nu(\tilde{a}_k^r \delta w,_r\,,\ \tilde{a}_k^s \delta w,_s)_{L^2(\Omega_0^f;\mathbb{R}^3)} = 0$$

which, with the condition $\delta w(0) = 0$, precisely proves that $\delta w = 0$, establishing the uniqueness of such a solution. □

We will also need information on $\tilde{w}_{tt}$.

Theorem 4. *Let* $\tilde{w} \in L^2(0,T;\mathcal{V}_{\tilde{v}}(\cdot))$ *denote the unique weak solution of* (53)*, whose existence is ensured by Theorem 3. Then* $\tilde{w}_t \in L^2(0,T;H_0^1(\Omega;\mathbb{R}^3))$ *and* $\tilde{w}_{tt} \in \mathcal{W}([0,T])$. *Furthermore,* $q_t \in L^2(0,T;L^2(\Omega_0^f;\mathbb{R}))$.

Proof. *Step 1. Limit as* $\varepsilon \to 0$ *in* (43). In order to get an estimate independent of ε from (43), we integrate by parts in time to remove the second time derivative on $q_{\varepsilon tt}$:

$$\begin{aligned}&\int_0^t q_{\varepsilon tt}\ [2(\tilde{a}_i^j)_t w_{\varepsilon t}^i,_j + (\tilde{a}_i^j)_{tt} w_\varepsilon{}^i,_j\,] \\ &= -\int_0^t q_{\varepsilon t}\ [2(\tilde{a}_i^j)_t w_{\varepsilon t}^i,_j + (\tilde{a}_i^j)_{tt} w_\varepsilon{}^i,_j\,]_t + q_{\varepsilon t}(t)[2(\tilde{a}_i^j)_t w_{\varepsilon t}^i,_j + (\tilde{a}_i^j)_{tt} w_\varepsilon{}^i,_j\,](t) \\ &\quad - q_1[2(\tilde{a}_i^j)_t(0) w_1{}^i,_j + (\tilde{a}_i^j)_{tt}(0) u_0{}^i,_j\,],\end{aligned} \tag{55}$$

($q_{\varepsilon t}(0) = q_1 - (1/\varepsilon)[\,(\tilde{a}_i^j)_t(0)u_0^i,_j + \tilde{a}_i^j(0)w_1^i,_j\,] = q_1$ in Ω_0^f by our compatibility conditions on the initial data), from which we then infer by the regularity of $\tilde{a}$ and of w_ε that

$$\begin{aligned}&\frac{1}{2}\|w_{\varepsilon tt}\|^2_{L^2(\Omega;\mathbb{R}^3)} + \nu\int_0^t (\tilde{a}_k^r w_{\varepsilon tt},_r\,,\ \tilde{a}_k^s w_{\varepsilon tt},_s)_{L^2(\Omega_0^f;\mathbb{R}^3)} \\ &\quad + \frac{1}{2}(c^{ijkl} w_{\varepsilon t}^k,_l\,, w_{\varepsilon t}^i,_j)_{L^2(\Omega_0^s;\mathbb{R})} \leqq \tilde{C}C_\delta \int_0^t \|q_{\varepsilon t}\|^2_{L^2(\Omega_0^f;\mathbb{R})} \\ &\quad + \delta\int_0^t \|\nabla w_{\varepsilon tt}\|^2_{L^2(\Omega_0^f;\mathbb{R}^9)} + \delta\,\|q_{\varepsilon t}(t)\|^2_{L^2(\Omega_0^f;\mathbb{R})}\end{aligned}$$

$$+\tilde{C}\, C_\delta \|\nabla w_{\varepsilon t}(t)\|^2_{L^2(\Omega_0^f;\mathbb{R}^9)} + \tilde{C}\, C_\delta\, \|\nabla w_\varepsilon(t)\|^2_{L^2(\Omega_0^f;\mathbb{R}^9)} + C_\delta \tilde{C} N(u_0, f)^2$$
$$+C \int_0^t \|w_{\varepsilon tt}\|^2_{L^2(\Omega;\mathbb{R}^3)},$$

where $\delta > 0$ is arbitrary.

Thus, for δ small enough,

$$\|w_{\varepsilon tt}(t)\|^2_{L^2(\Omega;\mathbb{R}^3)} + \int_0^t \|\nabla w_{\varepsilon tt}\|^2_{L^2(\Omega_0^f;\mathbb{R}^9)} + \|\nabla w_{\varepsilon t}(t)\|^2_{L^2(\Omega_0^s;\mathbb{R}^9)}$$
$$\leqq \tilde{C} \int_0^t \|q_{\varepsilon t}\|^2_{L^2(\Omega_0^f;\mathbb{R})} + \delta\, \|q_{\varepsilon t}(t)\|^2_{L^2(\Omega_0^f;\mathbb{R})} + \tilde{C}\, C_\delta \|\nabla w_{\varepsilon t}(t)\|^2_{L^2(\Omega_0^f;\mathbb{R}^9)}$$
$$+\tilde{C}\, N(u_0, f)^2 + C \int_0^t \|w_{\varepsilon tt}\|^2_{L^2(\Omega;\mathbb{R}^3)}. \tag{56}$$

By the Lagrange multiplier Lemma 13, we also have

$$\|q_{\varepsilon t}(t)\|^2_{L^2(\Omega_0^f;\mathbb{R})} \leqq C\Big[\|w_{\varepsilon tt}(t)\|^2_{L^2(\Omega;\mathbb{R}^3)} + \|\nabla w_{\varepsilon t}(t)\|^2_{L^2(\Omega_0^f;\mathbb{R}^9)}$$
$$+\tilde{C}|\nabla w_\varepsilon(t)\|^2_{L^2(\Omega_0^f;\mathbb{R}^9)} + \tilde{C}\|q_\varepsilon(t)\|^2_{L^2(\Omega_0^f;\mathbb{R})}$$
$$+\|\nabla w_\varepsilon(t)\|^2_{L^2(\Omega_0^s;\mathbb{R}^9)} + N(u_0, f)^2\Big],$$

and thus with (50), (51) and (56) for a choice of $\delta > 0$ small enough,

$$\|q_{\varepsilon t}(t)\|^2_{L^2(\Omega_0^f;\mathbb{R})} \leqq C \left[\tilde{C} \int_0^t \|q_{\varepsilon t}\|^2_{L^2(\Omega_0^f;\mathbb{R})} + \tilde{C}\, \|\nabla w_{\varepsilon t}(t)\|^2_{L^2(\Omega_0^f;\mathbb{R}^9)} \right.$$
$$+\|\nabla w_\varepsilon(t)\|^2_{L^2(\Omega_0^s;\mathbb{R}^9)} + \tilde{C}\, N(u_0, f)^2$$
$$\left. +C \int_0^t \|w_{\varepsilon tt}\|^2_{L^2(\Omega;\mathbb{R}^3)} \right],$$

which by Gronwall's inequality and (50), gives

$$\int_0^t \|q_{\varepsilon t}\|^2_{L^2(\Omega_0^f;\mathbb{R})} \leqq \tilde{C} \left[N(u_0, f)^2 + \int_0^t \|w_{\varepsilon tt}\|^2_{L^2(\Omega;\mathbb{R}^3)} \right],$$

and thus,

$$\|q_{\varepsilon t}(t)\|^2_{L^2(\Omega_0^f;\mathbb{R})} \leqq \tilde{C} \left[N(u_0, f)^2 + \|\nabla w_{\varepsilon t} t\|^2_{L^2(\Omega_0^f;\mathbb{R}^9)} + \int_0^t \|w_{\varepsilon tt}\|^2_{L^2(\Omega;\mathbb{R}^3)} \right]. \tag{57}$$

We then infer from (56) that

$$\|w_{\varepsilon tt}(t)\|^2_{L^2(\Omega;\mathbb{R}^3)} + \int_0^t \|\nabla w_{\varepsilon tt}\|^2_{L^2(\Omega_0^f;\mathbb{R}^9)} + \|\nabla w_{\varepsilon t}(t)\|^2_{L^2(\Omega_0^s;\mathbb{R}^9)}$$
$$\leqq \tilde{C} \left[\|\nabla w_{\varepsilon t}(t)\|^2_{L^2(\Omega_0^f;\mathbb{R}^9)} + N(u_0, f)^2 + \int_0^t \|w_{\varepsilon tt}\|^2_{L^2(\Omega;\mathbb{R}^3)} \right].$$

By the Gronwall inequality and (50), we first obtain an estimate for the term $\int_0^t \|w_{\varepsilon tt}(t)\|^2_{L^2(\Omega;\mathbb{R}^3)}$ which implies in turn that

$$\|w_{\varepsilon tt}(t)\|^2_{L^2(\Omega;\mathbb{R}^3)} + \int_0^t \|\nabla w_{\varepsilon tt}\|^2_{L^2(\Omega_0^f;\mathbb{R}^9)} + \|\nabla w_{\varepsilon t}(t)\|^2_{L^2(\Omega_0^s;\mathbb{R}^9)} \leqq \tilde{C}\ \|\nabla w_{\varepsilon t}(t)\|^2_{L^2(\Omega_0^f;\mathbb{R}^9)} + \tilde{C}\ N(u_0, f)^2.$$

Step 2. ε-independent estimate for $w_{\varepsilon tt}$ on $[0, T]$. In the same fashion as we derived (50) from (48), we can deduce that for all $t \in [0, T]$,

$$\|w_{\varepsilon tt}t\|^2_{L^2(\Omega;\mathbb{R}^3)} + \int_0^t \|\nabla w_{\varepsilon tt}\|^2_{L^2(\Omega_0^f;\mathbb{R}^9)} + \|\nabla w_{\varepsilon t}t\|^2_{L^2(\Omega_0^s;\mathbb{R}^9)} \leqq \tilde{C}N(u_0, f)^2. \tag{58}$$

From (57) and (58) we then infer

$$\|q_{\varepsilon t}(t)\|^2_{L^2(\Omega_0^f;\mathbb{R})} \leqq \tilde{C}\ N(u_0, f)^2. \tag{59}$$

We thus deduce that for the choice $\varepsilon = \frac{1}{m_l}$ there is a subsequence, still denoted $w_{\frac{1}{m_l}}$, such that

$$w_{\frac{1}{m_l}} \rightharpoonup \tilde{w} \text{ in } L^2(0, T; H^1(\Omega; \mathbb{R}^3)), \tag{60a}$$

$$w_{\frac{1}{m_l}t} \rightharpoonup \tilde{w}_t \text{ in } L^2(0, T; H^1(\Omega; \mathbb{R}^3)), \tag{60b}$$

$$w_{\frac{1}{m_l}tt} \rightharpoonup \tilde{w}_{tt} \text{ in } L^2(0, T; L^2(\Omega; \mathbb{R}^3)) \text{ and in } L^2(0, T; H^1(\Omega_0^f; \mathbb{R}^3)), \tag{60c}$$

$$q_{\frac{1}{m_l}} \rightharpoonup \tilde{q} \text{ in } L^2(0, T; L^2(\Omega_0^f; \mathbb{R})), \tag{60d}$$

$$q_{\frac{1}{m_l}t} \rightharpoonup \tilde{q}_t \text{ in } L^2(0, T; L^2(\Omega_0^f; \mathbb{R})). \tag{60e}$$

Step 3. Initial condition for $\tilde{w}_t$. By the weak convergence in (60), we infer from (38) that for each test function $\phi \in L^2(0, T; H^1_0(\Omega; \mathbb{R}^3))$,

$$\int_0^T (\tilde{w}_{tt}, \phi)_{L^2(\Omega;\mathbb{R}^3)}\, dt + \nu \int_0^T ((\tilde{a}^r_k \tilde{a}^s_k \tilde{w},_r)_t,\ \phi,_s)_{L^2(\Omega_0^f;\mathbb{R}^3)}\, dt + \int_0^T (c^{ijkl}\tilde{w}^k,_l, \phi^i,_j)_{L^2(\Omega_0^s;\mathbb{R})}\, dt - \int_0^T ((\tilde{a}^j_i \tilde{q})_t,\ \phi^i,_j)_{L^2(\Omega_0^f;\mathbb{R})} dt = \int_0^T (F_t, \phi)_{L^2(\Omega_0^f;\mathbb{R}^3)} + (f_t, \phi)_{L^2(\Omega_0^s;\mathbb{R}^3)}\, dt\ . \tag{61}$$

Now for the initial condition, we notice that $\tilde{w}_t \in C^0([0, T]; L^2(\Omega; \mathbb{R}^3))$. From the following identities which hold in $L^2(\Omega; \mathbb{R}^3)$, we find that

$$\tilde{w}_t(t) = \tilde{w}_t(0) + \int_0^t \tilde{w}_{tt}, \quad w_\varepsilon(t) = w_1 + \int_0^t w_{\varepsilon tt}.$$

Since $\int_0^\cdot w_{\varepsilon tt} \rightharpoonup \int_0^\cdot \tilde{w}_{tt}$ in $L^2(0, T; L^2(\Omega; \mathbb{R}^3))$ weakly, we deduce that $\tilde{w}_t(0) = w_1$ in $L^2(\Omega; \mathbb{R}^3)$.

Step 4. Strong convergence: the easy cases. We will also need the fact that the weak convergence in (60) is in fact strong. Notice that since $\tilde{w}_t \in L^2(0, T; H_0^1(\Omega; \mathbb{R}^3))$, we can use $\tilde{w}_t$ in (61) to get for any $t \in [0, T]$,

$$
\begin{aligned}
&\frac{1}{2}\|\tilde{w}_t(t)\|^2_{L^2(\Omega;\mathbb{R}^3)} + \nu \int_0^t ((\tilde{a}^r_k \tilde{a}^s_k \tilde{w},_r)_t,\ \tilde{w}_t,_s)_{L^2(\Omega_0^f;\mathbb{R}^3)} \\
&\quad - \int_0^t ((\tilde{a}^j_i \tilde{q})_t,\ \tilde{w}^i_t,_j)_{L^2(\Omega_0^f;\mathbb{R})} + \frac{1}{2}(c^{ijkl}\tilde{w}^k,_l(t), \tilde{w}^i,_j(t))_{L^2(\Omega_0^s;\mathbb{R})} \\
&= \frac{1}{2}\|w_1\|^2_{L^2(\Omega;\mathbb{R}^3)} + \frac{1}{2}(c^{ijkl}u^k_0,_l, u_0{}^i,_j)_{L^2(\Omega_0^s;\mathbb{R})} \\
&\quad + \int_0^t (F_t, \tilde{w}_t)_{L^2(\Omega_0^f;\mathbb{R}^3)} + (f_t, \tilde{w}_t)_{L^2(\Omega_0^s;\mathbb{R}^3)}\, dt.
\end{aligned}
$$

Since $\tilde{w}(t) \in \mathcal{V}_{\tilde{v}}(t)$ in $[0, T]$ implies $(\tilde{a}^j_i)_t \tilde{w}^i,_j = -\tilde{a}^j_i \tilde{w}^i_t,_j$, we then deduce that

$$
\begin{aligned}
&\frac{1}{2}\|\tilde{w}_t(t)\|^2_{L^2(\Omega;\mathbb{R}^3)} + \nu \int_0^t (\tilde{a}^r_k \tilde{a}^s_k \tilde{w}_t,_r,\ \tilde{w}_t,_s)_{L^2(\Omega_0^f;\mathbb{R}^3)} \\
&\quad +\nu \int_0^t ((\tilde{a}^r_k \tilde{a}^s_k)_t \tilde{w},_r,\ \tilde{w}_t,_s)_{L^2(\Omega_0^f;\mathbb{R}^3)} - \int_0^t ((\tilde{a}^j_i)_t \tilde{q},\ \tilde{w}^i_t,_j)_{L^2(\Omega_0^f;\mathbb{R})} \\
&\quad + \int_0^t ((\tilde{a}^j_i)_t \tilde{q}_t,\ \tilde{w}^i,_j)_{L^2(\Omega_0^f;\mathbb{R})} + \frac{1}{2}(c^{ijkl}\tilde{w}^k,_l(t), \tilde{w}^i,_j(t))_{L^2(\Omega_0^s;\mathbb{R})} \\
&= \frac{1}{2}\|w_1\|^2_{L^2(\Omega;\mathbb{R}^3)} + \frac{1}{2}(c^{ijkl}u^k_0,_l, u_0{}^i,_j)_{L^2(\Omega_0^s;\mathbb{R})} \\
&\quad + \int_0^t (F_t, \tilde{w}_t)_{L^2(\Omega_0^f;\mathbb{R}^3)} + (f_t, \tilde{w}_t)_{L^2(\Omega_0^s;\mathbb{R}^3)}\, dt. \qquad (62)
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
&\frac{1}{2}\|w_{\varepsilon t}(t)\|^2_{L^2(\Omega;\mathbb{R}^3)} + \nu \int_0^t ((\tilde{a}^r_k \tilde{a}^s_k w_\varepsilon,_r)_t,\ w_\varepsilon,_s)_{L^2(\Omega_0^f;\mathbb{R}^3)} \\
&\quad - \int_0^t ((\tilde{a}^j_i q_\varepsilon)_t,\ w_{\varepsilon t}^i,_j)_{L^2(\Omega_0^f;\mathbb{R})} + \frac{1}{2}(c^{ijkl} w_\varepsilon{}^k,_l(t), w_\varepsilon{}^i,_j(t))_{L^2(\Omega_0^s;\mathbb{R})} \\
&= \frac{1}{2}\|w_1\|^2_{L^2(\Omega;\mathbb{R}^3)} + \frac{1}{2}(c^{ijkl}u^k_0,_l, u_0{}^i,_j)_{L^2(\Omega_0^s;\mathbb{R})} \\
&\quad + \int_0^t (F_t, w_{\varepsilon t})_{L^2(\Omega_0^f;\mathbb{R}^3)} + (f_t, w_{\varepsilon t})_{L^2(\Omega_0^s;\mathbb{R}^3)}\, dt\ .
\end{aligned}
$$

From the definition of q_ε, $(\tilde{a}^j_i)_t w_\varepsilon{}^i,_j = -\tilde{a}^j_i w_{\varepsilon t}^i,_j - \varepsilon\,(q_{\varepsilon t} - q_1)$, and thus

$$
\begin{aligned}
&\frac{1}{2}\|w_{\varepsilon t}(t)\|^2_{L^2(\Omega;\mathbb{R}^3)} + \nu \int_0^t (\tilde{a}^r_k \tilde{a}^s_k w_{\varepsilon t},_r,\ w_{\varepsilon t},_s)_{L^2(\Omega_0^f;\mathbb{R}^3)} \\
&\quad +\nu \int_0^t ((\tilde{a}^r_k \tilde{a}^s_k)_t w_\varepsilon,_r,\ w_{\varepsilon t},_s)_{L^2(\Omega_0^f;\mathbb{R}^3)} - \int_0^t ((\tilde{a}^j_i)_t q_\varepsilon,\ w_{\varepsilon t}^i,_j)_{L^2(\Omega_0^f;\mathbb{R})} \\
&\quad + \int_0^t ((\tilde{a}^j_i)_t q_{\varepsilon t},\ w_\varepsilon{}^i,_j)_{L^2(\Omega_0^f;\mathbb{R})} + \varepsilon \int_0^t (q_{\varepsilon t},\ q_{\varepsilon t} - q_1)_{L^2(\Omega_0^f;\mathbb{R})}
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{1}{2}(c^{ijkl}w_\varepsilon{}^k{},_l(t),\ w_\varepsilon{}^i{},_j(t))_{L^2(\Omega_0^s;\mathbb{R})}\\
&=\frac{1}{2}\|w_1\|^2_{L^2(\Omega;\mathbb{R}^3)}+\frac{1}{2}(c^{ijkl}u_{0,l}^k,u_0{}^i{},_j)_{L^2(\Omega_0^s;\mathbb{R})}\\
&+\int_0^t(F_t,w_{\varepsilon t})_{L^2(\Omega_0^f;\mathbb{R}^3)}+\int_0^t(f_t,w_{\varepsilon t})_{L^2(\Omega_0^s;\mathbb{R}^3)}. \qquad (63)
\end{aligned}
$$

By integration by parts, since $q_\varepsilon(0)=q_0$,

$$
\begin{aligned}
\int_0^t((\tilde a_i^j)_t q_\varepsilon,\ w_{\varepsilon t}{}^i{},_j)_{L^2(\Omega_0^f;\mathbb{R})}\,dt=&-\int_0^t(((\tilde a_i^j)_t q_\varepsilon)_t,\ w_\varepsilon{}^i{},_j)_{L^2(\Omega_0^f;\mathbb{R})}\,dt\\
&+((\tilde a_i^j)_t q_\varepsilon(t),\ w_\varepsilon{}^i{},_j(t))_{L^2(\Omega_0^f;\mathbb{R})}\\
&-((\tilde a_i^j)_t(0)q_0,\ u_0{}^i{},_j)_{L^2(\Omega_0^f;\mathbb{R})}. \qquad (64)
\end{aligned}
$$

Now, since $q_\varepsilon(t)=q_0+\int_0^t q_{\varepsilon t}\,dt$ we deduce that for any $t\in[0,T]$, $q_\varepsilon(t)\rightharpoonup q_0+\int_0^t\tilde q_t\,dt$ in $L^2(\Omega_0^f;\mathbb{R})$, which proves that $\tilde q(t)=q_0+\int_0^t\tilde q_t$, and thus that $\tilde q\in C^0([0,T];L^2(\Omega_0^f;\mathbb{R}))$, with $\tilde q(0)=q_0$. Furthermore, for any $t\in[0,T]$,

$$
q_\varepsilon(t)\rightharpoonup\tilde q(t)\text{ in }L^2(\Omega_0^f;\mathbb{R})\text{ as }\varepsilon\to 0. \qquad (65)
$$

Similarly, since $w_\varepsilon(0)=\tilde w(0)=u_0$, we have for any $t\in[0,T]$,

$$
w_\varepsilon(t)\rightharpoonup\tilde w(t)\text{ in }H^1(\Omega_0^f;\mathbb{R}^3)\text{ as }\varepsilon\to 0\,.
$$

Moreover from

$$
\|w_\varepsilon(t)\|^2_{H^1(\Omega_0^f;\mathbb{R}^3)}=\|u_0\|^2_{H^1(\Omega_0^f;\mathbb{R}^3)}+2\int_0^t(w_{\varepsilon t},\ w_\varepsilon)_{H^1(\Omega_0^f;\mathbb{R}^3)}\,dt,
$$

we infer from the strong convergence in (54) and the weak convergence in (60) that for any $t\in[0,T]$,

$$
\|w_\varepsilon(t)\|^2_{H^1(\Omega_0^f;\mathbb{R}^3)}\to\|u_0\|^2_{H^1(\Omega_0^f;\mathbb{R}^3)}+2\int_0^t(\tilde w_t,\tilde w)_{H^1(\Omega_0^f;\mathbb{R}^3)}\,dt\text{ as }\varepsilon=\frac{1}{m_l}\to 0,
$$

from which we obtain the strong convergence

$$
w_\varepsilon(t)\to\tilde w(t)\text{ in }H^1(\Omega_0^f;\mathbb{R}^3)\text{ as }\varepsilon=\frac{1}{m_l}\to 0. \qquad (66)
$$

Thus, from (64), the strong convergence in (54) and (66) together with the weak convergence in (60) and (65) shows that

$$
\int_0^t((\tilde a_i^j)_t q_\varepsilon,w_{\varepsilon t}{}^i{},_j)_{L^2(\Omega_0^f;\mathbb{R})}\,dt\to\int_0^t((\tilde a_i^j)_t\tilde q,\tilde w_t^i{},_j)_{L^2(\Omega_0^f;\mathbb{R})}dt\text{ as }\varepsilon=\frac{1}{m_l}\to 0. \qquad (67)
$$

From (67), the weak convergence in (60) and the strong convergence in (54), we then deduce from (62) and (63), that as $\varepsilon = \frac{1}{m_l} \to 0$, for any $t \in [0, T]$,

$$\begin{aligned}
&\frac{1}{2}\|w_{\varepsilon t}(t)\|_{L^2(\Omega;\mathbb{R}^3)} + \nu \int_0^t (\tilde{a}_k^r \tilde{a}_k^s w_{\varepsilon t},_r\,,\ w_{\varepsilon t},_s\,)_{L^2(\Omega_0^f;\mathbb{R}^3)}\, dt \\
&\quad + \frac{1}{2}(c^{ijkl} w_\varepsilon{}^k,_l(t),\ w_\varepsilon{}^i,_j(t))_{L^2(\Omega_0^s;\mathbb{R})} \\
&\to \frac{1}{2}\|\tilde{w}_t(t)\|_{L^2(\Omega;\mathbb{R}^3)} + \nu \int_0^t (\tilde{a}_k^r \tilde{a}_k^s \tilde{w}_t,_r\,,\ \tilde{w}_t,_s\,)_{L^2(\Omega_0^f;\mathbb{R}^3)}\, dt \\
&\quad + \frac{1}{2}(c^{ijkl} \tilde{w}^k,_l(t),\ \tilde{w}^i,_j(t))_{L^2(\Omega_0^s;\mathbb{R})}\ ,
\end{aligned}$$

which implies the strong convergences

$$w_{\frac{1}{m_l'}}(t) \to \tilde{w}(t) \ \text{ in } H^1(\Omega;\mathbb{R}^3) \text{ for any } t \in [0,T], \tag{68a}$$

$$w_{\frac{1}{m_l'}t} \to \tilde{w}_t \ \text{ in } L^2(0,T;H^1(\Omega_0^f;\mathbb{R}^3)), \tag{68b}$$

$$w_{\frac{1}{m_l'}t}(t) \to \tilde{w}_t(t) \ \text{ in } L^2(\Omega;\mathbb{R}^3) \text{ for any } t \in [0,T]. \tag{68c}$$

From the strong convergence in (68) and Lemma 13, we also deduce that

$$\|q_\varepsilon - \tilde{q}\|_{L^2(0,T;L^2(\Omega_0^f;\mathbb{R}))} \to 0, \ \text{ as } \varepsilon \to 0\ . \tag{69}$$

Step 5. Strong convergence: the more delicate case of $\tilde{w}_{tt}$. Our main difficulty results from the fact that we cannot directly obtain an energy inequality for w_{tt} (from the limiting weak form of the twice time-differentiated problem). Rather, our starting point will be (43), from which we will get by weak lower semi-continuity the desired inequality, provided that we can prove that $w_{\varepsilon tt} \to \tilde{w}_{tt}$ in $L^2(0,T;L^2(\Omega_0^f;\mathbb{R}^3))$. To prove this result, let us first remind the reader that $w_{\varepsilon tt}$ satisfies, for all $\phi \in L^2(0,T;H_0^1(\Omega;\mathbb{R}^3))$,

$$\begin{aligned}
&\int_0^T (w_{\varepsilon ttt}, \phi)_{L^2(\Omega;\mathbb{R}^3)}\, dt + \nu \int_0^T ((\tilde{a}_k^s \tilde{a}_k^r w_\varepsilon,_r)_{tt},\ \phi,_s\,)_{L^2(\Omega_0^f;\mathbb{R}^3)}\, dt \\
&\quad + \int_0^T (c^{ijkl} w_{\varepsilon t}^k,_l\,, \phi^i,_j\,)_{L^2(\Omega_0^s;\mathbb{R})}\, dt - \int_0^T ((\tilde{a}_k^l q_\varepsilon)_{tt},\ \phi^k,_l\,)_{L^2(\Omega_0^f;\mathbb{R})} dt \\
&= \int_0^T (F_{tt}, \phi)_{L^2(\Omega_0^f;\mathbb{R}^3)} + (f_{tt}, \phi)_{L^2(\Omega_0^s;\mathbb{R}^3)}\, dt\ .
\end{aligned} \tag{70}$$

From the bounds associated with the weak convergence in (60), we then see that

$$\int_0^T \|w_{\varepsilon ttt}(t)\|^2_{\mathcal{V}_{\tilde{v}}(t)'}\, dt \leqq \check{C}, \tag{71}$$

where $\check{C}$ denotes a constant which depends on the data, the smoothing parameter implicit in $\tilde{a}$, but not on the penalization parameter ε. In the following, this letter will denote a generic constant depending on these variables. Let us fix $\delta > 0$, and let

$$\Omega_\delta^f = \{x \in \Omega_0^f \,|\, \mathrm{dist}(x, \Gamma_0) \geqq \delta,\ \mathrm{dist}(x, \partial\Omega) \geqq \delta\}.$$

Let us then denote on $[0, T]$,

$$\tilde{\Omega}_\delta(t) = \tilde{\eta}(t, \Omega_\delta^f).$$

For each $t \in (0, T)$, we have the existence of $\delta t > 0$ such that

$$\forall t' \in (t - \delta t, t + \delta t),\ \tilde{\Omega}_{2\delta}(t) \subset \tilde{\Omega}_\delta(t').$$

By a simple change of variables and (71), we then get

$$\left\| \det \tilde{a}\, w_{\varepsilon ttt} \circ \tilde{\eta}^{-1} \right\|_{L^2(t-\delta t, t+\delta t; H^1_{0,\mathrm{div}}(\tilde{\Omega}_{2\delta}(t);\mathbb{R}^3)')} \leqq \check{C}\,. \tag{72}$$

We set

$$u_\varepsilon = \det \tilde{a}\, w_{\varepsilon tt} \circ \tilde{\eta}^{-1}.$$

From (72) and (60), we then get

$$\|u_{\varepsilon t}\|_{L^2(t-\delta t, t+\delta t; H^1_{0,\mathrm{div}}(\tilde{\Omega}_{2\delta}(t);\mathbb{R}^3)')} \leqq \check{C},$$
$$\|u_\varepsilon\|_{L^2(t-\delta t, t+\delta t; H^1(\tilde{\Omega}_{2\delta}(t);\mathbb{R}^3))} \leqq \check{C}.$$

Thus, from the classical compactness results (since the domain $\tilde{\Omega}_{2\delta}(t)$ is fixed on $(t - \delta t, t + \delta t)$),

$$u_\varepsilon \to u = \det \tilde{a}\, w_{tt} \circ \tilde{\eta}^{-1} \text{ in } L^2(t - \delta t, t + \delta t; L^2(\tilde{\Omega}_{2\delta}(t); \mathbb{R}^3)),$$

which obviously gives (since on $(t - \delta t, t + \delta t)$, $\tilde{\eta}^{-1}(t', \tilde{\Omega}_{2\delta}(t)) \subset \Omega_\delta^f$),

$$w_{\varepsilon tt} \to w_{tt} \text{ in } L^2(t - \delta t, t + \delta t; L^2(\Omega_\delta^f; \mathbb{R}^3)).$$

By a finite covering argument, and the L^∞ bound (58) of $w_{\varepsilon tt}$ in $L^2(\Omega; \mathbb{R}^3)$, we then infer that

$$\limsup \int_0^T \|w_{\varepsilon tt}\|^2_{L^2(\Omega_\delta^f;\mathbb{R}^3)}\, dt = \int_0^T \|\tilde{w}_{tt}\|^2_{L^2(\Omega_\delta^f;\mathbb{R}^3)}\, dt. \tag{73}$$

Successively from the Cauchy-Schwarz and Sobolev inequalities,

$$\begin{aligned}\|w_{\varepsilon tt}\|^2_{L^2(0,T;L^2(\Omega_0^f \cap \Omega_\delta^{fc};\mathbb{R}^3))} &\leqq C\, |\Omega_0^f \cap \Omega_\delta^{fc}|\, \|\tilde{w}_{tt}\|^2_{L^2(0,T;H^1(\Omega_0^f \cap \Omega_\delta^{fc};\mathbb{R}^3))} \\ &\leqq \tilde{C}\, |\Omega_0^f \cap \Omega_\delta^{fc}|\, N(u_0, f)^2. \end{aligned} \tag{74}$$

From (73) and (74), we then infer

$$\begin{aligned}&\limsup \int_0^T \|w_{\varepsilon tt}\|^2_{L^2(\Omega_0^f;\mathbb{R}^3)}\, dt \\ &\quad \leqq \int_0^T \|\tilde{w}_{tt}\|^2_{L^2(\Omega_0^f;\mathbb{R}^3)}\, dt + \tilde{C}\, |\Omega_0^f \cap \Omega_\delta^{fc}|\, N(u_0, f)^2,\end{aligned}$$

which immediately shows that

$$\limsup \int_0^T \|w_{\varepsilon tt}\|^2_{L^2(\Omega_0^f;\mathbb{R}^3)}\, dt \leqq \int_0^T \|\tilde{w}_{tt}\|^2_{L^2(\Omega_0^f;\mathbb{R}^3)}\, dt,$$

thus establishing the strong convergence as $\varepsilon = \frac{1}{m_l} \to 0$,

$$w_{\frac{1}{m_l}\, tt} \to \tilde{w}_{tt} \text{ in } L^2(0,T;L^2(\Omega_0^f;\mathbb{R}^3)). \tag{75}$$

Next, we restrict our test function ϕ to be in the space $\{\phi \in \mathcal{V}_{\tilde{v}}(t) \mid \phi = 0 \text{ on } \overline{\Omega_0^s}\}$. For all such test functions and for a.e $t \in (0,T)$, $\phi \in \mathcal{V}_{\tilde{v}}(t)$,

$$\begin{aligned}(\tilde{w}_{tt}(t) - w_{\varepsilon tt}(t), \phi)_{L^2(\Omega;\mathbb{R}^3)} &+ \nu((\tilde{a}_k^r\tilde{a}_k^s(\tilde{w} - w_{\varepsilon},_r))_t(t),\ \phi,_s)_{L^2(\Omega_0^f;\mathbb{R}^3)} \\ &-((\tilde{a}_i^j)_t(\tilde{q} - q_\varepsilon)(t),\ \phi^i,_j)_{L^2(\Omega_0^f;\mathbb{R})} = 0\,,\end{aligned}$$

thus, the second Lagrange multiplier Lemma 15 ensures, from the strong convergence in (75), (69) and (68c), that

$$\|(\bar{q}_{\frac{1}{m_l}})_t - \bar{q}_t\|_{L^2(0,T;L^2(\Omega_0^f;\mathbb{R}))} \to 0\,, \tag{76}$$

where

$$\bar{q}_t = \tilde{q}_t - \frac{1}{|\Omega_0^f|}\int_{\Omega_0^f} \tilde{q}_t\ \det\nabla\tilde{\eta},$$

and a similar definition for $\bar{q}_\varepsilon$. In the following, we will denote

$$c = \frac{1}{|\Omega_0^f|}\int_{\Omega_0^f} \tilde{q}_t\ \det\nabla\tilde{\eta} \text{ and } c_\varepsilon = \frac{1}{|\Omega_0^f|}\int_{\Omega_0^f} q_{\varepsilon t}\ \det\nabla\tilde{\eta}.$$

Step 6. An inequality for $\tilde{w}_{tt}$ with a constant independent of the mollification parameter. Now, from the weak convergence (60) and the compactness of the trace operator, we then infer that as $\varepsilon = \frac{1}{m_l} \to 0$,

$$\|(\tilde{w} - w_\varepsilon)_{tt}\|^2_{L^2(0,T;L^2(\Gamma_0;\mathbb{R}^3))} \to 0. \tag{77}$$

We now note that from (43) and (55), for any $0 < t < T$, and $0 < \delta t < \text{Min}(t,\ T-t)$,

$$\begin{aligned}\frac{1}{2}\int_{t-\delta t}^{t+\delta t} \|w_{\varepsilon tt}\|^2_{L^2(\Omega;\mathbb{R}^3)} &+ \nu\int_{t-\delta t}^{t+\delta t}\int_0^{t'} (\tilde{a}_k^r w_{\varepsilon tt},_r\,,\ \tilde{a}_k^s w_{\varepsilon tt},_s)_{L^2(\Omega_0^f;\mathbb{R}^3)} \\ &+\frac{1}{2}\int_{t-\delta t}^{t+\delta t} (c^{ijkl} w_{\varepsilon t}{}^k,_l\,, w_{\varepsilon t}{}^i,_j)_{L^2(\Omega_0^s;\mathbb{R})} \\ &-\int_{t-\delta t}^{t+\delta t}\int_0^{t'}\int_{\Omega_0^f} q_{\varepsilon t}[2(\tilde{a}_i^j)_t w_{\varepsilon t}^i,_j + (\tilde{a}_i^j)_{tt} w_\varepsilon{}^i,_j]_t \\ &+\int_{t-\delta t}^{t+\delta t}\int_{\Omega_0^f} q_{\varepsilon t}\ [2\ (\tilde{a}_i^j)_t w_{\varepsilon t}^i,_j + (\tilde{a}_i^j)_{tt} w_\varepsilon{}^i,_j\,]\end{aligned}$$

$$
\begin{aligned}
&-2\int_{t-\delta t}^{t+\delta t}\int_0^{t'}\int_{\Omega_0^f}(\tilde a_i^j)_t q_{\varepsilon t} w_{\varepsilon tt}^i,_j - \int_{t-\delta t}^{t+\delta t}\int_0^{t'}\int_{\Omega_0^f}(\tilde a_i^j)_{tt} q_\varepsilon w_{\varepsilon tt}^i,_j \\
&+\nu\int_{t-\delta t}^{t+\delta t}\int_0^{t'}((\tilde a_k^r\tilde a_k^s)_{tt} w_\varepsilon,_r\,,\ w_{\varepsilon tt},_s)_{L^2(\Omega_0^f;\mathbb{R}^3)} \\
&+2\,\nu\int_{t-\delta t}^{t+\delta t}\int_0^{t'}((\tilde a_k^r\tilde a_k^s)_t w_{\varepsilon t},_r\,,\ w_{\varepsilon tt},_s)_{L^2(\Omega_0^f;\mathbb{R}^3)} \\
\leqq\ & C\ \delta t\ N(u_0, f)^2 + \int_{t-\delta t}^{t+\delta t}\int_0^{t'}(F_{tt}, w_{\varepsilon tt})_{L^2(\Omega_0^f;\mathbb{R}^3)} \\
&+\int_{t-\delta t}^{t+\delta t}\int_0^{t'}(f_{tt}, w_{\varepsilon tt})_{L^2(\Omega_0^s;\mathbb{R}^3)}.
\end{aligned}
\tag{78}
$$

The first three terms of the left-hand side of this inequality will be dealt with by weak lower semi-continuity. By the weak convergence in (60) and the strong convergence in (54), (68c), (66), and (69), we infer that all of the remaining terms, other than the term $\int_{t-\delta t}^{t+\delta t}\int_0^{t'}\int_{\Omega_0^f}(\tilde a_i^j)_t q_{\varepsilon t} w_{\varepsilon tt}^i,_j$, converge as $\varepsilon \to 0$ to the same expressions with the limits $\tilde w$ and $\tilde q$ replacing w_ε and q_ε. From the definitions of c and c_ε, we have

$$
\begin{aligned}
&\int_{t-\delta t}^{t+\delta t}\int_0^{t'}\int_{\Omega_0^f}(\tilde a_i^j)_t q_{\varepsilon t} w_{\varepsilon tt}^i,_j \\
&\quad = \int_{t-\delta t}^{t+\delta t}\int_0^{t'}\left[\int_{\Omega_0^f}(\tilde a_i^j)_t(\bar q_\varepsilon)_t w_{\varepsilon tt}^i,_j + c_\varepsilon\int_{\Omega_0^f}(\tilde a_i^j)_t w_{\varepsilon tt}^i,_j\right].
\end{aligned}
$$

From the strong convergence (76) and the weak convergence (60), we then deduce that the first term of the right-hand side of this inequality converges as $\varepsilon = \frac{1}{m_l} \to 0$ to the corresponding term where $\bar q_t$ replaces $q_{\varepsilon t}$ and $\tilde w_{tt}$ replaces $w_{\varepsilon tt}$.

For the second term of this right-hand side, we notice from a spatial integration by parts (since c_ε depends only on the time variable) that

$$
\begin{aligned}
&\int_{t-\delta t}^{t+\delta t}\int_0^{t'} c_\varepsilon\int_{\Omega_0^f}(\tilde a_i^j)_t w_{\varepsilon tt}^i,_j \\
&\quad = \int_{t-\delta t}^{t+\delta t}\int_0^{t'} c_\varepsilon\left[-\int_{\Omega_0^f}((\tilde a_i^j)_t),_j\, w_{\varepsilon tt}^i + \int_{\Gamma_0}(\tilde a_i^j)_t w_{\varepsilon tt}^i N_j\right]
\end{aligned}
$$

and thus from the weak convergence in (60) and the strong convergence in (75) and (77) we then get the convergence as $\varepsilon = \frac{1}{m_l} \to 0$ to the corresponding term where c replaces c_ε and $\tilde w_{tt}$ replaces $w_{\varepsilon tt}$. This implies that as $\varepsilon = \frac{1}{m_l} \to 0$,

$$
\int_{t-\delta t}^{t+\delta t}\int_0^{t'}\int_{\Omega_0^f}(\tilde a_i^j)_t q_{\varepsilon t} w_{\varepsilon tt}^i,_j \to \int_{t-\delta t}^{t+\delta t}\int_0^{t'}\int_{\Omega_0^f}(\tilde a_i^j)_t \tilde q_t \tilde w_{tt}^i,_j\,.
$$

Consequently, all the terms, except the three first ones, appearing in the inequality (78) are convergent as $\varepsilon = \frac{1}{m_l} \to 0$ to the same expression, where q_ε and w_ε are replaced respectively by $\tilde q$ and $\tilde w$.

By weak lower semi-continuity for the first three integrals, we then infer that as $\varepsilon = \frac{1}{m_l} \to 0$ the same inequality as the previous one holds with $\tilde{w}$ and $\tilde{q}$ replacing respectively w_ε and q_ε. By dividing those integrals by $2\delta t$ and passing to the limit as $\delta t \to 0$ (which is possible a.e. in $(0,T)$), we then find that a.e. in $(0,T)$,

$$\begin{aligned}
&\frac{1}{2}\|\tilde{w}_{tt}(t)\|^2_{L^2(\Omega;\mathbb{R}^3)} + \nu \int_0^t (\tilde{a}^r_k \tilde{w}_{tt},_r \,,\ \tilde{a}^s_k \tilde{w}_{tt},_s)_{L^2(\Omega^f_0;\mathbb{R}^3)} \\
&\quad + \frac{1}{2}(c^{ijkl}\tilde{w}^k_t,_l(t), \tilde{w}^i_t,_j(t))_{L^2(\Omega^s_0;\mathbb{R})} - \int_0^t \int_{\Omega^f_0} \tilde{q}_t[2(\tilde{a}^j_i)_t \tilde{w}^i_t,_j + (\tilde{a}^j_i)_{tt}\tilde{w}^i,_j]_t \\
&\quad + \int_{\Omega^f_0} \tilde{q}_t(t)[2(\tilde{a}^j_i)_t \tilde{w}^i_t,_j + (\tilde{a}^j_i)_{tt}\tilde{w}^i,_j](t) - 2\int_0^t \int_{\Omega^f_0} (\tilde{a}^j_i)_t \tilde{q}_t \tilde{w}^i_{tt},_j \\
&\quad + \nu \int_0^t ((\tilde{a}^r_k \tilde{a}^s_k)_{tt} \tilde{w},_r \,,\ \tilde{w}_{tt},_s)_{L^2(\Omega^f_0;\mathbb{R}^3)} \\
&\quad + 2\nu \int_0^t ((\tilde{a}^r_k \tilde{a}^s_k)_t \tilde{w}_t,_r \,,\ \tilde{w}_{tt},_s)_{L^2(\Omega^f_0;\mathbb{R}^3)} - \int_0^t \int_{\Omega^f_0} (\tilde{a}^j_i)_{tt} \tilde{q}\, \tilde{w}^i_{tt},_j \\
&\leqq C N(u_0, f)^2 + \int_0^t (F_{tt}, \tilde{w}_{tt})_{L^2(\Omega^f_0;\mathbb{R}^3)} + \int_0^t (f_{tt}, \tilde{w}_{tt})_{L^2(\Omega^s_0;\mathbb{R}^3)}, \qquad (79)
\end{aligned}$$

where (we recall) C does not depend on the smoothing parameter of $\tilde{a}$. □

9.5. Regularity for $\tilde{w}$ and its first and second time derivatives, dependent on the regularization parameter of $\tilde{a}$

As discussed in the introduction, we shall focus on the regularity near the interface, which will provide us with the trace estimates on the interface. Elliptic regularity for the Dirichlet problems will then yield the full regularity result in each interior component. In this subsection, $\tilde{C}$ continues to denote a generic constant which depends on the same variables as C and $C(M)$, and additionally on the regularization parameter. In Section 10, we obtain estimates independent of n, by interpolation mainly, which requires us to know *a priori* that the solution is smooth (without using the estimates that we get in this subsection, since they blow up with the regularization parameter).

We remind the reader that at this stage, we have already proved that $\tilde{w} \in L^2(0,T;\mathcal{V}_{\tilde{v}}(\cdot))$, $\tilde{w}_t \in L^2(0,T;H^1_0(\Omega;\mathbb{R}^3))$ and $\tilde{w}_{tt} \in \mathcal{W}([0,T])$, and that both $\tilde{q}$ and $\tilde{q}_t$ are in $L^2(0,T;L^2(\Omega^f_0;\mathbb{R}))$.

The missing regularity results will be recovered using difference quotients. Recall that if we consider the partition of the space $\mathbb{R}^3$ formed by the two half-spaces $\mathbb{R}^3_+ := \{(x^1,x^2,x^3) \in \mathbb{R}^3 \mid x^3 > 0\}$ and $\mathbb{R}^3_- := \{(x^1,x^2,x^3) \in \mathbb{R}^3 \mid x^3 < 0\}$ and the horizontal plane $\{x^3 = 0\}$ with the usual orthonormal basis (e_1,e_2,e_3), then we have

Definition 3. The first-order difference quotient of a function u of size h at x is given by

$$D_h u(x) = \frac{u(x+h) - u(x)}{|h|},$$

where h is any vector orthogonal to e_3. The second-order difference quotient of u of size h is defined as $D_{-h}D_h u(x)$, given explicitly by

$$D_{-h}D_h u(x) = \frac{u(x+h) + u(x-h) - 2u(x)}{|h|^2}.$$

We will denote

$$\nabla_0 u = (u,_1\,,\ u,_2\,).$$

Letting $A^{ij} = \tilde{a}^i_k \tilde{a}^j_k$, we write the weak form as

$$(\tilde{w}_t, \phi)_{L^2(\Omega;\mathbb{R}^3)} + \nu(A^{ij}\tilde{w},_i\,, \phi,_j\,)_{L^2(\Omega_0^f;\mathbb{R}^3)} + \left(c^{ijkl}\int_0^t \tilde{w}^k,_l\,, \phi^i,_j\right)_{L^2(\Omega_0^s;\mathbb{R})}$$
$$-(\tilde{q}, \tilde{a}^l_k \phi^k,_l\,)_{L^2(\Omega_0^f;\mathbb{R})} = (F, \phi)_{L^2(\Omega_0^f;\mathbb{R}^3)} + (f, \phi)_{L^2(\Omega_0^s;\mathbb{R}^3)}$$

for all $\phi \in H^1_0(\Omega;\mathbb{R}^3)$, for a.e. $0 \leqq t \leqq T$.

Next, assume that $\Omega = B(0,1)$, the unit ball centered at 0, and that $\Omega_0^f = \{x \in B(0,1) \mid x^3 > 0\}$ and $\Omega_0^s = \{x \in B(0,1) \mid x^3 < 0\}$. Select a smooth cut-off function ζ satisfying

$$\zeta = 1 \text{ on } B(0, \tfrac{1}{2}), \quad \zeta = 0 \text{ on } \mathbb{R}^3 - B(0,1), \text{ and } 0 \leqq \zeta \leqq 1.$$

Let $\phi = D_{-h}(\zeta^2 D_h \tilde{w})$; then clearly $\phi \in H^1_0(\Omega;\mathbb{R}^3)$ for a.e. $t \in [0,T]$. We may thus substitute ϕ into the above weak form to obtain

$$A_1 + A_2 + A_3 - A_4 = B,$$

where

$$A_1 = (D_h\tilde{w}_t, \zeta^2 D_h\tilde{w})_{L^2(\Omega;\mathbb{R}^3)},$$
$$A_2 = \nu(D_h(A^{ij}\tilde{w},_i\,), (\zeta^2 D_h\tilde{w}),_j\,)_{L^2(\Omega_0^f;\mathbb{R}^3)},$$
$$A_3 = \left(D_h(c^{ijkl}\int_0^t \tilde{w}^k,_l\,), (\zeta^2 D_h\tilde{w}^i),_j\right)_{L^2(\Omega_0^s;\mathbb{R})},$$
$$A_4 = (q, \tilde{a}^l_k(D_{-h}[\zeta^2 D_h\tilde{w}^k]),_l\,)_{L^2(\Omega_0^f;\mathbb{R})},$$
$$B = (F, D_{-h}(\zeta^2 D_h\tilde{w}))_{L^2(\Omega_0^f;\mathbb{R}^3)} + (D_h f, \zeta^2 D_h\tilde{w})_{L^2(\Omega_0^s;\mathbb{R}^3)}.$$

For the first two terms, we easily find that

$$A_1 = \frac{1}{2}\frac{d}{dt}\|\zeta D_h\tilde{w}\|^2_{L^2(\Omega;\mathbb{R}^3)},$$
$$A_2 \geqq C\|\zeta D_h\nabla\tilde{w}\|^2_{L^2(\Omega_0^f;\mathbb{R}^9)} - \tilde{C}\|\nabla\tilde{w}\|^2_{L^2(\Omega_0^f;\mathbb{R}^9)}.$$

For the remaining terms, we shall use the notation $c^h(x)$ to denote $c(x+h)$. Whereas the coefficients of the elasticity tensor are constant, it should be kept in mind that our assumption on the domain comes in fact from a change of variables which produces a non-constant elasticity tensor. It is the integral below with the $D_h c$ term which necessitates the hyperbolic scaling of our functional framework.

Expanding A_3, we have

$$A_3 = \left(\zeta^2 c^h : D_h \int_0^t \nabla\tilde{w},\, D_h \nabla\tilde{w}\right)_{L^2(\Omega_0^s;\mathbb{R}^9)} + \left(\zeta^2 D_h c : \int_0^t \nabla\tilde{w},\, D_h\nabla\tilde{w}\right)_{L^2(\Omega_0^s;\mathbb{R}^9)} + \left(2\zeta\nabla\zeta \otimes D_h\tilde{w},\, c^h : D_h \int_0^t \nabla\tilde{w}\right)_{L^2(\Omega_0^s;\mathbb{R}^9)} + \left(2\zeta\zeta,_j\, D_h c^{ijkl} \int_0^t \tilde{w}^k,_l\,,\, D_h\tilde{w}^i\right)_{L^2(\Omega_0^s;\mathbb{R})}.$$

The second term on the right-hand-side is

$$\left(D_{-h}\left(\zeta^2 D_h c : \int_0^t \nabla\tilde{w}\right), \nabla\tilde{w}\right)_{L^2(\Omega_0^s;\mathbb{R}^9)} = \left([\zeta^2 D_h c]^{-h} : D_{-h}\int_0^t \nabla\tilde{w} + D_{-h}(\zeta^2 D_h c) : \int_0^t \nabla\tilde{w}, \nabla\tilde{w}\right)_{L^2(\Omega_0^s;\mathbb{R}^9)}.$$

Hence for $\theta > 0$, we see that the A_3 term yields the following inequality:

$$\left| A_3 - \frac{1}{2}\frac{d}{dt}\left(\zeta^2 c^h : \int_0^t D_h\nabla\tilde{w}, \int_0^t D_h\nabla\tilde{w}\right)_{L^2(\Omega_0^s;\mathbb{R}^9)}\right| \leqq \theta \left\| \zeta D_{-h}\int_0^t \nabla\tilde{w}\right\|^2_{L^2(\Omega_0^s;\mathbb{R}^9)} + C_\theta \|\nabla\tilde{w}\|^2_{L^2(\Omega_0^s;\mathbb{R}^9)} + C\left\|\int_0^t \nabla\tilde{w}\right\|^2_{L^2(\Omega_0^s;\mathbb{R}^9)}.$$

For the A_4 term, we have

$$A_4 = (\tilde{q}\ ,\ \tilde{a}_k^l[\zeta^2]^{-h} D_{-h} D_h \tilde{w}^k,_l + \tilde{a}_k^l (D_{-h}\zeta^2) D_h\tilde{w}^k,_l + [2\zeta\zeta,_l]^{-h} D_{-h} D_h \tilde{w}^k \tilde{a}_k^l + 2\tilde{a}_k^l D_{-h}(\zeta\zeta,_l) D_h \tilde{w}^k)_{L^2(\Omega_0^f;\mathbb{R})}.$$

By the divergence-free condition, $\tilde{w} \in \mathcal{V}_{\tilde{v}}([0,T])$, we get, in Ω_0^f,

$$0 = D_{-h}([\tilde{a}_k^l]^h D_h\tilde{w}^k,_l + D_h\tilde{a}_k^l \tilde{w}^k,_l\,] = \tilde{a}_k^l D_{-h} D_h \tilde{w}^k,_l + D_{-h}[\tilde{a}_k^l]^h D_h\tilde{w}^k,_l + [D_h\tilde{a}_k^l]^{-h} D_{-h}\tilde{w}^k,_l + D_{-h} D_h \tilde{a}_k^l \tilde{w}^k,_l\,,$$

allowing us to eliminate the first term appearing in the expression of A_4, which gives for $\theta > 0$,

$$|A_4| \leqq \theta \|\zeta D_h \tilde{w}^k,_l\,\|^2_{L^2(\Omega_0^f;\mathbb{R}^3)} + C_\theta \|\tilde{q}\|^2_{L^2(\Omega_0^f;\mathbb{R})} + \tilde{C}\|\nabla\tilde{w}\|^2_{L^2(\Omega_0^f;\mathbb{R}^9)}.$$

Finally,

$$|B| \leqq \theta[\, \|\nabla \tilde{w}\|^2_{L^2(\Omega_0^s;\mathbb{R}^9)} + \|\zeta D_h \nabla \tilde{w}\|^2_{L^2(\Omega_0^f;\mathbb{R}^9)}\,]$$
$$+C\|\nabla \tilde{w}\|^2_{L^2(\Omega_0^f;\mathbb{R}^9)} + C_\theta[\, \|f\|^2_{H^1(\Omega_0^s;\mathbb{R}^3)} + \|f\|^2_{L^2(\Omega_0^f;\mathbb{R}^3)}\,].$$

Choosing $\theta > 0$ sufficiently small, we have the inequality

$$\frac{d}{dt}\left(\|\zeta D_h \tilde{w}\|^2_{L^2(\Omega;\mathbb{R}^3)} + \left(\zeta^2 c^h : \int_0^t D_h \nabla \tilde{w}, \int_0^t D_h \nabla \tilde{w}\right)_{L^2(\Omega_0^s;\mathbb{R}^9)} \right)$$
$$+\|\zeta D_h \nabla \tilde{w}\|^2_{L^2(\Omega_0^f;\mathbb{R}^9)}$$
$$\leqq \tilde{C}\left(\left\|\zeta \int_0^t D_h \nabla \tilde{w}\right\|^2_{L^2(\Omega_0^s;\mathbb{R}^9)} + \left\|\int_0^t \nabla \tilde{w}\right\|^2_{L^2(\Omega_0^s;\mathbb{R}^9)} + \|\nabla \tilde{w}\|^2_{L^2(\Omega_0^s;\mathbb{R}^9)} \right.$$
$$\left. +\|\nabla \tilde{w}\|^2_{L^2(\Omega_0^f;\mathbb{R}^9)} + \|\tilde{q}\|^2_{L^2(\Omega_0^f;\mathbb{R})} + \|f\|^2_{L^2(\Omega_0^f;\mathbb{R}^3)} + \|f\|^2_{H^1(\Omega_0^s;\mathbb{R}^3)} \right).$$

From Gronwall's inequality, it follows that $\partial_\alpha \partial_j \tilde{w} \in L^2(0,T; L^2(V^f;\mathbb{R}^3))$ where $V^f = \{x \in B(0,\frac{1}{2}) \mid x^3 > 0\}$, and where $\alpha = 1,2$ and $j = 1,2,3$. Hence, $\partial_\alpha \tilde{w} \in L^2(0,T; H^1(V^f;\mathbb{R}^3))$, so that by the trace theorem we obtain $\partial_\alpha \tilde{w} \in L^2(0,T; H^{0.5}(V^f \cap \{x^3 = 0\};\mathbb{R}^3))$. Thus,

$$\tilde{w} \in L^2(0,T; H^{1.5}(V^f \cap \{x^3 = 0\};\mathbb{R}^3)) \tag{80}$$

(with an estimate which blows up as the mollification parameter $n \to \infty$). Similarly,

$$\int_0^t \tilde{w} \in L^\infty(0,T; H^{1.5}(V^s \cap \{x^3 = 0\};\mathbb{R}^3)), \tag{81}$$

where $V^s = \{x \in B(0,\frac{1}{2}) \mid x^3 < 0\}$.

We now drop the assumption that Ω is the unit ball, and once again assume it is an open bounded subset of $\mathbb{R}^3$ with all of the smoothness assumption stated previously. We choose any point $x_0 \in \Gamma_0$ and assume that

$$\Omega_0^f \cap B(x_0,r) = \{x \in B(x_0,r) \mid x^3 > \gamma(x^1,x^2)\},$$
$$\Omega_0^s \cap B(x_0,r) = \{x \in B(x_0,r) \mid x^3 < \gamma(x^1,x^2)\}$$

for some $r > 0$ and some smooth function $\gamma : \mathbb{R}^2 \to \mathbb{R}$. We define the following change of variables:

$$y^i = x^i =: \Phi^i(x), \quad i = 1,2$$
$$y^3 = x^3 - \gamma(x^1,x^2) = \Phi^3(x),$$

and write

$$y = \Phi(x).$$

Similarly, we set

$$x^i = y^i =: \Psi^i(y), \quad i = 1, 2$$
$$x^3 = y^3 + \gamma(y^1, y^2) = \Psi^3(y),$$

and write

$$x = \Psi(y).$$

Then $\Phi = \Psi^{-1}$, and the mapping $x \mapsto \Phi(x) = y$ *straightens out* Γ_0 near x_0, and $\det \Phi = \det \Psi = 1$, (see [13]).

We can assume $0 = \Phi(x_0)$. Choose $s > 0$ so small that $B(0, s) \subset \Phi(B(x_0, r))$. Let

$$w'(t, y) = \tilde{w}(t, \Phi(y)), \;\; q'(t, y) = \tilde{q}(t, \Phi(y)), \;\; f'(t, y) = f(t, \Phi(y)).$$

Then w' and w'_t are in $L^2(0, T; H^1(\Omega'; \mathbb{R}^3))$, where $\Omega' = B(0, s)$. We also set

$$\Omega_0^{f'} = B(0, s) \cap \{y^3 > 0\}, \;\; \Omega_0^{s'} = B(0, s) \cap \{y^3 < 0\}.$$

Then, since $(\tilde{w}, \tilde{q})$ satisfy the weak formulation, applying the change of variables, we see that (w', q') satisfy

$$\begin{aligned}(w'_t, \phi')_{L^2(\Omega';\mathbb{R}^3)} &+ \nu(a'^{ij} w',_i, \phi',_j)_{L^2(\Omega_0^{f'};\mathbb{R}^3)} \\ &+ \left(c'^{ijkl} \int_0^t w'^k,_l, \phi'^i,_j\right)_{L^2(\Omega_0^{s'};\mathbb{R}^3)} \\ &-(q', [a_k^l \circ \Psi] g_l^r \phi'^k,_r)_{L^2(\Omega_0^{f'};\mathbb{R}^3)} = (f', \phi')_{L^2(\Omega';\mathbb{R}^3)} \end{aligned} \tag{82}$$

for all $\phi' \in H_0^1(\Omega'; \mathbb{R}^3)$, for a.e. $0 \leqq t \leqq T$, where

$$a'^{kl} = A^{ij} \circ \Psi \; g_i^k g_j^l, \;\; c'^{irks} = c^{ijkl} \; g_l^s g_j^r, \;\; g(y) = [\nabla\Psi(y)]^{-1}.$$

It is easy to verify that both a' and c' retain the uniform ellipticity conditions of the original operators A and c; Moreover, w' satisfies the divergence condition $a_i^j \circ \Phi w'^i,_j = 0$ in $[0, T] \times \Omega'$. Thus we may apply the results obtained above for the case where the domain is the unit ball to find that

$$w' \in L^2(0, T; H^{1.5}(V^{f'} \cap \{x^3 = 0\}; \mathbb{R}^3)),$$
$$\int_0^t w' \in L^\infty(0, T; H^{1.5}(V^{s'} \cap \{x^3 = 0\}; \mathbb{R}^3)),$$

where $V^{f'} = \{x \in B(0, \frac{s}{2}) \mid x^3 > 0\}$ and $V^{s'} = \{x \in B(0, \frac{s}{2}) \mid x^3 < 0\}$. Consequently,

$$\tilde{w} \in L^2(0, T; H^{1.5}(\partial V^f \cap \Gamma_0; \mathbb{R}^3)), \;\; \int_0^t \tilde{w} \in L^\infty(0, T; H^{1.5}(\partial V^s \cap \Gamma_0; \mathbb{R}^3)), \tag{83}$$

where $V^f = \Psi(V^{f'})$ and $V^s = \Psi(V^{s'})$.

Since Γ_0 is compact, we can as usual cover Γ_0 with finitely many sets of the type used above. Summing the resulting estimates, we find that we have, for the trace on Γ_0,

$$\tilde{w} \in L^2(0, T; H^{1.5}(\Gamma_0; \mathbb{R}^3)), \tag{84a}$$

$$\int_0^t \tilde{w} \in L^\infty(0, T; H^{1.5}(\Gamma_0; \mathbb{R}^3)). \tag{84b}$$

Converting the fluid equations into Eulerian variables by composing with $\tilde{\eta}^{-1}$, we obtain a Stokes problem in the domain $\tilde{\eta}(\Omega_0^f)$:

$$-\nu\Delta u + \nabla p = f - \tilde{w}_t \circ \tilde{\eta}^{-1} + \nu \tilde{a}_l^j,_j \circ \eta^{-1} u,_l - p\, (\tilde{a}_i^j),_j \circ \tilde{\eta}^{-1}, \tag{85a}$$

$$\operatorname{div} u = 0, \tag{85b}$$

with the boundary conditions that $u = 0$ on $\tilde{\eta}(\partial\Omega)$ and that $u \in L^2(0, T; H^{1.5}(\tilde{\eta}(\Gamma_0); \mathbb{R}^3))$, where $u = \tilde{w} \circ \tilde{\eta}^{-1}$ and $p = \tilde{q} \circ \tilde{\eta}^{-1}$. Since the domain $\tilde{\eta}(\Omega_0^f)$ is of class H^3, by the elliptic regularity of [8], (84a) implies that $u \in L^2(0, T; H^2(\tilde{\eta}(\Omega_0^f); \mathbb{R}^3))$ and $p \in L^2(0, T; H^1(\tilde{\eta}(\Omega_0^f); \mathbb{R}))$. It follows that

$$\tilde{w} \in L^2(0, T; H^2(\Omega_0^f; \mathbb{R}^3)), \quad \tilde{q} \in L^2(0, T; H^1(\Omega_0^f; \mathbb{R})). \tag{86}$$

Similarly, elliptic regularity of the elasticity problem shows that

$$\int_0^t \tilde{w} \in L^\infty(0, T; H^2(\Omega_0^s; \mathbb{R}^3)). \tag{87}$$

Next, we consider the weak form for the time derivate $\tilde{w}_t$ for all $\phi \in H_0^1(\Omega; \mathbb{R}^3)$:

$$(\tilde{w}_{tt}, \phi)_{L^2(\Omega;\mathbb{R}^3)} + \nu([A^{rs}\tilde{w},_r]_t, \phi,_s)_{L^2(\Omega_0^f;\mathbb{R}^3)} + (c^{ijkl}\tilde{w}^k,_l, \phi^i,_j)_{L^2(\Omega_0^s;\mathbb{R})}$$
$$-([\tilde{a}_i^j\tilde{q}]_t, \phi^i,_j)_{L^2(\Omega_0^f;\mathbb{R})} = (F_t, \phi)_{L^2(\Omega_0^f;\mathbb{R}^3)} + (f_t, \phi)_{L^2(\Omega_0^s;\mathbb{R}^3)} \text{ a.e. } t \in (0, T).$$

Expanding the time derivative, we see that there are two additional terms in the weak form given by $(A_t^{rs}\tilde{w},_r, \phi,_s)_{L^2(\Omega_0^f;\mathbb{R}^3)}$ and $((\tilde{a}_t)_i^j q, \phi_t{}^i,_j)_{L^2(\Omega_0^f;\mathbb{R}^3)}$. These additional terms are easy to handle, and by letting $\phi = D_{-h}(\zeta^2 D_h \tilde{w}_t)$, and following the identical procedure as above, since we also already know that $\tilde{w}_{tt} \in L^\infty(0, T; L^2(\Omega; \mathbb{R}^3))$, we find that

$$\tilde{w}_t \in L^2(0, T; H^2(\Omega_0^f; \mathbb{R}^3)),\ \tilde{q}_t \in L^2(0, T; H^1(\Omega_0^f; \mathbb{R})),$$
$$\tilde{w} \in L^\infty(0, T; H^2(\Omega_0^s; \mathbb{R}^3)). \tag{88}$$

Because of the assumptions on the forcing and these estimates for $\tilde{w}_t$, we may improve the regularity results (86) and (87). We apply the identical procedure, but this time we use $\phi = D_{-h}D_h(\zeta^2 D_{-h}D_h\tilde{w})$ as the test function. We find that

$$\tilde{w} \in L^2(0, T; H^3(\Omega_0^f; \mathbb{R}^3)),\ \tilde{q} \in L^2(0, T; H^2(\Omega_0^f; \mathbb{R})),$$
$$\int_0^t \tilde{w} \in L^\infty(0, T; H^3(\Omega_0^s; \mathbb{R}^3)). \tag{89}$$

Moreover, $\|(\tilde{w}, \tilde{q})\|_{Z_T} \leqq \tilde{C} N(u_0, f)$, where the constant $\tilde{C} \to \infty$ as the mollification parameter $n \to \infty$.

In the following section, we will use a different form of (82). If we denote by ζ a smooth cut-off function, equal to 1 in a neighborhood of 0 contained in Ω' and 0 outside Ω', and denote $W = \zeta^2 w'$, $Q = \zeta^2 q'$, $\tilde{b}_l^j = \tilde{a}_l^k \circ \Psi\, g_k^j$, $C^{ijkl} = c'^{ijkl}$, we then obtain for any $\varphi \in H^1(\mathbb{R}^3; \mathbb{R}^3)$,

$$\begin{aligned}(W_t,\ \varphi)_{L^2(\mathbb{R}^3;\mathbb{R}^3)} &+ \nu\ (\tilde{b}_l^j \tilde{b}_l^k W,_k\ ,\ \varphi,_j\ )_{L^2(\mathbb{R}^3_+;\mathbb{R}^3)} \\ &+ \left(C^{ijkl} \int_0^t W^k,_l\ ,\ \varphi^i,_j \right)_{L^2(\mathbb{R}^3_-;\mathbb{R}^3)} - (Q, \tilde{b}_k^r \varphi^k,_r\ )_{L^2(\mathbb{R}^3_+;\mathbb{R})} \\ = (F_1,\ \varphi)_{L^2(\mathbb{R}^3_+;\mathbb{R}^3)} &+ (H_j,\ \varphi,_j\ )_{L^2(\mathbb{R}^3_+;\mathbb{R}^3)} \\ &+ (F_2,\ \varphi)_{L^2(\mathbb{R}^3_-;\mathbb{R}^3)} + (K_j,\ \varphi,_j\ )_{L^2(\mathbb{R}^3_+;\mathbb{R}^3)}\ ,\end{aligned} \tag{90}$$

where

$$F_1^i = \zeta^2 F'^i - \nu\, \zeta^2,_j\, \tilde{b}_l^j \tilde{b}_l^k w'^i,_k + q' \tilde{b}_i^r \zeta^2,_r\ , \tag{91a}$$

$$H_j = \nu\, \tilde{b}_l^j \tilde{b}_l^k \zeta^2,_k\, w', \tag{91b}$$

$$F_2^i = \zeta^2 f'^i - C^{ijkl} \zeta^2,_j \int_0^t w'^k,_l\ , \tag{91c}$$

$$K_j^i = C^{ijkl} \zeta^2,_l \int_0^t w'^k. \tag{91d}$$

Moreover, W satisfies the divergence condition

$$\tilde{b}_i^j W^i,_j = \mathfrak{a} = \tilde{b}_i^j \zeta^2,_j\, W^i \text{ in } [0, T] \times \mathbb{R}^3. \tag{92}$$

Note that we consider the above inner-products over all of $\mathbb{R}^3$ since W and its derivatives are compactly supported in Ω'; the contribution outside Ω' is zero regardless of the way in which we extend $\tilde{b}$ and g to $[\Omega']^c$. This same remark also applies to (92).

10. Estimate for (20): the case of the actual coefficients

10.1. Energy estimate for $\tilde{w}_{tt}$ independent of the regularization parameter for $\tilde{a}$

We are now going to use the regularity results (88) and (89) in the energy inequality (79) (which was bounded by a constant that does not dependent on the mollification parameter). Our approach now will be to use interpolation inequalities to obtain an estimate which is independent of the regularization parameter.

This section will be divided into eight steps, each of which is devoted to the estimation of the various integral terms in (79).

In what follows, $\delta > 0$ is a given positive number; the choice of δ will be made precise later, as it will have to be chosen sufficiently small.

Step 1. Let $I_1 = \int_0^t \int_{\Omega_0^f} \tilde{q}_t (\tilde{a}_i^j)_t \tilde{w}_{tt}^i,_j$. Then, by the Cauchy-Schwarz inequality and by interpolation,

$$I_1 \leqq \delta \int_0^t \|\nabla \tilde{w}_{tt}\|^2_{L^2(\Omega_0^f;\mathbb{R}^9)} + C_\delta \int_0^t \|\tilde{a}_t\|^2_{L^4(\Omega_0^f;\mathbb{R}^9)} \|\tilde{q}_t\|^2_{L^4(\Omega_0^f;\mathbb{R})}$$
$$\leqq \delta \int_0^t \|\nabla \tilde{w}_{tt}\|^2_{L^2(\Omega_0^f;\mathbb{R}^9)} + C_\delta C(M) \int_0^t \|\tilde{q}_t\|^{0.5}_{L^2(\Omega_0^f;\mathbb{R})} \|\tilde{q}_t\|^{1.5}_{H^1(\Omega_0^f;\mathbb{R})} ,$$

where we have used (12) for the L^∞ control of $\tilde{a}_t$ in H^1. Thus,

$$I_1 \leqq \delta \int_0^t \|\nabla \tilde{w}_{tt}\|^2_{L^2(\Omega_0^f;\mathbb{R}^9)} + C_\delta C(M) \sup_{(0,t)} \|\tilde{q}_t\|^{0.5}_{L^2(\Omega_0^f;\mathbb{R})}$$
$$\times \left[\int_0^t \|\tilde{q}_t\|^2_{H^1(\Omega_0^f;\mathbb{R})} \right]^{\frac{3}{4}} T^{\frac{1}{4}}.$$

By Lemma 13 applied to (61), and (12),

$$\|\tilde{q}_t\|^2_{L^2(\Omega_0^f;\mathbb{R})} \leqq C \; [\; \|\tilde{w}_{tt}\|^2_{L^2(\Omega;\mathbb{R}^3)} + \|\tilde{q}\; \tilde{a}_t\|^2_{L^2(\Omega_0^f;\mathbb{R}^9)} + \|F_t\|^2_{L^2(\Omega_0^f;\mathbb{R}^3)}$$
$$+ \|f_t\|^2_{L^2(\Omega_0^s;\mathbb{R}^3)} + \|\tilde{w}_t\|^2_{H^1(\Omega_0^f;\mathbb{R}^3)} + \|\tilde{w}\|^2_{H^2(\Omega_0^f;\mathbb{R}^3)}$$
$$+ \|\tilde{w}\|^2_{H^1(\Omega_0^s;\mathbb{R}^3)} \;]. \tag{93}$$

Thus, with $\tilde{w}_t(t) = w_1 + \int_0^t \tilde{w}_{tt}$, $\tilde{w}(t) = u_0 + \int_0^t \tilde{w}_t$ and $\tilde{q} = q_0 + \int_0^t \tilde{q}_t$ respectively in $H^1(\Omega_0^f;\mathbb{R}^3)$, $H^2(\Omega_0^f;\mathbb{R}^3)$, and $H^1(\Omega_0^f;\mathbb{R})$,

$$I_1 \leqq \; \delta \int_0^t \|\nabla \tilde{w}_{tt}\|^2_{L^2(\Omega_0^f;\mathbb{R}^9)}$$
$$+ C_\delta C(M)\; T^{\frac{1}{4}} \Bigg[N(u_0, f)^2 + T \int_0^t \|\tilde{q}_t\|^2_{H^1(\Omega_0^f;\mathbb{R})} + \int_0^t \|\tilde{q}_t\|^2_{H^1(\Omega_0^f;\mathbb{R})}$$
$$+ T \left[\int_0^t \|\tilde{w}_{tt}\|^2_{H^1(\Omega_0^f;\mathbb{R}^3)} + \int_0^t \|\tilde{w}_t\|^2_{H^2(\Omega_0^f;\mathbb{R}^3)} \right]$$
$$+ \sup_{[0,T]} \|\tilde{w}\|^2_{H^1(\Omega_0^s;\mathbb{R}^3)} + \sup_{[0,T]} \|\tilde{w}_{tt}\|^2_{L^2(\Omega;\mathbb{R}^3)} \Bigg]. \tag{94}$$

Step 2. Let $I_2 = \int_0^t \int_{\Omega_0^f} \tilde{q}_t (\tilde{a}_i^j)_{tt} \tilde{w}_t^i,_j$. Then,

$$I_2 \leqq \int_0^t \|(\tilde{a}_i^j)_{tt} \tilde{w}_t^i,_j\|_{L^{\frac{6}{5}}(\Omega_0^f;\mathbb{R})} \|\tilde{q}_t\|_{L^6(\Omega_0^f;\mathbb{R})}$$
$$\leqq \delta \int_0^t \|\tilde{q}_t\|^2_{H^1(\Omega_0^f;\mathbb{R})} + C_\delta \int_0^t \|\nabla \tilde{w}_t\|^2_{L^3(\Omega_0^f;\mathbb{R}^9)} \|\tilde{a}_{tt}\|^2_{L^2(\Omega_0^f;\mathbb{R}^9)}.$$

Thus, when we use (12) for the L^∞ control of $\tilde{a}_{tt}$ in L^2,

$$I_2 \leqq \delta \int_0^t \|\tilde{q}_t\|^2_{H^1(\Omega_0^f;\mathbb{R})} + C_\delta \; C(M) \; \int_0^t \|\nabla \tilde{w}_t\|^{0.5}_{L^2(\Omega_0^f;\mathbb{R}^9)} \|\nabla \tilde{w}_t\|^{1.5}_{H^1(\Omega_0^f;\mathbb{R}^9)},$$

which with $\tilde{w}_t(t) = \tilde{w}_1 + \int_0^t \tilde{w}_{tt}$ gives

$$I_2 \leqq \delta \int_0^t \|\tilde{q}_t\|^2_{H^1(\Omega_0^f;\mathbb{R})} + C_\delta\, T^{\frac{1}{4}} \left[N(u_0, f)^2 + \int_0^t \|\nabla \tilde{w}_t\|^2_{H^1(\Omega_0^f;\mathbb{R}^9)} + T \int_0^t \|\nabla \tilde{w}_{tt}\|^2_{L^2(\Omega_0^f;\mathbb{R}^9)} \right]. \tag{95}$$

Step 3. Let $I_3 = \int_0^t \int_{\Omega_0^f} \tilde{q}_t (\tilde{a}_i^j)_{ttt} \tilde{w}^i,_j$. Then,

$$\begin{aligned} I_3 &\leqq \delta \int_0^t \|\tilde{a}_{ttt}\|^2_{L^2(\Omega_0^f;\mathbb{R}^9)} \|\nabla \tilde{w}\|^2_{L^4(\Omega_0^f;\mathbb{R}^9)} + C_\delta \int_0^t \|\tilde{q}_t\|^2_{L^4(\Omega_0^f;\mathbb{R})} \\ &\leqq \delta \int_0^t \|\tilde{a}_{ttt}\|^2_{L^2(\Omega_0^f;\mathbb{R}^9)} \left[N(u_0, f)^2 + T \int_0^t \|\tilde{w}_t\|^2_{H^2(\Omega_0^f;\mathbb{R}^3)} \right] \\ &\quad + C_\delta \int_0^t \|\tilde{q}_t\|^{0.5}_{L^2(\Omega_0^f;\mathbb{R})} \|\tilde{q}_t\|^{1.5}_{H^1(\Omega_0^f;\mathbb{R})}, \end{aligned}$$

where we have used $\tilde{w} = u_0 + \int_0^t \tilde{w}_t$. Thus, by (12), and (93),

$$\begin{aligned} I_3 &\leqq \delta C(M) \left[N(u_0, f)^2 + T \int_0^t \|\tilde{w}_t\|^2_{H^2(\Omega_0^f;\mathbb{R}^3)} \right] + C_\delta T^{\frac{1}{4}} \int_0^t \|\tilde{q}_t\|^2_{H^1(\Omega_0^f;\mathbb{R}^3)} \\ &\quad + C_\delta C(M)\, T^{\frac{1}{4}} \left[N(u_0, f)^2 + T \int_0^t \|\tilde{q}_t\|^2_{H^1(\Omega_0^f;\mathbb{R})} + T \int_0^t \|\tilde{w}_{tt}\|^2_{H^1(\Omega_0^f;\mathbb{R}^3)} \right. \\ &\quad \left. + T \int_0^t \|\tilde{w}_t\|^2_{H^2(\Omega_0^f;\mathbb{R}^3)} + \sup_{[0,T]} \|\tilde{w}\|^2_{H^1(\Omega_0^s;\mathbb{R}^3)} + \sup_{[0,T]} \|\tilde{w}_{tt}\|^2_{L^2(\Omega;\mathbb{R}^3)} \right]. \end{aligned} \tag{96}$$

Step 4. Let $I_4 = \int_0^t \int_{\Omega_0^f} \tilde{q} (\tilde{a}_i^j)_{tt} \tilde{w}^i_{tt},_j$. Then,

$$\begin{aligned} I_4 &\leqq \delta \int_0^t \|\nabla \tilde{w}_{tt}\|^2_{L^2(\Omega_0^f;\mathbb{R}^9)} + C_\delta \int_0^t \|\tilde{a}_{tt}\|^2_{L^2(\Omega_0^f;\mathbb{R}^9)} \|\tilde{q}\|^2_{W^{1,4}(\Omega_0^f;\mathbb{R})} \\ &\leqq \delta \int_0^t \|\nabla \tilde{w}_{tt}\|^2_{L^2(\Omega_0^f;\mathbb{R}^9)} + C_\delta C(M) \int_0^t \|\tilde{q}\|^{0.5}_{H^1(\Omega_0^f;\mathbb{R})} \|\tilde{q}\|^{1.5}_{H^1(\Omega_0^f;\mathbb{R})} \\ &\leqq \delta \int_0^t \|\nabla \tilde{w}_{tt}\|^2_{L^2(\Omega_0^f;\mathbb{R}^9)} \\ &\quad + C_\delta C(M)\, T^{\frac{1}{4}} \left[N(u_0, f)^2 + T \int_0^t \|\tilde{q}_t\|^2_{H^1(\Omega_0^f;\mathbb{R})} + \int_0^t \|\tilde{q}\|^2_{H^2(\Omega_0^f;\mathbb{R})} \right]. \end{aligned} \tag{97}$$

The next two steps will require the introduction of $\delta_1 > 0$, which is different than δ and will also be made precise later.

Step 5. Let $I_5 = -\int_{\Omega_0^f} \tilde{q}_t(t) (\tilde{a}_i^j)_{tt} \tilde{w}^i,_j(t)$. We first notice that

$$I_5 = -\int_{\Omega_0^f} \tilde{q}_t(t) (\tilde{a}_i^j)_{tt} (\tilde{w}^i,_j(t) - u_0^i,_j) - \int_{\Omega_0^f} \tilde{q}_t(t) (\tilde{a}_i^j)_{tt} u_0^i,_j .$$

For the second term of the right-hand side of this equality,

$$-\int_{\Omega_0^f} \tilde{q}_t(t)(\tilde{a}_i^j)_{tt} u_0^i,_j \leqq \delta_1 \|\tilde{q}_t(t)\|^2_{L^2(\Omega_0^f;\mathbb{R})} + C_{\delta_1} \|\tilde{a}_{tt}(t)\|^2_{L^2(\Omega_0^f;\mathbb{R}^9)} \|u_0\|^2_{H^3(\Omega_0^f;\mathbb{R}^3)},$$

and thus by (17), since $T \leqq T_M$,

$$-\int_{\Omega_0^f} \tilde{q}_t(t)(\tilde{a}_i^j)_{tt} u_0^i,_j \leqq \delta_1 \|\tilde{q}_t(t)\|^2_{L^2(\Omega_0^f;\mathbb{R})} + C_{\delta_1} C(M)\, N(u_0, f)^2. \tag{98}$$

For the other term,

$$-\int_{\Omega_0^f} \tilde{q}_t(t)(\tilde{a}_i^j)_{tt} (\tilde{w}^i,_j(t) - u_0^i,_j) \\ \leqq \delta\, \|\tilde{q}_t(t)\|^2_{L^2(\Omega_0^f;\mathbb{R})} + C_\delta \|\tilde{a}_{tt}(t)\|^2_{L^3(\Omega_0^f;\mathbb{R}^9)} \|\nabla\tilde{w} - \nabla u_0\|^2_{L^6(\Omega_0^f;\mathbb{R}^9)}, \tag{99}$$

and thus, by the L^∞ control in L^3 provided by (19),

$$\int_{\Omega_0^f} \tilde{q}_t(t)(\tilde{a}_i^j)_{tt} (\tilde{w}^i,_j(t) - u_0^i,_j) \\ \leqq \delta\, \|\tilde{q}_t(t)\|^2_{L^2(\Omega_0^f;\mathbb{R})} + C_\delta\, C(M)\, T \int_0^t \|\nabla\tilde{w}_t\|^2_{H^1(\Omega_0^f;\mathbb{R}^9)}. \tag{100}$$

By (93), (98) and (100), we finally have

$$I_5 \leqq (\delta + \delta_1)\Bigg[N(u_0, f)^2 + C(M)T \left[\int_0^t \|\tilde{q}_t\|^2_{H^1(\Omega_0^f;\mathbb{R})} + \int_0^t \|\tilde{w}_{tt}\|^2_{H^1(\Omega_0^f;\mathbb{R}^3)}\right] \\ + C(M)T \int_0^T \|\tilde{w}_t\|^2_{H^2(\Omega_0^f;\mathbb{R}^3)} + \sup_{[0,T]} \|\tilde{w}\|^2_{H^1(\Omega_0^s;\mathbb{R}^3)} + \sup_{[0,T]} \|\tilde{w}_{tt}\|^2_{L^2(\Omega;\mathbb{R}^3)}\Bigg] \\ + C_\delta\, C(M)\, T \int_0^t \|\nabla\tilde{w}_t\|^2_{H^1(\Omega_0^f;\mathbb{R}^9)} + C_{\delta_1} C\, N(u_0, f)^2. \tag{101}$$

Remark 20. Note that L^3 and L^6 in (99) are limit cases for both (19) and the Sobolev embeddings in dimension three. In dimension $\geqq 4$, this would no longer be possible and we would be required to introduce a smoother functional framework.

Step 6. Let $I_6 = -\int_{\Omega_0^f} \tilde{q}_t(t)(\tilde{a}_i^j)_t w_t^i,_j(t)$. Similarly to our previous step, we first notice that

$$I_6 = -\int_{\Omega_0^f} \tilde{q}_t(t)((\tilde{a}_i^j)_t(t) - (\tilde{a}_i^j)_t(0))\tilde{w}_t^i,_j(t) - \int_{\Omega_0^f} \tilde{q}_t(t)(\tilde{a}_i^j)_t(0)\tilde{w}_t^i,_j(t).$$

For the second term of the right-hand side of this equality,

$$-\int_{\Omega_0^f} \tilde{q}_t(t)(\tilde{a}_i^j)_t(0)\tilde{w}_t^i,_j(t) \leqq \delta_1\, \|\tilde{a}_t(0)\|^2_{H^2(\Omega_0^f;\mathbb{R}^9)} \|\tilde{q}_t(t)\|^2_{L^2(\Omega_0^f;\mathbb{R})} \\ + C_{\delta_1} \|\nabla\tilde{w}_t(t)\|^2_{L^2(\Omega_0^f;\mathbb{R}^9)},$$

and thus,

$$-\int_{\Omega_0^f} \tilde{q}_t(t)(\tilde{a}_i^j)_t(0)\tilde{w}_t^i,_j(t) \leqq C\delta_1 N(u_0, f)^2 \|\tilde{q}_t(t)\|^2_{L^2(\Omega_0^f;\mathbb{R})} + C_{\delta_1}\left[\|\nabla \tilde{w}_1\|^2_{L^2(\Omega_0^f;\mathbb{R}^9)} + T \int_0^t \|\nabla \tilde{w}_{tt}\|^2_{L^2(\Omega_0^f;\mathbb{R}^9)} \right]. \tag{102}$$

For the other term,

$$-\int_{\Omega_0^f} \tilde{q}_t(t)((\tilde{a}_i^j)_t(t) - (\tilde{a}_i^j)_t(0))\tilde{w}_t^i,_j(t) \leqq \delta \|\tilde{q}_t(t)\|^2_{L^2(\Omega_0^f;\mathbb{R})} + C_\delta \|\tilde{a}_t(t) - \tilde{a}_t(0)\|^2_{L^6(\Omega_0^f;\mathbb{R}^9)} \|\nabla \tilde{w}_t\|^2_{L^3(\Omega_0^f;\mathbb{R}^9)},$$

and by (16),

$$-\int_{\Omega_0^f} \tilde{q}_t(t)((\tilde{a}_i^j)_t(t) - (\tilde{a}_i^j)_t(0))\tilde{w}_t^i,_j(t) \leqq \delta \|\tilde{q}_t(t)\|^2_{L^2(\Omega_0^f;\mathbb{R})} + C_\delta\, C(M)\, T\, \|\nabla \tilde{w}_t(t)\|^2_{L^3(\Omega_0^f;\mathbb{R}^9)}.$$

In the same fashion as we proved (19), we use the L^∞ control in L^3:

$$\|\nabla \tilde{w}_t(t)\|^2_{L^3(\Omega_0^f;\mathbb{R}^9)} \leqq \|\nabla w_1\|^2_{L^3(\Omega_0^f;\mathbb{R}^9)} + C \left[\int_0^t \|\nabla \tilde{w}_t\|^2_{H^1(\Omega_0^f;\mathbb{R}^9)} + \int_0^t \|\nabla \tilde{w}_{tt}\|^2_{L^2(\Omega_0^f;\mathbb{R}^9)} \right],$$

which combined with the previous inequality provides us with

$$\begin{aligned} -\int_{\Omega_0^f} \tilde{q}_t(t)((\tilde{a}_i^j)_t(t) - (\tilde{a}_i^j)_t(0))\tilde{w}_t^i,_j(t) &\leqq \delta \|\tilde{q}_t(t)\|^2_{L^2(\Omega_0^f;\mathbb{R})} + C_\delta\, C(M)\, T\, \|\nabla \tilde{w}_1\|^2_{H^1(\Omega_0^f;\mathbb{R}^9)} \\ &\quad + C_\delta C(M) T \left[\int_0^t \|\nabla \tilde{w}_t\|^2_{H^1(\Omega_0^f;\mathbb{R}^9)} + \int_0^t \|\nabla \tilde{w}_{tt}\|^2_{L^2(\Omega_0^f;\mathbb{R}^9)} \right]. \end{aligned} \tag{103}$$

By (102) and (103), we finally have

$$\begin{aligned} I_6 \leqq (\delta + \delta_1) \Bigg[& C N(u_0, f)^2 + C(M) T \int_0^t \|\tilde{w}_{tt}\|^2_{H^1(\Omega_0^f;\mathbb{R}^3)} \\ & + C(M) T \left[\int_0^t \|\tilde{q}_t\|^2_{H^1(\Omega_0^f;\mathbb{R})} + \int_0^t \|\tilde{w}_t\|^2_{H^2(\Omega_0^f;\mathbb{R}^3)} \right] + \sup_{[0,t]} \|\tilde{w}_{tt}\|^2_{L^2(\Omega;\mathbb{R}^3)} \\ & + \sup_{[0,t]} \|\tilde{w}\|^2_{H^1(\Omega_0^s;\mathbb{R}^3)} \Bigg] + C_{\delta_1} \left[N(u_0, f)^2 + T \int_0^t \|\nabla \tilde{w}_{tt}\|^2_{L^2(\Omega_0^f;\mathbb{R}^9)} \right] \\ & + C_\delta C(M) T \left[N(u_0, f)^2 + \int_0^t \|\nabla \tilde{w}_{tt}\|^2_{L^2(\Omega_0^f;\mathbb{R}^9)} + \int_0^t \|\nabla \tilde{w}_t\|^2_{H^1(\Omega_0^f;\mathbb{R}^9)} \right]. \end{aligned} \tag{104}$$

Step 7. Let $I_7 = -\int_0^t ((\tilde{a}_k^r \tilde{a}_k^s)_{tt} \tilde{w},_r \,,\ \tilde{w}_{tt},_s)_{L^2(\Omega_0^f;\mathbb{R}^3)}$. Then,

$$I_7 \leqq \delta \int_0^t \|\nabla \tilde{w}_{tt}\|^2_{L^2(\Omega_0^f;\mathbb{R}^9)} + C_\delta \int_0^t \|\tilde{(\tilde{a}_k^r \tilde{a}_k^s)}_{tt}\|^2_{L^2(\Omega_0^f;\mathbb{R})} \|\nabla \tilde{w}\|^2_{W^{1,4}(\Omega_0^f;\mathbb{R}^9)}$$
$$\leqq \delta \int_0^t \|\nabla \tilde{w}_{tt}\|^2_{L^2(\Omega_0^f;\mathbb{R}^9)} + C_\delta C(M) \int_0^t \|\nabla \tilde{w}\|^{0.5}_{H^1(\Omega_0^f;\mathbb{R}^9)} \|\nabla \tilde{w}\|^{1.5}_{H^2(\Omega_0^f;\mathbb{R}^9)} ,$$

where we have used (12) for the L^∞ control of $\tilde{a}_{tt}$, $\tilde{a}_t$ and $\tilde{a}$ respectively in L^2, H^1 and H^2. Thus,

$$I_7 \leqq \delta \int_0^t \|\nabla \tilde{w}_{tt}\|^2_{L^2(\Omega_0^f;\mathbb{R}^9)} + C_\delta C(M) T^{\frac{1}{4}} \Big[N(u_0, f)^2$$
$$+T \int_0^t \|\nabla \tilde{w}_t\|^2_{H^1(\Omega_0^f;\mathbb{R}^9)} + \int_0^t \|\nabla \tilde{w}\|^2_{H^2(\Omega_0^f;\mathbb{R}^9)} \Big]. \tag{105}$$

Step 8. Let $I_8 = -\int_0^t ((\tilde{a}_k^r \tilde{a}_k^s)_t \tilde{w}_t,_r \,,\ \tilde{w}_{tt},_s)_{L^2(\Omega_0^f;\mathbb{R}^3)}$. Then,

$$I_8 \leqq \delta \int_0^t \|\nabla \tilde{w}_{tt}\|^2_{L^2(\Omega_0^f;\mathbb{R}^9)} + C_\delta \int_0^t \|\tilde{(\tilde{a}_k^r \tilde{a}_k^s)}_t\|^2_{L^4(\Omega_0^f;\mathbb{R})} \|\nabla \tilde{w}_t\|^2_{L^4(\Omega_0^f;\mathbb{R}^9)}$$
$$\leqq \delta \int_0^t \|\nabla \tilde{w}_{tt}\|^2_{L^2(\Omega_0^f;\mathbb{R}^9)} + C_\delta C(M) \int_0^t \|\nabla \tilde{w}_t\|^{0.5}_{L^2(\Omega_0^f;\mathbb{R}^9)} \|\nabla \tilde{w}_t\|^{1.5}_{H^1(\Omega_0^f;\mathbb{R}^9)} .$$

Consequently,

$$I_8 \leqq \delta \int_0^t \|\nabla \tilde{w}_{tt}\|^2_{L^2(\Omega_0^f;\mathbb{R}^9)} + C_\delta C(M)\, T^{\frac{1}{4}} \Big[N(u_0, f)$$
$$+T \int_0^t \|\nabla \tilde{w}_{tt}\|^2_{L^2(\Omega_0^f;\mathbb{R}^9)} + \int_0^t \|\nabla \tilde{w}_t\|^2_{H^1(\Omega_0^f;\mathbb{R}^9)} \Big]. \tag{106}$$

Step 9. Thus, from (79), and estimates (94)–(106), we then obtain the inequality

$$\sup_{[0,T]} \|\tilde{w}_{tt}(t)\|^2_{L^2(\Omega;\mathbb{R}^3)} + \int_0^T \|\tilde{w}_{tt}\|^2_{H^1(\Omega_0^f;\mathbb{R}^3)} + \sup_{[0,T]} \|\tilde{w}_t\|^2_{H^1(\Omega_0^s;\mathbb{R}^3)}$$
$$\leqq [\, C\delta(1 + C(M))N(u_0, f)^2 + C\delta_1(1 + TC(M) + N(u_0, f)^2)] \|(\tilde{w}, \tilde{q})\|^2_{Z_T}$$
$$+C_{\delta_1} N(u_0, f)^2 + C_{\delta_1} T^{\frac{1}{4}} \|(\tilde{w}, \tilde{q})\|^2_{Z_T}$$
$$+C_\delta C(M) T^{\frac{1}{4}} (\|(\tilde{w}, \tilde{q})\|^2_{Z_T} + N(u_0, f)^2). \tag{107}$$

We then infer from (107), (93), and (12) that $\|\tilde{q}_t\|_{L^\infty(0,T;L^2(\Omega_0^f;\mathbb{R}))}$ is also bounded by the right-hand side of (107); this bound is important for elliptic estimates that follow.

10.2. Estimate of $\tilde{w}_t$ independent of the regularization of $\tilde{a}$

In this subsection, let Ψ_i be one of the H^3 charts defining a neighborhood of Ω_0^s and let $W_i = \zeta_i^2\ \tilde{w} \circ \Psi_i$. Since the estimates that follow do not depend on the choice of Ψ_i, we will denote W_i simply by W.

Recall that for all $\phi \in L^2(0,T;H^1(\mathbb{R}^3;\mathbb{R}^3))$,

$$\begin{aligned}\int_0^T (W_{tt},\phi)_{L^2(\mathbb{R}^3;\mathbb{R}^3)}\,dt + \nu\int_0^T ((\tilde b_k^r \tilde b_k^s W,_r)_t,\ \phi,_s)_{L^2(\mathbb{R}^3_+;\mathbb{R}^3)}\,dt \\ + \int_0^T (C^{irks}W^k,_r,\phi^i,_s)_{L^2(\mathbb{R}^3_-;\mathbb{R})}\,dt - \int_0^T ((\tilde b_i^j Q)_t,\ \phi^i,_j)_{L^2(\mathbb{R}^3_+;\mathbb{R})}dt \\ = \int_0^T (F_{1t},\phi)_{L^2(\mathbb{R}^3_+;\mathbb{R}^3)} + (H_{it},\phi,_i)_{L^2(\mathbb{R}^3_+;\mathbb{R}^3)}\,dt \\ + \int_0^T (F_{2t},\phi)_{L^2(\mathbb{R}^3_-;\mathbb{R}^3)} + (K_{it},\phi,_i)_{L^2(\mathbb{R}^3_-;\mathbb{R}^3)}\,dt\ .\end{aligned}$$

With the choice of $\phi = D_{-h}D_hW_t$ in this variational formulation, which is possible since $\tilde w_t \in L^2(0,T;H^1_0(\Omega;\mathbb{R}^3))$, we then get

$$\begin{aligned}\frac{1}{2}\|D_hW_t(T)\|^2_{L^2(\mathbb{R}^3;\mathbb{R}^3)} + \nu\int_0^T (\tilde b_k^r\tilde b_k^s D_hW_t,_r\ ,\ D_hW_t,_s)_{L^2(\mathbb{R}^3_+;\mathbb{R}^3)} \\ +\frac{1}{2}(C^{irks}D_hW^k,_r(T),\ D_hW^i,_s(T))_{L^2(\mathbb{R}^3_-;\mathbb{R})} \\ -\int_0^T (D_h(\tilde b_i^j Q)_t,\ D_hW_t^i,_j)_{L^2(\mathbb{R}^3_+;\mathbb{R})} \\ +\nu\int_0^T (D_h(\tilde b_k^r\tilde b_k^s)W_t,_r^h\ ,\ D_hW_t,_s)_{L^2(\mathbb{R}^3_+;\mathbb{R}^3)} \\ +\nu\int_0^T ((\tilde b_k^r\tilde b_k^s)_t D_hW,_r\ ,\ D_hW_t,_s)_{L^2(\mathbb{R}^3_+;\mathbb{R}^3)} \\ +\nu\int_0^T (D_h(\tilde b_k^r\tilde b_k^s)_t W,_r^h\ ,\ D_hW_t,_s)_{L^2(\mathbb{R}^3_+;\mathbb{R}^3)} \\ +\int_0^T (D_hC^{irks}W^k,_r\ ,D_hW_t^i,_s)_{L^2(\mathbb{R}^3_-;\mathbb{R})} \\ \leqq C\ N(u_0,f)^2 \\ +\int_0^T (F_{1t},D_{-h}D_hW_t)_{L^2(\mathbb{R}^3_+;\mathbb{R}^3)} + (D_hH_{it},\ D_hW_t,_i)_{L^2(\mathbb{R}^3_+;\mathbb{R}^3)} \\ +\int_0^T (F_{2t},D_{-h}D_hW_t)_{L^2(\mathbb{R}^3_-;\mathbb{R}^3)} + (D_hH_{it},D_hW_t,_i)_{L^2(\mathbb{R}^3_-;\mathbb{R}^3)}. \qquad (108)\end{aligned}$$

Since the estimation of the integrals with the indefinite sign in this inequality does not create any new difficulty with respect to the estimates that we have obtained in the previous subsection (they are even easier since the more difficult integrals I_5 and I_6 do not have an analogue here), we provide the details in the appendix. With $\delta > 0$ to be fixed later, this leads us to

$$\begin{aligned}\int_0^T \|D_h\nabla W_t\|^2_{L^2(\mathbb{R}^3_+;\mathbb{R}^9)} + \sup_{[0,T]}\|D_h\nabla W\|^2_{L^2(\mathbb{R}^3_+;\mathbb{R}^9)} \\ \leqq [C\delta(1+C(M))N(u_0,f)^2 + C\delta_1(1+TC(M)+N(u_0,f)^2)]\|(\tilde w,\tilde q)\|^2_{Z_T}\end{aligned}$$

$$+C_{\delta_1} N(u_0, f)^2 + C_{\delta_1} T^{\frac{1}{4}} \|(\tilde{w}, \tilde{q})\|_{Z_T}^2$$
$$+C_\delta C(M) T^{\frac{1}{4}} (\|(\tilde{w}, \tilde{q})\|_{Z_T}^2 + N(u_0, f)^2).$$

We remark here that the estimates obtained in this section could have been performed with $t \in (0, T)$ generically replacing T; this explains the presence of

$$\sup_{[0,T]} \|D_h \nabla W\|_{L^2(\mathbb{R}^3_+;\mathbb{R}^9)}$$

on the left-hand side of this inequality. As this inequality is independent of h, we then deduce that

$$\int_0^T \|\nabla_0 \nabla W_t\|^2_{L^2(\mathbb{R}^3_+;\mathbb{R}^{18})} + \sup_{[0,T]} \|\nabla_0 \nabla W\|^2_{L^2(\mathbb{R}^3_+;\mathbb{R}^{18})}$$
$$\leqq [C\delta(1 + C(M))N(u_0, f)^2 + C\delta_1(1 + TC(M) + N(u_0, f)^2)]\|(\tilde{w}, \tilde{q})\|^2_{Z_T}$$
$$+C_{\delta_1} N(u_0, f)^2 + C_{\delta_1} T^{\frac{1}{4}} \|(\tilde{w}, \tilde{q})\|^2_{Z_T}$$
$$+C_\delta C(M) T^{\frac{1}{4}} (\|(\tilde{w}, \tilde{q})\|^2_{Z_T} + N(u_0, f)^2),$$

and thus for the trace, where we will write for notational convenience $\mathbb{R}^2 = \{x_3 = 0\}$,

$$\int_0^T \|\nabla_0 W_t\|^2_{H^{0.5}(\mathbb{R}^2;\mathbb{R}^6)} + \sup_{[0,T]} \|\nabla_0 W\|^2_{H^{0.5}(\mathbb{R}^2;\mathbb{R}^6)}$$
$$\leqq [C\delta(1 + C(M))N(u_0, f)^2 + C\delta_1(1 + TC(M) + N(u_0, f)^2)]\|(\tilde{w}, \tilde{q})\|^2_{Z_T}$$
$$+C_{\delta_1} N(u_0, f)^2 + C_{\delta_1} T^{\frac{1}{4}} \|(\tilde{w}, \tilde{q})\|^2_{Z_T}$$
$$+C_\delta C(M) T^{\frac{1}{4}} (\|(\tilde{w}, \tilde{q})\|^2_{Z_T} + N(u_0, f)^2),$$

which implies that

$$\int_0^T \|W_t\|^2_{H^{1.5}(\mathbb{R}^2;\mathbb{R}^3)} + \sup_{[0,T]} \|W\|^2_{H^{1.5}(\mathbb{R}^2;\mathbb{R}^3)}$$
$$\leqq [C\delta(1 + C(M))N(u_0, f)^2 + C\delta_1(1 + TC(M) + N(u_0, f)^2)]\|(\tilde{w}, \tilde{q})\|^2_{Z_T}$$
$$+C_{\delta_1} N(u_0, f)^2 + C_{\delta_1} T^{\frac{1}{4}} \|(\tilde{w}, \tilde{q})\|^2_{Z_T}$$
$$+C_\delta C(M) T^{\frac{1}{4}} (\|(\tilde{w}, \tilde{q})\|^2_{Z_T} + N(u_0, f)^2).$$

Since this has been done for any $W = \zeta_i^2\, \tilde{w} \circ \Psi_i$, we then deduce by the finite-covering argument and the fact that each Ψ_i is of class H^3 that

$$\int_0^T \|\tilde{w}_t\|^2_{H^{1.5}(\Gamma_0;\mathbb{R}^3)} + \sup_{[0,T]} \|\tilde{w}\|^2_{H^{1.5}(\Gamma_0;\mathbb{R}^3)}$$
$$\leqq [C\delta(1 + C(M))N(u_0, f)^2 + C\delta_1(1 + TC(M) + N(u_0, f)^2)]\|(\tilde{w}, \tilde{q})\|^2_{Z_T}$$
$$+C_{\delta_1} N(u_0, f)^2 + C_{\delta_1} T^{\frac{1}{4}} \|(\tilde{w}, \tilde{q})\|^2_{Z_T}$$
$$+C_\delta C(M) T^{\frac{1}{4}} (\|(\tilde{w}, \tilde{q})\|^2_{Z_T} + N(u_0, f)^2). \qquad (109)$$

Elliptic regularity for the Stokes problem (see [8]) (for $t \in [0, T]$ considered as fixed)

$$
\begin{aligned}
-\nu\triangle[\tilde{w}_t^i \circ \tilde{\eta}^{-1}] + (\tilde{q}_t \circ \tilde{\eta}^{-1}),_i &= -\tilde{w}_{tt}^i \circ \tilde{\eta}^{-1} + F_t^i \circ \tilde{\eta}^{-1} + \nu(\tilde{a}_l^j,_j \tilde{a}_l^k \tilde{w}^i,_k)_t \circ \tilde{\eta}^{-1} \\
&\quad -[\tilde{a}_i^j,_j \tilde{q}]_t \circ \tilde{\eta}^{-1} - [(\tilde{a}_i^k)_t \tilde{q},_k] \circ \tilde{\eta}^{-1}, \\
&\quad +\nu[(\tilde{a}_l^j \tilde{a}_l^k)_t \tilde{w}^i,_k],_j \circ \tilde{\eta}^{-1} \text{ in } \tilde{\eta}(t, \Omega_0^f), \\
\operatorname{div}(\tilde{w}_t \circ \tilde{\eta}^{-1})(t, \cdot) &= -[(\tilde{a}_i^j)_t w^i,_j] \circ \tilde{\eta}^{-1} \text{ in } \tilde{\eta}(t, \Omega_0^f), \\
\tilde{w}_t \circ \tilde{\eta}^{-1}(t, \cdot) &= 0 \text{ on } \tilde{\eta}(t, \partial\Omega), \\
\tilde{w}_t \circ \tilde{\eta}^{-1}(t, \cdot) &= \tilde{w}_t \circ \tilde{\eta}^{-1}(t, \cdot) \text{ on } \tilde{\eta}(t, \Gamma_0),
\end{aligned}
$$

then implies with the L^∞-in-time estimate (17) of η (and thus of $\tilde{\eta}$) into H^3, and the fact that $\|q_t(t)\|_{L^2(\Omega_0^f;\mathbb{R})}$ is bounded,

$$
\begin{aligned}
&\|\tilde{w}_t \circ \tilde{\eta}^{-1}(t)\|_{H^2(\tilde{\eta}(t,\Omega_0^f);\mathbb{R}^3)} + \|\tilde{q}_t \circ \tilde{\eta}^{-1}(t)\|_{H^1(\tilde{\eta}(t,\Omega_0^f);\mathbb{R})} \\
&\quad \leqq C[\| - \tilde{w}_{tt}^i \circ \tilde{\eta}^{-1} + F_t^i \circ \tilde{\eta}^{-1} - [(\tilde{a}_i^k)_t \tilde{q},_k] \circ \tilde{\eta}^{-1} \\
&\qquad +\nu[(\tilde{a}_l^j \tilde{a}_l^k)_t \tilde{w}^i,_k],_j \circ \tilde{\eta}^{-1}\|_{L^2(\tilde{\eta}(t,\Omega_0^f);\mathbb{R}^3)} + \|(\tilde{a}_l^j,_j \tilde{a}_l^k \tilde{w},_k)_t\|^2_{L^2(\tilde{\eta}(t,\Omega_0^f);\mathbb{R}^3)} \\
&\qquad +\|\tilde{w}_t \circ \tilde{\eta}^{-1}(t)\|_{H^{1.5}(\tilde{\eta}(t,\Gamma_0);\mathbb{R}^3)} + \|[\tilde{a}_i^j,_j \tilde{q}]_t \circ \tilde{\eta}^{-1}\|^2_{L^2(\tilde{\eta}(t,\Omega_0^f);\mathbb{R})}].
\end{aligned}
$$

Thus, once again with (17) and (16),

$$
\begin{aligned}
&\|\tilde{w}_t(t)\|_{H^2(\Omega_0^f;\mathbb{R}^3)} - C\,\|\nabla\tilde{w}_t(t)\|_{L^4(\Omega_0^f;\mathbb{R}^9)} + \|\tilde{q}_t(t)\|_{H^1(\Omega_0^f;\mathbb{R})} \\
&\quad \leqq C[\|\tilde{w}_{tt}\|_{L^2(\Omega_0^f;\mathbb{R}^3)} + \|\nabla\tilde{w}\|_{W^{1,4}(\Omega_0^f;\mathbb{R}^9)} + \|\nabla\tilde{w}\|_{L^4(\Omega_0^f;\mathbb{R}^9)} + \|\nabla\tilde{w}_t\|_{L^4(\Omega_0^f;\mathbb{R}^9)} \\
&\qquad +\|\nabla\tilde{q}\|_{L^4(\Omega_0^f;\mathbb{R}^3)} + \|f_t\|_{L^2(\Omega;\mathbb{R}^3)} + \|\tilde{w}_t\|_{H^{1.5}(\Gamma_0;\mathbb{R}^3)} + \sqrt{T}\|\tilde{q}_t\|_{L^2(\Omega_0^f;\mathbb{R})}],
\end{aligned}
$$

from which we immediately infer that

$$
\begin{aligned}
&\int_0^T \|\tilde{w}_t(t)\|^2_{H^2(\Omega_0^f;\mathbb{R}^3)} + \int_0^T \|\tilde{q}_t(t)\|^2_{H^1(\Omega_0^f;\mathbb{R})} \\
&\quad \leqq C\Bigg[\int_0^T \|\tilde{w}_{tt}\|^2_{L^2(\Omega_0^f;\mathbb{R}^3)} + \int_0^T \|f_t\|^2_{L^2(\Omega;\mathbb{R}^3)} + \int_0^T \|\tilde{w}_t\|^2_{H^{1.5}(\Gamma_0;\mathbb{R}^3)} \\
&\qquad +T^{\frac{1}{4}}\left[\|\nabla\tilde{w}_1\|^2_{H^1(\Omega_0^f;\mathbb{R}^9)} + T\int_0^T \|\nabla\tilde{w}_{tt}\|^2_{H^1(\Omega_0^f;\mathbb{R}^9)} + \int_0^T \|\nabla\tilde{w}_t\|^2_{H^1(\Omega_0^f;\mathbb{R}^9)}\right] \\
&\qquad +T^{\frac{1}{4}}\left[\|\nabla u_0\|^2_{H^1(\Omega_0^f;\mathbb{R}^9)} + T\int_0^T \|\nabla\tilde{w}_t\|^2_{H^1(\Omega_0^f;\mathbb{R}^9)} + \int_0^T \|\nabla\tilde{w}\|^2_{H^2(\Omega_0^f;\mathbb{R}^9)}\right] \\
&\qquad +T^{\frac{1}{4}}\left[N(u_0, f)^2 + T\int_0^T \|\tilde{q}_t\|^2_{H^1(\Omega_0^f;\mathbb{R})} + \int_0^T \|\tilde{q}\|^2_{H^2(\Omega_0^f;\mathbb{R})}\right]\Bigg].
\end{aligned}
$$

Thus, with the trace estimate (109) and (107),

$$
\begin{aligned}
&\int_0^T \|\tilde{w}_t\|^2_{H^2(\Omega_0^f;\mathbb{R}^3)} + \int_0^T \|\tilde{q}_t\|^2_{H^1(\Omega_0^f;\mathbb{R})} \\
&\quad \leqq [C\delta(1 + C(M))N(u_0, f)^2 + C\delta_1(1 + TC(M) + N(u_0, f)^2)]\|(\tilde{w}, \tilde{q})\|^2_{Z_T}
\end{aligned}
$$

$$+C_{\delta_1} N(u_0, f)^2 + C_\delta C(M) T^{\frac{1}{4}} (\|(\tilde{w}, \tilde{q})\|^2_{Z_T} + N(u_0, f)^2)$$
$$+C_{\delta_1} T^{\frac{1}{4}} \|(\tilde{w}, \tilde{q})\|^2_{Z_T}. \tag{110}$$

Similarly, the classical elliptic-regularity theory for the elasticity problem (for $t \in [0, T]$)

$$-[c^{ijkl} \tilde{w}^k,_l],_j = -\tilde{w}_{tt} + f_t \quad \text{in } \Omega_0^s,$$
$$\tilde{w}(t, \cdot) = \tilde{w}(t, \cdot) \quad \text{on } \Gamma_0 = \partial\Omega_0^s,$$

implies that $\|\tilde{w}\|_{H^2(\Omega_0^s;\mathbb{R}^3)} \leqq C\,[\,\| -\tilde{w}_{tt} + f_t\|_{L^2(\Omega_0^s;\mathbb{R}^3)} + \|\tilde{w}\|_{H^{1.5}(\Gamma_0;\mathbb{R}^3)}\,]$, which with (109) and (107) provides us with the estimate

$$\sup_{[0,T]} \|\tilde{w}\|^2_{H^2(\Omega_0^s;\mathbb{R}^3)} \leqq [C\delta(1 + C(M))N(u_0, f)^2$$
$$+C\delta_1(1 + TC(M) + N(u_0, f)^2)]\|(\tilde{w}, \tilde{q})\|^2_{Z_T}$$
$$+C_{\delta_1} N(u_0, f)^2 + C_\delta C(M) T^{\frac{1}{4}} (\|(\tilde{w}, \tilde{q})\|^2_{Z_T} + N(u_0, f)^2)$$
$$+C_{\delta_1} T^{\frac{1}{4}} \|(\tilde{w}, \tilde{q})\|^2_{Z_T}. \tag{111}$$

10.3. Estimate of $\tilde{w}$ independent of the regularization of $\tilde{a}$

Just as in the previous subsection, W again denotes $W_i = \zeta^2\, \tilde{w} \circ \Psi_i$, where we recall that Ψ_i denotes the ith chart.

Choosing $\phi = D_{-h} D_h D_{-h} D_h W$ in the variational formulation (90), we then find that

$$\frac{1}{2}\|D_{-h} D_h W(T)\|^2_{L^2(\mathbb{R}^3;\mathbb{R}^3)} + \nu \int_0^T (\tilde{b}_k^r \tilde{b}_k^s D_{-h} D_h W,_r\,,\; D_{-h} D_h W,_s\,)_{L^2(\mathbb{R}^3_+;\mathbb{R}^3)}$$
$$+\frac{1}{2}(C^{irks} D_{-h} D_h \int_0^T W^k,_r\,,\; D_{-h} D_h \int_0^T W^i,_s\,)_{L^2(\mathbb{R}^3_-;\mathbb{R})}$$
$$-\int_0^T (D_{-h} D_h(\tilde{b}_i^j Q),\; D_{-h} D_h W^i,_j\,)_{L^2(\mathbb{R}^3_+;\mathbb{R})}$$
$$+\nu \int_0^T (D_{-h} D_h(\tilde{b}_k^r \tilde{b}_k^s) W,_r\,,\; D_{-h} D_h W,_s\,)_{L^2(\mathbb{R}^3_+;\mathbb{R}^3)}$$
$$-\sum_{p=0}^{1} \nu \int_0^T (D_{(-1)^p h}(\tilde{b}_k^r \tilde{b}_k^s) D_{(-1)^p h} W,_r\,,\; D_{-h} D_h W,_s\,)_{L^2(\mathbb{R}^3_+;\mathbb{R}^3)}$$
$$+\int_0^T (D_{-h} D_h[C^{irks}] \int_0^T W^k,_r\,, D_{-h} D_h W^i,_s\,)_{L^2(\mathbb{R}^3_-;\mathbb{R})}$$
$$-\sum_{p=0}^{1} \int_0^T (D_{(-1)^p h}[C^{irks}] \int_0^T D_{(-1)^p h} W^k,_r\,, D_{-h} D_h W^i,_s\,)_{L^2(\mathbb{R}^3_-;\mathbb{R})}$$
$$\leqq C\, N(u_0, f)^2 + \int_0^T (D_{-h} F_1, D_h D_{-h} D_h W)_{L^2(\mathbb{R}^3_+;\mathbb{R}^3)}$$

$$+\int_0^T (D_{-h}D_h H_i,\ D_{-h}D_h W,_i)_{L^2(\mathbb{R}^3_+;\mathbb{R}^3)}$$
$$+\int_0^T (D_{-h}D_h F_2, D_{-h}D_h W)_{L^2(\mathbb{R}^3_-;\mathbb{R}^3)} + (D_{-h}D_h K_i, D_h W,_i)_{L^2(\mathbb{R}^3_-;\mathbb{R}^3)}. \quad (112)$$

Similarly as in the previous subsection, the estimates provided in the appendix yield (with $\delta > 0$ to be fixed later)

$$\begin{aligned}\int_0^T \|D_{-h}D_h \nabla W\|^2_{L^2(\mathbb{R}^3_+;\mathbb{R}^9)} &+ \sup_{[0,T]} \|D_{-h}D_h \nabla \int_0^\cdot W\|^2_{L^2(\mathbb{R}^3_+;\mathbb{R}^9)}\\ &\leqq C\delta\,(1+C(M))N(u_0, f)^2\,\|(\tilde{w},\tilde{q})\|^2_{Z_T} + CN(u_0,f)^2\\ &\quad + C_\delta C(M)T^{\frac14}\,(\|(\tilde{w},\tilde{q})\|^2_{Z_T} + N(u_0,f)^2).\end{aligned}$$

As this inequality is independent of h, we deduce just as in the previous section that

$$\begin{aligned}\int_0^T \|\tilde{w}\|^2_{H^{2.5}(\Gamma_0;\mathbb{R}^3)} &+ \sup_{[0,T]} \|\int_0^\cdot \tilde{w}\|^2_{H^{2.5}(\Gamma_0;\mathbb{R}^3)}\\ &\leqq C\delta\,(1+C(M))N(u_0, f)^2\,\|(\tilde{w},\tilde{q})\|^2_{Z_T} + CN(u_0,f)^2\\ &\quad + C_\delta C(M)T^{\frac14}\,(\|(\tilde{w},\tilde{q})\|^2_{Z_T} + N(u_0,f)^2). \quad (113)\end{aligned}$$

Elliptic regularity for the Stokes problem (see [8]) for $t \in [0, T]$,

$$\begin{aligned}-\nu\triangle[\tilde{w}\circ\tilde{\eta}^{-1}](t,\cdot) + \nabla(\tilde{q}\circ\tilde{\eta}^{-1})(t,\cdot) &= -\tilde{w}_t\circ\tilde{\eta}^{-1} + f + \nu\tilde{a}^j_l,_j \circ\eta^{-1}(\tilde{w}\circ\eta^{-1}),_l\\ &\quad -(\tilde{a}^j_i,_j\,\tilde{q})\circ\tilde{\eta}^{-1} \text{in } \tilde{\eta}(t,\Omega^f_0),\\ \operatorname{div}(\tilde{w}\circ\tilde{\eta}^{-1})(t,\cdot) &= 0 \text{ in } \tilde{\eta}(t,\Omega^f_0),\\ (\tilde{w}\circ\tilde{\eta}^{-1})(t,\cdot) &= 0 \text{ on } \tilde{\eta}(t,\partial\Omega),\\ (\tilde{w}\circ\tilde{\eta}^{-1})(t,\cdot) &= (\tilde{w}\circ\tilde{\eta}^{-1})(t,\cdot) \text{ on } \tilde{\eta}(t,\Gamma_0),\end{aligned}$$

then implies with (17)

$$\begin{aligned}\|\tilde{w}\circ\tilde{\eta}^{-1}(t)\|_{H^3(\tilde{\eta}(t,\Omega^f_0);\mathbb{R}^3)} &+ \|\tilde{q}\circ\tilde{\eta}^{-1}(t)\|_{H^2(\tilde{\eta}(t,\Omega^f_0);\mathbb{R})}\\ &\leqq C\,[\,\|-\tilde{w}_t\circ\tilde{\eta}^{-1} + f + \nu\tilde{a}^j_l,_j\circ\eta^{-1}(\tilde{w}\circ\eta^{-1}),_l\|_{H^1(\tilde{\eta}(t,\Omega^f_0);\mathbb{R}^3)}\\ &\quad + \|(\tilde{a}^j_i,_j\,\tilde{q})\circ\tilde{\eta}^{-1}\|_{H^1(\tilde{\eta}(t,\Omega^f_0);\mathbb{R})} + \|\tilde{w}\circ\tilde{\eta}^{-1}(t)\|_{H^{2.5}(\tilde{\eta}(t,\Gamma_0);\mathbb{R}^3)}].\end{aligned}$$

Thus, with (17),

$$\begin{aligned}\|\tilde{w}(t)\|_{H^3(\Omega^f_0;\mathbb{R}^3)} &- C\|\tilde{w}(t)\|_{W^{2,4}(\Omega^f_0;\mathbb{R}^3)} + \|\tilde{q}(t)\|_{H^2(\Omega^f_0;\mathbb{R})} - C\|\tilde{q}(t)\|_{W^{1,4}(\Omega^f_0;\mathbb{R})}\\ &\leqq C[\|\tilde{w}_t\|_{H^1(\Omega^f_0;\mathbb{R}^3)} + \|\tilde{w}\|_{W^{2,4}(\Omega^f_0;\mathbb{R}^3)} + \|f\|_{H^1(\Omega;\mathbb{R}^3)} + \|\tilde{w}\|_{H^{2.5}(\Gamma_0;\mathbb{R}^3)}],\end{aligned}$$

from which we immediately infer,

$$\int_0^T \|\tilde{w}(t)\|^2_{H^3(\Omega_0^f;\mathbb{R}^3)} + \int_0^T \|\tilde{q}(t)\|^2_{H^2(\Omega_0^f;\mathbb{R})}$$
$$\leqq C\left[TN(u_0, f)^2 + T^2\int_0^T \|\tilde{w}_t\|^2_{H^1(\Omega_0^f;\mathbb{R}^3)} \right.$$
$$+\int_0^T \|f\|^2_{H^1(\Omega;\mathbb{R}^3)} + \int_0^T \|\tilde{w}\|^2_{H^{2.5}(\Gamma_0;\mathbb{R}^3)}$$
$$+C\,T^{\frac{1}{4}}\left[N(u_0, f)^2 + T\int_0^T \|\tilde{q}_t\|^2_{H^1(\Omega_0^f;\mathbb{R}^3)} + \int_0^T \|\tilde{q}\|^2_{H^2(\Omega_0^f;\mathbb{R}^3)}\right]$$
$$\left.+C\,T^{\frac{1}{4}}\left[\|u_0\|^2_{H^2(\Omega_0^f;\mathbb{R}^3)} + T\int_0^T \|\tilde{w}_t\|^2_{H^2(\Omega_0^f;\mathbb{R}^3)} + \int_0^T \|\tilde{w}\|^2_{H^3(\Omega_0^f;\mathbb{R}^3)}\right]\right].$$

Thus, with the trace estimate (113),

$$\int_0^T \|\tilde{w}\|^2_{H^3(\Omega_0^f;\mathbb{R}^3)} + \int_0^T \|\tilde{q}\|^2_{H^2(\Omega_0^f;\mathbb{R})}$$
$$\leqq [C\delta(1 + C(M))N(u_0, f)^2 + C\delta_1(1 + TC(M) + N(u_0, f)^2)]\|(\tilde{w}, \tilde{q})\|^2_{Z_T}$$
$$+C_{\delta_1}N(u_0, f)^2 + C_\delta C(M)T^{\frac{1}{4}}\,(\|(\tilde{w}, \tilde{q})\|^2_{Z_T} + N(u_0, f)^2)$$
$$+C_{\delta_1}T^{\frac{1}{4}}\|(\tilde{w}, \tilde{q})\|^2_{Z_T}. \tag{114}$$

Similarly, elliptic regularity for the elasticity problem (for $t \in [0, T]$)

$$-[c^{ijkl}\int_0^t \tilde{w}^k{}_{,l}\,]_{,j} = -\tilde{w}_t + f \quad \text{in } \Omega_0^s$$
$$\int_0^t \tilde{w}(t,\cdot) = \int_0^t \tilde{w}(t,\cdot) \quad \text{on } \Gamma_0 = \partial\Omega_0^s,$$

implies that

$$\left\|\int_0^t \tilde{w}\right\|_{H^3(\Omega_0^s;\mathbb{R}^3)} \leqq C\left[\|-\tilde{w}_t + f\|_{H^1(\Omega_0^s;\mathbb{R})} + \left\|\int_0^t \tilde{w}\right\|_{H^{2.5}(\Gamma_0;\mathbb{R}^3)}\right],$$

which with (113) and (107) provides the inequality

$$\sup_{[0,T]}\left\|\int_0^{\cdot} \tilde{w}\right\|^2_{H^3(\Omega_0^s;\mathbb{R}^3)}$$
$$\leqq [C\delta(1 + C(M))N(u_0, f)^2 + C\delta_1(1 + TC(M) + N(u_0, f)^2)\,]\,\|(\tilde{w}, \tilde{q})\|^2_{Z_T}$$
$$+C_{\delta_1}N(u_0, f)^2 + C_\delta C(M)T^{\frac{1}{4}}\,(\|(\tilde{w}, \tilde{q})\|^2_{Z_T} + N(u_0, f)^2)$$
$$+C_{\delta_1}T^{\frac{1}{4}}\|(\tilde{w}, \tilde{q})\|^2_{Z_T}. \tag{115}$$

10.4. Existence and uniqueness of a smooth solution for the non-regularized system (20).

We now infer from (107), (110), (111), (114) and (115) that

$$\begin{aligned}\|(\tilde{w},\tilde{q})\|^2_{Z_T} \leqq & [\, C\delta(1+C(M)+N(u_0,f)^2) \\ & +C\delta_1(1+TC(M)+N(u_0,f)^2)\,]\,\|(\tilde{w},\tilde{q})\|^2_{Z_T} \\ & +C_{\delta_1}N(u_0,f)^2+C_\delta C(M)T^{\frac14}\,(\|(\tilde{w},\tilde{q})\|^2_{Z_T}+N(u_0,f)^2) \\ & +C_{\delta_1}T^{\frac14}\|(\tilde{w},\tilde{q})\|^2_{Z_T},\end{aligned}$$

this inequality being independent of the smoothing parameter of $\tilde{a}$.

We will call the constant C in this inequality C_1 to indicate that at this stage it is a constant given by our successive estimates which, for the sake of conciseness, we have yet to make explicit.

First, we fix δ_1 so that

$$C_1\delta_1 \leqq \tfrac18 \text{ and } C_1\delta_1 N(u_0,f)^2 \leqq \tfrac18.$$

The constant C_{δ_1} becomes thus determined, and we have

$$\begin{aligned}\|(\tilde{w},\tilde{q})\|^2_{Z_T} \leqq & \; [C_1\delta\,(1+C(M))N(u_0,f)^2+C_1\delta_1 TC(M)]\,\|(\tilde{w},\tilde{q})\|^2_{Z_T} \\ & +[C_{\delta_1}+C_\delta C(M)]\,T^{\frac14}\|(\tilde{w},\tilde{q})\|^2_{Z_T}+C_{\delta_1}N(u_0,f)^2 \\ & +C_\delta C(M)N(u_0,f)^2T^{\frac14}+\frac14\|(\tilde{w},\tilde{q})\|^2_{Z_T}.\end{aligned}$$

Now let

$$M=\sup(M_0,\ 2\,[C_1+C_{\delta_1}]\,N(u_0,f)^2)\,. \tag{116}$$

Consequently, $C(M)$ becomes a fixed constant. Now, let us fix $\delta>0$ small enough so that

$$\begin{aligned}\|(\tilde{w},\tilde{q})\|^2_{Z_T} \leqq & \; [\tfrac18+C_1\delta_1 TC(M)+[C_{\delta_1}+C_\delta C(M)]T^{\frac14}]\,\|(\tilde{w},\tilde{q})\|^2_{Z_T} \\ & +C_\delta C(M)\,T^{\frac14}\,N(u_0,f)^2+C_{\delta_1}\,N(u_0,f)^2+\tfrac14\|(\tilde{w},\tilde{q})\|^2_{Z_T}.\end{aligned}$$

Now, let $T\in(0,T_M)$ be small enough so that

$$\tfrac34\|(\tilde{w},\tilde{q})\|^2_{Z_T} \leqq \tfrac14\,\|(\tilde{w},\tilde{q})\|^2_{Z_T}+C_1\,N(u_0,f)^2+C_{\delta_1}\,N(u_0,f)^2,$$

which implies

$$\|(\tilde{w},\tilde{q})\|^2_{Z_T} \leqq M\,. \tag{117}$$

Henceforth, we revert to our original notation, denoting $\tilde{w}$ and $\tilde{a}$ by the sequential notation w_n and a_n, respectively. The uniform bound (117) ensures the existence of a weakly convergent subsequence $(w_{\sigma(n)}, q_{\sigma(n)})$ in the reflexive Hilbert space Y_T such that

$$(w_{\sigma(n)}, q_{\sigma(n)}) \rightharpoonup (w, q) \text{ in } Y_T.$$

The usual compactness arguments then provide the strong convergence

$$(w_{\sigma(n)}, q_{\sigma(n)}) \to (w, q) \text{ in } L^2(0, T; H^2(\Omega_0^f; \mathbb{R}^3)) \times L^2(0, T; H^1(\Omega_0^f; \mathbb{R})) . \tag{118}$$

Combined with the strong convergence

$$a_n \to a \text{ in } L^2(0, T; H^2(\Omega_0^f; \mathbb{R}^9))$$

(which follows from the mollification process), the Sobolev embeddings provide the strong convergence

$$\begin{aligned} a_n a_n^T \nabla w_n &\to a a^T \nabla w \text{ in } L^2(0, T; L^2(\Omega_0^f; \mathbb{R}^9)), \\ a_n^T q_n &\to a^T q \text{ in } L^2(0, T; L^2(\Omega_0^f; \mathbb{R}^9)). \end{aligned}$$

We then deduce from (53) that, for each $\phi \in L^2(0, T; H_0^1(\Omega_0^f; \mathbb{R}^3))$,

$$\begin{aligned} &\int_0^T (w_t, \phi)_{L^2(\Omega;\mathbb{R}^3)}\, dt + \nu \int_0^T (a_k^r w,_r \,,\ a_k^s \phi,_s)_{L^2(\Omega_0^f;\mathbb{R}^3)}\, dt \\ &\quad + \int_0^T \left(c^{ijkl} \int_0^t w^k,_l \,, \phi^i,_j \right)_{L^2(\Omega_0^s;\mathbb{R})} dt + \int_0^T (q,\ a_k^l \phi^k,_l)_{L^2(\Omega_0^f;\mathbb{R})} dt \\ &= \int_0^T (f \circ \eta, \phi)_{L^2(\Omega_0^f;\mathbb{R}^3)} + (f, \phi)_{L^2(\Omega_0^s;\mathbb{R}^3)}\, dt, \end{aligned} \tag{119}$$

which combined with the fact that, from (118), for all $t \in [0, T]$, $w(t) \in \mathcal{V}_v(t)$, proves that w is a weak solution of (20).

Since $w \in L^2(0, T; H_0^1(\Omega; \mathbb{R}^3))$ we infer the uniqueness of a solution in Y_T to this system in the same classical fashion as for the solution $\tilde{w}$ of the regularized problem.

Moreover, it is also obvious that we have from (117) the estimate

$$\|w\|_{W_T}^2 \leqq M. \tag{120}$$

This concludes the proof of Theorem 2.

Henceforth, M is given by (116) and T is chosen such that (117) holds.

11. The fixed-point scheme for the nonlinear problem

We will make use of the Tychonoff fixed-point Theorem in our fixed-point procedure (see, for example, [5]). Recall that this states that for a reflexive separable Banach space X, and $C \subset X$ a closed, convex, bounded subset, if $F : C \to C$ is weakly sequentially continuous into X, then F has at least one fixed point.

With the quantities M and T being defined as in the previous section, we make the following

Definition 4. We define a mapping Θ_T from $C_T(M)$ into itself (from estimate (120)), which associates $w \in W_T$, the unique solution in Y_T of (20), with a given element $v \in W_T$.

We next have the following weak sequential continuity result.

Lemma 16. *The mapping Θ_T associating w with $v \in C_T(M)$ is weakly sequentially continuous from $C_T(M)$ into $C_T(M)$ (endowed with the norm of X_T).*

Proof. Let $(v_p)_{p\in\mathbb{N}}$ be a given sequence of elements of $C_T(M)$ weakly convergent (in X_T) toward a given element $v \in C_T(M)$ ($C_T(M)$ is sequentially weakly closed as a closed convex set) and let $(v_{\sigma(p)})_{p\in\mathbb{N}}$ be any subsequence of this sequence.

Since $V_f^3(T)$ is compactly embedded into $L^2(0, T; H^2(\Omega_0^f; \mathbb{R}^3))$, we deduce the following strong convergence results in $L^2(0, T; L^2(\Omega_0^f; \mathbb{R}))$ as p goes to ∞:

$$(a_l^j)_p(a_l^k)_p \to a_l^j a_l^k, \tag{121a}$$

$$[(a_l^j)_p(a_l^k)_p]_{,j} \to (a_l^j a_l^k)_{,j}, \tag{121b}$$

$$(a_i^k)_p \to a_i^k, \tag{121c}$$

$$f^i \circ \eta_p \to f^i \circ \eta. \tag{121d}$$

Now, let $w_p = \Theta_T(v_p)$ and let q_p be the associated pressure, so that $(q_p)_{p\in\mathbb{N}}$ is in a bounded set of $V_f^2(T)$. Since $Y_T = X_T \times V_f^2(T)$ is a reflexive Hilbert space, let $(w_{\sigma'(p)}, q_{\sigma'(p)})_{p\in\mathbb{N}}$ be a subsequence weakly converging in Y_T toward an element $(w, q) \in Y_T$. Since $C_T(M)$ is weakly closed in X_T, we also have $w \in C_T(M)$.

We can then infer in a similar fashion as for the proof of (119) in the previous section, that for each $\phi \in L^2(0, T; H_0^1(\Omega_0^f; \mathbb{R}^3))$,

$$\begin{aligned}\int_0^T (w_t, \phi)_{L^2(\Omega;\mathbb{R}^3)}\, dt &+ \nu \int_0^T (a_k^r w,_r\, ,\ a_k^s \phi,_s\,)_{L^2(\Omega_0^f;\mathbb{R}^3)}\, dt \\ &+ \int_0^T \left(c^{ijkl} \int_0^t w^k,_l\, , \phi^i,_j \right)_{L^2(\Omega_0^s;\mathbb{R})} dt + \int_0^T (q,\ a_k^l \phi^k,_l\,)_{L^2(\Omega_0^f;\mathbb{R})} dt \\ = \int_0^T (f \circ \eta, \phi)_{L^2(\Omega_0^f;\mathbb{R}^3)} &+ (f, \phi)_{L^2(\Omega_0^s;\mathbb{R}^3)}\, dt,\end{aligned}$$

which combined with the fact that, from (121), for all $t \in [0, T]$, $w(t) \in \mathcal{V}_v(t)$, shows that w is a weak solution of (20) in $C_T(M)$, i.e., $w = \Theta_T(v)$.

Therefore, we deduce that the whole sequence $(\Theta_T(v_n))_{n\in\mathbb{N}}$ weakly converges in $C_T(M)$ toward $\Theta_T(v)$, which concludes the proof. □

12. Proof of Theorem 1

The mapping Θ being weakly continuous from the closed bounded convex set $C_T(M)$ into itself from Lemma 16, we infer from the Tychonoff fixed-point theorem (see, for instance [5]) that it admits (at least) one fixed point $v = \Theta(v)$ in

C_T. Moreover, since $T \leqq T_M$, Lemma 5 ensures that there is no collision between solids or between a solid and $\partial\Omega$. Thus, (v, q) is a solution of (4). Note that the continuity of the Lagrangian velocities $v^f = v^s$ at the interface Γ_0 is ensured by our functional framework, since $(v, q) \in X_T$ implies $v \in L^2(0, T; H_0^1(\Omega; \mathbb{R}^3))$, which provides the equality $v^f = v^s$ in $H^{\frac{1}{2}}(\Gamma_0; \mathbb{R}^3)$.

13. Uniqueness

Uniqueness will be obtained under stronger assumptions than the ones used to establish existence, for reasons that will be explained hereafter.

If $(\tilde{v}, \tilde{q}) \in Y_T$ is another solution of (4), then

$$(v-\tilde{v})^i_t - \nu(a^j_l a^k_l (v^i,_k - \tilde{v}^i,_k)),_j + a^k_i (q,_k - \tilde{q},_k) = \delta f^i \text{ in } (0,T)\times\Omega^f_0, \tag{122a}$$

$$a^k_i (v-\tilde{v})^i,_k = \delta a \text{ in } (0,T)\times\Omega^f_0, \tag{122b}$$

$$(v-\tilde{v})^i_t - \left[c^{ijkl}\int_0^t (v-\tilde{v})^k,_l\right],_j = 0 \text{ in } (0,T)\times\Omega^s_0, \tag{122c}$$

$$\nu(v^f-\tilde{v}^f)^i,_k a^k_l a^j_l N_j - (q-\tilde{q}) a^j_i N_j = c^{ijkl}\int_0^t (v^s-\tilde{v}^s)^k,_l\, N_j + \delta g^i \text{ on } (0,T)\times\Gamma_0, \tag{122d}$$

$$v-\tilde{v} = 0 \text{ on } (0,T)\times\partial\Omega, \tag{122e}$$

$$v-\tilde{v} = 0 \text{ on } \{0\}\times\Omega, \tag{122f}$$

with

$$\delta f^i = -\nu((a^j_l a^k_l - \tilde{a}^j_l \tilde{a}^k_l)\, \tilde{v}^i,_k)),_j + f\circ\eta - f\circ\tilde{\eta} + (-a^k_i + \tilde{a}^k_i)\tilde{q},_k \text{ in } (0,T)\times\Omega^f_0, \tag{123a}$$

$$\delta a = (\tilde{a}^k_i - a^k_i)\tilde{v}^i,_k \text{ in } (0,T)\times\Omega^f_0, \tag{123b}$$

$$\delta g^i = \nu(\tilde{v}^f,^i_k \tilde{a}^k_l \tilde{a}^j_l N_j - \tilde{v}^f,^i_k a^k_l a^j_l N_j) - \tilde{q}(\tilde{a}^j_i - a^j_i) N_j \text{ on } (0,T)\times\Gamma_0. \tag{123c}$$

If we view this problem with $v - \tilde{v}$ as the unknown velocity and $q - \tilde{q}$ as the associated pressure in the fluid, this problem looks similar to the linear problem (20); it is tempting to conclude that similar estimates as in the study of the regularity of (20) would yield a differential inequality that would provide uniqueness. It appears, however, that such a procedure fails because, due to the Dirichlet boundary condition on $\partial\Omega$ for the velocity, we are not able to get the necessary information on the second derivative of the pressure function. Such information is crucial since δf_{tt} contains $\tilde{q}_{tt}$, which makes the second time differentiated problem impossible to estimate.

For this reason, we will need to impose more regularity on the data and forcing functions, so that we, in turn, have enough information on $\tilde{q}_{tt}$, which will then be viewed as a coefficient in the study of the regularity of (122) in Y_T.

We first update the functional framework. Let us define

$$V_f^4(T) = \{u \in L^2(0,T; H^4(\Omega_0^f; \mathbb{R}^3)) |\, u_t \in V_f^3(T)\},$$
$$V_s^4(T) = \{u \in L^2(0,T; H^4(\Omega_0^s; \mathbb{R}^3)) |\, u_t \in V_s^3(T)\}.$$

Let us define the reflexive separable Hilbert space as

$$X_T^u = \left\{ u \in L^2(0,T; H_0^1(\Omega; \mathbb{R}^3)) |\, \left(u^f, \int_0^\cdot u^s \right) \in V_f^4(T) \times V_s^4(T) \right\},$$

endowed with its natural norm

$$\|u\|^2_{X_T^u} = \|u_t\|^2_{X_T} + \|u\|^2_{L^2(0,T;H^4(\Omega_0^f;\mathbb{R}^3))} + \left\| \int_0^\cdot u \right\|^2_{L^2(0,T;H^4(\Omega_0^s;\mathbb{R}^3))}.$$

In a similar fashion, we introduce

$$\begin{aligned} Y_T^u = \{(u,p) |\, & u \in X_T^u,\ p \in L^2(0,T; H^3(\Omega_0^f; \mathbb{R})),\ p_t \in L^2(0,T; H^2(\Omega_0^f; \mathbb{R})), \\ & p_{tt} \in L^2(0,T; H^1(\Omega_0^f; \mathbb{R}))\}, \end{aligned}$$

endowed with its natural norm

$$\begin{aligned} \|(u,p)\|^2_{Y_T^u} = {} & \|u\|^2_{X_T^u} + \|p\|^2_{L^2(0,T;H^3(\Omega_0^f;\mathbb{R}))} + \|p_t\|^2_{L^2(0,T;H^2(\Omega_0^f;\mathbb{R}))} \\ & + \|p_{tt}\|^2_{L^2(0,T;H^1(\Omega_0^f;\mathbb{R}))}. \end{aligned}$$

We will also need

$$\begin{aligned} W_T^u = \Big\{ & u \in X_T^u |\, u_{ttt} \in L^\infty(0,T; L^2(\Omega; \mathbb{R}^3)),\ \int_0^\cdot u \in L^\infty(0,T; H^4(\Omega_0^s; \mathbb{R}^3) \\ & u \in L^\infty(0,T; H^3(\Omega_0^s; \mathbb{R}^3),\ u_t \in L^\infty(0,T; H^2(\Omega_0^s; \mathbb{R}^3) \\ & u_{tt} \in L^\infty(0,T; H^1(\Omega_0^s; \mathbb{R}^3) \Big\}, \end{aligned}$$

endowed with its natural norm

$$\begin{aligned} \|u\|^2_{W_T^u} = {} & \|u\|^2_{X_T^u} + \|u_{ttt}\|^2_{L^\infty(0,T;L^2(\Omega;\mathbb{R}^3))} + \left\| \int_0^\cdot u \right\|^2_{L^\infty(0,T;H^4(\Omega_0^s;\mathbb{R}^3))} \\ & + \|u\|^2_{L^\infty(0,T;H^3(\Omega_0^s;\mathbb{R}^3))} + \|u_t\|^2_{L^\infty(0,T;H^2(\Omega_0^s;\mathbb{R}^3))} \\ & + \|u_{tt}\|^2_{L^\infty(0,T;H^1(\Omega_0^s;\mathbb{R}^3))}, \end{aligned}$$

as well as $Z_T^u(T) = \{(u,p) \in Y_T^u |\, u \in W_T^u\}$ endowed with its natural norm,

$$\begin{aligned} \|(u,p)\|^2_{Z_T^u} = {} & \|u\|^2_{W_T^u} + \|p\|^2_{L^2(0,T;H^3(\Omega_0^f;\mathbb{R}))} + \|p_t\|^2_{L^2(0,T;H^2(\Omega_0^f;\mathbb{R}))} \\ & + \|p_{tt}\|^2_{L^2(0,T;H^1(\Omega_0^f;\mathbb{R}))}. \end{aligned}$$

We can then define the convex set $C_M^u(T)$ in the same fashion as $C_M(T)$, with W_T^U replacing W_T and with the additional condition $w_{tt}(0) = w_2$ where w_2 has been defined in (28).

We can then prove, in a way similar to the proof of Theorem 2 (with the introduction of the penalized problems, time-differentiated three times now) that the following holds:

Theorem 5. *With the same assumptions as in Theorem 1, and with the additional conditions that* Ω *is of class* H^4, Ω_0^s *is of class* H^5, *the initial data* $u_0 \in H^7(\Omega_0^f;\mathbb{R}^3)\cap H^4(\Omega_0^s;\mathbb{R}^3)\cap H_0^1(\Omega;\mathbb{R}^3)\cap L^2_{\mathrm{div,f}}$, $f(0) \in H^5(\Omega;\mathbb{R}^3)$, *satisfying the supplementary compatibility conditions* (*recall the assumption of Section* 2)

$$\begin{aligned}
&[(\nu[\nabla w_2^f\ N]^i + 2\nu w_1^f{,}_k^i\,(a_l^k a_l^j)_t(0)N_j + \nu u_0^f{,}_k^i\,(a_l^k a_l^j)_{tt}(0))_{i=1}^3]_{\tan}\\
&\quad = [\ (c^{ijkl} w_1^s{,}_l^k\ N_j)_{i=1}^3\ ]_{\tan} \quad \text{on } \Gamma_0,\\
&w_2 \in H_0^1(\Omega;\mathbb{R}^3),
\end{aligned}$$

and the supplementary assumption on the forcing function that

$$\begin{aligned}
f &\in L^2(0,\bar{T};H^3(\Omega;\mathbb{R}^3)),\ \ f_t \in L^2(0,\bar{T};H^2(\Omega;\mathbb{R}^3)),\\
f_{tt} &\in L^2(0,\bar{T};H^1(\Omega;\mathbb{R}^3))\\
f_{ttt} &\in L^2(0,\bar{T};L^2(\Omega;\mathbb{R}^3)),
\end{aligned}$$

there is a $T > 0$ *such that there exists a solution* $(v,q) \in Y_T^u$ *of* (4). *Furthermore,* $v \in C_M^u(T)$ *for* M *appropriate.*

We can now get estimates for (122) which will give an appropriate differential inequality, in the space Z_T used to prove Theorem 2. We notice that this problem is similar to (20) except for the divergence-type condition which is not set to zero, and the boundary forcing on the interface.

The Neumann forcing does not give any specific difficulty, and can be handled without modification of our previous estimates.

The divergence-type condition does not bring any difficulty either because we do not need to establish the existence of a solution to (122), since it comes *de facto* from the definition of v and $\tilde{v}$; we can directly use this condition in the steps where we obtained ε-independent estimates for w_{tt}, w_t and w. We also do not have to regularize the coefficients, since the regularity of w is a given. Those three steps would provide us in the same fashion as for the proof of Theorem 20 with the appropriate estimates to be made precise later. Note that this process works because the right-hand side of the divergence condition for w in (122) has (roughly speaking) the term $\nabla\eta - \nabla\tilde{\eta}$, which has one time derivative less than the left-hand side $\nabla v - \nabla\tilde{v}$ (the term ∇v on the right-hand side being viewed as a coefficient whose regularity is given by Theorem 5).

We are now in a position to state the uniqueness result.

Theorem 6. *With the same assumptions as in Theorem 5, and with the additional assumption that there exists* $K > 0$ *such that*

$$\forall t \leqq \bar{T},\ \forall (x, y) \in \Omega \times \Omega, |f(t, x) - f(t, y)| + |\nabla f(t, x) \\ -\nabla f(t, y)| + |f_t(t, x) - f_t(t, y)| + |\nabla f_t(t, x) \\ -\nabla f_t(t, y)| + |f_{tt}(t, x) - f_{tt}(t, y)| + |\nabla^2 f(t, x) - \nabla^2 f(t, y)| \\ \leqq K\ |x - y|, \tag{124}$$

i.e., f, ∇f, $\nabla^2 f$, f_t, ∇f_t and f_{tt} are uniformly Lipschitz continuous in the spatial variable, then the solution is unique.

Proof. With those assumptions on f, we have for the forcing $f \circ \eta - f \circ \tilde{\eta}$ appearing in (122), an estimate

$$\|f \circ \eta - f \circ \tilde{\eta}\|_{L^2(0,T;H^2(\Omega_0^f;\mathbb{R}^3))} + \|(f \circ \eta - f \circ \tilde{\eta})_t\|_{L^2(0,T;H^1(\Omega_0^f;\mathbb{R}^3))} \\ + \|(f \circ \eta - f \circ \tilde{\eta})_{tt}\|_{L^2(0,T;L^2(\Omega_0^f;\mathbb{R}^3))} \leqq C\ \|\eta - \tilde{\eta}\|_{V_f^3(T)}.$$

The other terms associated with δf, δg, δa have the same effect in the integral estimates for w_{tt}, w_t and w. This leads us to

$$\forall t \in (0, T),\ \|(v - \tilde{v}, q - \tilde{q})\|_{Z_t} \leqq C_1\ \|\eta - \tilde{\eta}\|_{V_f^3(t)}, \tag{125}$$

where the constant C_1 depends here on the same variables as the generic constant C as well as on the initial data. This thus implies

$$\forall t \in (0, T),\ \|v - \tilde{v}\|_{V_f^3(t)} \leqq C_1\ \|\eta - \tilde{\eta}\|_{V_f^3(t)},$$

from which we infer

$$\forall t \in (0, T),\ \|v - \tilde{v}\|_{V_f^3(t)} \leqq C_1 \int_0^t \|v - \tilde{v}\|_{V_f^3(t)} \leqq C_1 t\ \|v - \tilde{v}\|_{V_f^3(t)},$$

which shows that for $T_1 = \frac{1}{2C_1}$, we have $v = \tilde{v}$ on $[0, T_1]$. We can then iterate this, starting from the initial time set at T_1, which shows in a similar fashion, since $v(T_1) = \tilde{v}(T_1)$, that $v = \tilde{v}$ on $[T_1, 2T_1]$. By induction, we get $v = \tilde{v}$ on $[0, T]$. □

14. Concluding remarks

Whereas the fluid-solid interaction is indeed a moving interface problem, it appears that the methods for its analysis differ drastically from the classical methods developed for the Navier-Stokes fluid interfaces independently by Solonnikov (see [15] and references therein) and Beale [1].

First, our functional framework scales in a *hyperbolic* fashion in both the parabolic (fluid) and hyperbolic (solid) phases.

Second, whereas the fixed-point problem (20) is inspired by the classical fixed-point problem used in parabolic-type interface problems, the Fourier-transform technique used to get regularity in parabolic theories requires the introduction of the problem with constant coefficients (for which there are explicit solutions), with the forcing functions containing the difference (small in a neighborhood of a point on the interface) between the actual coefficient and this constant coefficient. Whereas this procedure is contractive for parabolic problems, the hyperbolic part is problematic in the sense that the difference between the actual and the constant hyperbolic viscosity is not regular enough to get these contractive estimates (those coefficients are not constant after the truncation and change of variables to the full space problem).

Third, whereas energy methods without the use of Fourier techniques are indeed known for incompressible fluid interfaces, the highest-order time derivative of the pressure is known in $L^2(0, T; L^2(\Omega; \mathbb{R}^3))$ in that case, which allows the use of an iterative method from the constant-coefficient problem in energy spaces similar to the ones described in [15]. In the fluid-solid problem, the knowledge of the highest-order time derivative of the pressure is not known, which prevents such an iterative procedure from the constant-coefficient problem to get regularity. Instead, we are forced to work directly with the Lagrangian formulation (20), which requires the introduction of the penalized problems for reasons explained previously about the pressure. In turn, working with the Lagrangian formulation (20) requires us to first smooth the coefficients, and then to obtain estimates independent of the smoothing parameter by using interpolation inequalities.

Fourth, we clearly identify in our method the central and sufficient role of the *trace* of the velocity on the material interface Γ_0, whereas classical regularity results in interface problems involve the study of the regularity in the interior.

Fifth, once again regarding the pressure estimate, obtaining a contractive fixed-point scheme does not seem possible for the hyperbolic-parabolic problem (even with data arbitrarily smooth), whereas it is indeed the best-known method for the parabolic interface case. Note, however, that this later point is associated with the *incompressibility* of the fluid and does not seem to appear without this constraint.

This last remark is not without consequences for the numerical analysis of the problem, which we shall develop in future work. As for the question of the convergence of solutions of certain regularized models considered by other authors, it seems that the evidently natural approach of taking an elasticity law with a finite number N of modes introduced in [7] and letting $N \to \infty$ leads to some difficulties, as there is no elliptic operator for the discrete elasticity problem for which H^3 regularity may be used independently of the number of modes. On the other hand, it can be shown that the addition of a hyperviscosity to the solid problem (similar in spirit to the hyperviscous plate problem introduced in [2]) would indeed converge to the solution of the actual problem as the hyperviscosity parameter tends to zero, since we can apply the methods constructed here to this family of problems and obtain estimates that are independent of the hyperviscous parameter in the correct norms:

Appendix A. Some additional estimates

Appendix A.1. Estimates for (108)

In this section, $\delta > 0$ is assumed given and we now proceed to the estimate of the terms of (108) whose sign is indefinite. Recall that from (12), $\tilde{a}$, $\tilde{a}_t$, and thus

$\tilde{b}$, $\tilde{b}_t$ are controlled respectively in $L^\infty(H^2)$ and $L^\infty(H^1)$, independently of the regularizing parameter n associated with $\tilde{a}$.

Step 1. Let $J_1 = \int_0^T (D_h\tilde{b}_i^j\ Q_t,\ D_h W_t^i,_j)_{L^2(\mathbb{R}^3_+;\mathbb{R})}$. Then,

$$
\begin{aligned}
|J_1| &\leqq \delta \int_0^T \|D_h\nabla W_t\|^2_{L^2(\mathbb{R}^3_+;\mathbb{R}^9)} + C_\delta \int_0^T \|D_h\tilde{b}\|^2_{L^4(\mathbb{R}^3_+;\mathbb{R}^9)}\|Q_t\|^2_{L^4(\mathbb{R}^3_+;\mathbb{R})} \\
&\leqq \delta \int_0^T \|D_h\nabla \tilde{W}_t\|^2_{L^2(\mathbb{R}^3_+;\mathbb{R}^9)} + C_\delta C(M) \int_0^T \|Q_t\|^{0.5}_{L^2(\mathbb{R}^3_+;\mathbb{R})}\|Q_t\|^{1.5}_{H^1(\mathbb{R}^3_+;\mathbb{R})} \\
&\leqq \delta \int_0^T \|D_h\nabla W_t\|^2_{L^2(\mathbb{R}^3_+;\mathbb{R}^9)} \\
&\quad + C_\delta C(M)\ T^{\frac{1}{4}} \left[\sup_{(0,T)} \|Q_t\|^2_{L^2(\mathbb{R}^3_+;\mathbb{R})} + \int_0^T \|Q_t\|^2_{H^1(\mathbb{R}^3_+;\mathbb{R})} \right].
\end{aligned}
$$

From (93) and the definitions of W and Q, we then infer

$$
\begin{aligned}
|J_1| \leqq\ & C\delta \int_0^T \|\tilde{w}_t\|^2_{H^2(\Omega_0^f;\mathbb{R}^3)} \\
& + C_\delta C(M)\ T^{\frac{1}{4}} \Bigg[N(u_0, f)^2 + T\int_0^T \|\tilde{q}_t\|^2_{H^1(\Omega_0^f;\mathbb{R})} + \sup_{[0,T]} \|\tilde{w}_{tt}\|^2_{L^2(\Omega;\mathbb{R}^3)} \\
& + T \left[\int_0^T \|\tilde{w}_t\|^2_{H^2(\Omega_0^f;\mathbb{R}^3)} + \int_0^T \|\tilde{w}_{tt}\|^2_{H^1(\Omega_0^f;\mathbb{R}^3)} \right] \\
& + \sup_{[0,T]} \|\tilde{w}\|^2_{H^1(\Omega_0^s;\mathbb{R}^3)} + \int_0^T \|\tilde{q}_t\|^2_{H^1(\Omega_0^f;\mathbb{R})} \Bigg]. \qquad \text{(A.126)}
\end{aligned}
$$

Step 2. Let $J_2 = \int_0^T (D_h\tilde{b}_{t\,i}^{\ j}\ Q,\ D_h W_t^i,_j)_{L^2(\mathbb{R}^3_+;\mathbb{R})}$. Similarly as for J_1,

$$
\begin{aligned}
|J_2| &\leqq \delta \int_0^T \|D_h\nabla W_t\|^2_{L^2(\mathbb{R}^3_+;\mathbb{R}^9)} + C_\delta \int_0^T \|D_h\tilde{b}_t\|^2_{L^2(\mathbb{R}^3_+;\mathbb{R}^9)}\|Q\|^2_{W^{1,4}(\mathbb{R}^3_+;\mathbb{R})} \\
&\leqq \delta \int_0^T \|D_h\nabla W_t\|^2_{L^2(\mathbb{R}^3_+;\mathbb{R}^9)} + C_\delta C(M) \int_0^T \|Q\|^{0.5}_{H^1(\mathbb{R}^3_+;\mathbb{R})}\|Q\|^{1.5}_{H^2(\mathbb{R}^3_+;\mathbb{R})} \\
&\leqq C\delta \int_0^T \|\tilde{w}_t\|^2_{H^2(\Omega_0^f;\mathbb{R}^3)} + C_\delta C(M)\ T^{\frac{1}{4}} \\
&\quad \times \left[N(u_0, f)^2 + T\int_0^T \|\tilde{q}_t\|^2_{H^1(\Omega_0^f;\mathbb{R})} + \int_0^T \|\tilde{q}\|^2_{H^2(\Omega_0^f;\mathbb{R})} \right]. \qquad \text{(A.127)}
\end{aligned}
$$

Step 3. Let $J_3 = \int_0^T (\tilde{b}_{t\,i}^{\ j}\ D_h Q,\ D_h W_t^i,_j)_{L^2(\mathbb{R}^3_+;\mathbb{R})}$. Then,

$$
|J_3| \leqq \delta \int_0^T \|D_h\nabla W_t\|^2_{L^2(\mathbb{R}^3_+;\mathbb{R}^9)} + C_\delta \int_0^T \|\tilde{b}_t\|^2_{L^4(\mathbb{R}^3_+;\mathbb{R}^9)}\|D_h Q\|^2_{L^4(\mathbb{R}^3_+;\mathbb{R})}.
$$

Thus, similarly as for (A.127),

$$|J_3| \leqq C\delta \int_0^T \|\tilde{w}_t\|^2_{H^2(\Omega_0^f;\mathbb{R}^3)} + C_\delta C(M) T^{\frac{1}{4}} \times \left[N(u_0, f)^2 + T \int_0^T \|\tilde{q}_t\|^2_{H^1(\Omega_0^f;\mathbb{R})} + \int_0^T \|\tilde{q}\|^2_{H^2(\Omega_0^f;\mathbb{R})} \right]. \quad \text{(A.128)}$$

Step 4. Let $J_4 = \int_0^T (\tilde{b}_i^j \; D_h Q_t, \; D_h W_t^i,_j)_{L^2(\mathbb{R}^3_+;\mathbb{R})}$. This term will require more care. We first notice that

$$J_4 = \int_0^T (D_h Q_t, \; D_h[\tilde{b}_i^j W_t^i,_j])_{L^2(\mathbb{R}^3_+;\mathbb{R})} - \int_0^T (D_h Q_t, \; D_h \tilde{b}_i^j \; W_t^i,_j (\cdot + h))_{L^2(\mathbb{R}^3_+;\mathbb{R})},$$

which with the divergence relation (92) leads us to

$$J_4 = \int_0^T (D_h Q_t, \; D_h \mathfrak{a}_t)_{L^2(\mathbb{R}^3_+;\mathbb{R})} - \int_0^T (D_h Q_t, \; D_h[\tilde{b_t}_i^j W^i,_j])_{L^2(\mathbb{R}^3_+;\mathbb{R})} - \int_0^T (D_h Q_t, \; D_h \tilde{b}_i^j \; W_t^i,_j (\cdot + h))_{L^2(\mathbb{R}^3_+;\mathbb{R})}. \quad \text{(A.129)}$$

For the first integral of this identity, $J_4^1 = \int_0^T (D_h Q_t, \; D_h \mathfrak{a}_t)_{L^2(\mathbb{R}^3_+;\mathbb{R})}$, we have

$$\begin{aligned}
|J_4^1| \leqq\ & \delta \int_0^T \|D_h Q_t\|^2_{L^2(\mathbb{R}^3_+;\mathbb{R})} + C_\delta \Bigg(\int_0^T \|D_h \tilde{b}_t\|^2_{L^2(\mathbb{R}^3_+;\mathbb{R}^9)} \|\tilde{w}\|^2_{W^{1,4}(\Omega_0^f;\mathbb{R}^3)} \\
& + \int_0^T \|D_h \tilde{b}\|^2_{L^4(\mathbb{R}^3_+;\mathbb{R}^9)} \|\tilde{w}_t\|^2_{L^4(\Omega_0^f;\mathbb{R}^3)} \\
& + \int_0^T \|\tilde{b}\|^2_{W^{1,4}(\mathbb{R}^3_+;\mathbb{R}^9)} \|\nabla \tilde{w}_t\|^2_{L^2(\Omega_0^f;\mathbb{R}^9)} \\
& + \int_0^T \|\tilde{b}_t\|^2_{L^4(\mathbb{R}^3_+;\mathbb{R}^9)} \|\nabla \tilde{w}\|^2_{L^4(\Omega_0^f;\mathbb{R}^9)} \Bigg) \\
\leqq\ & \delta \int_0^T \|D_h Q_t\|^2_{L^2(\mathbb{R}^3_+;\mathbb{R})} \\
& + C_\delta C(M) \; T \; [\sup_{[0,T]} \|\tilde{w}\|^2_{H^2(\Omega_0^f;\mathbb{R}^3)} + \sup_{[0,T]} \|\tilde{w}_t\|^2_{H^1(\Omega_0^f;\mathbb{R}^3)}] \, , \\
\leqq\ & C\delta \int_0^T \|\tilde{q}_t\|^2_{H^1(\Omega_0^f;\mathbb{R})} + C_\delta C(M) \; T \; \Bigg[N(u_0, f)^2 + T \int_0^T \|\tilde{w}_t\|^2_{H^2(\Omega_0^f;\mathbb{R}^3)} \\
& + T \int_0^T \|\tilde{w}_{tt}\|^2_{H^1(\Omega_0^f;\mathbb{R}^3)} \Bigg]. \qquad \text{(A.130)}
\end{aligned}$$

Next, for $J_4^2 = \int_0^T (D_h Q_t,\ D_h[\tilde{b_t}{}_i^j W^i,_j])_{L^2(\mathbb{R}^3_+;\mathbb{R})}$,

$$|J_4^2| \leqq \delta \int_0^T \|D_h Q_t\|^2_{L^2(\mathbb{R}^3_+;\mathbb{R})} + C_\delta \int_0^T \|D_h \tilde{b}_t\|^2_{L^2(\mathbb{R}^3_+;\mathbb{R}^9)} \|\nabla W\|^2_{W^{1,4}(\mathbb{R}^3_+;\mathbb{R}^9)} + C_\delta \int_0^T \|\tilde{b}_t\|^2_{L^4(\mathbb{R}^3_+;\mathbb{R}^9)} \|D_h \nabla W\|^2_{L^4(\mathbb{R}^3_+;\mathbb{R}^9)}\ .$$

Therefore,

$$|J_4^2| \leqq \delta \int_0^t \|D_h Q_t\|^2_{L^2(\mathbb{R}^3_+;\mathbb{R})} + C_\delta C(M)\ T^{\frac{1}{4}} \left[\sup_{[0,T]} \|\nabla W\|^2_{H^1(\mathbb{R}^3_+;\mathbb{R}^9)} + \int_0^T \|\nabla W\|^2_{H^2(\mathbb{R}^3_+;\mathbb{R}^9)} \right],$$

which with the definition of W and Q provides

$$|J_4^2| \leqq C\delta \int_0^t \|\tilde{q}_t\|^2_{H^1(\Omega_0^f;\mathbb{R})} + C_\delta C(M)\ T^{\frac{1}{4}} \left[N(u_0, f)^2 + T \int_0^T \|\nabla \tilde{w}_t\|^2_{H^1(\Omega_0^f;\mathbb{R}^9)} + \int_0^T \|\nabla \tilde{w}\|^2_{H^2(\Omega_0^f;\mathbb{R}^9)} \right]. \tag{A.131}$$

Similarly, for $J_4^3 = \int_0^T (D_h Q_t,\ D_h \tilde{b}_i^j\ W_t{}^i,_j (\cdot + h))_{L^2(\mathbb{R}^3_+;\mathbb{R})}$,

$$\begin{aligned} |J_4^3| &\leqq \delta \int_0^T \|D_h Q_t\|^2_{L^2(\mathbb{R}^3_+;\mathbb{R})} + C_\delta \int_0^T \|D_h \tilde{b}\|^2_{L^4(\mathbb{R}^3_+;\mathbb{R}^9)} \|\nabla W_t\|^2_{L^4(\mathbb{R}^3_+;\mathbb{R}^9)} \\ &\leqq C\delta \int_0^t \|\nabla \tilde{q}_t\|^2_{L^2(\Omega_0^f;\mathbb{R}^3)} + C_\delta C(M)\ T^{\frac{1}{4}} \\ &\quad \times \left[N(u_0, f)^2 + T \int_0^T \|\nabla \tilde{w}_{tt}\|^2_{L^2(\Omega_0^f;\mathbb{R}^9)} + \int_0^T \|\nabla \tilde{w}_t\|^2_{H^1(\Omega_0^f;\mathbb{R}^9)} \right]. \end{aligned} \tag{A.132}$$

Step 5. Let $J_5 = \int_0^T (D_h(\tilde{b}_k^r \tilde{b}_k^s) W_t,_r (\cdot + h),\ D_h W_t,_s)_{L^2(\mathbb{R}^3_+;\mathbb{R}^3)}$. Then,

$$\begin{aligned} |J_5| &\leqq \delta \int_0^T \|D_h \nabla W_t\|^2_{L^2(\mathbb{R}^3_+;\mathbb{R}^9)} + C_\delta \int_0^T \|D_h(\tilde{b}\tilde{b}^T)\|^2_{L^4(\mathbb{R}^3_+;\mathbb{R}^9)} \|\nabla W_t\|^2_{L^4(\mathbb{R}^3_+;\mathbb{R}^9)} \\ &\leqq C\delta \int_0^t \|\tilde{w}_t\|^2_{H^2(\Omega_0^f;\mathbb{R}^3)} + C_\delta C(M)\ T^{\frac{1}{4}} \left[N(u_0, f)^2 \right. \\ &\quad \left. + T \int_0^T \|\nabla \tilde{w}_{tt}\|^2_{L^2(\Omega_0^f;\mathbb{R}^9)} + \int_0^T \|\nabla \tilde{w}_t\|^2_{H^1(\Omega_0^f;\mathbb{R}^9)} \right]. \end{aligned} \tag{A.133}$$

Step 6. Let $J_6 = \int_0^T (D_h(\tilde{b}_k^r \tilde{b}_k^s)_t W,_r (\cdot + h),\ D_h W_t,_s)_{L^2(\mathbb{R}^3_+;\mathbb{R}^3)}$. Similarly,

$$\begin{aligned} |J_6| \leqq\ & \delta \int_0^T \|D_h \nabla W_t\|^2_{L^2(\mathbb{R}^3_+;\mathbb{R}^9)} \\ & + C_\delta \int_0^T \|D_h(\tilde{b}\tilde{b}^T)_t\|^2_{L^2(\mathbb{R}^3_+;\mathbb{R}^9)} \|\nabla W\|^2_{W^{1,4}(\mathbb{R}^3_+;\mathbb{R}^9)} \\ \leqq\ & C\delta \int_0^t \|\tilde{w}_t\|^2_{H^2(\Omega_0^f;\mathbb{R}^3)} + C_\delta C(M)\, T^{\frac{1}{4}} \Big[N(u_0, f)^2 \\ & + T \int_0^T \|\nabla \tilde{w}_t\|^2_{H^1(\Omega_0^f;\mathbb{R}^9)} + \int_0^T \|\nabla \tilde{w}\|^2_{H^2(\Omega_0^f;\mathbb{R}^9)} \Big]. \end{aligned} \tag{A.134}$$

Step 7. For $J_7 = \int_0^T ((\tilde{b}_k^r \tilde{b}_k^s)_t\ D_h W,_r\ ,\ D_h W_t,_s)_{L^2(\mathbb{R}^3_+;\mathbb{R}^3)}$, we have

$$\begin{aligned} |J_7| \leqq\ & \delta \int_0^T \|D_h \nabla W_t\|^2_{L^2(\mathbb{R}^3_+;\mathbb{R}^9)} + C_\delta \int_0^T \|(\tilde{b}\tilde{b}^T)_t\|^2_{L^4(\mathbb{R}^3_+;\mathbb{R}^9)} \|D_h \nabla W\|^2_{L^4(\mathbb{R}^3_+;\mathbb{R}^9)} \\ \leqq\ & C\delta \int_0^T \|w_t\|^2_{H^2(\Omega_0^f;\mathbb{R}^3)} + C_\delta C(M)\, T^{\frac{1}{4}} \Big[N(u_0, f)^2 \\ & + T \int_0^T \|\nabla \tilde{w}_t\|^2_{H^1(\Omega_0^f;\mathbb{R}^9)} + \int_0^T \|\nabla \tilde{w}\|^2_{H^2(\Omega_0^f;\mathbb{R}^9)} \Big]. \end{aligned} \tag{A.135}$$

For the next step, we introduce $\delta_1 > 0$, which is different from δ.

Step 8. Let $J_8 = \int_0^T (D_h C^{irks} W^k,_r\ ,\ D_h W_t^i,_s)_{L^2(\mathbb{R}^3_-;\mathbb{R})}$.
An integration by parts in time gives

$$\begin{aligned} J_8 =\ & -\int_0^T (D_h C^{irks}\ W_t^k,_r\ ,\ D_h W^i,_s)_{L^2(\mathbb{R}^3_-;\mathbb{R})} \\ & + [\ (\ D_h C^{irks}\ W^k,_r (\cdot),\ D_h W^i(\cdot),_s)_{L^2(\mathbb{R}^3_-;\mathbb{R})}\ ]_0^T\ . \end{aligned}$$

Since Ω_0^s is of class H^4,

$$\begin{aligned} |J_8| \leqq\ & CT\ [\ \sup_{[0,T]} \|\nabla W_t\|^2_{L^2(\mathbb{R}^3_-;\mathbb{R}^9)} + \sup_{[0,T]} \|W\|^2_{H^2(\mathbb{R}^3_-;\mathbb{R}^3)}\ ] + CN(u_0, f)^2 \\ & + C_{\delta_1}\ [\ \|\nabla W(T) - \nabla W(0)\|^2_{L^2(\mathbb{R}^3_-;\mathbb{R}^9)} + \|\nabla W(0)\|^2_{L^2(\mathbb{R}^3_-;\mathbb{R}^9)}\ ] \\ & + \delta_1 \sup_{[0,T]} \|D_h \nabla W\|^2_{L^2(\mathbb{R}^3_-;\mathbb{R}^9)}, \end{aligned}$$

and thus,

$$\begin{aligned} |J_8| \leqq\ & CT[\sup_{[0,T]} \|\nabla \tilde{w}_t\|^2_{L^2(\Omega_0^s;\mathbb{R}^9)} + \sup_{[0,T]} \|\tilde{w}\|^2_{H^2(\Omega_0^s;\mathbb{R}^3)}\ ] + C_{\delta_1}\ N(u_0, f)^2 \\ & + C_{\delta_1} T^2 \sup_{[0,T]} \|\nabla \tilde{w}_t\|^2_{L^2(\Omega_0^s;\mathbb{R}^9)} + C\delta_1 \sup_{[0,T]} \|\tilde{w}\|^2_{H^2(\Omega_0^s;\mathbb{R}^3)}. \end{aligned} \tag{A.136}$$

Step 9. Let $J_9 = \int_0^T (F_{1t},\ D_{-h} D_h W_t)_{L^2(\mathbb{R}^3_+;\mathbb{R}^3)}$. Then

$$
\begin{aligned}
|J_9| \leqq\ & \delta \int_0^T \|W_t\|^2_{H^2(\mathbb{R}^3_+;\mathbb{R}^3)} + C_\delta \int_0^T \|(\tilde{b}\tilde{b}^T)_t\|^2_{L^4(\mathbb{R}^3_+;\mathbb{R}^9)} \|\nabla W\|^2_{L^4(\mathbb{R}^3_+;\mathbb{R}^9)} \\
& + C_\delta \int_0^T \|(\tilde{b}\tilde{b}^T)\|^2_{W^{1,4}(\mathbb{R}^3_+;\mathbb{R}^9)} \|\nabla W_t\|^2_{L^2(\mathbb{R}^3_+;\mathbb{R}^9)} + C_\delta\ N(u_0, f)^2 \\
& + C_\delta \int_0^T \|\tilde{b}_t\|^2_{L^4(\mathbb{R}^3_+;\mathbb{R}^9)} \|q\|^2_{L^4(\Omega_0^f;\mathbb{R})} + C_\delta \int_0^T \|\tilde{b}\|^2_{L^\infty(\mathbb{R}^3_+;\mathbb{R}^9)} \|q_t\|^2_{L^2(\Omega_0^f;\mathbb{R})} \\
\leqq\ & C\delta \int_0^T \|\tilde{w}_t\|^2_{H^2(\Omega_0^f;\mathbb{R}^3)} + C_\delta N(u_0, f)^2 \\
& + C_\delta C(M) T \Bigg[N(u_0, f)^2 + T \int_0^T \|\tilde{w}_t\|^2_{H^2(\Omega_0^f;\mathbb{R}^3)} \\
& + T \int_0^T \|\tilde{w}_{tt}\|^2_{H^1(\Omega_0^f;\mathbb{R}^3)} \Bigg] + C_\delta C(M) T \\
& \times \left[N(u_0, f)^2 + T \int_0^T \|\tilde{q}_t\|^2_{H^1(\Omega_0^f;\mathbb{R})} + \sup_{[0,T]} \|\tilde{q}_t\|^2_{L^2(\Omega_0^f;\mathbb{R})} \right].
\end{aligned}
$$

Thus, with (93),

$$
\begin{aligned}
|J_9| \leqq\ & C\delta \int_0^T \|\tilde{w}_t\|^2_{H^2(\Omega_0^f;\mathbb{R}^3)} + C_\delta\ N(u_0, f)^2 \\
& + C_\delta C(M)\ T\ \Bigg[N(u_0, f)^2 + T \int_0^T \|\tilde{w}_t\|^2_{H^2(\Omega_0^f;\mathbb{R}^3)} \\
& + T \int_0^T \|\tilde{w}_{tt}\|^2_{H^1(\Omega_0^f;\mathbb{R}^3)} \Bigg] + C_\delta C(M) T \\
& \times \left[T \int_0^T \|\tilde{q}_t\|^2_{H^1(\Omega_0^f;\mathbb{R})} + \sup_{[0,T]} \|\tilde{w}_{tt}\|^2_{L^2(\Omega;\mathbb{R}^3)} + \sup_{[0,T]} \|\tilde{w}\|^2_{H^1(\Omega_0^s;\mathbb{R}^3)} \right].
\end{aligned}
\tag{A.137}
$$

Step 10. For $J_{10} = \int_0^T (D_h H_{i\,t},\ D_h W_{t,i})_{L^2(\mathbb{R}^3_+;\mathbb{R}^3)}$, we have

$$
\begin{aligned}
|J_{10}| \leqq\ & \delta \int_0^T \|W_t\|^2_{H^2(\mathbb{R}^3_+;\mathbb{R}^3)} + C_\delta \int_0^T \|(\tilde{b}\tilde{b}^T)_t\|^2_{L^4(\mathbb{R}^3_+;\mathbb{R}^9)} \|\nabla W\|^2_{L^4(\mathbb{R}^3_+;\mathbb{R}^9)} \\
& + C_\delta \int_0^T \|D_h(\tilde{b}\tilde{b}^T)_t\|^2_{L^2(\mathbb{R}^3_+;\mathbb{R}^9)} \|W\|^2_{W^{1,4}(\mathbb{R}^3_+;\mathbb{R}^3)} \\
& + C_\delta \int_0^T \|\tilde{b}\tilde{b}^T\|^2_{W^{1,4}(\mathbb{R}^3_+;\mathbb{R}^9)} \|\nabla W_t\|^2_{L^2(\mathbb{R}^3_+;\mathbb{R}^9)} \\
& + C_\delta \int_0^T \|D_h(\tilde{b}\tilde{b}^T)\|^2_{L^4(\mathbb{R}^3_+;\mathbb{R}^9)} \|W_t\|^2_{L^4(\mathbb{R}^3_+;\mathbb{R}^3)}
\end{aligned}
$$

$$\leqq C\delta \int_0^T \|\tilde{w}_t\|^2_{H^2(\Omega_0^f;\mathbb{R}^3)} + C_\delta C(M)\ T\ nonumber$$
$$\times\left[N(u_0, f)^2 + T\int_0^T \|\tilde{w}_t\|^2_{H^2(\Omega_0^f;\mathbb{R}^3)} + T\int_0^T \|\tilde{w}_{tt}\|^2_{H^1(\Omega_0^f;\mathbb{R}^3)}\right]. \tag{A.138}$$

Step 11. Let

$$J_{11} = \int_0^T (D_h F_{2t},\ D_h W_t)_{L^2(\mathbb{R}^3_-;\mathbb{R}^3)} + \int_0^T (D_{-h} D_h K_{i\,t},\ W_{t},_i)_{L^2(\mathbb{R}^3_-;\mathbb{R}^3)}.$$

Then,

$$|J_{11}| \leqq\ C\ T\ [\sup_{[0,T]} \|\tilde{w}_t\|^2_{H^1(\Omega_0^s;\mathbb{R}^3)} + \sup_{[0,T]} \|\tilde{w}\|^2_{H^2(\Omega_0^s;\mathbb{R}^3)}] + CN(u_0, f)^2. \tag{A.139}$$

Appendix A.2. Estimates for (112)

As in the previous section, recall that from (12), $\tilde{a}$, and thus $\tilde{b}$, is bounded in $L^\infty(H^2)$ independently of the parameter n associated with $\tilde{a}$.

Step 1. For $K_1 = \int_0^T (D_{-h} D_h(\tilde{b}_i^j)\ Q,\ D_{-h} D_h W^i,_j)_{L^2(\mathbb{R}^3_+;\mathbb{R})}$, we have

$$\begin{aligned}
|K_1| &\leqq \delta \int_0^T \|D_{-h} D_h \nabla W\|^2_{L^2(\mathbb{R}^3_+;\mathbb{R}^9)} \\
&\quad + C_\delta \int_0^T \|D_{-h} D_h \tilde{b}\|^2_{L^2(\mathbb{R}^3_+;\mathbb{R}^9)} \|Q\|^2_{W^{1,4}(\mathbb{R}^3_+;\mathbb{R})} \\
&\leqq \delta \int_0^T \|D_{-h} D_h \nabla W\|^2_{L^2(\mathbb{R}^3_+;\mathbb{R}^9)} + C_\delta C(M) \int_0^T \|Q\|^{0.5}_{H^1(\mathbb{R}^3_+;\mathbb{R})} \|Q\|^{1.5}_{H^2(\mathbb{R}^3_+;\mathbb{R})} \\
&\leqq\ C\delta \int_0^T \|\tilde{w}\|^2_{H^3(\Omega_0^f;\mathbb{R}^3)} \\
&\quad + C_\delta C(M)\ T^{\frac{1}{4}} \left[\ N(u_0, f)^2 + T\int_0^T \|q_t\|^2_{H^1(\Omega_0^f;\mathbb{R})} + \int_0^T \|q\|^2_{H^2(\Omega_0^f;\mathbb{R})}\right].
\end{aligned} \tag{A.140}$$

Step 2. Let $K_2 = \sum_{p=0}^1 \int_0^T (D_{(-1)^p h} \tilde{b}_i^j\ D_{(-1)^p h} Q,\ D_{-h} D_h W^i,_j)_{L^2(\mathbb{R}^3_+;\mathbb{R})}$. Then,

$$\begin{aligned}
|K_2| &\leqq \delta \int_0^T \|D_{-h} D_h \nabla W\|^2_{L^2(\mathbb{R}^3_+;\mathbb{R}^9)} + C_\delta \int_0^T \|D_h \tilde{b}\|^2_{L^4(\mathbb{R}^3_+;\mathbb{R}^9)} \|D_h Q\|^2_{L^4(\mathbb{R}^3_+;\mathbb{R})} \\
&\leqq C\delta \int_0^T \|\tilde{w}\|^2_{H^3(\Omega_0^f;\mathbb{R}^3)} + C_\delta C(M) T^{\frac{1}{4}} \left[\ N(u_0, f)^2 \right.\\
&\quad \left. + T\int_0^T \|q_t\|^2_{H^1(\Omega_0^f;\mathbb{R})} + \int_0^T \|q\|^2_{H^2(\Omega_0^f;\mathbb{R})}\right].
\end{aligned} \tag{A.141}$$

Step 3. Let $K_3 = \int_0^T (\tilde{b}_i^j \; D_{-h} D_h Q, \; D_{-h} D_h W^i,_j)_{L^2(\mathbb{R}^3_+;\mathbb{R})}$.

We first notice that

$$\begin{aligned} K_3 = & \int_0^T (D_{-h} D_h Q, \; D_{-h} D_h [\tilde{b}_i^j W^i,_j])_{L^2(\mathbb{R}^3_+;\mathbb{R})} \\ & + \sum_{p=0}^{1} \int_0^T (D_{-h} D_h Q, \; D_{(-1)^p h} \tilde{b}_i^j \; D_{(-1)^p h} W^i,_j)_{L^2(\mathbb{R}^3_+;\mathbb{R})} \\ & - \int_0^T (D_{-h} D_h Q, \; D_h D_{-h} [\tilde{b}_i^j] \; W^i,_j)_{L^2(\mathbb{R}^3_+;\mathbb{R})}, \end{aligned}$$

which with the divergence relation (92) leads us to

$$\begin{aligned} K_3 = & \int_0^T (D_{-h} D_h Q, \; D_{-h} D_h \mathfrak{a})_{L^2(\mathbb{R}^3_+;\mathbb{R})} \\ & + \sum_{p=0}^{1} \int_0^T (D_{-h} D_h Q, \; D_{(-1)^p h} \tilde{b}_i^j \; D_{(-1)^p h} W^i,_j)_{L^2(\mathbb{R}^3_+;\mathbb{R})} \\ & - \int_0^T (D_{-h} D_h Q, \; D_h D_{-h} [\tilde{b}_i^j] \; W^i,_j)_{L^2(\mathbb{R}^3_+;\mathbb{R})}. \end{aligned}$$

We then have

$$\begin{aligned} |K_3| \leqq & \; \delta \int_0^T \|D_{-h} D_h Q\|^2_{L^2(\mathbb{R}^3_+;\mathbb{R})} + C_\delta \int_0^T \|\tilde{b}\|^2_{H^2(\mathbb{R}^3_+;\mathbb{R}^9)} \|\tilde{w}\|^2_{H^2(\Omega_0^f;\mathbb{R}^3)} \\ & + C_\delta \int_0^T \|D_h \tilde{b}\|^2_{L^4(\mathbb{R}^3_+;\mathbb{R}^9)} \|D_h \nabla W\|^2_{L^4(\mathbb{R}^3_+;\mathbb{R}^9)} \\ & + C_\delta \int_0^T \|\tilde{b}\|^2_{H^2(\mathbb{R}^3_+;\mathbb{R}^9)} \|\nabla W\|^2_{W^{1,4}(\mathbb{R}^3_+;\mathbb{R}^9)} \\ \leqq & \; C\delta \int_0^T \|\tilde{q}\|^2_{H^2(\Omega_0^f;\mathbb{R})} + C_\delta C(M) T \left[N(u_0, f)^2 + T \int_0^T \|\tilde{w}_t\|^2_{H^2(\Omega_0^f;\mathbb{R}^3)} \right] \\ & + C_\delta C(M) T^{\frac{1}{4}} \left[N(u_0, f)^2 + T \int_0^T \|\tilde{w}_t\|^2_{H^2(\Omega_0^f;\mathbb{R}^3)} + \int_0^T \|\tilde{w}\|^2_{H^3(\Omega_0^f;\mathbb{R}^3)} \right]. \end{aligned} \tag{A.142}$$

Step 4. Let $K_4 = \int_0^T (D_{-h} D_h (\tilde{b}_k^r \tilde{b}_k^s) W,_r \,, \; D_{-h} D_h W,_s)_{L^2(\mathbb{R}^3_+;\mathbb{R}^3)}$. Then,

$$\begin{aligned} |K_4| \leqq & \; \delta \int_0^T \|D_{-h} D_h \nabla W\|^2_{L^2(\mathbb{R}^3_+;\mathbb{R}^9)} + C_\delta \int_0^T \|\tilde{b}\tilde{b}^T\|^2_{H^2(\mathbb{R}^3_+;\mathbb{R}^9)} \|\nabla W\|^2_{W^{1,4}(\mathbb{R}^3_+;\mathbb{R}^9)} \\ \leqq & \; C\delta \int_0^T \|\tilde{w}\|^2_{H^3(\Omega_0^f;\mathbb{R}^3)} + C_\delta C(M) \, T^{\frac{1}{4}} \left[N(u_0, f)^2 \right. \\ & \left. + T \int_0^T \|\nabla \tilde{w}_t\|^2_{H^1(\Omega_0^f;\mathbb{R}^9)} + \int_0^T \|\nabla \tilde{w}\|^2_{H^2(\Omega_0^f;\mathbb{R}^9)} \right]. \end{aligned} \tag{A.143}$$

Step 5. For $K_5 = \sum_{p=0}^{1} \int_0^T (D_{(-1)^p h}(\tilde{b}_k^r \tilde{b}_k^s) D_{(-1)^p h} W,_r \,,\ D_{-h} D_h W,_s)_{L^2(\mathbb{R}^3_+;\mathbb{R}^3)}$,

$$\begin{aligned}
|K_5| &\leqq \delta \int_0^T \|D_{-h} D_h \nabla W\|^2_{L^2(\mathbb{R}^3_+;\mathbb{R}^9)} \\
&\quad + C_\delta \int_0^T \|D_h(\tilde{b}\tilde{b}^T)\|^2_{L^4(\mathbb{R}^3_+;\mathbb{R}^9)} \|D_h \nabla W\|^2_{L^4(\mathbb{R}^3_+;\mathbb{R}^9)} \\
&\leqq \delta \int_0^t \|\tilde{w}\|^2_{H^3(\Omega_0^f;\mathbb{R}^3)} + C_\delta C(M)\, T^{\frac{1}{4}} \Big[N(u_0, f)^2 \\
&\quad + T \int_0^T \|\tilde{w}_t\|^2_{H^2(\Omega_0^f;\mathbb{R}^3)} + \int_0^T \|\tilde{w}\|^2_{H^3(\Omega_0^f;\mathbb{R}^3)} \Big]. \qquad \text{(A.144)}
\end{aligned}$$

Step 6. For $K_6 = \int_0^T (D_{-h} D_h [C^{irks}] \int_0^\cdot W^k,_r \,,\ D_{-h} D_h W^i,_s)_{L^2(\mathbb{R}^3_-;\mathbb{R})}$, an integration by parts in time gives

$$\begin{aligned}
K_6 = &- \int_0^T \Big(D_{-h} D_h [C^{irks}]\, W^k,_r \,,\ D_{-h} D_h \int_0^\cdot W^i,_s \Big)_{L^2(\mathbb{R}^3_-;\mathbb{R})} \\
&+ \Big(D_{-h} D_h [C^{irks}] \int_0^T W^k,_r \,,\ D_{-h} D_h \int_0^T W^i,_s \Big)_{L^2(\mathbb{R}^3_-;\mathbb{R})},
\end{aligned}$$

from which we infer from the H^4 regularity of Ω_0^s,

$$\begin{aligned}
K_6 \leqq\ & C \int_0^T \|\nabla W\|_{L^4(\mathbb{R}^3_-;\mathbb{R}^9)} \Big\| D_{-h} D_h \int_0^\cdot \nabla W \Big\|_{L^2(\mathbb{R}^3_-;\mathbb{R}^9)} \\
&+ \Big\| \int_0^T \nabla W \Big\|_{L^4(\mathbb{R}^3_-;\mathbb{R}^9)} \Big\| D_{-h} D_h \int_0^T \nabla W \Big\|_{L^2(\mathbb{R}^3_-;\mathbb{R}^9)},
\end{aligned}$$

leading us to

$$|K_6| \leqq\ CT \Big[\sup_{[0,T]} \|\tilde{w}\|^2_{H^2(\Omega_0^s;\mathbb{R}^3)} + \sup_{[0,T]} \Big\| \int_0^\cdot \tilde{w} \Big\|^2_{H^3(\Omega_0^s;\mathbb{R}^3)} \Big]. \qquad \text{(A.145)}$$

Step 7. Let

$$K_7 = \sum_{p=0}^{1} \int_0^T \Big(D_{(-1)^p h} [C^{irks}] \int_0^\cdot D_{(-1)^p h} W^k,_r \,,\ D_{-h} D_h W^i,_s \Big)_{L^2(\mathbb{R}^3_-;\mathbb{R})}.$$

An integration by parts in time gives

$$\begin{aligned}
K_7 = &- \sum_{p=0}^{1} \int_0^T \Big(D_{(-1)^p h} [C^{irks}]\, D_{(-1)^{p+1} h} W^k,_r \,,\ D_{-h} D_h \int_0^\cdot W^i,_s \Big)_{L^2(\mathbb{R}^3_-;\mathbb{R})} \\
&+ \Big(D_{(-1)^p h} [C^{irks}]\, D_{(-1)^p h} \int_0^T W^k,_r \,,\ D_{-h} D_h \int_0^T W^i,_s \Big)_{L^2(\mathbb{R}^3_-;\mathbb{R})},
\end{aligned}$$

and thus, from the H^4 regularity of Ω_0^s,

$$K_7 \leqq C \int_0^T \|D_h \nabla W\|_{L^2(\mathbb{R}^3_-;\mathbb{R}^9)} \left\| D_{-h} D_h \int_0^\cdot \nabla W \right\|_{L^2(\mathbb{R}^3_-;\mathbb{R}^9)} + \left\| D_h \int_0^T \nabla W \right\|_{L^2(\mathbb{R}^3_-;\mathbb{R}^9)} \left\| D_{-h} D_h \int_0^T \nabla W \right\|_{L^2(\mathbb{R}^3_-;\mathbb{R}^9)}.$$

Therefore,

$$|K_7| \leqq CT \left[\sup_{[0,T]} \|\tilde{w}\|^2_{H^2(\Omega_0^s;\mathbb{R}^3)} + \sup_{[0,T]} \left\| \int_0^\cdot \tilde{w} \right\|^2_{H^3(\Omega_0^s;\mathbb{R}^3)} \right]. \quad \text{(A.146)}$$

Remark 21. The H^4 regularity of Ω_0^s is used only for proving (A.145) and (A.146). As a matter of fact, $W^{3,p}$ for $p > 3$ would have been sufficient.

Step 8. Let $K_8 = \int_0^T (D_{-h} F_1,\ D_h D_{-h} D_h W)_{L^2(\mathbb{R}^3_+;\mathbb{R}^3)}$. Then

$$\begin{aligned}
|K_8| \leqq \delta \int_0^T \|W\|^2_{H^3(\mathbb{R}^3_+;\mathbb{R}^3)} &+ C_\delta \int_0^T \|D_h(\tilde{b}\tilde{b}^T)\|^2_{L^4(\mathbb{R}^3_+;\mathbb{R}^9)} \|\nabla \tilde{w}\|^2_{L^4(\Omega_0^f;\mathbb{R}^9)} \\
&+ C_\delta \int_0^T \|\tilde{b}\tilde{b}^T\|^2_{L^\infty(\mathbb{R}^3_+;\mathbb{R}^9)} \|D_h \nabla \tilde{w}\|^2_{L^2(\Omega_0^f;\mathbb{R}^9)} + CN(u_0, f)^2 \\
&+ C_\delta \int_0^T \|\nabla \tilde{b}\|^2_{L^4(\mathbb{R}^3_+;\mathbb{R}^{27})} \|q\|^2_{L^4(\Omega_0^f;\mathbb{R})} \\
&+ C_\delta \int_0^T \|\tilde{b}\|^2_{L^\infty(\mathbb{R}^3_+;\mathbb{R}^9)} \|\nabla q\|^2_{L^2(\Omega_0^f;\mathbb{R}^3)} \\
\leqq \delta \int_0^T \|\tilde{w}\|^2_{H^3(\Omega_0^f;\mathbb{R}^3)} &+ C_\delta C(M) T \left[N(u_0, f)^2 + T \int_0^T \|\tilde{w}_t\|^2_{H^2(\Omega_0^f;\mathbb{R}^3)} \right] \\
&+ C_\delta C(M) T \left[N(u_0, f)^2 + T \int_0^T \|q_t\|^2_{H^1(\Omega_0^f;\mathbb{R})} \right] + CN(u_0, f)^2. \quad \text{(A.147)}
\end{aligned}$$

Step 9. For $K_9 = \int_0^T (D_{-h} D_h H_i,\ D_{-h} D_h W,_i\,)_{L^2(\mathbb{R}^3_+;\mathbb{R}^3)}$, we have

$$\begin{aligned}
|K_9| \leqq \delta \int_0^T \|W\|^2_{H^3(\mathbb{R}^3_+;\mathbb{R}^3)} &+ C_\delta \int_0^T \|D_h(\tilde{b}\tilde{b}^T)\|^2_{L^4(\mathbb{R}^3_+;\mathbb{R}^9)} \|\nabla \tilde{w}\|^2_{L^4(\Omega_0^f;\mathbb{R}^9)} \\
&+ C_\delta \int_0^T \|D_{-h} D_h(\tilde{b}\tilde{b}^T)\|^2_{L^2(\mathbb{R}^3_+;\mathbb{R}^9)} \|\tilde{w}\|^2_{W^{1,4}(\Omega_0^f;\mathbb{R}^3)} \\
&+ C_\delta \int_0^T \|\tilde{b}\tilde{b}^T\|^2_{L^\infty(\mathbb{R}^3_+;\mathbb{R}^9)} \|\tilde{w}\|^2_{H^2(\Omega_0^f;\mathbb{R}^3)} \\
\leqq \delta \int_0^T \|\tilde{w}\|^2_{H^3(\Omega_0^f;\mathbb{R}^3)} &+ C_\delta C(M) T \left[N(u_0, f)^2 + T \int_0^T \|\tilde{w}_t\|^2_{H^2(\Omega_0^f;\mathbb{R}^3)} \right].
\end{aligned}$$

(A.148)

Step 10. Let

$$K_{10} = \int_0^T (D_{-h} D_h F_2,\ D_{-h} D_h W)_{L^2(\mathbb{R}^3_-;\mathbb{R}^3)} + \int_0^T (D_{-h} D_h D_{-h} K_i,\ D_h W,_i)_{L^2(\mathbb{R}^3_-;\mathbb{R}^3)}.$$

Then

$$|K_{10}| \leqq CT \left[\sup_{[0,T]} \|\tilde{w}\|^2_{H^2(\Omega_0^s;\mathbb{R}^3)} + \sup_{[0,T]} \left\| \int_0^\cdot \tilde{w} \right\|^2_{H^3(\Omega_0^s;\mathbb{R}^3)} \right] + C\ N(u_0, f)^2. \tag{A.149}$$

Acknowledgements. SS was partially supported by National Science Foundation under grants DMS-0105004 and NSF ITR-0313370.

References

1. BEALE, J.T.: Large-time regularity of viscous surface waves. *Arch. Rational Mech. Anal.* **84**, 307–352 (1983/84)
2. CHAMBOLLE, A., DESJARDINS, B., ESTEBAN, M.J., GRANDMONT, C.: Existence of weak solutions for an unsteady fluid-plate interaction problem. Preprint
3. CIARLET, P.G.: Mathematical Elasticity, Volume II: Theory of Plates. North-Holland, Amsterdam, 1997
4. CONCA, C., SAN MARTIN, J., TUCSNAK, M.: Existence of solutions for the equations modelling the motion of a rigid body in a viscous fluid. *Commun. Partial Differential Equations* **25**, 1019–1042 (2000)
5. DEIMLING, K.: Nonlinear Functional Analysis. Springer-Verlag, 1985
6. DESJARDINS, B., ESTEBAN, M.J.: Existence of weak solutions for the motion of rigid bodies in a viscous fluid. *Arch. Rational Mech. Anal.* **146**, 59–71 (1999)
7. DESJARDINS, B., Esteban, M.J., GRANDMONT, C., LE TALLEC, P.: Weak solutions for a fluid-structure interaction problem. *Rev. Mat. Complut.* **14**, 523–538 (2001)
8. EBENFELD, S.: L^2-regularity theory of linear strongly elliptic Dirichlet systems of order $2m$ with minimal regularity in the coefficients. *Quat. Appl. Math.* **60**, 547–576 (2002)
9. EVANS, L.C.: Partial Differential Equations. Graduate Studies in Mathematics, 19 American Mathematical Society, Providence, RI, 1998
10. FLORI, F., ORENGA, P.: Fluid-structure interaction: analysis of a 3-D compressible model. *Ann. Inst. H. Poincaré Anal. Non Linéaire* **17**, 753–777 (2000)
11. GRANDMONT, C.: Existence et unicité de solutions d'un problème de couplage fluide-structure bidimensionnel stationnaire. *C. R. Acad. Sci. Paris Sér. I.* **326**, 651–656 (1998)
12. GRANDMONT, C., MADAY, Y.: Existence for unsteady fluid-structure interaction problem. *Math. Model. Numer. Anal.* **34**, 609–636 (2000)
13. GUNZBURGER, M.D., LEE, H-C., SEREGIN, G.A.: Global existence of weak solutions for viscous incompressible flows around a moving rigid body in three dimensions. *J. Math. Fluid Mech.* **2**, 219–266 (2000)
14. SERRE, D.: Chute libre d'un solide dans un fluide visqueux incompressible: Existence. *Japan J. Appl. Math.* **4**, 33–73 (1987)
15. SOLONNIKOV, V.A.: Solvability of the problem of evolution of a viscous incompressible fluid bounded by a free surface on a finite time interval. *St. Petersburg Math. J.* **3**, 189–220 (1992)

16. SOLONNIKOV, V.A., ŠČADILOV, V.E.: On a boundary value problem for a stationary system of Navier-Stokes equations. *Proc. Steklov Inst. Math.* **125**, 186–199 (1973)
17. WEINBERGER, H.F.: Variational properties of steady fall in Stokes flow. *J. Fluid Mech.* **52**, 321–344 (1972)

Department of Mathematics
University of California at Davis
Davis, CA 95616
e-mails: coutand@math.ucdavis.edu
shkoller@math.ucdavis.edu

(*Accepted July 22, 2004*)
Published online November 2, 2004 – © Springer-Verlag (*2004*)