JOURNAL OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 20, Number 3, July 2007, Pages 829–930
S 0894-0347(07)00556-5
Article electronically published on March 5, 2007

WELL-POSEDNESS OF THE FREE-SURFACE INCOMPRESSIBLE EULER EQUATIONS WITH OR WITHOUT SURFACE TENSION

DANIEL COUTAND AND STEVE SHKOLLER

1. Introduction

1.1. **The problem statement and background.** For $\sigma \geq 0$ and for arbitrary initial data, we prove local existence and uniqueness of solutions in Sobolev spaces to the free boundary incompressible Euler equations in vacuum:

$$\partial_t u + \nabla_u u + \nabla p = 0 \quad \text{in } Q\,, \tag{1.1a}$$

$$\operatorname{div} u = 0 \quad \text{in } Q\,, \tag{1.1b}$$

$$p = \sigma\, H \quad \text{on } \partial Q\,, \tag{1.1c}$$

$$(\partial_t + \nabla_u)|_{\partial Q} \in T(\partial Q)\,, \tag{1.1d}$$

$$u = u_0 \quad \text{on } Q_{t=0}\,, \tag{1.1e}$$

$$Q_{t=0} = \Omega\,, \tag{1.1f}$$

where $Q = \bigcup_{0\leq t\leq T}\{t\} \times \Omega(t)$, $\Omega(t) \subset \mathbb{R}^n$, $n = 2$ or 3, $\partial Q = \bigcup_{0\leq t\leq T}\{t\} \times \partial\Omega(t)$, $\nabla_u u = u^j \partial u^i/\partial x^j$, and where Einstein's summation convention is employed. The vector field u is the Eulerian or spatial velocity field defined on the time-dependent domain $\Omega(t)$, p denotes the pressure function, H is twice the mean curvature of the boundary of the fluid $\partial\Omega(t)$, and σ is the surface tension. Equation (1.1a) is the conservation of momentum, (1.1b) is the conservation of mass, (1.1c) is the well-known Laplace-Young boundary condition for the pressure function, (1.1d) states that the free boundary moves with the velocity of the fluid, (1.1e) specifies the initial velocity, and (1.1f) fixes the initial domain Ω.

Almost all prior well-posedness results were focused on *irrotational* fluids (potential flow), wherein the additional constraint $\operatorname{curl} u = 0$ is imposed; with the irrotationality constraint, the Euler equations (1.1) reduce to the well-known water-waves equations, wherein the motion of the interface is decoupled from the rest of the fluid and is governed by singular boundary integrals that arise from the use of complex variables and the equivalence of incompressibility and irrotationality with the Cauchy-Riemann equations. For 2D fluids (and hence 1D interfaces), the earliest local existence results were obtained by Nalimov [14], Yosihara [22], and Craig [5] for initial data near equilibrium. Beale, Hou, and Lowengrub [4] proved that the linearization of the 2D water-wave problem is well-posed if a Taylor sign condition is added to the problem formulation, thus preventing Rayleigh-Taylor instabilities. Using the Taylor sign condition, Wu [20] proved local existence for

Received by the editors November 9, 2005.
2000 *Mathematics Subject Classification.* Primary 35Q35, 35R35, 35Q05, 76B03.
Key words and phrases. Euler equations, free boundary problems, surface tension.

©2007 American Mathematical Society
Reverts to public domain 28 years from publication

the 2D water-wave problem for arbitrary (sufficiently smooth) initial data. Later Ambrose [2] and Ambrose and Masmoudi [3] proved local well-posedness of the 2D water-wave problem with surface tension on the boundary replacing the Taylor sign condition.

In 3D, Wu [21] used Clifford analysis to prove local existence of the full water-wave problem with *infinite depth*, showing that the Taylor sign condition is always satisfied in the irrotational case by virtue of the maximum principle holding for the potential flow. Lannes [11] provided a proof for the *finite depth case with varying bottom* by implementing a Nash-Moser iteration. The first well-posedness result for the full Euler equations with zero surface tension, $\sigma = 0$, is due to Lindblad [13] with the additional "physical condition" that

$$\nabla p \cdot n < 0 \text{ on } \partial Q, \tag{1.2}$$

where n denotes the exterior unit normal to $\partial\Omega(t)$. The condition (1.2) is equivalent to the Taylor sign condition and provided Christodoulou and Lindblad [6] with enough boundary regularity to establish *a priori* estimates for smooth solutions to (1.1) together with (1.2) and $\sigma = 0$. (Ebin [10] provided a counterexample to well-posedness when (1.2) is not satisfied.) Nevertheless, local existence did not follow in [6], as finding approximations of the Euler equations for which existence and uniqueness is known and which retain the transport-type structure of the Euler equations is highly nontrivial, and this geometric transport-type structure is crucial for the a priori estimates. In [12], Lindblad proved well-posedness of the linearized Euler equations, but the estimates were not sufficient for well-posedness of the nonlinear problem. The estimates were improved in [13], wherein Lindblad implemented a Nash-Moser iteration to deal with the manifest loss of regularity in his linearized model and thus established the well-posedness result in the case that (1.2) holds and $\sigma = 0$.

Local existence for the case of positive surface tension, $\sigma > 0$, remained open. Although the Laplace-Young condition (1.1c) provides improved regularity for the boundary, the required nonlinear estimates are more difficult to close due to the complexity of the mean curvature operator and the need to study time-differentiated problems which do not arise in the $\sigma = 0$ case. It appears that the use of the time-differentiated problem in Lindblad's paper [13] is due to the use of certain tangential projection operators, but this is not necessary. We note that our energy function is different from that in [13] and provides better control of the Lagrangian coordinate.

After completing this work, we were informed of the paper of Schweizer [16] who studies the Euler equations for $\sigma > 0$ in the case that the free-surface is a graph over the two-torus. In that paper, he obtains a priori estimates under a smallness assumption for the initial surface; well-posedness follows under the additional assumption that there is no vorticity on the boundary. We also learned of the paper by Shatah and Zeng [15] who establish a priori estimates for both the $\sigma = 0$ and $\sigma > 0$ cases without any restrictions on the initial data.

1.2. **Main results.** We prove two main theorems concerning the well-posedness of (1.1). The first theorem, for the case of positive surface tension $\sigma > 0$, is new; for our second theorem, corresponding to the zero surface tension case, we present a new proof that does not require a Nash-Moser procedure and has optimal regularity.

Theorem 1.1 (Well-posedness with surface tension)**.** *Suppose that $\sigma > 0$, Γ is of class $H^{5.5}$, and $u_0 \in H^{4.5}(\Omega)$. Then, there exists $T > 0$ and a solution*

$(u(t),p(t),\Omega(t))$ of (1.1) with $u \in L^\infty(0,T;H^{4.5}(\Omega(t)))$, $p \in L^\infty(0,T;H^4(\Omega(t)))$, and $\partial\Omega(t) \in H^{5.5}$. The solution is unique if $u_0 \in H^{5.5}(\Omega)$ and $\partial\Omega \in H^{6.5}$.

Theorem 1.2 (Well-posedness with Taylor sign condition)**.** *Suppose $\sigma = 0$, $\partial\Omega$ is of class H^3, and $u_0 \in H^3(\Omega)$ and condition (1.2) holds at $t = 0$. Then, there exists $T > 0$ and a unique solution $(u(t),p(t),\Omega(t))$ of (1.1) with $u \in L^\infty(0,T;H^3(\Omega(t)))$, $p \in L^\infty(0,T;H^{3.5}(\Omega(t)))$, and $\partial\Omega(t) \in H^3$.*

1.3. **Lagrangian representation of the Euler equations.** The Eulerian problem (1.1), set on the moving domain $\Omega(t)$, is converted to a PDE on the fixed domain Ω, by the use of Lagrangian variables. Let $\eta(\cdot,t) : \Omega \to \Omega(t)$ be the solution of

$$\partial_t\eta(x,t) = u(\eta(x,t),t), \quad \eta(x,0) = \mathrm{Id}$$

and set

$$v(x,t) := u(\eta(x,t),t), \quad q(x,t) := p(\eta(x,t),t), \quad \text{and} \quad a(x,t) := [\nabla\eta(x,t)]^{-1}\,.$$

The variables v, q and a are functions of the fixed domain Ω and denote the material velocity, pressure, and inverse Jacobian, respectively. Thus, on the fixed domain, (1.1) transforms to

$$\eta = \mathrm{Id} + \int_0^t v \quad \text{in } \Omega\times(0,T]\,, \tag{1.3a}$$

$$\partial_t v + a\,\nabla q = 0 \quad \text{in } \Omega\times(0,T]\,, \tag{1.3b}$$

$$\mathrm{Tr}(a\,\nabla v) = 0 \quad \text{in } \Omega\times(0,T]\,, \tag{1.3c}$$

$$q\,a^T N/|a^T N| = -\sigma\,\Delta_g(\eta) \quad \text{on } \Gamma\times(0,T]\,, \tag{1.3d}$$

$$(\eta, v) = (\mathrm{Id}, u_0) \quad \text{on } \Omega\times\{t=0\}\,, \tag{1.3e}$$

where N denotes the unit normal to Γ and Δ_g is the surface Laplacian with respect to the induced metric g on Γ, written in local coordinates as

$$\Delta_g = \sqrt{g}^{-1}\partial_\alpha[\sqrt{g}g^{\alpha\beta}\partial_\beta]\,,\ g^{\alpha\beta} = [g_{\alpha\beta}]^{-1}\,,\ g_{\alpha\beta} = \eta_{,\alpha}\cdot\eta_{,\beta}\,,\ \text{and}\ \sqrt{g} = \sqrt{\det g}\,. \tag{1.4}$$

Theorem 1.3 ($\sigma > 0$)**.** *Suppose that $\sigma > 0$, $\partial\Omega$ is of class $H^{5.5}$, and $u_0 \in H^{4.5}(\Omega)$ with $\operatorname{div} u_0 = 0$. Then, there exists $T > 0$ and a solution (v,q) of (1.3) with $v \in L^\infty(0,T;H^{4.5}(\Omega))$, $q \in L^\infty(0,T;H^4(\Omega))$, and $\Gamma(t) \in H^{5.5}$. The solution satisfies*

$$\sup_{t\in[0,T]}\left(|\partial\Omega(t)|^2_{5.5} + \sum_{k=0}^{3}\|\partial_t^k v(t)\|^2_{4.5-1.5k} + \sum_{k=0}^{2}\|\partial_t^k q(t)\|^2_{4-1.5k}\right) \le \tilde{M}_0$$

where $\tilde{M}_0$ denotes a polynomial function of $\|\Gamma\|_{5.5}$ and $\|u_0\|_{4.5}$. The solution is unique for $u_0 \in H^{5.5}(\Omega)$ and $\Gamma \in H^{6.5}$.

Remark 1. Our theorem is stated for a fluid in vacuum, but the analogous theorem holds for a vortex sheet, i.e., for the motion of the interface separating two inviscid immiscible incompressible fluids; the boundary condition (1.1c) is replaced by $[p]_\pm = \sigma H$, where $[p]_\pm$ denotes the jump in pressure across the interface.

For the zero surface tension case, we have

Theorem 1.4 ($\sigma = 0$ and condition (1.2))**.** *Suppose that $\sigma = 0$, Γ is of class H^3, $u_0 \in H^3(\Omega)$, and condition (1.2) holds at $t = 0$. Then, there exists $T > 0$ and a unique solution (v,q) of (1.3) with $v \in L^\infty(0,T;H^3(\Omega))$, $q \in L^\infty(0,T;H^3(\Omega))$, and $\Gamma(t) \in H^3$.*

Because of the regularity of the solutions, Theorems 1.3 and 1.4 imply Theorems 1.1 and 1.2, respectively.

Remark 2. Note that in 3D, we require less regularity on the initial data than [13].

Remark 3. Since the vorticity satisfies the equation $\partial_t \operatorname{curl} u + \mathcal{L}_u \operatorname{curl} u = 0$, where $\mathcal{L}_u$ denotes the Lie derivative in the direction u, it follows that if $\operatorname{curl} u_0 = 0$, then $\operatorname{curl} u(t) = 0$. Thus our result also covers the simplified case of irrotational flow. In particular, Theorem 1.3 shows that the 3D irrotational water-wave problem with surface tension is well-posed. In the zero surface tension case, our result improves the regularity of the data required by Wu [21].

1.4. **General methodology and outline of the paper.**

1.4.1. *Artificial viscosity and the smoothed κ-problem.* Our methodology begins with the introduction of a smoothed or *approximate* problem (4.1), wherein two basic ideas are implemented: first, we smooth the transport velocity using a new tool which we call *horizontal convolution by layers*; second, we introduce an *artificial viscosity* term in the Laplace-Young boundary condition ($\sigma > 0$) which simultaneously preserves the transport-type structure of the Euler equations, provides a PDE for which we can prove the existence of unique smooth solutions, and for which there exist a priori estimates which are independent of the artificial viscosity parameter κ. With the addition of the artificial viscosity term, the dispersive boundary condition is converted into a parabolic-type boundary condition, and thus finding solutions of the smoothed problem becomes an easier matter. On the other hand, the a priori estimates for the κ problem are more difficult than the formal estimates for the Euler equations.

The horizontal convolution is defined in Section 2. The domain Ω is partitioned into coordinate charts, each the image of the unit cube in $\mathbb{R}^3$. A double convolution is performed in the horizontal direction only (this is equivalent to the tangential direction in coordinate patches near the boundary). While there is no smoothing in the vertical direction, our horizontal convolution commutes with the trace operator and avoids the need to introduce an extension operator, the latter destroying the natural transport structure. The development of the horizontal convolution by layers is absolutely crucial in proving the regularity of the weak solutions that we discuss below. Furthermore, it is precisely this tool which enables us to prove Theorem 1.2 without the use of Nash-Moser iteration. To reiterate, this horizontal smoothing operator preserves the essential transport-type structure of the Euler equations.

1.4.2. *Weak solutions in a variational framework and a fixed point, $\sigma > 0$.* The solution to the smoothed κ-problem (4.1) is obtained via a topological fixed-point procedure, founded upon the analysis of the linear problem (4.2). To solve the linear problem, we introduce a few new ideas. First, we penalize the pressure function; in particular, with $\epsilon > 0$ the penalization parameter, we introduce the penalized pressure function $q_\epsilon = \frac{1}{\epsilon}\operatorname{Tr}(a\,\nabla w)$. Second, we find a new class of $[H^{\frac{3}{2}}(\Omega)]'$-weak solutions of the penalized and linearized smoothed κ-problem in a variational formulation. The penalization allows us to perform difference quotient analysis in order to prove regularity of our weak solutions; without penalization, difference quotients of weak solutions do not satisfy the "divergence-free" constraint and as such cannot be used as test functions. Furthermore, the penalization of the pressure function avoids the need to analyze the highest-order time derivative of the pressure, which would otherwise be highly problematic. In the setting of the penalized problem, we crucially rely on the horizontal convolution by layers to establish regularity of our weak penalized solution. Third, we introduce the Lagrange multiplier lemmas, which associate a pressure function to the weak solution of a variational problem for which the test functions satisfy the incompressibility constraint. These lemmas allow us to pass to the limit as the penalization parameter tends to zero, and thus, together with the Tychonoff fixed-point theorem, establish solutions to the smoothed problem (4.1). At this stage, however, the time interval of existence and the bounds for the solution depend on the parameter κ.

1.4.3. *Solutions of the κ-problem for $\sigma = 0$ via transport.* For the $\sigma = 0$ problem, we use horizontal convolution to smooth the transport velocity as well as the moving domain. Existence and uniqueness of this smoothed κ problem (17.1) is found using simple transport-type arguments that rely on the pressure gaining regularity just as in the fixed-domain case. Once again, the time interval of existence and the bounds for the solution a priori depend on κ.

1.4.4. *A priori estimates and κ-asymptotics.* We develop a priori estimates which show that the energy function $E_\kappa(t)$ in Definition 10.1 associated to our smoothed problem (4.1) is bounded by a constant depending only on the initial data and not on κ. The estimates rely on the Hodge decomposition elliptic estimate (5.1).

In Section 10, we obtain estimates for the divergence and curl of η, v and their space and time derivatives. The main novelty lies in the curl estimate for η. The remaining portion of the energy is obtained by studying boundary regularity via energy estimates.

These nonlinear boundary estimates for the surface tension case $\sigma > 0$ are more complicated than the ones for the $\sigma = 0$ case with the Taylor sign condition (1.2) since it is necessary to analyze the time-differentiated Euler equations, which is not essential in the $\sigma = 0$ case (unless optimal regularity is sought).

We note that the use of the smoothing operator in Definition 2.1, where a double convolution is employed, is necessary in order to find exact (or perfect) derivatives for the highest-order error terms. The idea is that one of the convolution operators is moved onto a function which is a priori not smoothed, and commutation-type lemmas are developed for this purpose.

We obtain the a priori estimate

$$\sup_{t\in[0,T]} E_\kappa(t) \leq M_0 + TP(\sup_{t\in[0,T]} E_\kappa(t))\,,$$

where M_0 depends only on the data and P is a polynomial. The addition of the artificial viscosity term allows us to prove that $E_\kappa(t)$ is continuous; thus, following the development in [8], there exists a sufficiently small time T, which is independent of κ, such that $\sup_{t\in[0,T]} E_\kappa(t) < \tilde{M}_0$ for $\tilde{M}_0 > M_0$.

We then find κ-independent nonlinear estimates for the $\sigma = 0$ case for the energy function (20.1).

Outline. Sections 2–15 are devoted to the case of positive surface tension $\sigma > 0$. Sections 16–27 concern the problem with zero surface tension $\sigma = 0$ together with the Taylor sign condition (1.2) imposed.

1.5. **Notation.** Throughout the paper, we shall use the Einstein convention with respect to repeated indices or exponents. We specify here our notation for certain vector and matrix operations.

We write the Euclidean inner-product between two vectors x and y as $x \cdot y$, so that $x \cdot y = x^i\, y^i$.
The transpose of a matrix A will be denoted by A^T, *i.e.*, $(A^T)^i_j = A^j_i$.
We write the product of a matrix A and a vector b as $A\,b$, *i.e.*, $(A\,b)^i = A^i_j b^j$.
The product of two matrices A and S will be denoted by $A{\cdot}S$, *i.e.*, $(A{\cdot}S)^i_j = A^i_k\; S^k_j$.
The trace of the product of two matrices A and S will be denoted by $A : S$, *i.e.*, $(A : S)^i_j = A^i_k\; S^k_i$.

For Ω, a domain of class H^s $(s \geq 2)$, there exists a well-defined extension operator that we shall make use of later.

Lemma 1.1. *There exists $E(\Omega)$, a linear and continuous operator from $H^r(\Omega)$ into $H^r(\mathbb{R}^3)$ $(0 \leq r \leq s)$, such that for any $v \in H^r(\Omega)$ $(0 \leq r \leq s)$, $E(\Omega)(v) = v$ in Ω.*

We will use the notation $H^s(\Omega)$ to denote either $H^s(\Omega;\mathbb{R})$ (for a pressure function, for instance) or $H^s(\Omega;\mathbb{R}^3)$ (for a velocity vector field) and we denote the standard norm of $H^s(\Omega)$ $(s \geq 0)$ by $\|\cdot\|_s$. The $H^s(\Omega)$ inner-product will be denoted $(\cdot,\cdot)_s$.

We shall use the following notation for derivatives: ∂_t or $(\cdot)_t$ denotes the partial time derivative, ∂ denotes the tangential derivative on Γ (or in a small enough neighborhood of Γ), and ∇ denotes the three-dimensional gradient.

Letting (x^1, x^2) denote a local coordinate system on Γ, for $\alpha = 1, 2$, we let either ∂_α or $(\cdot)_{,\alpha}$ denote $\frac{\partial}{\partial x^\alpha}$. We define

$$\partial^\alpha := g_0^{\alpha\beta}\partial_\beta\,, \quad |\partial^k\phi|^2 = \partial^{\alpha_1}\partial^{\alpha_2}\cdots\partial^{\alpha_k}\partial_{\alpha_1}\partial_{\alpha_2}\cdots\partial_{\alpha_k}$$

for integers $k \geq 0$, where $g_0 = g_{t=0}$ is the (induced) metric on Γ. In particular, $|\partial^0\phi| = |\phi|$, $|\partial^1\phi|^2 = |\partial\phi|^2 = \partial^\alpha\phi\partial_\alpha\phi$ and $\partial^k\phi$ will mean any kth tangential derivative of ϕ.

The area element on Γ in local coordinates is $dS_0 = \sqrt{g_0}dx^1 \wedge dx^2$ and the pull-back of the area element dS on $\Gamma(t) = \eta(\Gamma)$ is given by $\eta^*(dS) = \sqrt{g}dS_0$. Let $\{U_i\}_{i=1}^K$ denote an open covering of Γ, and let $\{\xi_i\}_{i=1}^K$ denote the partition of unity subordinate to this cover. The $L^2(\Gamma)$ norm is

$$|\phi|_0 := \|\phi\|_{L^2(\Gamma)} = \left(\int_\Gamma \phi^2 dS_0\right)^{\frac{1}{2}}\,,$$

and the $H^k(\Gamma)$ norm for integers $k \geq 1$ is

$$|\phi|_k := \|\phi\|_{H^k(\Gamma)} = \left(\sum_{i=1}^k\sum_{l=1}^K |\xi_l\partial^i\phi|_0^2\right)^2\,.$$

Similarly, for the Hilbert space inner-products, we use

$$[\phi,\psi]_0 := [\phi,\psi]_{L^2(\Gamma)} = \int_\Gamma \phi\,\psi\,dS_0,$$

$$[\phi,\psi]_k := [\phi,\psi]_{H^k(\Gamma)} = [\phi,\psi]_0 + \sum_{i=1}^k\sum_{l=1}^K [\xi_l\partial^i\phi, \xi_l\partial_i\psi]_0\,.$$

Fractional-order spaces are defined via interpolation using the trace spaces of Lions (see, for example, [1]).

The dual of a Banach space X is denoted by X', and the corresponding norm in X' will be denoted $\|\cdot\|_{X'}$. For $L \in H^s(\Omega)'$ and $v \in H^s(\Omega)$, the duality pairing between L and v is denoted by $\langle L, v\rangle_s$.

Throughout the paper, we shall use C to denote a generic constant, which may possibly depend on the coefficient σ or on the initial geometry given by Ω (such as a Sobolev constant or an elliptic constant), and we use $P(\cdot)$ to denote a generic polynomial function of $(\cdot)$. For the sake of notational convenience, we will often write $u(t)$ for $u(t,\cdot)$.

2. Convolution by horizontal layers and the smoothed transport velocity

Let $\Omega \subset \mathbb{R}^n$ denote an open subset of class H^6, and let $\{U_i\}_{i=1}^K$ denote an open covering of $\Gamma := \partial\Omega$, such that for each $i \in \{1, 2, ..., K\}$,

$$\theta_i : (0,1)^2 \times (-1,1) \to U_i \quad \text{is an } H^6 \text{ diffeomorphism}\,,$$
$$U_i \cap \Omega = \theta_i((0,1)^3) \quad \text{and} \quad U_i \cap \Gamma = \theta_i((0,1)^2 \times \{0\})\,,$$
$$\theta_i(x_1, x_2, x_3) = (x_1, x_2, \psi_i(x_1, x_2) + x_3) \text{ and } \det \nabla\theta_i = 1 \text{ in } (0,1)^3\,.$$

Next, for $L > K$, let $\{U_i\}_{i=K+1}^L$ denote a family of open sets contained in Ω such that $\{U_i\}_{i=1}^L$ is an open cover of Ω. Let $\{\alpha_i\}_{i=1}^L$ denote the partition of unity subordinate to this covering.

Thus, each coordinate patch is locally represented by the unit cube $(0,1)^3$ and for the first K patches (near the boundary), the tangential (or horizontal) direction is represented by $(0,1)^2 \times \{0\}$.

Definition 2.1 (Horizontal convolution)**.** Let $0 \le \rho \in \mathcal{D}((0,1)^2)$ denote an even Friederich mollifier, normalized so that $\int_{(0,1)^2} \rho = 1$, with corresponding dilated function

$$\rho_{\frac{1}{\delta}}(x) = \frac{1}{\delta^2}\rho\left(\frac{x}{\delta}\right), \quad \delta > 0.$$

For $w \in H^1((0,1)^3)$ such that $\operatorname{supp}(w) \subset [\delta, 1-\delta]^2 \times (0,1)$, set

$$\rho_{\frac{1}{\delta}} \star_h w(x_H, x_3) = \int_{\mathbb{R}^2} \rho_{\frac{1}{\delta}}(x_H - y_H) w(y_h, x_3) dy_H\,, \quad y_H = (y_1, y_2)\,.$$

We then have the tangential integration by parts formula

$$\rho_{\frac{1}{\delta}} \star_h w,_\alpha (x_H, x_3) = \int_{\mathbb{R}^2} \rho_{\frac{1}{\delta}},_\alpha (x_H - y_H) w(y_h, x_3) dy_H\,, \quad \alpha = 1, 2\,,$$

while

$$\rho_{\frac{1}{\delta}} \star_h w,_3 (x_H, x_3) = \int_{\mathbb{R}^2} \rho_{\frac{1}{\delta}}(x_H - y_H) w,_3 (y_h, x_3) dy_H\,.$$

It should be clear that $\star_h$ smooths w in the horizontal directions, but not in the vertical direction. Fubini's theorem ensures that

$$\|\rho_{\frac{1}{\delta}} \star_h w\|_{s,(0,1)^3} \le C_s \|w\|_{s,(0,1)^3} \text{ for any } s \ge 0\,, \tag{2.1}$$

and we shall often make implicit use of this inequality.

Remark 4. The horizontal convolution $\star_h w$ does not smooth w in the vertical direction; however, it does commute with the trace operator, so that

$$\left(\rho_{\frac{1}{\delta}} \star_h w\right)\Big|_{(0,1)^2\times\{0\}} = \rho_{\frac{1}{\delta}} \star_h w|_{(0,1)^2\times\{0\}} ,$$

which is essential for our methodology. Also, note that $\star_h$ smooths without the introduction of an extension operator, required by standard convolution operators on bounded domains; the extension to the full space would indeed be problematic for the transport structure of the divergence and curl of solutions to the Euler-type PDEs that we introduce.

Definition 2.2 (Smoothing the velocity field)**.** For $v \in L^2(\Omega)$ and any $\kappa \in (0, \frac{\kappa_0}{2})$ with

$$\kappa_0 = \min_{i=1}^{K} \text{dist}\left(\text{supp}(\alpha_i \circ \theta_i) , [(0,1)^2 \times \{0\}]^c \cap \partial[0,1]^3\right) ,$$

set

$$v_\kappa = \sum_{i=1}^{K} \sqrt{\alpha_i} \left[\rho_{\frac{1}{\kappa}} \star_h [\rho_{\frac{1}{\kappa}} \star_h ((\sqrt{\alpha_i} v) \circ \theta_i)]\right] \circ \theta_i^{-1} + \sum_{i=K+1}^{L} \alpha_i v .$$

It follows from (2.1) that there exists a constant $C > 0$ which is independent of κ such that for any $v \in H^s(\Omega)$ for $s \geq 0$,

$$\|v_\kappa\|_s \leq C\|v\|_s \quad\text{and}\quad |v_\kappa|_{s-1/2} \leq C|v|_{s-1/2} . \tag{2.2}$$

The smoothed particle displacement field is given by

$$\eta_\kappa = \text{Id} + \int_0^t v_\kappa . \tag{2.3}$$

For each $x \in U_i$, let $\tilde{x} = \theta_i^{-1}(x)$. The difference of the velocity field and its smoothed counterpart along the boundary Γ then takes the form

(2.4)

$$\begin{aligned} &v_\kappa(x) - v(x) \\ &\quad = \sum_{i=1}^{K} \int\int_{B(0,\kappa)^2} \zeta_i(x) \rho_{\frac{1}{\kappa}}(\tilde{y}) \rho_{\frac{1}{\kappa}}(\tilde{z}) \left[(\zeta_i v)(\theta_i(\tilde{x} - (\tilde{y} + \tilde{z}))) - (\zeta_i v)(\theta_i(\tilde{x}))\right] d\tilde{z}\, d\tilde{y} , \end{aligned}$$

where $\zeta_i(x) = \sqrt{\alpha_i(\theta_i(\tilde{x}))}$. Combining (1.1a), (2.3), and (2.4),

(2.5)

$$\begin{aligned} &\eta_\kappa(x) - \eta(x) \\ &\quad = \sum_{i=1}^{K} \int\int_{B(0,\kappa)^2} \zeta_i(x) \rho_{\frac{1}{\kappa}}(\tilde{y}) \rho_{\frac{1}{\kappa}}(\tilde{z}) \left[(\zeta_i \eta)(\theta_i(\tilde{x} - (\tilde{y} + \tilde{z}))) - (\zeta_i \eta)(\theta_i(\tilde{x}))\right] d\tilde{z}\, d\tilde{y} . \end{aligned}$$

For any $u \in H^{1.5}(\Gamma)$ and for $y \in B(x, \kappa)$, where $B(x, \kappa)$ denotes the disk of radius κ centered at x, the mean value theorem shows that

$$|u(y) - u(x)| \leq C|r^{-1}|_{L^q(B(x,\kappa))} |\partial u|_{L^p(B(x,\kappa))}, \qquad r = \text{radial coordinate},$$

so that in particular, with $p = 4$ and $q = \frac{4}{3}$,

$$|u(y) - u(x)| \leq C\sqrt{\kappa} |\partial u|_{L^4} \leq C\sqrt{\kappa} |u|_{1.5} ,$$

the last inequality following from the Sobolev embedding theorem. Hence, for $U \in H^{1.5}(\Gamma)$,

$$|U_\kappa(x) - U(x)|_{L^\infty} \le C\sqrt{\kappa}|U|_{1.5}\,. \tag{2.6}$$

Note that the constant C depends on $\max_{i\in\{1,\dots,K\}} |\theta_i|_{5.5}$.

Letting $\zeta_i = \sqrt{\alpha_i}$ and $R = (0,1)^2$, we also have that for any $\phi \in L^2(\Gamma)$,

$$\int_\Gamma v_\kappa\,\phi = \sum_{i=1}^K \int_R \rho_{\frac{1}{\kappa}} \star_h \rho_{\frac{1}{\kappa}} \star_h \zeta_i v(x)\,\zeta_i\phi(x) = \sum_{i=1}^K \int_R \rho_{\frac{1}{\kappa}} \star_h \zeta_i v(x)\,\rho_{\frac{1}{\kappa}} \star_h \zeta_i\phi(x)$$

$$= \int_\Gamma \sum_{i=1}^K [\rho_{\frac{1}{\kappa}} \star_h (\zeta_i v\circ\theta_i)]\circ\theta_i^{-1}\,[\rho_{\frac{1}{\kappa}} \star_h (\zeta_i\phi\circ\theta_i)]\circ\theta_i^{-1}\,. \tag{2.7}$$

Finally, we need the following

Lemma 2.1 (Commutation-type lemma)**.** *Let $g \in L^2(\Gamma)$ satisfy $dist(supp(g), \partial R) < \kappa_0$ and let $f \in H^s(\Gamma)$ for $s > 1$. Then independently of $\kappa \in (0,\kappa_0)$, there exists a constant $C > 0$ such that*

$$\left|\rho_{\frac{1}{\kappa}} \star_h [fg] - f\rho_{\frac{1}{\kappa}} \star_h g\right|_{0,R} \le C\,\kappa|f|_{s+1,R}\,|g|_{0,R}\,.$$

We also have

$$\left\|\rho_{\frac{1}{\kappa}} \star_h [fg] - f\rho_{\frac{1}{\kappa}} \star_h g\right\|_{0,[0,1]^3} \le C\,\kappa\|f\|_{s+\frac{3}{2},[0,1]^3}\,\|g\|_{0,[0,1]^3}$$

whenever $g \in L^2(\Omega)$, $f \in H^s(\Omega)$ and

$$\frac{\kappa}{2} < \min(dist(supp\ fg, \{1\}\times[0,1]^2), dist(supp\ fg, \{0\}\times[0,1]^2)).$$

Proof. Let $\triangle = \rho_{\frac{1}{\kappa}} \star_h [fg] - f\rho_{\frac{1}{\kappa}} \star_h g$. Then

$$|\triangle(x)| = \left|\int_{B(x,\kappa)} \rho_{\frac{1}{\kappa}}(x-y)[f(y)-f(x)]g(y)dy\right|$$

$$\le C\,\kappa|f|_{s+1,R}\int_{B(x,\kappa)} \rho_{\frac{1}{\kappa}}(x-y)|g(y)|dy\,,$$

so that

$$|\triangle|_{0,R} \le C\,\kappa|f|_{s+1,R}\,\left|\rho_{\frac{1}{\kappa}} \star_h |g|\,\right|_{0,R} \le C\,\kappa|f|_{s+1,R}\,|g|_{0,R}\,.$$

The inequality on $[0,1]^3$ follows the identical argument with an additional integration over the vertical coordinate. The hypothesis on the support of fg makes the integral well-defined. □

Remark 5. Higher-order commutation-type lemmas will be developed for the case of zero surface tension in Section 21.

3. Closed convex set used for the fixed point for $\sigma > 0$

In order to construct solutions for our approximate model (4.1), we use a topological fixed-point argument which necessitates the use of high-regularity Sobolev spaces. In particular, we shall assume that the initial velocity u_0 is in $H^{13.5}(\Omega)$ and that Ω is of class C^∞; after establishing our result for the smoothed initial domain and velocity, we will show that both Ω and u_0 can be taken with the optimal regularity stated in Theorem 1.3.

For $T > 0$, we define the following closed convex set of the Hilbert space $L^2(0,T;H^{13.5}(\Omega))$:

$$C_T = \{v \in L^2(0,T;H^{13.5}(\Omega)) | \sup_{[0,T]} \|v\|_{13.5} \le 2\|u_0\|_{13.5} + 1\}.$$

It is clear that C_T is nonempty, since it contains the constant (in time) function u_0, and is a convex, bounded and closed subset of the separable Hilbert space $L^2(0,T;H^{13.5}(\Omega))$.

Let $v \in C_T$ be given, and define η by (1.3a), the Bochner integral being taken in the separable Hilbert space $H^{13.5}(\Omega)$.

Henceforth, we assume that $T > 0$ is given such that independently of the choice of $v \in C_T$, we have the injectivity of $\eta(t)$ on $\overline{\Omega}$, the existence of a normal vector to $\eta(\Omega,t)$ at any point of $\eta(\Gamma,t)$, and the invertibility of $\nabla\eta(t)$ for any point of $\overline{\Omega}$ and for any $t \in [0,T]$. Such a condition can be achieved by selecting T small enough so that

$$\|\nabla\eta - \mathrm{Id}\|_{L^\infty(0,T;H^{13.5}(\Omega))} \le \epsilon_0 \,, \tag{3.1}$$

for $\epsilon_0 > 0$ taken sufficiently small. Condition (3.1) holds if $T\|\nabla u_0\|_{H^2} \le \epsilon_0$. Thus,

$$a = [\nabla\eta]^{-1} \tag{3.2}$$

is well-defined.

Then choosing $T > 0$ even smaller, if necessary, there exists $\kappa_0 > 0$ such that for any $\kappa \in (0, \frac{\kappa_0}{2})$, we have the injectivity of $\eta_\kappa(t)$ on Ω for any $t \in [0,T]$; furthermore, $\nabla\eta_\kappa$ satisfies the condition (3.1) with η_κ replacing η. We let $n_\kappa(\eta_\kappa(x))$ denote the exterior unit normal to $\eta_\kappa(\Omega)$ at $\eta_\kappa(x)$ with $x \in \Gamma$.

Our notational convention will be as follows: if we choose $\bar{v} \in C_T$, then $\bar{\eta}$ is the flow map coming from (1.3a), and $\bar{a}$ is the associated pull-back, $\bar{a} = [\nabla\bar{\eta}]^{-1}$. Thus, a bar over the velocity field will imply a bar over the Lagrangian variable and the associated pull-back.

For a given v_κ, our notation is as follows:

$$\eta_\kappa(t) = \mathrm{Id} + \int_0^t v_\kappa \quad \text{and} \quad \eta_\kappa(0) = \mathrm{Id}\,,$$
$$a_\kappa = \mathrm{Cof}\,\nabla\eta_\kappa\,, \quad J_\kappa = \det\nabla\eta_\kappa\,, \quad g_{\kappa\alpha\beta} = \partial_\alpha\eta_\kappa \cdot \partial_\beta\eta_\kappa\,.$$

We take T (which a priori depends on κ) even smaller if necessary to ensure that for $t \in [0,T]$,

$$\sqrt{g(t)}^{-1} \le 2\sqrt{g_0}^{-1}\,, \tag{3.3a}$$

$$\sqrt{g_\kappa(t)}^{-1} \le 2\sqrt{g_0}^{-1}\,, \tag{3.3b}$$

$$\frac{1}{2} \le J_\kappa(t) \le \frac{3}{2}\,. \tag{3.3c}$$

Lemma 3.1. *For $v \in C_T$ and for any $s \ge 0$, we have independently of the choice of $v \in C_T$ that*

$$\sup_{[0,T]} |v_\kappa|_s \le C_{\kappa,s}\, P(\|u_0\|_{13.5})\,.$$

Proof. By the standard properties of the convolution a.e. in $[0,T]$:

$$|v_\kappa|_s \le [\frac{C}{\kappa^{s-13}} + 1]|v|_{13} \le [\frac{C}{\kappa^{s-13}} + 1][2\|u_0\|_{13.5} + 1], \tag{3.4}$$

where we have used the definition of C_T for the second inequality. $\square$

Recall that $\{\theta_i\}_{i=1}^K$ is our open cover of Γ. Given $\bar{v} \in C_T$, we define the matrix $\bar{b}_\kappa^l = [\nabla(\bar{\eta}_\kappa \circ \theta_l)]^{-1}$ and assume that $T > 0$ is sufficiently small so that independently of $\bar{v} \in C_T$, we have the following determinant-type condition for $\bar{b}_\kappa^l$:

$$\frac{1}{2} \leq (\bar{b}_\kappa^l)_3^3 \sum_{i=1}^3 [(\bar{b}_\kappa^l)_i^3]^2, \text{ in } (0,1)^3. \tag{3.5}$$

Such a condition is indeed possible since at time $t = 0$ we have $(\bar{b}_\kappa^l)_3^3 \sum_{i=1}^3 [(\bar{b}_\kappa^l)_i^3]^2 = 1 + \psi_l,_1^2 + \psi_l,_2^2$.

4. The smoothed κ-problem and its linear fixed point formulation

Unlike the case of zero surface tension, for $\sigma > 0$ there does not appear to be a simple sequence of approximate problems for the Euler equations (1.1) which can be solved only with simple transport-type arguments. For the surface tension case, the problem is crucially variational in nature, and the addition of an artificial viscosity term on the boundary Γ seems unavoidable in order to be able to construct a sequence of approximate or smoothed solutions.

As we shall make precise below, our construction of the approximating sequence of problems is based on smoothing the transport velocity by use of the horizontal convolution by layers (see Definition 2.2), and hence smoothing the Lagrangian flow map and associated pull-back. Simultaneously, we introduce a new type of parabolic-type artificial viscosity boundary operator on Γ (of the same order in space as the surface tension operator). Note that unlike the case of interface motion in the fluid-structure interaction problem that we studied in [8], there is not a unique choice of the artificial viscosity term; in particular, other choices of artificial viscosity are possible for the asymptotic limit as the artificial viscosity is taken to zero.

We can now define our sequence of smoothed κ-problems. For our artificial viscosity parameter $\kappa \in (0, \frac{\kappa_0}{2})$, let (v, q) be the solution of

$$\eta = \mathrm{Id} + \int_0^t v \quad \text{in } \Omega \times (0,T]\,, \tag{4.1a}$$

$$\partial_t v + J_\kappa^{-1} a_\kappa \nabla q = 0 \qquad \text{in } \Omega \times (0,T]\,, \tag{4.1b}$$

$$\mathrm{Tr}(a_\kappa \nabla v) = 0 \qquad \text{in } \Omega \times (0,T]\,, \tag{4.1c}$$

(4.1d)

$$-\sigma \frac{\sqrt{g}}{\sqrt{g_\kappa}} \Delta_g(\eta) \cdot n_k(\eta_\kappa)\, n_k(\eta_\kappa) - \kappa \Delta_0 [v \cdot n_\kappa(\eta_\kappa)] n_\kappa(\eta_\kappa) = q\, n_\kappa(\eta_\kappa) \qquad \text{on } \Gamma \times (0,T]\,,$$

$$(\eta, v) = (\mathrm{Id}, u_0) \quad \text{on } \Omega \times \{t = 0\}, \tag{4.1e}$$

where $n_\kappa(\eta_\kappa) = \frac{(a_\kappa)^T N}{|(a_\kappa)^T N|}$ and $\Delta_0 = \sqrt{g_\kappa}^{-1} \partial_\alpha [\sqrt{g_0} g_0^{\alpha\beta} \partial_\beta]$. Note that on Γ, $\sqrt{g_\kappa} = |a_\kappa^T N|$ and that $(g_\kappa)_{\alpha\beta} = \eta_\kappa,_\alpha \cdot \eta_\kappa,_\beta$.

In order to obtain solutions to the sequence of approximate κ-problems (4.1), we study a linear problem whose fixed point will provide the desired solutions. If we denote by $\bar{v}$ an arbitrary element of C_T and if $\bar{\eta}_\kappa$, $\bar{a}_\kappa$, and $\bar{J}_\kappa$ are the associated smoothed Lagrangian variables given by Definition 2.2, then we define w to be the solution of

$$\partial_t w + \bar{J}_\kappa^{-1} \bar{a}_\kappa \nabla q = 0 \qquad \text{in } \Omega \times (0,T]\,, \tag{4.2a}$$

$$\operatorname{Tr}(\bar{a}_\kappa \nabla w) = 0 \qquad \text{in } \Omega \times (0,T]\,, \tag{4.2b}$$

$$-\sigma \frac{\sqrt{\bar{g}}}{\sqrt{\bar{g}_\kappa}} [\Delta_{\bar{g}}(\bar{\eta}) \cdot \bar{n}_\kappa(\bar{\eta}_\kappa)]\, \bar{n}_\kappa(\bar{\eta}_\kappa) - \kappa \Delta_{\bar{0}}[w \cdot \bar{n}_\kappa(\bar{\eta}_\kappa)]\, \bar{n}_\kappa(\bar{\eta}_\kappa) = q\, \bar{n}_\kappa(\bar{\eta}_\kappa) \quad \text{on } \Gamma \times (0,T]\,, \tag{4.2c}$$

$$(\eta, w) = (\mathrm{Id}, u_0) \ \text{ on } \Omega \times \{t=0\}, \tag{4.2d}$$

where $\bar{g}_{\alpha\beta} = \bar{\eta},_\alpha \cdot \bar{\eta},_\beta$ and $\Delta_{\bar{0}} = \sqrt{\bar{g}_\kappa}^{-1} \partial_\alpha [\sqrt{g_0} g_0^{\alpha\beta} \partial_\beta]$.

For a solution w to (4.2), a fixed point of the map $\bar{v} \mapsto w$ provides a solution of our smoothed problem (4.1).

In the following sections, we assume that $\bar{v} \in C_T$ is given and κ is in $(0, \frac{\kappa_0}{2})$. Until Section 10, wherein we study the asymptotic behavior of the problem (4.1) as $\kappa \to 0$, the parameter κ is fixed.

5. Hodge decomposition elliptic estimates

Our estimates are based on the following standard elliptic estimate:

Proposition 5.1. *For an H^r domain Ω, $r \geq 3$, if $v \in L^2(\Omega)$ with $\operatorname{curl} v \in H^{s-1}(\Omega)$, $\operatorname{div} v \in H^{s-1}(\Omega)$, and $v \cdot N|_\Gamma \in H^{s-\frac{1}{2}}(\Gamma)$ for $1 \leq s \leq r$, then there exists a constant $C > 0$ depending only on Ω such that*

$$\begin{aligned} \|v\|_s &\leq C \left(\|v\|_0 + \|\operatorname{curl} v\|_{s-1} + \|\operatorname{div} v\|_{s-1} + |v \cdot N|_{s-\frac{1}{2}} \right), \\ \|v\|_s &\leq C \left(\|v\|_0 + \|\operatorname{curl} v\|_{s-1} + \|\operatorname{div} v\|_{s-1} + |v \cdot T_\alpha|_{s-\frac{1}{2}} \right), \end{aligned} \tag{5.1}$$

where T_α, $\alpha = 1, 2$, are the tangent vectors to Γ.

The first estimate with $V \cdot N$ is standard (see, for example, [19]), while the second with $V \cdot T_\alpha$ follows from the fact that $T_\alpha \cdot N = 0$.

6. Weak solutions for the penalized problem and their regularity

The aim of this section is to establish the existence of the solution w_ϵ to the penalized version (of the divergence-free condition) of the linearized and smoothed κ-problem (4.2). In particular, we study the weak form of this problem with the pressure function q, approximated by the penalized pressure

$$q^\epsilon = -\frac{1}{\epsilon} \operatorname{Tr}(\bar{a}_\kappa \nabla w) \quad \text{for } 0 < \epsilon << 1\,.$$

In this section, as well as in Sections 8 and 9, we let

$$N(u_0, x, y) = P(\|u_0\|_{13.5}, x, y) \tag{6.1}$$

denote a generic polynomial function of $\|u_0\|_{13.5}$, x, and y, where x and y will typically denote norms of various quantities.

6.1. **Step 1. Galerkin sequence.** By introducing a basis $(e_l)_{l=1}^{\infty}$ of $H^1(\Omega)$ and $L^2(\Omega)$ and taking the approximation at rank $l \geq 2$ under the form $w_l(t,x) = \sum_{k=1}^{l} y_k(t)\ e_k(x)$, satisfying on $[0,T]$ the system of ordinary differential equations

$$\begin{aligned}&\text{(i) } (\bar{J}_\kappa\ w_{l\,t}, \phi)_0 + \kappa[w_l \cdot \bar{n}_\kappa(\bar{\eta}_\kappa), \phi \cdot \bar{n}_\kappa(\bar{\eta}_\kappa)]_1 - \sigma[L_{\bar{g}}\bar{\eta} \cdot \bar{n}_\kappa(\bar{\eta}_\kappa), \phi \cdot \bar{n}_\kappa(\bar{\eta}_\kappa)]_0\\ &\qquad - ((\bar{a}_\kappa)_i^j q_l,\ \phi^i,_j)_0 = 0,\ \forall \phi \in span(e_1, ..., e_l)\,,\\ &\text{(ii) } w_l(0) = (u_0)_l,\ \text{ in } \Omega\ ,\end{aligned}$$

where $L_{\bar{g}} = \frac{\sqrt{\bar{g}}}{\sqrt{g_0}}\Delta_{\bar{g}}$, $q_l = -\frac{1}{\epsilon}(\bar{a}_\kappa)_i^j w_l^i,_j$, and $(u_0)_l$ denotes the $L^2(\Omega)$ projection of u_0 on $span(e_1, ..., e_l)$, we see that the Cauchy-Lipschitz theorem gives us the local well-posedness for w_l on some $[0, T_{max}]$. The use of the test function w_l in this system of ODEs (which is allowed as it belongs to $span(e_1, ..., e_l)$) gives us in turn the energy law for any $t \in (0, T_{\max})$,

$$\begin{aligned}&\frac{1}{2}\|\bar{J}_\kappa^{\frac{1}{2}} w_l(t)\|_0^2 + \kappa\int_0^t [w_l \cdot \bar{n}_\kappa(\bar{\eta}_\kappa), w_l \cdot \bar{n}_\kappa(\bar{\eta}_\kappa)]_1 + \epsilon\int_0^t \|q_l\|_0^2\\ &-\frac{1}{2}\int_0^t ((\bar{J}_\kappa)_t w_l, w_l)_0 = \frac{1}{2}\|(u_0)_l\|_0^2 + \sigma\int_0^t [L_{\bar{g}}\bar{\eta} \cdot \bar{n}_\kappa(\bar{\eta}_\kappa), w_l \cdot \bar{n}_\kappa(\bar{\eta}_\kappa)]_0,\end{aligned}$$

which, with the control of $\bar{\eta}^\kappa$ provided by the definition of C_T gives the bound

$$\frac{1}{4}\|w_l(t)\|_0^2 + C\kappa\int_0^t |w_l \cdot \bar{n}_\kappa(\bar{\eta}_\kappa)|_1^2 + \epsilon\int_0^t \|q_l\|_0^2 \leq CN(u_0). \tag{6.2}$$

6.2. **Step 2. Weak solution w_ϵ of the penalized problem.** We then infer from (6.2) that w_l is defined on $[0,T]$ and that there is a subsequence (still denoted with the subscript l) satisfying

$$w_l \rightharpoonup w_\epsilon \ \text{ in } L^2(0,T; L^2(\Omega)), \tag{6.3a}$$

$$q_l \rightharpoonup q_\epsilon \ \text{ in } L^2(0,T; L^2(\Omega)), \tag{6.3b}$$

where

$$q_\epsilon = -\frac{1}{\epsilon}(\bar{a}_\kappa)_i^j w_\epsilon,_j^i\,. \tag{6.4}$$

We can also rewrite (6.3) as

$$w_l \rightharpoonup w_\epsilon \ \text{ in } L^2(0,T; L^2(\Omega)), \tag{6.5a}$$

$$\operatorname{div}(w_l \circ \bar{\eta}_\kappa^{-1})(\bar{\eta}_\kappa) \rightharpoonup \operatorname{div}(w_\epsilon \circ \bar{\eta}_\kappa^{-1})(\bar{\eta}_\kappa) \ \text{ in } L^2(0,T; L^2(\Omega))\ , \tag{6.5b}$$

which with the bound (6.2) and the definition of the normal $\bar{n}_\kappa$ provides

$$w_l \cdot \bar{n}_\kappa(\bar{\eta}_\kappa) \rightharpoonup w_\epsilon \cdot \bar{n}_\kappa(\bar{\eta}_\kappa) \ \text{ in } L^2(0,T; H^1(\Gamma)). \tag{6.6}$$

It follows from standard arguments and the ODE defining w_l that $w_{\epsilon t} \in L^2(0,T; H^{\frac{3}{2}}(\Omega)')$, $w_\epsilon \in \mathcal{C}^0([0,T]; H^{\frac{3}{2}}(\Omega)')$ with $w_\epsilon(0) = u_0$, and that for $\phi \in L^2(0,T; H^{\frac{3}{2}}(\Omega))$,

$$\begin{aligned}&\int_0^T \langle \bar{J}_\kappa\, w_{\epsilon t}, \phi\rangle_{\frac{3}{2}} + \kappa\int_0^T [\partial(w_\epsilon \cdot \bar{n}_\kappa(\bar{\eta}_\kappa)), \partial(\phi \cdot \bar{n}_\kappa(\bar{\eta}_\kappa))]_0\\ &\qquad - \int_0^T \langle q_\epsilon, (\bar{a}_\kappa)_i^j \phi^i,_j\rangle_{\frac{1}{2}} = \sigma\int_0^T [L_{\bar{g}}\bar{\eta} \cdot \bar{n}_\kappa(\bar{\eta}_\kappa), \phi \cdot \bar{n}_\kappa(\bar{\eta}_\kappa)]_0\ .\end{aligned} \tag{6.7}$$

Since by definition $\bar a_\kappa = \text{Cof}\nabla\bar\eta_\kappa$, this implies that in Ω,

$$w_{\epsilon t} + \nabla p_\epsilon(\bar\eta_\kappa) = 0, \tag{6.8}$$

where $p_\epsilon \circ \bar\eta_\kappa = q_\epsilon$ in Ω. Since $\nabla p_\epsilon(\bar\eta_\kappa) \in L^2(0,T;H^{-1}(\Omega))$, this equality is true in $L^2(0,T;H^{-1}(\Omega))$ as well.

6.3. **Step 3.** w_ϵ **is bounded in** $L^2(0,T;H^1(\Omega))$ **independently of** ϵ**.** Denoting $u_\epsilon = w_\epsilon \circ \bar\eta_\kappa^{-1}$, by integrating (6.8) in time from 0 to t, we obtain the important formula

$$\operatorname{curl} u_\epsilon(\bar\eta_\kappa) = \operatorname{curl} u_0 + \int_0^t B(\bar u_\kappa, u_\epsilon)(\bar\eta_\kappa) \quad \text{ in } L^2(0,T;H^{-1}(\Omega))\,, \tag{6.9}$$

with

$$B(\bar u_\kappa, u_\epsilon) = -(\bar u^i_\kappa,_2\, u_{\epsilon 3},_i - \bar u^i_\kappa,_3\, u_{\epsilon 2},_i\,,\ \bar u^i_\kappa,_3\, u_{\epsilon 1},_i - \bar u^i_\kappa,_1\, u_{\epsilon 3},_i\,,\ \bar u^i_\kappa,_1\, u_{\epsilon 2},_i - \bar u^i_\kappa,_2\, u_{\epsilon 1},_i\,).$$

Remark 6. Note well that our approximated and penalized κ-problem preserves the structure of the original Euler equations as can be seen by (6.8). As a result, (6.9) contains only first-order derivatives of the velocity.

Our next task is to prove that w_ϵ is in $L^2(0,T;H^1(\Omega))$. For suppose that this were the case; then, (6.9) together with bounds on the divergence of w_ϵ and $w_\epsilon \cdot N$ on Γ provide bounds for w_ϵ in $L^2(0,T;H^1(\Omega))$ (by the Hodge elliptic estimate (5.1)) which are independent of $\epsilon > 0$.

We proceed by showing that appropriately convolved velocity fields are bounded independently of the parameter of convolution in $L^2(0,T;H^1(\Omega))$. This is the first instance that our horizontal convolution by layers is crucially required.

6.3.1. *For any subdomain* $\omega \subset\subset \Omega$, $w_\epsilon \in L^2(0,T;H^1(\omega))$. We analyze the third component of (6.9), the other components being treated similarly. This leads us to the following equality in $L^2(0,T;H^{-1}(\Omega))$:

$$\begin{aligned}&\bar J_\kappa^{-1}[(\bar a_\kappa)^j_2 w_\epsilon,^1_j - (\bar a_\kappa)^j_1 w_\epsilon,^2_j\,]\\ &\quad = -\operatorname{curl} u_0^3 + \int_0^t \bar J_\kappa^{-2}[\,-\bar v,^i_j\,(\bar a_\kappa)^j_2 w_\epsilon,^1_l\,(\bar a_\kappa)^l_i + \bar v,^i_j\,(\bar a_\kappa)^j_1 w_\epsilon,^2_l\,(\bar a_\kappa)^l_i].\end{aligned}$$

Our goal is to prove that $w_\epsilon \in H^1(\Omega)$. To proceed, we let σ_p denote a standard sequence of Friederich's mollifier in $\mathbb{R}^3$ with support $B(0,1/p)$, and we establish that $\sigma_p \star w_\epsilon$ is bounded in $H^1(\omega)$ for any $\omega \subset\subset \Omega$. For this purpose, we choose $\psi \in \mathcal{D}(\Omega)$ and find that

$$\begin{aligned}&\bar J_\kappa^{-1}[(\bar a_\kappa)^j_2(\psi w_\epsilon),^1_j - (\bar a_\kappa)^j_1(\psi w_\epsilon),^2_j\,]\\ &\quad = -\psi \operatorname{curl} u_0^3 + \bar J_\kappa^{-1}(\bar a_\kappa)^j_2\psi,_j\, {w_\epsilon}^1 - \bar J_\kappa^{-1}(\bar a_\kappa)^j_1\psi,_j\, {w_\epsilon}^2\\ &\qquad + \int_0^t \bar J_\kappa^{-2}[\,-\bar v,^i_j\,(\bar a_\kappa)^j_2(\psi w_\epsilon),^1_l\,(\bar a_\kappa)^l_i + \bar v,^i_j\,(\bar a_\kappa)^j_1(\psi w_\epsilon),^2_l\,(\bar a_\kappa)^l_i]\\ &\qquad - \int_0^t \bar J_\kappa^{-2}[\,-\bar v,^i_j\,(\bar a_\kappa)^j_2\psi,_l\, {w_\epsilon}^1(\bar a_\kappa)^l_i + \bar v,^i_j\,(\bar a_\kappa)^j_1\psi,_l\, {w_\epsilon}^2(\bar a_\kappa)^l_i].\end{aligned} \tag{6.10}$$

In order to proceed, we shall need to identify curl-type structures (in Lagrangian variables) for $\sigma_p \star w_\epsilon$; this requires the following: for $\dfrac{1}{p} \le \text{dist}(\text{supp}\psi, \Omega^c)$ and

$f \in C^\infty(\Omega)$, we have the equality in $H^{-1}(\Omega)$

$$\sigma_p \star [f(\psi w_\epsilon),_j] - f\sigma_p \star [(\psi w_\epsilon),_j] = \int_{\mathbb{R}^3} (\sigma_p),_j (x-y)(f(y)-f(x))\psi w_\epsilon(y)dy - \int_{\mathbb{R}^3} \sigma_p(x-y) f,_j (y)\psi w_\epsilon(y)dy\,,$$

showing that $\sigma_p \star [f(\psi w_\epsilon),_j] - f\ \sigma_p \star [(\psi w_\epsilon),_j] \in L^2(\Omega)$, with

$$\big\|\sigma_p \star [f(\psi w_\epsilon),_j] - f\ \sigma_p \star [(\psi w_\epsilon),_j]\big\|_0 \leq C[\ \|\sigma,_j\|_{0,\mathbb{R}^3} + \|\sigma\|_{0,\mathbb{R}^3}]\|\nabla f\|_{L^\infty(\Omega)}\|w_\epsilon\|_0. \tag{6.11}$$

We thus infer from (6.10) and (6.11) that the vorticity structure satisfies

$$\bar{J}_\kappa^{-1}[(\bar{a}_\kappa)_2^j \sigma_p \star (\psi w_\epsilon),_j^1 - (\bar{a}_\kappa)_1^j \sigma_p \star (\psi w_\epsilon),_j^2] = \int_0^t \bar{J}_\kappa^{-2}[\ -\bar{v},_j^i (\bar{a}_\kappa)_2^j \sigma_p \star (\psi w_\epsilon),_l^1 (\bar{a}_\kappa)_i^l + \bar{v},_j^i (\bar{a}_\kappa)_1^j \sigma_p \star (\psi w_\epsilon),_l^2 (\bar{a}_\kappa)_i^l] + R_1, \tag{6.12}$$

with $\|R_1\|_{L^2(0,T;L^2(\Omega))} \leq N(u_0)$, where $N(u_0)$ is defined in (6.1). Next, we infer from (6.5) and (6.11) that the divergence structure satisfies

$$(\bar{a}_\kappa)_i^j \sigma_p \star (\psi w_\epsilon),_j^i = R_2, \tag{6.13}$$

with $\|R_2\|_{L^2(0,T;L^2(\Omega))} \leq N(u_0)$. Since we also have $\psi w_\epsilon = 0$ on Γ, so that with (5.1), we have a.e. in $(0,T)$

$$\|\sigma_p \star (\psi w_\epsilon)(t)\|_1 \leq \|R_1(t)\|_0 + \|R_2(t)\|_0 + N(u_0) \int_0^t \|\sigma_p \star (\psi w_\epsilon)\|_1\,,$$

and thus

$$\int_0^T \|\sigma_p \star (\psi w_\epsilon)\|_1^2 \leq N(u_0). \tag{6.14}$$

Since this inequality does not depend on p, this implies that $\psi w_\epsilon \in L^2(0,T;H^1(\Omega))$, and therefore $w_\epsilon \in L^2(0,T;H^1(\omega))$, with an estimate depending *a priori* on $\omega \subset\subset \Omega$.

6.3.2. *The horizontal convolved-by-layers velocity fields are in* $L^2(0,T;H^1(\Omega))$. Fix $l \in \{1,...,K\}$, and set

$$W(l) = w_\epsilon \circ \theta_l \quad \text{and} \quad \bar{b}_\kappa^l = [\nabla(\bar{\eta}_\kappa \circ \theta_l)]^{-1}.$$

Hence, in $(0,1)^2 \times (\frac{1}{p},1)$ for $p>1$, the Lagrangian "divergence-free" constraint is given by

$$(\bar{b}_\kappa^l)_i^j (\alpha_l W(l)),_j^i = -(\bar{b}_\kappa^l)_i^j \alpha_l,_j\ W(l)^i - \alpha_l \epsilon\, q_\epsilon(\theta_l)\,, \tag{6.15}$$

where the crucial observation is that the right-hand side of equation (6.15) is in $L^2(0,T;L^2([0,1]^3))$.

Now for $\dfrac{1}{m} \leq \text{dist}(\text{supp}\alpha_l, \partial(0,1)\times(0,1)^2)$ and f smooth in $[0,1]^3$, we have by Lemma 2.1 that for $\beta = 1,2$, $\rho_m \star_h [f(\alpha_l W(l)),_\beta] - f\rho_m \star_h [(\alpha_l W(l)),_\beta] \in L^2((0,1)^2)$, with the estimate a.e. in $(\dfrac{1}{p},1)$:

$$\big|\rho_m \star_h [f(\alpha_l W(l)),_\beta] - f\ \rho_m \star_h [(\alpha_l W(l)),_\beta]\big|_{0,(0,1)^2\times\{y\}} \leq C_\rho |\nabla f|_{L^\infty((0,1)^2\times\{y\})}\ |W(l)|_{0,(0,1)^2\times\{y\}}.$$

This leads to

$$\big\|\rho_m \star_h [f(\alpha_l W(l)),_\beta] - f\ \rho_m \star_h [(\alpha_l W(l)),_\beta]\big\|_{0,(0,1)^3} \tag{6.16}$$
$$\leq C_{\rho_m} \|\nabla f\|_{L^\infty((0,1)^3)} \|W(l)\|_{0,(0,1)^3}.$$

Now, for the case of the vertical derivative, we will need to express $W(l),_3$ in terms of $W(l),_1$, $W(l),_2$, $\operatorname{curl}_{\bar\eta_\kappa\circ\theta_l} W(l)$ and $\operatorname{div}_{\bar\eta_\kappa\circ\theta_l} W(l)$, where

$$\begin{aligned}
\operatorname{div}_{\bar\eta_\kappa\circ\theta_l} W(l) &= (\bar b^l_\kappa)^j_i W(l),^i_j\,,\\
\operatorname{curl}^1_{\bar\eta_\kappa\circ\theta_l} W(l) &= (\bar b^l_\kappa)^i_2 W(l),^3_i - (\bar b^l_\kappa)^i_3 W(l),^2_i\,,\\
\operatorname{curl}^2_{\bar\eta_\kappa\circ\theta_l} W(l) &= (\bar b^l_\kappa)^i_3 W(l),^1_i - (\bar b^l_\kappa)^i_1 W(l),^3_i\,,\\
\operatorname{curl}^3_{\bar\eta_\kappa\circ\theta_l} W(l) &= (\bar b^l_\kappa)^i_1 W(l),^2_i - (\bar b^l_\kappa)^i_2 W(l),^1_i\,.
\end{aligned}$$

Notice that the first three lines above can be written as the following vector field:

$$(\operatorname{div}_{\bar\eta_\kappa\circ\theta_l} W(l), \operatorname{curl}^1_{\bar\eta_\kappa\circ\theta_l} W(l), \operatorname{curl}^2_{\bar\eta_\kappa\circ\theta_l} W(l)) = \sum_{i=1}^3 M^\kappa_i W(l),_i\,,$$

where the M^κ_i are smooth matrix fields depending on $\bar b^l_\kappa$. From condition (3.5), since

$$\det M^\kappa_3 = (\bar b^l_\kappa)^3_3 \ \sum_{i=1}^3 [(\bar b^l_\kappa)^3_i]^2 \geq \frac{1}{2},$$

we see that M^κ_3 is invertible on $[0,T]$ (regardless of the choice of $\bar v \in C_T$). Therefore,

$$W(l),_3 = \operatorname{div}_{\bar\eta_\kappa\circ\theta_l} W(l)\ V^\kappa + M^\kappa \operatorname{curl}_{\bar\eta_\kappa\circ\theta_l} W(l) + \sum_{i=1}^2 A^\kappa_i W(l),_i\,, \tag{6.17}$$

where M^κ and the A^κ_i are smooth matrix fields depending on $\bar b^l_\kappa$ and V^κ is a vector field depending on $\bar b^l_\kappa$. From (6.10), we have that

$$\operatorname{curl}_{\bar\eta_\kappa\circ\theta_l} W(l) = \operatorname{curl} u_0(\theta_l) + \sum_{i=1}^3 \int_0^t N^\kappa_i W(l),_i$$

where the N^κ_i are smooth matrix fields depending on $\bar b^l_\kappa$. By using (6.17) and the fact that $\operatorname{div}_{\bar\eta_\kappa\circ\theta_l} W(l) \in L^2(0,T;L^2(\Omega))$ from (6.5), we obtain after time differentiating that

$$[\operatorname{curl}_{\bar\eta_\kappa\circ\theta_l} W(l)]_t - N^\kappa_3 M^\kappa \operatorname{curl}_{\bar\eta_\kappa\circ\theta_l} W(l) = \sum_{\beta=1}^2 P^\kappa_\beta W(l),_\beta\,, + N^\kappa_3 V^\kappa \operatorname{div}_{\bar\eta_\kappa\circ\theta_l} W(l)$$

where P^κ_β, $\beta = 1,2$, are smooth matrix fields depending on $\bar b^l_\kappa$. Therefore,

$$\operatorname{curl}_{\bar\eta_\kappa\circ\theta_l} W(l) = A^\kappa \operatorname{curl} u_0(\theta_l) + A^\kappa \int_0^t (B^\kappa_\beta W(l),_\beta\,, + \operatorname{div}_{\bar\eta_\kappa\circ\theta_l} W(l)) \tag{6.18}$$

where A^κ and B^κ_β, $\beta = 1,2$, are smooth matrix fields depending on $\bar b^l_\kappa$. With (6.16) and (6.18), we infer in a similar way as for (6.13) that on $(0,1)^2 \times (\frac{1}{p}, 1)$ we have

$$\operatorname{curl}_{\bar\eta_\kappa\circ\theta_l} \rho_m \star_h [\alpha_l W(l)] = A^\kappa \sum_{\beta=1}^2 \int_0^t B^\kappa_\beta \rho_m \star_h [\alpha_l \circ \theta_l W(l)],_\beta + R_3, \tag{6.19}$$

with $\|R_3\|_{L^2(0,T;L^2((0,1)^3))} \leq N(u_0)$. Therefore, with (6.17) and (6.18), we have that

(6.20)
$$[\alpha_l(\theta_l)W(l)],_3 = M^\kappa A^\kappa \sum_{\beta=1}^{2} \int_0^t B^\kappa_\beta [\alpha_l(\theta_l)W(l)],_\beta + \sum_{\beta=1}^{2} A^\kappa_\beta [\alpha_l(\theta_l)W(l)],_\beta + R_4,$$

with $\|R_4\|_{L^2(0,T;L^2((0,1)^3))} \leq N(u_0)$. Thus, for any test function $\varphi \in H^1((0,1)^3)$,

$$\int_{(0,1)^2\times\frac{1}{p}} \alpha_l(\theta_l)W(l)\cdot\varphi = \int_{(0,1)^2\times(\frac{1}{p},0)} [\alpha_l(\theta_l)W(l)],_3 \cdot\varphi + \int_{(0,1)^2\times(\frac{1}{p},0)} \alpha_l(\theta_l)W(l)\cdot\varphi,_3\,.$$

Now, since for $\beta = 1,2$, we have

$$\int_{(0,1)^2\times(\frac{1}{p},0)} [\alpha_l(\theta_l)W(l)],_\beta \cdot\varphi = -\int_{(0,1)^2\times(\frac{1}{p},0)} [\alpha_l(\theta_l)W(l)]\cdot\varphi,_\beta\,,$$

using (6.20), we infer that

$$\Big|\int_{(0,1)^2\times\frac{1}{p}} \alpha_l(\theta_l)W(l)\cdot\varphi\Big| \leq C\ (\|W(l)\|_{0,(0,1)^3} + \|R_4\|_{0,(0,1)^3})\|\varphi\|_{1,(0,1)^3},$$

implying (independently of $p>1$) the following trace estimate for $W(l)$ (not just its normal component):

$$\int_0^T |\alpha_l(\theta_l)\ W(l)|^2_{-\frac{1}{2},(0,1)^2\times\frac{1}{p}} \leq N(u_0). \tag{6.21}$$

Similarly as (6.19), we also have the divergence relation

$$\operatorname{div}_{\bar\eta_\kappa\circ\theta_l}\,\rho_m \star_h [\alpha_l W(l)] = C^\kappa \sum_{\beta=1}^{2}\int_0^t D^\kappa_\beta \rho_m \star_h [\alpha_l\circ\theta_l W(l)],_\beta + R_5, \tag{6.22}$$

where $\|R_5\|_{L^2(0,T;L^2((0,1)^3))} \leq N(u_0)$ and C^κ and D^κ_β, $\beta=1,2$, are smooth matrix fields in terms of $\bar b^l_\kappa$. ¿From (6.19) and (6.22), we then infer, just as in (6.14), that

$$\int_0^T \|\rho_m \star_h [\alpha_l W(l)]\circ(\bar\eta_\kappa\circ\theta_l)^{-1}\|^2_{1,\Omega^l_p} \leq\ N(u_0) + \int_0^T |\rho_m \star_h [\alpha_l W(l)]\circ(\bar\eta_\kappa\circ\theta_l)^{-1}\cdot \bar n_\kappa|^2_{\frac{1}{2},\partial\Omega^l_p},$$

where $\Omega^l_p = \theta_l((0,1)^2\times(\frac{1}{p},1))$. Thus,

$$\int_0^T \|\rho_m \star_h [\alpha_l(\theta_l)W(l)]\|^2_{1,(0,1)^2\times(\frac{1}{p},1)} \leq N(u_0) + \int_0^T |\rho_m \star_h [\alpha_l(\theta_l)W(l)]|^2_{\frac{1}{2},(0,1)^2\times\frac{1}{p}}.$$

Now, from the properties of the convolution,

$$\frac{1}{m}\,|\rho_m \star_h [\alpha_l(\theta_l)W(l)]\,|_{\frac{1}{2},(0,1)^2\times\frac{1}{p}} \leq C\,|\rho_m \star_h [\alpha_l(\theta_l)W(l)]\,|_{-\frac{1}{2},(0,1)^2\times\frac{1}{p}}\,,$$

which, with (6.21), leads us (independently of $p>1$) to

$$\frac{1}{m^2}\int_0^T \|\rho_m \star_h [\alpha_l(\theta_l)W(l)]\|^2_{1,(0,1)^2\times(\frac{1}{p},1)} \leq N(u_0),$$

for any $0 < \dfrac{1}{m} \le \operatorname{dist}(\operatorname{supp}\alpha_l(\theta_l), \partial(0,1)^2 \times (0,1))$. Since this estimate holds for any $p > 1$, we then infer that

$$\frac{1}{m^2} \int_0^T \|\rho_m \star_h [\alpha_l(\theta_l) W(l)]\|^2_{1,(0,1)^3} \le N(u_0), \tag{6.23}$$

for any $0 < \dfrac{1}{m} \le \operatorname{dist}(\operatorname{supp}\alpha_l(\theta_l), \partial(0,1)^2 \times (0,1))$. Therefore, $\rho_m \star_h [\alpha_l(\theta_l) W(l)] \in H^1((0,1)^3)$ (which was not a priori known since our convolution smooths only in the horizontal directions), with a bound depending *a priori* on m.

6.3.3. *Control of the horizontal convolved-by-layers velocity fields independently of m.* From (6.19) and (6.22), we infer that

$$\begin{aligned}\int_0^T &\|\rho_m \star_h [\alpha_l W(l)] \circ (\bar\eta_\kappa \circ \theta_l)^{-1}\|^2_{1,\bar\eta_\kappa(\Omega)} \\ &\le N(u_0) + \int_0^T |\rho_m \star_h [\alpha_l(\theta_l) W(l)] \circ (\bar\eta_\kappa \circ \theta_l)^{-1} \cdot \bar n_\kappa|^2_{\frac12,\partial\bar\eta_\kappa(\Omega)},\end{aligned}$$

and thus,

$$\begin{aligned}\int_0^T &\|\rho_m \star_h [\alpha_l(\theta_l) W(l)]\|^2_{1,(0,1)^3} \\ &\le N(u_0) + \int_0^T |\rho_m \star_h [\alpha_l(\theta_l) W(l)] \cdot \bar n_\kappa(\bar\eta_\kappa \circ \theta_l)|^2_{\frac12,(0,1)^2\times\{0\}}.\end{aligned} \tag{6.24}$$

Next, we have for any $x \in (0,1)^2 \times \{0\}$:

$$\rho_m \star_h [\alpha_l(\theta_l) W(l)] \cdot \bar n_\kappa(\bar\eta_\kappa \circ \theta_l)(x) = \rho_m \star_h [\alpha_l(\theta_l) W(l) \cdot \bar n_\kappa(\bar\eta_\kappa \circ \theta_l)](x) + f(x),$$

with

$$\begin{aligned}f(x) = \int_{\mathbb{R}^2} &\rho_m(x_H - y_H) \alpha_l(\theta_l) W(l)(y_H, x_3) \\ &\cdot [\bar n_\kappa(\bar\eta_\kappa \circ \theta_l)(x_H, x_3) - \bar n_\kappa(\bar\eta_\kappa \circ \theta_l)(y_H, x_3)] dy_H.\end{aligned}$$

Therefore, with (6.24), we obtain

$$\begin{aligned}\int_0^T &\|\rho_m \star_h [\alpha_l(\theta_l) W(l)]\|^2_{1,(0,1)^3} \\ &\le N(u_0) + |f|_{\frac12,(0,1)^2\times\{0\}} + \int_0^T |\rho_m \star_h [\alpha_l(\theta_l) W(l) \cdot \bar n_\kappa(\bar\eta_\kappa \circ \theta_l)]|^2_{\frac12,(0,1)^2\times\{0\}} \\ &\le N(u_0) + |f|_{\frac12,(0,1)^2\times\{0\}} + \int_0^T |\alpha_l(\theta_l) W(l) \cdot \bar n_\kappa(\bar\eta_\kappa \circ \theta_l)|^2_{\frac12,(0,1)^2\times\{0\}} \\ &\le N(u_0) + |f|_{\frac12,(0,1)^2\times\{0\}} + \int_0^T |\alpha_l w_\epsilon \cdot \bar n_\kappa(\bar\eta_\kappa)|^2_{\frac12,\Gamma} \\ &\le N(u_0) + |f|_{\frac12,(0,1)^2\times\{0\}},\end{aligned} \tag{6.25}$$

where we have used the trace control (6.6) for the last inequality in (6.25). We now turn our attention to $|f|_{\frac12,(0,1)^2\times\{0\}}$. We first have that

$$\|f\|_{0,(0,1)^3} \le \frac{C}{m} \|\bar n_\kappa(\bar\eta_\kappa)\|_{H^3(\Omega)} \|\rho_m \star_h \alpha_l(\theta_l) |W(l)|\|_{0,(0,1)^3} \le \frac{C}{m} N(u_0), \tag{6.26}$$

where we have used the definition of C_T to bound $\|\bar{n}_\kappa(\bar{\eta}_\kappa)\|_{H^3(\Omega)}$. Next, we have for $\beta = 1, 2$ that

$$f,_\beta(x) = \int_{\mathbb{R}^2} \rho_m,_\beta(x_H - y_H)\alpha_l(\theta_l)W(l)(y_H, x_3) \cdot [\bar{n}_\kappa(\bar{\eta}_\kappa \circ \theta_l)(x_H, x_3) - \bar{n}_\kappa(\bar{\eta}_\kappa \circ \theta_l)(y_H, x_3)]dy_H + \int_{\mathbb{R}^2} \rho_m(x_H - y_H)\alpha_l(\theta_l)W(l)(y_H, x_3)dy \cdot \bar{n}_\kappa(\bar{\eta}_\kappa \circ \theta_l),_\beta(x),$$

showing that

(6.27)

$$\begin{aligned}\|f,_\beta\|_{0,(0,1)^3} &\leq \frac{C}{m}\|\bar{n}_\kappa(\bar{\eta}_\kappa)\|_{H^3(\Omega)}\sum_{i=1}^3 \|\,|\rho_m,_\beta| \star_h \alpha_l(\theta_l)|W(l)^i|\|_{0,(0,1)^3} \\ &\quad + \|\bar{n}_\kappa(\bar{\eta}_\kappa)\|_{H^3(\Omega)}\sum_{i=1}^3 \|\rho_m \star_h \alpha_l(\theta_l)|W(l)^i|\|_{0,(0,1)^3} \\ &\leq C\|\bar{n}_\kappa(\bar{\eta}_\kappa)\|_{H^3(\Omega)}\sum_{i=1}^3 \||(\rho,_\beta)_m| \star_h \alpha_l(\theta_l)|W(l)^i|\|_{0,(0,1)^3} \\ &\quad + \|\bar{n}_\kappa(\bar{\eta}_\kappa)\|_{H^3(\Omega)}\sum_{i=1}^3 \|\rho_m \star_h \alpha_l(\theta_l)|W(l)^i|\|_{0,(0,1)^3} \\ &\leq C\sum_{i=1}^3 \||(\rho,_\beta)_m| \star_h \alpha_l(\theta_l)|W(l)^i|\|_{0,(0,1)^3} \\ &\quad + C\sum_{i=1}^3 \|\rho_m \star_h \alpha_l(\theta_l)|W(l)^i|\|_{0,(0,1)^3} \\ &\leq C\|\alpha_l(\theta_l)W(l)\|_{0,(0,1)^3} \leq N(u_0).\end{aligned}$$

Next, for the vertical derivative,

$$f,_3(x) = \int_{\mathbb{R}^2} \rho_m(x_H - y_H)[\alpha_l(\theta_l)W(l)],_3(y_H, x_3) \cdot [\bar{n}_\kappa(\bar{\eta}_\kappa \circ \theta_l)]^{(x_H,x_3)}_{(y_H,x_3)}dy_H + \int_{\mathbb{R}^2} \rho_m(x_H - y_H)[\alpha_l(\theta_l)W(l)](y_H, x_3) \cdot [\bar{n}_\kappa(\bar{\eta}_\kappa \circ \theta_l),_3]^{(x_H,x_3)}_{(y_H,x_3)}dy_H\,,$$

where $[\cdot]^{(x_H,x_3)}_{(y_H,x_3)} = [\cdot](x_H, x_3) - [\cdot](y_H, x_3)$. Notice that for a smooth matrix field A in $(0,1)^3$ and for $\beta = 1, 2$,

$$G(x) = \int_{\mathbb{R}^2} \rho_m(x_H - y_H)A(y_h, x_3)[\alpha_l(\theta_l)W(l)],_\beta(y_H, x_3) \cdot [\bar{n}_\kappa(\bar{\eta}_\kappa \circ \theta_l)]^{(x_H,x_3)}_{(y_H,x_3)}dy_H$$

satisfies

$$G(x) =$$

$$- m \int_{\mathbb{R}^2} (\rho,_\beta)_m(x_H - y_H) A(y_h, x_3)[\alpha_l(\theta_l) W(l)](y_H, x_3) \cdot [\bar{n}_\kappa(\bar{\eta}_\kappa \circ \theta_l)]^{(x_H, x_3)}_{(y_H, x_3)} dy_H$$

$$- \int_{\mathbb{R}^2} \rho_m(x_H - y_H) A,_\beta (y_h, x_3)[\alpha_l(\theta_l) W(l)](y_H, x_3) \cdot [\bar{n}_\kappa(\bar{\eta}_\kappa \circ \theta_l)]^{(x_H, x_3)}_{(y_H, x_3)} dy_H$$

$$- \int_{\mathbb{R}^2} \rho_m(x_H - y_H) A(y_h, x_3)[\alpha_l(\theta_l) W(l)](y_H, x_3) \cdot [\bar{n}_\kappa(\bar{\eta}_\kappa \circ \theta_l),_\beta]^{(x_H, x_3)}_{(y_H, x_3)} dy_H,$$

showing, just as for (6.27), that $\|G\|_{0,(0,1)^3} \leq N(u_0)$. Therefore, with (6.20), we see that the first integral term appearing in the expression of $f,_3$ is bounded in a similar way, implying that

$$\|f,_3\|_{0,(0,1)^3} \leq N(u_0). \tag{6.28}$$

Consequently, with (6.26), (6.27), (6.28), we obtain that

$$\|f\|_{1,(0,1)^3} \leq N(u_0). \tag{6.29}$$

Therefore, (6.25) implies that

$$\int_0^T \|\rho_m \star_h [\alpha_l(\theta_l) W(l)]\|^2_{1,(0,1)^3} \leq N(u_0). \tag{6.30}$$

6.3.4. *Control of w_ϵ in $L^2(0,T;H^1(\Omega))$.* Since (6.30) holds independently of m sufficiently large, this implies that

$$\int_0^T \|\alpha_l(\theta_l) W(l)\|^2_{1,(0,1)^3} \leq N(u_0).$$

Since we proved in subsection 6.3.1 that w_ϵ is bounded in $L^2(0,T;H^1(\omega))$ independently of ϵ for each domain $\omega \subset\subset \Omega$, this provides us with the estimate

$$\int_0^T \|w_\epsilon\|^2_1 \leq N(u_0), \tag{6.31}$$

independently of $\epsilon > 0$.

Remark 7. In the two-dimensional case, a simpler proof of Step 3 is possible, founded upon a scalar potential function for the velocity field. For conciseness, we consider a simply connected domain, the nonsimply connected case being treated similarly by local charts. Once again, we let $u_\epsilon = w_\epsilon \circ \bar{\eta}_\kappa^{-1}$. From (6.5b) and (6.6), let $w^\epsilon_\tau \in L^2(0,T;H^1(\Omega))$ such that

$$\operatorname{div}(w^\epsilon_\tau(\bar{\eta}_\kappa^{-1}))(\bar{\eta}_\kappa) = \operatorname{div}(w_\epsilon(\bar{\eta}_\kappa^{-1}))(\bar{\eta}_\kappa) \ \text{ in } L^2(0,T;L^2(\Omega)),$$
$$w^\epsilon_\tau \cdot \bar{n}_\kappa(\bar{\eta}_\kappa) = w_\epsilon \cdot \bar{n}_\kappa(\bar{\eta}_\kappa) \ \text{ in } L^2(0,T;H^1(\Gamma)) \ .$$

We infer the existence of $\psi^\epsilon \in L^2(0,T;H^1_0(\bar{\eta}_\kappa(\Omega)))$ such that $u_\epsilon = w^\epsilon_\tau(\bar{\eta}_\kappa^{-1}) + (-\psi^\epsilon,_2, \psi^\epsilon,_1)$. Now, from (6.10), we see that in $L^2(0,T;H^{-1}(\Omega))$, we have for $\bar{\psi}^\epsilon = \psi^\epsilon \circ \bar{\eta}_\kappa$,

$$-(\bar{a}_\kappa)^k_i((\bar{a}_\kappa)^j_i \bar{\psi}^\epsilon,_j),_k = f^\epsilon_\tau - \int_0^t A^\kappa_{ij} \bar{\psi}^\epsilon,_{ij}, \tag{6.32}$$

where f^ϵ_τ is bounded in $L^2(0,T;L^2(\Omega))$. It is readily seen that $\bar{\psi}^\epsilon$ is the unique solution of this equation in $L^2(0,T;H^1_0(\Omega))$. We now establish that this uniqueness provides extra regularity for $\bar{\psi}^\epsilon$. By defining the mapping Θ from $L^2(0,T;H^2(\Omega) \cap$

$H^1_0(\Omega)$) into itself by associating to any ξ in this space the solution $\Theta\xi$ (for almost all $t\in[0,T]$) of

$$-(\bar a_\kappa)^k_i((\bar a_\kappa)^j_i\Theta\xi,_j),_k=f_\tau-\int_0^t A^\kappa_{ij}\xi,_{ij}\,,$$

we see that for t_1 small enough (depending on Sobolev constants and on $\|\bar u_\kappa\|_{L^\infty(0,T;H^3(\Omega))}$) Θ is contractive from $L^2(0,t_1;H^2(\Omega)\cap H^1_0(\Omega))$ into itself, which provides a fixed point for Θ in this space. It is thus a solution of (6.32) on $[0,t_1]$. By uniqueness of such a solution, we have that $\bar\psi^\epsilon\in L^2(0,t_1;H^2(\Omega))$ and thus that $w_\epsilon\in L^2(0,t_1;H^1(\Omega))$. By defining a mapping similar to Θ, but this time starting from $t_2\in[\frac{t_1}{2},t_1]$ such that $w_\epsilon(t_2)\in H^1(\Omega)$ instead of u_0 (which ensures that the new f_τ is still in $L^2(0,t_2;L^2(\Omega))$), we obtain the same conclusion on $[t_2,t_2+t_1]$, leading us to $w_\epsilon\in L^2(0,\frac{3}{2}t_1;H^1(\Omega))$. By induction, we then find $w_\epsilon\in L^2(0,T;H^1(\Omega))$.

Remark 8. Whereas Hodge decompositions with vector potentials ψ are possible in higher dimension, it turns out that a Dirichlet condition $\psi=0$ for the associated elliptic problem is not possible. This in turn is problematic for any uniqueness argument in $L^2(0,T;H^1(\Omega))$ for ψ, since it does not seem possible to find a boundary condition that would be naturally associated to the second-order operators appearing on both sides of the three-dimensional analogous of (6.32).

7. Pressure as a Lagrange multiplier

We will need two Lagrange multiplier lemmas for our pressure function in our analysis as the penalization parameter $\epsilon\to0$. We begin with a lemma that is necessary for a new Hodge-type decomposition of the velocity field.

Lemma 7.1. *For all $l\in H^{\frac12}(\Omega)$, $t\in[0,T]$, there exists a constant $C>0$ and $\phi(l)\in H^{\frac32}(\Omega)$ such that $(\bar a_\kappa)^j_i(t)\phi^i,_j=l$ in Ω and*

$$\|\phi(l)\|^2_{\frac32}\le C\|l\|^2_{\frac12}. \tag{7.1}$$

Proof. Let $\psi(l)$ be the solution of

$$(\bar a_\kappa)^j_i[(\bar a_\kappa)^k_i\psi(l),_k],_j=l\text{ in }\Omega, \tag{7.2a}$$

$$\psi(l)=0\text{ on }\Gamma. \tag{7.2b}$$

We then see that $\phi^i(l)=(\bar a_\kappa)^j_i\psi(l),_j$ satisfies the statement of the lemma. The inequality (7.1) is a simple consequence of the properties of l and of the condition $\bar v\in C_T$. ☐

We can now follow [18]. For $p\in H^{\frac12}(\Omega)'$, define the linear functional on $H^{\frac32}(\Omega)$ by $\langle p,(\bar a_\kappa)^j_i(t)\varphi^i,_j\rangle_{\frac12}$, where $\varphi\in H^{\frac32}(\Omega)$. By the Riesz representation theorem, there is a bounded linear operator $Q(t):(H^{\frac12}(\Omega))'\to H^{\frac32}(\Omega)$ such that

$$\forall\varphi\in H^{\frac32}(\Omega),\ \langle p,\ (\bar a_\kappa)^j_i(t)\varphi^i,_j\rangle_{\frac12}=(Q(t)p,\ \varphi)_{\frac32}.$$

Letting $\varphi=Q(t)p$ shows that

$$\|Q(t)p\|_{\frac32}\le C\|p\|_{H^{\frac12}(\Omega)'} \tag{7.3}$$

for some constant $C>0$. Using Lemma 7.1, we see that

$$\forall l\in H^{\frac12}(\Omega),\ \langle p,\ l\rangle_{\frac12}=(Q(t)p,\ \phi(l))_{\frac32},$$

and thus

$$\|p\|_{H^{\frac{1}{2}}(\Omega)'} \le C\|Q(t)p\|_{\frac{3}{2}}, \tag{7.4}$$

which shows that $R(Q(t))$ is closed in $H^{\frac{3}{2}}(\Omega)$. Let

$$\mathcal{V}_{\bar v}(t) = \{v \in L^2(\Omega) \mid (\bar a_\kappa)_i^j(t) v^i,_j(t) = 0\}.$$

Since $\mathcal{V}_{\bar v}(t) \cap H^{\frac{3}{2}}(\Omega) = R(Q(t))^\perp$, it follows that

$$H^{\frac{3}{2}}(\Omega) = R(Q(t)) \oplus_{H^{\frac{3}{2}}(\Omega)} \mathcal{V}_{\bar v}(t) \cap H^{\frac{3}{2}}(\Omega). \tag{7.5}$$

We can now introduce our first Lagrange multiplier.

Lemma 7.2. *Let $\mathfrak{L}(t) \in H^{\frac{3}{2}}(\Omega)'$ be such that $\mathfrak{L}(t)\varphi = 0$ for any $\varphi \in \mathcal{V}_{\bar v}(t) \cap H^{\frac{3}{2}}(\Omega)$. Then there exists a unique $q(t) \in H^{\frac{1}{2}}(\Omega)'$, which is termed the pressure function, satisfying*

$$\forall \varphi \in H^{\frac{3}{2}}(\Omega), \quad \mathfrak{L}(t)(\varphi) = \langle q(t),\ (\bar a_\kappa)_i^j \varphi^i,_j \rangle_{\frac{1}{2}}.$$

Moreover, there is a $C > 0$ (which does not depend on $t \in [0,T]$ and on the choice of $\bar v \in C_T$) such that

$$\|q(t)\|_{H^{\frac{1}{2}}(\Omega)'} \le C\ \|\mathfrak{L}(t)\|_{H^{\frac{3}{2}}(\Omega)'}.$$

Proof. By the decomposition (7.5), for $\varphi \in H^{\frac{3}{2}}(\Omega, \mathbb{R}^3)$, we let $\varphi = v_1 + v_2$, where $v_1 \in \mathcal{V}_{\bar v}(t) \cap H^{\frac{3}{2}}(\Omega)$ and $v_2 \in R(Q(t))$. From our assumption, it follows that

$$\mathfrak{L}(t)(\varphi) = \mathfrak{L}(t)(v_2) = (\psi(t), v_2)_{H^{\frac{3}{2}}(\Omega)} = (\psi(t), \varphi)_{H^{\frac{3}{2}}(\Omega)},$$

for a unique $\psi(t) \in R(Q(t))$.

From the definition of $Q(t)$ we then get the existence of a unique $q(t) \in H^{\frac{1}{2}}(\Omega)'$ such that

$$\forall \varphi \in H^{\frac{3}{2}}(\Omega), \quad \mathfrak{L}(t)(\varphi) = \langle q(t),\ (\bar a_\kappa)_i^j \varphi^i,_j \rangle_{\frac{1}{2}}.$$

The estimate stated in the lemma is then a simple consequence of (7.4). □

We also need the case where the pressure function is in $H^{\frac{1}{2}}(\Omega)$. We start, as above, with a simple elliptic result:

Lemma 7.3. *For all $l \in H^{\frac{1}{2}}(\Omega)'$, $t \in [0,T]$, there exists a constant $C > 0$ and $\phi(l) \in H^{\frac{1}{2}}(\Omega)$ such that $(\bar a_\kappa)_i^j(t)\phi^i,_j = l$ in Ω and*

$$\|\phi(l)\|^2_{\frac{1}{2}} \le C\|l\|^2_{H^{\frac{1}{2}}(\Omega)'}. \tag{7.6}$$

Proof. Let $\psi(l)$ be the solution of (7.2). Since ψ is linear and continuous from $H^1(\Omega)'$ into $H^1(\Omega)$ and from $L^2(\Omega)$ into $H^2(\Omega)$, by interpolation, we have that ψ is linear and continuous from $H^{\frac{1}{2}}(\Omega)'$ into $H^{\frac{3}{2}}(\Omega)$. We then see that $\phi^i(l) = (\bar a_\kappa)_i^j \psi(l),_j$ satisfies the statement of the lemma. □

For $p \in H^{\frac{1}{2}}(\Omega)$, we define the linear functional on $X(t)$ by $\langle (\bar a_\kappa)_i^j(t)\varphi^i,_j, p\rangle_{\frac{1}{2}}$, where $\varphi \in X(t) = \{\psi \in H^{\frac{1}{2}}(\Omega) |\ (\bar a_\kappa)_i^j \psi^i,_j \in H^{\frac{1}{2}}(\Omega)'\}$. By the Riesz representation theorem, there is a bounded linear operator $Q(t) : H^{\frac{1}{2}}(\Omega) \to X(t)$ such that

$$\forall \varphi \in X(t),\ \langle (\bar a_\kappa)_i^j(t)\varphi^i,_j, p\rangle_{\frac{1}{2}} = (Q(t)p,\ \varphi)_{X(t)}.$$

Letting $\varphi = Q(t)p$ shows that

$$\|Q(t)p\|_{X(t)} \le C\|p\|_{H^{\frac{1}{2}}(\Omega)} \tag{7.7}$$

for some constant $C > 0$. Using Lemma 7.3, we see that

$$\forall l \in H^{\frac{1}{2}}(\Omega)',\ \langle l,\ p\rangle_{\frac{1}{2}} = (Q(t)p,\ \phi(l))_{X(t)},$$

and thus

$$\|p\|_{H^{\frac{1}{2}}(\Omega)} \le C\|Q(t)p\|_{X_t}, \tag{7.8}$$

which shows that $R(Q(t))$ is closed in $X(t)$. Since $\mathcal{V}_{\bar{v}}(t) \cap X(t) = R(Q(t))^\perp$, it follows that

$$X(t) = R(Q(t)) \oplus_{X(t)} \mathcal{V}_{\bar{v}}(t) \cap X(t). \tag{7.9}$$

Our second Lagrange multiplier lemma can now be stated.

Lemma 7.4. *Let $\mathfrak{L}(t) \in X(t)'$ be such that $\mathfrak{L}(t)\varphi = 0$ for any $\varphi \in \mathcal{V}_{\bar{v}}(t) \cap H^{\frac{1}{2}}(\Omega)$. Then there exists a unique $q(t) \in H^{\frac{1}{2}}(\Omega)$, which is termed the pressure function, satisfying*

$$\forall \varphi \in X(t), \quad \mathfrak{L}(t)(\varphi) = \langle (\bar{a}_\kappa)^j_i \varphi^i{}_{,j}\, , q(t)\rangle_{\frac{1}{2}}.$$

Moreover, there is a $C > 0$ (which does not depend on $t \in [0, T]$ and on the choice of $\bar{v} \in C_T$) such that

$$\|q(t)\|_{H^{\frac{1}{2}}(\Omega)} \le C\ \|\mathfrak{L}(t)\|_{X(t)'}.$$

Proof. By the decomposition (7.9), for $\varphi \in X(t)$, we let $\varphi = v_1 + v_2$, where $v_1 \in \mathcal{V}_{\bar{v}}(t) \cap H^{\frac{1}{2}}(\Omega)$ and $v_2 \in R(Q(t))$. From our assumption, it follows that

$$\mathfrak{L}(t)(\varphi) = \mathfrak{L}(t)(v_2) = (\psi(t), v_2)_{X(t)} = (\psi(t), \varphi)_{X(t)},$$

for a unique $\psi(t) \in R(Q(t))$.

From the definition of $Q(t)$ we then get the existence of a unique $q(t) \in H^{\frac{1}{2}}(\Omega)$ such that

$$\forall \varphi \in X(t), \quad \mathfrak{L}(t)(\varphi) = \langle (\bar{a}_\kappa)^j_i \varphi^i{}_{,j}\, ,\ q(t)\rangle_{\frac{1}{2}}.$$

The estimate stated in the lemma is then a simple consequence of (7.8). □

8. Existence of a solution to the linearized smoothed κ-problem (4.2)

In this section, we prove the existence of a solution w to the linear problem (4.2), constructed as the limit $\epsilon \to 0$.

The analysis requires establishing the regularity of the weak solution. Note that the extra regularity on u_0 is needed in order to ensure the regularity property for w, q, and their time derivatives as stated in the next theorem, without having to consider the variational limits of the time-differentiated penalized problems.

Theorem 8.1. *Suppose that $u_0 \in H^{13.5}(\Omega)$ and Ω is of class C^∞. Then, there exists a unique weak solution w to the linear problem (4.2), which is moreover in $L^2(0, T; H^{13.5}(\Omega))$. Furthermore,*

$$\partial_t^i w \in L^2(0, T; H^{13.5-3i}(\Omega)) \cap L^\infty(0, T; H^{12.5-3i}(\Omega)), \quad i = 1, 2, 3, 4\,,$$
$$\partial_t^i q \in L^2(0, T; H^{11.5-3i}(\Omega)) \cap L^\infty(0, T; H^{10.5-3i}(\Omega)), \quad i = 0, 1, 2, 3\,.$$

Proof. **Step 1. The limit as $\epsilon \to 0$.**

Let $\epsilon = \frac{1}{m}$; we first pass to the weak limit as $m \to \infty$. The inequality (6.2) provides the following bound, independent of ϵ:

$$\int_0^T \frac{1}{\epsilon}\|(\bar{a}_\kappa)^j_i w_\epsilon{}^i{}_{,j}\,\|_0^2 + |w_\epsilon \cdot \bar{n}_\kappa(\bar{\eta}_\kappa)|_1^2 + \|w_\epsilon\|_0^2\ dt \le N(u_0)$$

which provides a subsequence $\{w_{\frac{1}{m_l}}\}$ such that

$$w_{\frac{1}{m_l}} \rightharpoonup w \quad \text{in} \quad L^2(0,T;L^2(\Omega)) \ , \tag{8.1a}$$

$$(\bar{a}_\kappa)_i^j w^i_{\frac{1}{m_l}},_j \rightharpoonup (\bar{a}_\kappa)_i^j w^i,_j \quad \text{in} \quad L^2(0,T;L^2(\Omega)) \ , \tag{8.1b}$$

$$w_{\frac{1}{m_l}} \cdot \bar{n}_\kappa(\bar{\eta}_\kappa) \rightharpoonup w \cdot \bar{n}_\kappa(\bar{\eta}_\kappa) \quad \text{in} \quad L^2(0,T;H^1(\Gamma)) \ . \tag{8.1c}$$

The justification for $w \cdot \bar{n}_\kappa(\bar{\eta}_\kappa)$ being the third weak limit in (8.1) comes from the identity $(\bar{a}_\kappa)_i^j {w_\epsilon}^i,_j = \operatorname{div}(w_\epsilon \circ \bar{\eta}_\kappa^{-1})(\bar{\eta}_\kappa)$ and the fact that $\bar{n}_\kappa$ is the normal to $\bar{\eta}_\kappa(\Omega)$.

Moreover, since (6.2) also shows that $\|(\bar{a}_\kappa)_i^j w^i_{\frac{1}{m}},_j\|_{L^2(0,T;L^2(\Omega))} \to 0$ as $m \to \infty$, we then have $\|(\bar{a}_\kappa)_i^j w^i,_j\|_{L^2(0,T;L^2(\Omega))} = 0$, *i.e.*,

$$(\bar{a}_\kappa)_i^j w^i,_j = 0 \text{ in } L^2(0,T;L^2(\Omega)). \tag{8.2}$$

Now, let us denote $u = w \circ \bar{\eta}_\kappa^{-1}$, so that thanks to (8.2) and (6.9) we have

$$\operatorname{div} u = 0 \text{ in } \bar{\eta}_\kappa(\Omega) \ , \tag{8.3a}$$

$$\operatorname{curl} u(\bar{\eta}_\kappa) = \operatorname{curl} u_0 + \int_0^t B(\nabla \bar{u}_\kappa, \nabla u) \text{ in } H^{-1}(\Omega). \tag{8.3b}$$

By proceeding as in Step 3 of Section 6, the trace regularity $(u \cdot \bar{n}_\kappa)(\bar{\eta}_\kappa) \in L^2(0,T;H^1(\Gamma))$ and the system (8.3) then yield

$$\|w\|_{L^2(0,T;H^{\frac{3}{2}}(\Omega))} \le N(u_0),$$

where $N(u_0)$ is defined in (6.1).

Step 2. The equation for w and the pressure.

Now, for any $y \in L^2(0,T;H^{\frac{3}{2}}(\Omega))$ and $l = (\bar{a}_\kappa)_i^j y^i,_j$, we see that for a solution φ almost everywhere on $(0,T)$ of the elliptic problem

$$\begin{aligned} (\bar{a}_\kappa)_i^j [\bar{J}_\kappa^{-1} (\bar{a}_\kappa)_i^k \varphi,_k],_j &= l \text{ in } \Omega, \\ \varphi &= 0 \text{ on } \Gamma, \end{aligned}$$

if we let $e^i = \bar{J}_\kappa^{-1}(\bar{a}_\kappa)_i^k \varphi,_k$ and set $v = y - e$, we have that e and v are both in $L^2(0,T;H^{\frac{3}{2}}(\Omega))$, with

$$\begin{aligned} \int_0^T [\|e\|^2_{\frac{3}{2}} + \|v\|^2_{\frac{3}{2}}] \le& C \int_0^T \|y\|^2_{\frac{3}{2}}, \\ (\bar{a}_\kappa)_i^j v^i,_j =& 0. \end{aligned}$$

Since $(\bar{a}_\kappa)_i^j w^i,_j = 0$ in $L^2(0,T;L^2(\Omega))$, we infer that $(\bar{a}_\kappa)_i^j w^i_t,_j = -[(\bar{a}_\kappa)_i^j]_t w^i,_j \in L^2(0,T;H^{\frac{1}{2}}(\Omega))$ and that

$$\langle \bar{J}_\kappa \ w_t, e \rangle_{\frac{3}{2}} = ([(\bar{a}_\kappa)_i^j]_t w^i,_j \ , \varphi)_0.$$

But v also satisfies the variational equation

$$\begin{aligned} \int_0^T \langle \bar{J}_\kappa \ w_{\epsilon t}, v \rangle_{\frac{3}{2}} + \kappa \int_0^T [w_\epsilon \cdot \bar{n}_\kappa(\bar{\eta}_\kappa), v \cdot \bar{n}_\kappa(\bar{\eta}_\kappa)]_1 \\ = \sigma \int_0^T [L_{\bar{g}} \bar{\eta} \cdot \bar{n}_\kappa(\bar{\eta}_\kappa), v \cdot \bar{n}_\kappa(\bar{\eta}_\kappa)]_0 \ , \end{aligned}$$

leading to

$$\lim_{\epsilon\to 0}\int_0^T \langle \bar{J}_\kappa\ w_{\epsilon t}, y\rangle_{\frac{3}{2}} = \int_0^T ([(\bar{a}_\kappa)_i^j]_t w^i,_j, \varphi)_0 + \sigma\int_0^T [L_{\bar{g}}\bar{\eta}\cdot\bar{n}_\kappa(\bar{\eta}_\kappa), v\cdot\bar{n}_\kappa(\bar{\eta}_\kappa)]_0 \\ - \kappa\int_0^T [w\cdot\bar{n}_\kappa(\bar{\eta}_\kappa), v\cdot\bar{n}_\kappa(\bar{\eta}_\kappa)]_1.$$

We then see that as $\epsilon\to 0$,

$$\int_0^T \|w_{\epsilon t}\|^2_{H^{\frac{3}{2}}(\Omega)'} \le N(u_0). \tag{8.4}$$

By standard arguments, we infer that $w_{\epsilon t}\rightharpoonup w_t$ in $L^2(0,T;H^{\frac{3}{2}}(\Omega)')$. This ensures that $w\in\mathcal{C}^0([0,T];L^2(\Omega))$, and the condition $w_\epsilon(0)=u_0$ provides $w(0)=u_0$. Furthermore, we also have for any $\phi\in L^2(0,T;H^{\frac{3}{2}}(\Omega))$ such that $(\bar{a}_\kappa)_i^j\phi^i,_j=0$ in $(0,T)\times\Omega$ the variational equation

$$\int_0^T \langle \bar{J}_\kappa\ w_t, \phi\rangle_{\frac{3}{2}} + \kappa\int_0^T [w\cdot\bar{n}_\kappa(\bar{\eta}_\kappa), \phi\cdot\bar{n}_\kappa(\bar{\eta}_\kappa)]_1 = \sigma\int_0^T [L_{\bar{g}}\bar{\eta}\cdot\bar{n}_\kappa(\bar{\eta}_\kappa), \phi\cdot\bar{n}_\kappa(\bar{\eta}_\kappa)]_0\ .$$

Next, since $w_t\in L^2(0,T;H^{\frac{3}{2}}(\Omega)')$, the Lagrange multiplier Lemma 7.2 shows that there exists $q\in L^2(0,T;H^{\frac{1}{2}}(\Omega)')$ such that for any $\phi\in L^2(0,T;H^{\frac{3}{2}}(\Omega))$,

$$\int_0^T \langle \bar{J}_\kappa\ w_t, \phi\rangle_{\frac{3}{2}} + \kappa\int_0^T [w\cdot\bar{n}_\kappa(\bar{\eta}_\kappa), \phi\cdot\bar{n}_\kappa(\bar{\eta}_\kappa)]_1 \\ - \int_0^T \langle q, (\bar{a}_\kappa)_i^j\phi^i,_j\rangle_{\frac{1}{2}} = \sigma\int_0^T [L_{\bar{g}}\bar{\eta}\cdot\bar{n}_\kappa(\bar{\eta}_\kappa), \phi\cdot\bar{n}_\kappa(\bar{\eta}_\kappa)]_0\ . \tag{8.5}$$

Now, if we have another solution $\tilde{w}\in L^2(0,T;H^{\frac{3}{2}}(\Omega))$ such that $\tilde{w}(0)=u_0$ and $\tilde{w}_t\in L^2(0,T;H^{\frac{3}{2}}(\Omega)')$, we then see, by using $w-\tilde{w}$ as a test function in the difference between (8.5) and its counterpart with $\tilde{w}$, that we get $w-\tilde{w}=0$, ensuring uniqueness to the weak solution of (4.2).

Step 3. Regularity of w. We can now study the regularity of w via difference quotient techniques. We will denote $\mathbb{R}^3_+=\{x\in\mathbb{R}^3|\ x_3>0\}$, $S_0=B(0,1)\cap\{x\in\mathbb{R}^3|\ x_3=0\}$ and $B_+(0,r)=B(0,r)\cap\mathbb{R}^3_+$. We denote by θ a C^∞ diffeomorphism from $B(0,1)$ into a neighborhood V of a point $x_0\in\Gamma$ such that $\theta(B(0,1)\cap\mathbb{R}^3_+)=V\cap\Omega$, with $\det\nabla\theta=1$. We consider the smooth cut-off function $\psi(x)=e^{\frac{1}{|x|^2-\frac{1}{2}}}$ if $x\in B(0,\frac{1}{2})$ and $\psi(x)=0$ elsewhere, and with the use of the test function $[D_{-h}[\psi D_h(w\circ\theta)]]\circ\theta^{-1}\in L^2(0,T;H^{\frac{3}{2}}(\Omega))$ in (8.5), with $h=|h|e_\alpha(\alpha=1,2)$, we obtain

$$I_1+\kappa I_2+I_3 = \sigma\int_0^T [L_{\bar{g}}\bar{\eta}\cdot\bar{n}_\kappa(\bar{\eta}_\kappa), [D_{-h}[\psi D_h(w\circ\theta)]]\circ\theta^{-1}\cdot\bar{n}_\kappa(\bar{\eta}_\kappa)]_0\ ,$$

with

$$I_1 = \int_0^T \langle \bar{J}_\kappa\ w_t, [D_{-h}[\psi D_h(w\circ\theta)]]\circ\theta^{-1}\rangle_{\frac{3}{2}},$$

$$I_2 = \int_0^T [\partial(w\cdot\bar{n}_\kappa(\bar{\eta}_\kappa)), \partial([D_{-h}[\psi D_h(w\circ\theta)]]\circ\theta^{-1}\cdot\bar{n}_\kappa(\bar{\eta}_\kappa))]_0,$$

$$I_3 = -\int_0^T \langle q, (\bar{a}_\kappa)_i^j[D_{-h}[\psi D_h(w\circ\theta),^i_j]]\circ\theta^{-1}\rangle_{\frac{1}{2}}.$$

For I_1, we simply have

(8.6)
$$\begin{aligned} I_1 =& \|\sqrt{\psi}\ w\circ\theta(t)\|^2_{L^2(B_+(0,1))} - \|\sqrt{\psi}\ u_0\circ\theta\|^2_{L^2(B_+(0,1))} \\ &+ \int_0^T \langle D_h[\bar{J}_\kappa(\theta)]\ w_t\circ\theta, \psi D_h(w\circ\theta)\rangle_{\frac{3}{2}} \\ \geq& \|\sqrt{\psi}\ w\circ\theta(t)\|^2_{L^2(B_+(0,1))} - N(u_0) - \int_0^T \|w_t\|_{H^{\frac{3}{2}}(\Omega)'} \|D_h[\bar{J}_\kappa(\theta)]\psi D_h(w\circ\theta)\|_{\frac{3}{2}} \\ \geq& \|\sqrt{\psi}\ w\circ\theta(t)\|^2_{L^2(B_+(0,1))} - C_\delta N(u_0) - \delta\int_0^T \|\sqrt{\psi} D_h(w\circ\theta)\|^2_{\frac{3}{2}}, \end{aligned}$$

where we have used (8.4) for the last inequality and where the choice of $\delta > 0$ will be made precise later.

For I_2, we have, if we define in $B_+(0,1)$, $W = w\circ\theta$ and $N^\kappa = \bar{n}_\kappa(\bar{\eta}_\kappa)(\theta)$,

$$\begin{aligned} I_2 &= \int_0^T \int_{S_0} \frac{G_{\alpha\beta}}{\sqrt{a_0}} [W\cdot N^\kappa],_\alpha [D_{-h}[\psi D_h W]\cdot N^\kappa],_\beta \\ &= \int_0^T \int_{S_0} \frac{G_{\alpha\beta}}{\sqrt{a_0}} [D_h W\cdot N^\kappa],_\alpha [\psi D_h W\cdot N^\kappa],_\beta \\ &\quad + \int_0^T \int_{S_0} [D_h[\frac{G_{\alpha\beta}}{\sqrt{a_0}}[W\cdot N^\kappa],_\alpha] - \frac{G_{\alpha\beta}}{\sqrt{a_0}}[D_h W\cdot N^\kappa],_\alpha][\psi D_h W\cdot N^\kappa],_\beta \\ &\quad + \int_0^T \int_{S_0} [\frac{G_{\alpha\beta}}{\sqrt{a_0}}[W\cdot N^\kappa],_\alpha](\cdot + h)[\psi D_h W\cdot D_h N^\kappa],_\beta\,, \end{aligned} \tag{8.7}$$

where $G_{\alpha\beta} = \theta,_\alpha\cdot\theta,_\beta$ and $a_0 = \det G$.

In this section, we will denote by $\|\cdot\|_{s,\Theta}$ and $|\cdot|_{s,\partial\Theta}$ the standard norms of $H^s(\Theta)$ and $H^s(\partial\Theta)$.

For the first term appearing in the right-hand side of the second inequality, we have

$$\begin{aligned} \frac{g_{0\alpha\beta}}{\sqrt{a_0}}[D_h W\cdot N^\kappa],_\alpha [\psi D_h W\cdot N^\kappa],_\beta =& \frac{g_{0\alpha\beta}}{\sqrt{a_0}}\psi[D_h W\cdot N^\kappa],_\alpha [D_h W\cdot N^\kappa],_\beta \\ &+ \frac{g_{0\alpha\beta}}{\sqrt{a_0}}\sqrt{\psi}[D_h W\cdot N^\kappa],_\alpha D_h W\cdot N^\kappa \frac{\psi,_\beta}{\sqrt{\psi}}, \end{aligned}$$

and thus, since ψ, $\sqrt{\psi}$ and $\dfrac{\nabla\psi}{\sqrt{\psi}}$ are chosen smooth, we infer that

$$\begin{aligned} \int_0^T \int_{S_0} \frac{g_{0\alpha\beta}}{\sqrt{a}}[D_h W\cdot N^\kappa],_\alpha [\psi D_h W\cdot N^\kappa],_\beta \geq C\int_0^T \|\sqrt{\psi} D_h W\cdot N^\kappa\|^2_{1,S_0} \\ - N(u_0). \end{aligned}$$

The other terms in (8.7) are easily estimated leading to the estimate

$$I_2 \geq C\int_0^T \|\sqrt{\psi} D_h W\cdot N^\kappa\|^2_{1,S_0} - N(u_0). \tag{8.8}$$

Concerning I_3, we have

$$I_3 = -\int_0^T \langle q, (\bar{b}_\kappa)^j_i [D_{-h}[\psi D_h(W)]^i,_j \circ\theta^{-1}]\rangle_{\frac{1}{2}},$$

with $\bar b_\kappa = [\nabla(\bar\eta_\kappa\circ\theta)]^{-1}$. Now since $(\bar b_\kappa)^j_i W^i,_j = 0$, we obtain

$$\begin{aligned}(\bar b_\kappa)^j_i D_{-h}D_h W^i,_j =& - D_{-h}D_h(\bar b_\kappa)^j_i W^i,_j - D_h[(\bar b_\kappa)^j_i](\cdot - h)D_{-h}W^i,_j \\ & - D_{-h}[(\bar b_\kappa)^j_i(\cdot + h)]D_h W^i,_j\,,\end{aligned}$$

and thus

$$\begin{aligned}|I_3| &\le C\int_0^T \|q\|_{H^{\frac12}(\Omega)'}\Big[\,\|\psi D_h W^i,_j\|_{\frac12,B_+(0,1)} + \Big\|\frac{D_h\psi}{\sqrt{\psi}}\,\sqrt{\psi}D_h W^i,_j\Big\|_{\frac12,B_+(0,1)} \\ &\qquad + \|\sqrt{\psi}D_h W\|_{\frac32,B_+(0,1)}\Big] \\ &\le C_\delta N(u_0) + \delta\int_0^T |\sqrt{\psi}D_h W^i,_j|^2_{\frac12,B_+(0,1)}, \qquad (8.9)\end{aligned}$$

where $\delta>0$ is arbitrary. Now, let Θ be a smooth domain included in $B_+(0,1)$ and containing $B_+(0,\frac12)$. The inequalities (8.6), (8.8) and (8.9) yield

$$\int_0^T |\sqrt{\psi}D_h W\cdot N^\kappa|^2_{1,\partial\Theta} \le C_\kappa\ N(u_0) + \delta\int_0^T \|\sqrt{\psi}D_h W\|^2_{\frac32,\Theta}. \tag{8.10}$$

We now define in $B_+(0,1)$

$$\begin{aligned}\operatorname{div}_{\bar\eta_\kappa\circ\theta} W &= \operatorname{div}(W\circ\theta^{-1}\circ\bar\eta_\kappa^{-1})(\bar\eta_\kappa\circ\theta) = \operatorname{div}(u)(\bar\eta_\kappa\circ\theta),\\ \operatorname{curl}_{\bar\eta_\kappa\circ\theta} W &= \operatorname{curl}(u)(\bar\eta_\kappa\circ\theta).\end{aligned}$$

Thus, (8.3) translates in $B_+(0,1)$ to

$$\begin{aligned}&\operatorname{div}_{\bar\eta_\kappa\circ\theta} W = 0,\\ &[\operatorname{curl}_{\bar\eta_\kappa\circ\theta} W](t) = [\operatorname{curl} u_0]\circ\theta + \int_0^t B(\nabla\bar u_\kappa,\nabla u)(\bar\eta_\kappa\circ\theta)\\ &\qquad = [\operatorname{curl} u_0]\circ\theta + \int_0^t B(\nabla\bar u_\kappa,\nabla W(\bar\eta_\kappa\circ\theta)^{-1}\nabla(\bar\eta_\kappa\circ\theta)^{-1})(\bar\eta_\kappa\circ\theta),\end{aligned}$$

and thus
(8.11a)

$$\operatorname{div}_{\bar\eta_\kappa\circ\theta}(\sqrt{\psi}D_h W) = -\sqrt{\psi}D_h(\bar b_\kappa)^j_i\ W^i,_j(\cdot+h) + \frac12\frac{\psi,_j}{\sqrt{\psi}}\ (\bar b_\kappa)^j_i D_h W^i,$$

(8.11b)

$$[\operatorname{curl}_{\bar\eta_\kappa\circ\theta}(\sqrt{\psi}D_h W)](t) = R(W) + \int_0^t B(\nabla\bar u_\kappa(\bar\eta_\kappa\circ\theta),\nabla[\sqrt{\psi}D_h W][\nabla(\bar\eta_\kappa\circ\theta)]^{-1}),$$

with

$$\int_0^T \|R(W)\|^2_{\frac12,\Theta} \le CN(u_0).$$

With the trace estimate (8.10) and the control of W in $L^2(0,T;H^{\frac32}(\Theta))$, we can then infer as we did in Step 3 of Section 6 that

$$\int_0^T \|\sqrt{\psi}D_h W\|^2_{\frac32,\Theta} \le C_\kappa\ N(u_0) + C_\kappa\delta\int_0^T \|\sqrt{\psi}D_h W\|^2_{\frac32,\Theta},$$

and thus with a choice of δ small enough,

$$\int_0^T \|\sqrt{\psi}D_h W\|^2_{\frac32,\Theta} \le C_\kappa\ N(u_0),$$

yielding

$$\int_0^T |\sqrt{\psi} D_h W|^2_{1,\partial\Theta} \le C_\kappa\ N(u_0).$$

Since this estimate is independent of h, we get the trace estimate

$$\int_0^T |\sqrt{\psi} W|^2_{2,\partial\Theta} \le C_\kappa\ N(u_0),$$

and thus with this trace estimate and the div and curl system (8.11), still with arguments similar to those in Step 2 of Section 6,

$$\int_0^T |\sqrt{\psi} W|^2_{\frac{5}{2},\Theta} \le C_\kappa\ N(u_0).$$

By patching together all the estimates obtained on each chart defining Ω, we thus deduce that

$$\int_0^T \|w\|^2_{\frac{5}{2}} \le C_\kappa\ N(u_0). \tag{8.12}$$

Now, for the pressure, we see that for any $y \in X^{\frac{1}{2}}(t) = \{\phi \in H^{\frac{1}{2}}(\Omega) |\ (\bar{a}_\kappa)^k_i(t)\phi^i,_k \in H^{\frac{1}{2}}(\Omega)'\}$, for φ a solution of the elliptic problem

$$(\bar{a}_\kappa)^j_i[(\bar{a}_\kappa)^k_i \varphi^i,_k],_j = (\bar{a}_\kappa)^k_i(t) y^i,_k \ \text{ in } (H^{\frac{1}{2}})'(\Omega),$$
$$\varphi = 0 \text{ on } \Gamma,$$

we have by interpolation that $\varphi \in H^{\frac{3}{2}}(\Omega)$. If we once again let $e = (\bar{a}_\kappa)^k_i \varphi,_k$ and set $v := y - e$, we have that $e \in H^{\frac{1}{2}}(\Omega)$, $v \in V(t) = \{\phi \in H^{\frac{1}{2}}(\Omega) |\ (\bar{a}_\kappa)^k_i(t)\phi^i,_k = 0\}$, with $\|e\|_{\frac{1}{2}} + \|v\|_{\frac{1}{2}} \le C\|y\|_{X^{\frac{1}{2}}(t)}$. Now, by proceeding in the same fashion as in Step 2 above, we see that thanks to our decomposition and the regularity (8.12), $w_t \in L^2(0,T; X^{\frac{1}{2}}(t)')$ with

$$\int_0^T \|w_t\|^2_{X^{\frac{1}{2}}(t)'} \le N(u_0).$$

By the Lagrange multiplier Lemma 7.4, we then infer

$$\int_0^T \|q\|^2_{\frac{1}{2}} \le N(u_0). \tag{8.13}$$

Next, by using $D_{-h} D_h[\psi D_{-h} D_h w]$ as a test function in (8.5), we infer, similarly to how we obtained (8.12), that the estimates (8.12) and (8.13) imply that

$$\int_0^T \|w\|^2_{\frac{7}{2}} \le N(u_0). \tag{8.14}$$

We now explain the additional estimates employed for this higher-order differencing. We need the fact that independently of any horizontal vector h, there exists a constant $C > 0$ such that for $\text{Supp}\psi + h \subset \Theta$, we have that

$$\forall f \in H^{\frac{3}{2}}(\Theta),\ \|\sqrt{\psi} D_h f\|_{\frac{1}{2},\Theta} \le C\ \|f\|_{\frac{3}{2},\Theta},$$
$$\forall f \in H^{\frac{1}{2}}(\Theta),\ \|\sqrt{\psi} D_h f\|_{H^{\frac{1}{2}}(\Theta)'} \le C\ \|f\|_{\frac{1}{2},\Theta}. \tag{8.15}$$

The first inequality easily follows by interpolation. For the second one, if $f \in L^2(\Theta)$, we notice that for any $\phi \in H^1(\Theta)$, since the difference quotients are in a horizontal direction,

$$\int_\Theta \sqrt{\psi} D_h f \ \phi = \int_\Theta \sqrt{\psi} f D_{-h}\phi + \int_\Theta D_{-h}\sqrt{\psi} f \phi(\cdot - h)$$
$$\leq C\|f\|_{0,\Theta}\|\phi\|_{1,\Theta},$$

which shows that there exists $C > 0$ such that

$$\forall f \in L^2(\Theta),\ \|\sqrt{\psi} D_h f\|_{H^1(\Theta)'} \leq C\ \|f\|_{0,\Theta}.$$

By interpolating with the obvious inequality (for some $C > 0$)

$$\forall f \in H^1(\Theta),\ \|\sqrt{\psi} D_h f\|_{0,\Theta} \leq C\ \|f\|_{1,\Theta},$$

we then get (8.15).

Now, the pressure solves the elliptic equation

$$\Delta p = -(\bar{u}_\kappa)^i,_j\, w^j,_i \ \text{ in } \bar{\eta}_\kappa(\Omega), \tag{8.16a}$$

$$p = -[\sigma \Delta_{\bar{g}} \bar{\eta} \cdot \bar{n}_\kappa(\bar{\eta}_\kappa)\ \bar{n}_\kappa(\bar{\eta}_\kappa) + \kappa \Delta_{\bar{0}}(w \cdot \bar{n}_\kappa(\bar{\eta}_\kappa))\ \bar{n}_\kappa(\bar{\eta}_\kappa)](\bar{\eta}_\kappa^{-1}). \tag{8.16b}$$

Using the same change of variables that provides the pressure estimate (18.4) and using the elliptic estimates for coefficients with Sobolev class regularity as in [9], we find that

$$\int_0^t \|q\|^2_{\frac{3}{2}} \leq N(u_0, \sup_{[0,t]} |\bar{w}_\kappa|_4, \int_0^t \|w\|^2_{\frac{7}{2}}),$$

where the right-hand side is defined in (6.1). Therefore with (8.14),

$$\int_0^t \|q\|^2_{\frac{3}{2}} \leq N(u_0, \sup_{[0,t]} \|\bar{w}_\kappa\|_4).$$

Higher-order regularity results follow successively by appropriate higher-order difference quotients, leading to, for $n \geq 1$,

$$\int_0^t \|w\|^2_{n+\frac{3}{2}} + \int_0^t \|q\|^2_{n-\frac{1}{2}} \leq CN(u_0, \sup_{[0,t]} |\bar{w}_\kappa|_{n+2}). \tag{8.17}$$

Now, since $w_t = -(\bar{a}^\kappa)^j_i q,_j$ in Ω, we then infer that for $n \geq 2$,

$$\int_0^t \|w_t\|^2_{n-\frac{3}{2}} \leq N(u_0, \sup_{[0,t]} |\bar{w}_\kappa|_{n+2}), \tag{8.18}$$

and thus in $[0, t]$,

$$\|w(t)\|_{13.5} \leq \|u_0\|_{13.5} + \sqrt{t}\ N(u_0, \sup_{[0,t]} |\bar{w}_\kappa|_{17}).$$

By Lemma 3.1 (for the smoothing operation given in Definition 2.2 on C_T), we have that

$$\|w(t)\|_{13.5} \leq \|u_0\|_{13.5} + \sqrt{t}\ N_0(u_0, C^0_\kappa), \tag{8.19}$$

where we use C^0_κ to denote a fixed (nongeneric) constant which depends on κ.

9. Existence of a fixed-point solution of the smoothed κ-problem with surface tension

Let $A : (\bar{w} \in B_0) \mapsto w$, with w a solution of (4.2). By the relation (8.19), we see that if we take $T_\kappa \in (0,T)$ such that

$$\sqrt{T_\kappa} N_0(u_0, C^0_\kappa) \le 1,$$

then

$$A(C_{T_\kappa}) \subset C_{T_\kappa}. \tag{9.1}$$

We now prove that A is weakly lower semi-continuous in C_{T_κ}. To this end, let $(\bar{w}^n)_{n=0}^\infty$ be a weakly convergent sequence (in $L^2(0,T_\kappa; H^{13.5}(\Omega))$) toward a weak limit $\bar{w}$. Necessarily, $\bar{w} \in C_{T_\kappa}$.

By the usual compactness theorems, we have the successive strong convergent sequences

$$\begin{aligned} \bar{\eta}^n &\to \bar{\eta} \ \text{ in } L^2(0,T_\kappa; H^{12.5}(\Omega)),\\ (\bar{\eta}^n)_\kappa &\to \bar{\eta}_\kappa \ \text{ in } L^2(0,T_\kappa; H^{12.5}(\Omega)). \end{aligned}$$

Now, if we let $w^n = A(\bar{w}^n)$, we obtain from the stability of C_{T_κ} by A and (8.18) the following bounds:

$$\begin{aligned} &\int_0^T \|w^n_t\|^2_{10.5} \le CN(u_0),\\ &\sup_{[0,T]} \|w_n\|_{13.5} \le 2\|u_0\|_{13.5} + 1. \end{aligned}$$

We thus have the existence of a weakly convergent subsequence $(w^{\sigma(n)})$ in the space $L^2(0,T_\kappa; H^{13.5}(\Omega))$, to a limit $l \in C_{T_\kappa}$. By compactness, from our bound on w^n_t,

$$w^{\sigma(n)} \to l \ \text{ in } L^2(0,T_\kappa; H^{12.5}(\Omega)).$$

From the strong convergence of $(\bar{\eta}^n)^\kappa$, we then infer from the relations $(\bar{a}^n_\kappa)^i_j\, w^n{,}^i_j = 0$ in Ω that

$$(\bar{a}_\kappa)^i_j\, l{,}^i_j = 0 \ \text{ in } \Omega. \tag{9.2}$$

Moreover, we see that

$$p^{\sigma(n)} \rightharpoonup p \text{ in } L^2(0,T_\kappa; H^{11.5}(\Omega)),$$

with p the solution of

$$\begin{aligned} \triangle p &= -(\bar{u}_\kappa)^i{,}_j\, l^j{,}_i \ \text{ in } \bar{\eta}_\kappa(\Omega),\\ p &= -[\sigma \Delta_{\bar{g}} \bar{\eta} \cdot \bar{n}_\kappa(\bar{\eta}_\kappa) + \kappa \Delta_0 (w \cdot \bar{n}_\kappa(\bar{\eta}_\kappa))](\bar{\eta}_\kappa^{-1}). \end{aligned}$$

From the relations (8.5) for each n, we see from the previous weak and strong convergence that

$$\begin{aligned} &\int_0^T \langle \bar{J}_\kappa l_t, \phi\rangle_{\frac{3}{2}} + \kappa \int_0^T [l \cdot \bar{n}_\kappa(\bar{\eta}_\kappa), \phi \cdot \bar{n}_\kappa(\bar{\eta}_\kappa)]_1\\ &\qquad - \int_0^T \langle q, (\bar{a}_\kappa)^j_i \phi^i{,}_j \rangle_{\frac{1}{2}} = \sigma \int_0^T [L_{\bar{g}} \bar{\eta} \cdot \bar{n}_\kappa(\bar{\eta}_\kappa), \phi \cdot \bar{n}_\kappa(\bar{\eta}_\kappa)]_0 \ , \end{aligned} \tag{9.3}$$

which together with (9.2) and the fact that $l \in C_{T_\kappa}$ implies that $l = A(\bar{w})$. By uniqueness of the limit, we then infer that

$$w^n \rightharpoonup w \text{ in } L^2(0,T_\kappa;H^{13.5}(\Omega)).$$

By the Tychonoff fixed-point theorem, we then conclude the existence of a fixed point $\bar{w} = w$ in the closed convex set C_{T_κ} of the separable Banach space $L^2(0,T_\kappa;H^{13.5}(\Omega))$. This fixed point satisfies the smoothed system (4.1), if we denote $\eta = \mathrm{Id} + \int_0^\cdot w$ and $u = w \circ \eta^{\kappa-1}$. It is also readily seen that w, q and their time derivatives have the regularity stated in Theorem 8.1. ☐

10. Estimates for the divergence and curl

Definition 10.1 (Energy function for the smoothed κ-problem)**.** We set

$$E_\kappa^{2D}(t) = \sum_{k=0}^{3} \|\partial_t^k \eta(t)\|^2_{4.5-k} + \|v_{ttt}(t)\|^2_0 + \|\sqrt{\kappa}\eta(t)\|^2_{5.5} + \sum_{k=0}^{3}\int_0^t \|\partial_t^k v(t)\|^2_{4.5-k}$$

and

$$E_\kappa^{3D}(t) = \sum_{k=0}^{4} \|\partial_t^k \eta(t)\|^2_{5.5-k} + \|v_{tttt}(t)\|^2_0 + \|\sqrt{\kappa}\eta(t)\|^2_{6.5} + \sum_{k=0}^{4}\int_0^t \|\partial_t^k v(t)\|^2_{5.5-k}\,.$$

We use $E_\kappa(t)$ to denote the energy function when the dimension is clear.

We use these energy functions to construct solutions for the Euler equations. The increase in the derivative count from the 2D case to the 3D case is necessitated by the Sobolev embedding theorem. We will show that solutions of the κ-problem (4.1) have bounded energy $E_\kappa(t)$ for $t \in [0,T]$ when T is taken sufficiently small and that the bound is, in fact, independent of κ; as such, we will prove that the limit as $\kappa \to 0$ of the sequence of solutions to the κ-problem converges to a solution of the Euler equations.

Our estimates begin with the following

Lemma 10.1 (Divergence and curl estimates)**.** *Let* $\mathfrak{n} := \dim(\Omega) = 2$ *or* 3. *Let* $L_1 = \mathrm{curl}$ *and* $L_2 = \mathrm{div}$, *and let* $\eta_0 := \eta(0)$ *and*

$$M_0 := P(\|u_0\|_{2.5+\mathfrak{n}}, |\Gamma|_{4+\mathfrak{n}}, \sqrt{\kappa}\|u_0\|_{1.5+3\mathfrak{n}}, \sqrt{\kappa}|\Gamma|_{1+3\mathfrak{n}})$$

denote a polynomial function of its arguments. Then for $j = 1, 2$,

$$\sup_{t\in[0,T]} \|\sqrt{\kappa}L_j\eta(t)\|^2_{2.5+\mathfrak{n}} + \sum_{k=0}^{\mathfrak{n}+1}\left(\sup_{t\in[0,T]}\|L_j\partial_t^k\eta(t)\|^2_{1.5+\mathfrak{n}-k} + \int_0^T \|\sqrt{\kappa}L_j\partial_t^k v\|^2_{2.5+\mathfrak{n}-k}\right) \leq M_0 + C\,T\,P(\sup_{t\in[0,T]} E_\kappa(t))\,.$$

Proof. In Eulerian variables, equation (4.1b) is written as $u_t^i + u^i{,_l}\,(u_\kappa)^l + p{,_i} = 0$, where the transport velocity is the horizontally smoothed vector u_κ. Taking the curl of this equation and using the formula $(\mathrm{curl}\,u)^i = \epsilon_{ijk}u^k{,_j}$ with ε_{ijk} denoting the permutation symbol, we see that $\varepsilon_{ijk}[\partial_t u^k{,_j} + u^k{,_{jl}}\,u^l_\kappa + u^k{,_l}\,u^l_\kappa{,_j}] = 0$. Thus, defining the bilinear form $B^i(\nabla u, \nabla u_\kappa) = \epsilon_{ijk}u^k{,_l}\,(u_\kappa)^l{,_j}$, we can write the vorticity equation as $\frac{D}{Dt}\,\mathrm{curl}\,u = B(\nabla u, \nabla u_\kappa)$. (When the transport velocity is divergence-free, then B is the familiar vortex-stretching term.) Composing this equation with

η_κ, switching to Lagrangian variables via the chain rule, and integrating this from 0 to t, we have

(10.1)
$$\varepsilon_{ijk} v^k,_r A_{\kappa j}^r = \operatorname{curl} u_0 + \int_0^t B_{a_\kappa}(\tau) d\tau, \qquad B^i_{a_\kappa} := \varepsilon_{ijk} J_\kappa^{-2} v^k,_r a_{\kappa l}^r (v_\kappa)^l,_m a_{\kappa j}^m,$$

where $A_\kappa = J_\kappa^{-1} a_\kappa$. This is the time-integrated Lagrangian form of the vorticity equation. We will need to space-differentiate this equation once more for the estimate on $\operatorname{curl} \eta$. Hence,

$$\varepsilon_{ijk} \nabla v^k,_r A_{\kappa j}^r = \nabla \operatorname{curl} u_0^i + \varepsilon_{ikj} v^k,_r \nabla A_{\kappa j}^r + \int_0^t \nabla B_{a_\kappa}(\tau) d\tau \,. \tag{10.2}$$

We begin with the estimates for the case that $\mathfrak{n} = 2$; we set $E_\kappa(t) = E_\kappa^{2D}(t)$ and proceed with the estimate for $\operatorname{curl} \eta$. Using that $\nabla v^k,_r A_{\kappa j}^r = \partial_t (\nabla \eta^k,_r A_{\kappa j}^r) - \nabla \eta^k,_r \partial_t A_{\kappa j}^r$, we see that

$$\varepsilon_{ijk} \partial_t (\nabla \eta^k,_r A_{\kappa j}^r) = \nabla \operatorname{curl} u_0^i + \varepsilon_{ijk} \nabla \eta^k,_r \partial_t A_{\kappa j}^r + \varepsilon_{ikj} v^k,_r \nabla A_{\kappa j}^r + \int_0^t \nabla B_{a_\kappa}(\tau) d\tau \,.$$

Integrating once again in time from 0 to t yields

$$\varepsilon_{ijk} \nabla \eta^k,_r A_{\kappa j}^r = t \nabla \operatorname{curl} u_0^i + \varepsilon_{ijk} \int_0^t (\nabla \eta^k,_r \partial_t A_{\kappa j}^r + v^k,_r \nabla A_{\kappa j}^r) + \int_0^t \int_0^{t'} \nabla B_{a_\kappa} \,,$$

where

$$\begin{aligned} \nabla B_{a_\kappa} = \varepsilon_{ijk} \big[& J_\kappa^{-2} (\nabla v^k,_r v_\kappa{}^l,_m + v^k,_r \nabla v_\kappa{}^l,_m) a_{\kappa l}^r a_{\kappa j}^m \\ & + J_\kappa^{-2} v^k,_r v_\kappa{}^l,_m (\nabla a_{\kappa l}^r a_{\kappa j}^m + a_{\kappa l}^r \nabla a_{\kappa j}^m) + (\nabla J_\kappa^{-2}) v^k,_r v_\kappa{}^l,_m a_{\kappa l}^r a_{\kappa j}^m \big] \end{aligned} \tag{10.3}$$

and

$$\begin{aligned} \nabla a_{\kappa j}^m &= J_\kappa^{-1} (a_{\kappa r}^s a_{\kappa j}^m - a_{\kappa r}^m a_{\kappa j}^s) \nabla \eta_\kappa{}^r,_s \,, \\ \partial_t a_{\kappa j}^m &= J_\kappa^{-1} (a_{\kappa r}^s a_{\kappa j}^m - a_{\kappa r}^m a_{\kappa j}^s) v_\kappa{}^r,_s \,, \\ \nabla J_\kappa &= a_{\kappa s}^r \nabla \eta_\kappa{}^r,_s \,. \end{aligned} \tag{10.4}$$

Since $\|v_\kappa\|_s \le C \|v\|_s$ (and similarly for η_κ), we will write (10.3) and (10.4) in the following way:

$$\begin{aligned} \nabla B_{a_\kappa} &\sim J_\kappa^{-2} a_\kappa^2 \nabla v \nabla^2 v + J_\kappa^{-3} a_\kappa^3 (\nabla v)^2 \nabla^2 \eta \,, \\ \nabla a_\kappa &\sim J_\kappa^{-1} a_\kappa^2 \nabla^2 \eta \,, \\ \partial_t a_\kappa &\sim J_\kappa^{-1} a_\kappa^2 \nabla v \,, \\ \nabla J_\kappa &\sim a_\kappa \nabla^2 \eta \,, \end{aligned}$$

where note that we are not distinguishing between η_κ and η or between v_κ and v in the highest-order terms. The point is that the precise structure of these equations is not important for our estimates; we need only to be careful with the derivative count appearing in these expressions. The power on each expression is merely to indicate the number of times such a term appears.

Next, with the fundamental theorem of calculus,

$$\varepsilon_{ijk} \nabla \eta^k,_r A_{\kappa j}^r = \nabla \operatorname{curl} \eta^i + \varepsilon_{ijk} \nabla \eta^k,_r \int_0^t \partial_t A_{\kappa j}^r \,,$$

so that

(10.5)

$$\nabla(\operatorname{curl}\eta^i - t\operatorname{curl}u_0^i) = \varepsilon_{ijk}\left[\nabla\eta^k,_r\int_0^t \partial_t a_\kappa{}_j^r + \int_0^t(\nabla\eta^k,_r\partial_t a_\kappa{}_j^r + v^k,_r\nabla a_\kappa{}_j^r)\right] + \int_0^t\int_0^{t'}\nabla B_{a_\kappa}\,.$$

Let

$$F := P(J_\kappa^{-1}, a_\kappa, \nabla v) \text{ and } F_1 := P(F, \nabla^2\eta, \nabla^2 v)$$

denote polynomial functions of their arguments. We then express (10.5) as

$$\nabla\operatorname{curl}\eta^i \sim t\nabla\operatorname{curl}u_0^i + \nabla^2\eta\int_0^t F + \int_0^t F\,\nabla^2\eta + \int_0^t\int_0^{t'} F\,(\nabla^2\eta + \nabla^2 v)\,,$$

and taking two more spatial derivatives yields

$$\begin{aligned}\nabla^3\operatorname{curl}\eta^i \sim{}& t\nabla^3\operatorname{curl}u_0^i + \nabla^4\eta\int_0^t F + \nabla^3\eta\int_0^t F_1 + \nabla^2\eta\int_0^t F_1\\ &+\left(\int_0^t\int_0^{t'} + \int_0^t\right)\left[F_1\,(\nabla^3\eta + \nabla^3 v) + F\,\nabla^4\eta\right]\\ &+\int_0^t\int_0^{t'}\left[F_1\,\nabla^3 v + F\,\nabla^4 v\right]\,.\end{aligned}$$

Since $\int_0^t\int_0^{t'} F\,\nabla^4 v = -\int_0^t\int_0^{t'} F_t\,\nabla^4\eta + \int_0^t F\nabla^4\eta$,

(10.6)

$$\begin{aligned}\nabla^3\operatorname{curl}\eta^i \sim{}& t\nabla^3\operatorname{curl}u_0^i + \nabla^4\eta\int_0^t F + (\nabla^3\eta + \nabla^2\eta)\int_0^t F_1\\ &+\left(\int_0^t\int_0^{t'} + \int_0^t\right)\left[F_1\,(\nabla^3\eta + \nabla^3 v) + (F + F_t)\,\nabla^4\eta\right] + \int_0^t\int_0^{t'} F_1\,\nabla^3 v\,.\end{aligned}$$

We use interpolation to compute $\|\nabla^3\operatorname{curl}\eta\|_{0.5} = \|\operatorname{curl}\eta\|_{3.5}$. We begin with the highest-order term:

$$\left\|\int_0^t (F+F_t)\,\nabla^4\eta\right\|_0 \le \sup_{t\in[0,T]}\|F+F_t\|_{L^\infty}\left\|\int_0^t\eta\right\|_4 \le C\sup_{t\in[0,T]}\|F+F_t\|_{1.5}\left\|\int_0^t\eta\right\|_4,$$

$$\begin{aligned}\left\|\int_0^t (F+F_t)\,\nabla^4\eta\right\|_1 &\le \sup_{t\in[0,T]}\|F+F_t\|_{L^\infty}\left\|\int_0^t\eta\right\|_5 + \sup_{t\in[0,T]}\|\nabla(F+F_t)\|_{L^4}\left\|\int_0^t\nabla^4\eta\right\|_{L^4}\\ &\le C\sup_{t\in[0,T]}\|F+F_t\|_{1.5}\left\|\int_0^t\eta\right\|_5\,.\end{aligned}$$

Since $\|F+F_t\|_{1.5} \le C\|F\|_{L^\infty}\|v_t\|_{2.5}$, by the interpolation Theorem 7.17 in [1],

$$\begin{aligned}\left\|\int_0^t (F+F_t)\,\nabla^4\eta\right\|_{0.5} &\le C\sup_{t\in[0,T]}\|F\|\,\|v_t\|_{2.5}\left\|\int_0^t\eta\right\|_{4.5}\\ &\le C\,T\sup_{t\in[0,T]}\|F\|\,\|v_t\|_{2.5}\,\|\eta\|_{4.5}.\end{aligned}$$

The other terms have similar estimates in the $H^{0.5}(\Omega)$-norm, so that

$$\sup_{t\in[0,T]} \|\operatorname{curl}\eta\|^2_{3.5} \le T\|u_0\|^2_{4.5} + t \sup_{t'\in[0,t]} \|\int_0^t v\|^2_{3.5} + T \sup_{t\in[0,T]} \|v_t\|_{2.5}\|\eta\|^2_{4.5}\,. \tag{10.7}$$

By differentiating (10.6) once more in space, the same interpolation estimates show that

$$\sup_{t\in[0,T]} \|\sqrt{\kappa}\operatorname{curl}\eta\|^2_{4.5} \le T\|\sqrt{\kappa}u_0\|^2_{5.5} + T\int_0^T \|v\|^2_{4.5} + T \sup_{t\in[0,T]} \|v_t\|_{2.5}\|\sqrt{\kappa}\eta\|^2_{5.5}\,.$$

Next, we rewrite (10.1) as

$$\operatorname{curl} v = \operatorname{curl} u_0 + \varepsilon_{ijk} v^k{}_{,r} \int_0^t \partial_t A_{\kappa}{}^r_j + \int_0^t B_{a_\kappa}\,. \tag{10.8}$$

Using the fact that $H^s(\Omega)$ is a multiplicative algebra for $s > 1$, it follows from (10.3) and (10.4) that $\sup_{t\in[0,T]} \|\operatorname{curl} v(t)\|_{2.5} \le \|u_0\|_{4.5} + C\,T\,P(\sup_{t\in[0,T]} E_\kappa(t))$. Differentiating the above expression for $\operatorname{curl} v$ yields

$$\operatorname{curl} v_t = \varepsilon_{ijk} v^k{}_{,r}\, \partial_t a_{\kappa}{}^r_j + B_{a_\kappa} + \varepsilon_{ijk}\partial_t v^k{}_{,r} \int_0^t \partial_t A_{\kappa}{}^r_j\,,$$

so with the fundamental theorem of calculus and our generic polynomial function F,

$$\operatorname{curl} v_t \sim P(\nabla u_0) + \nabla v_t \int_0^t F + \int_0^t F_t, \quad F_t \sim F\,\nabla v_t\,. \tag{10.9}$$

Again using the properties of the multiplicative algebra, we see that

$$\sup_{t\in[0,T]} \|\operatorname{curl} v_t(t)\|^2_{1.5} \le P(\|u_0\|_{4.5}) + C\,T\,P(\sup_{t\in[0,T]} E_\kappa(t)).$$

From the time differentiation of (10.9),

$$\begin{aligned} \operatorname{curl} v_{tt} &\sim \nabla v_t F + \nabla v_{tt} \int_0^t F \\ &\sim \nabla v_t(0) P(\nabla u_0) + \nabla v_{tt} \int_0^t F + \int_0^t [F\nabla v_{tt} + F\nabla v_t \nabla v_t]\,. \end{aligned} \tag{10.10}$$

We must estimate the $H^{0.5}(\Omega)$-norm of the three terms on the right-hand side of (10.10) by using interpolation. Let L denote the linear form given by $L(w) = \int_0^t Fw$. Then

$$\|L(w)\|_0 \le C_0\|\int_0^t w\|_0, \quad C_0 = \sup_{[0,t]} \|F\|_{L^\infty}.$$

Letting $F_1 := P(J_\kappa^{-1}, a_\kappa, \nabla v, \nabla^2\eta, \nabla^2 v)$, it is easy to check that

$$\|L(w)\|_1 \le C_1\|\int_0^t w\|_1, \quad C_1 = \sup_{[0,t]} \|F_1\|_{L^\infty}.$$

By the interpolation Theorem 7.17 in [1],

$$\|\int_0^t F\nabla v_{tt}\|_{0.5} \le \sqrt{C_0}\sqrt{C_1}\,\|\int_0^t v_{tt}\|_{1.5}\,,$$

so that by Jensen's inequality and Sobolev embedding,

$$\| \int_0^t F\nabla v_{tt} \|^2_{0.5} \le C\,T\,P(\sup_{t\in[0,T]} E_\kappa(t)).$$

All of the other time-dependent terms in (10.10) have the same bound by the same interpolation procedure. For the time $t=0$ term, interpolation provides the estimate

$$\|P(\nabla u_0)\nabla v_t(0)\|_{0.5} \le CP(\|u_0\|_{4.5})\|v_t(0)\|_{1.5} \le CP(\|u_0\|_{4.5})\|q(0)\|_{2.5} \le M_0.$$

The initial pressure $q(0)$ solves the Dirichlet problem

$$\begin{aligned} \Delta q(0) &= \mathfrak{i}_0 := (u_0)^i,_j\,(u_0)_\kappa)^j,_i \ \text{ in } \Omega\,, \\ q(0) &= \mathfrak{b}_0 := \frac{\sqrt{g_0}}{\sqrt{g_0}_\kappa}\Pi_0{}^i_j g_0^{\alpha\beta}\eta_0{}^i,_{\alpha\beta}\,N^i_\kappa + \kappa\Delta_0(u_0\cdot N_\kappa) \text{ on } \Gamma\,. \end{aligned}$$

Since $\|q(0)\|_{2.5} \le C(\|\mathfrak{i}_0\|_{0.5} + |\mathfrak{b}_0|_2) \le M_0$, we see that $\sup_{t\in[0,T]} \|\operatorname{curl} v_{tt}(t)\|^2_{0.5} \le M_0 + C\,T\,P(\sup_{t\in[0,T]} E_\kappa(t))$.

Differentiating (10.10) with respect to time, we see that $\operatorname{curl} v_{ttt} \sim \nabla v_{ttt}\int_0^t F + F\,\nabla v_{tt} + F\,\nabla v_t\,\nabla v_t$ so that by the fundamental theorem of calculus

$$\begin{aligned} F\,\nabla v_{tt} + F\,\nabla v_t\,\nabla v_t = F(_0)[\nabla v_{tt}(0) + \nabla v_t(0)\,\nabla v_t(0)] \\ + \int_0^t [F\,\nabla v_{ttt} + F\,\nabla v_t\,\nabla v_{tt} + F\,\nabla v_t\,\nabla v_t\,\nabla v_t]\,, \end{aligned}$$

so that

(10.11)

$$\begin{aligned} &\int_0^T \|\sqrt{\kappa}\operatorname{curl} v_{ttt}\|^2_{0.5} \\ &\quad \le \int_0^T \left\|\sqrt{\kappa}F(0)[\nabla v_{tt}(0) + \nabla v_t(0)\,\nabla v_t(0)]\right\|^2_{0.5} \int_0^T \left\|\sqrt{\kappa}\nabla v_{ttt}\int_0^t F\right\|^2_{0.5} \\ &\quad\quad + \int_0^T \left\|\int_0^t [F\,\sqrt{\kappa}\nabla v_{ttt} + F\,\nabla v_t\,\sqrt{\kappa}\nabla v_{tt} + F\,\nabla v_t\,\nabla v_t\,\sqrt{\kappa}\nabla v_t]\right\|^2_{0.5}. \end{aligned}$$

We repeat the interpolation estimates between $L^2(\Omega)$ and $H^1(\Omega)$ just as for the estimates for $\operatorname{curl} v_{tt}$; for example,

$$\left\|\int_0^t F\sqrt{\kappa}\nabla v_{ttt}\right\|_{0.5} \le \sqrt{C_0}\sqrt{C_1}\ \left\|\int_0^t \sqrt{\kappa}v_{ttt}\right\|_{1.5},$$

so that by Jensen's inequality and Sobolev embedding,

$$\| \int_0^t F\sqrt{\kappa}\nabla v_{ttt}\|^2_{0.5} \le C\,t\,\sup_{[0,t]} E_\kappa\,\|\sqrt{\kappa}v_{ttt}\|^2_{0.5}\,.$$

Thus, integrating from 0 to T gives the estimate

$$\begin{aligned} \int_0^T \|\int_0^t F\sqrt{\kappa}\nabla v_{ttt}\|^2_{0.5} &\le C\,T\,P(\sup_{t\in[0,T]} E_\kappa(t)) \int_0^T \|\sqrt{\kappa}v_{ttt}\|^2_{0.5} \\ &\le C\,T\,P(\sup_{t\in[0,T]} E_\kappa(t))\,. \end{aligned}$$

The other time dependent terms in (10.11) have the same bound by the same argument. The time $t=0$ terms require analysis of the elliptic problem for $q_t(0)$:

$$\begin{aligned}\Delta q_t(0) &\sim \mathfrak{i}_1 := \nabla[P(\nabla u_0)\,\nabla q(0)] + F\,\nabla v_t(0) \text{ in } \Omega\,,\\ q_t(0) &\sim \mathfrak{b}_1 := Q(\partial\eta_0)\partial^2 u_0 + Q(\partial\eta_0)\partial u_0 \partial^2\eta_0\\ &\qquad\qquad + \kappa\Delta_0(v_t(0)\cdot N_\kappa) + \kappa\Delta_0(u_0\cdot Q(\partial{\eta_0}_\kappa)\partial {u_0}_\kappa) \text{ on } \Gamma\,.\end{aligned}$$

By interpolation estimates (as above),

$$\int_0^T \left\|\sqrt{\kappa}F(0)[\nabla v_{tt}(0) + \nabla v_t(0)\,\nabla v_t(0)]\right\|^2_{0.5} \le \kappa T\,\|P(\nabla u_0)\|^2_{L^\infty}\|v_{tt}(0)\|^2_{1.5},$$

and since time differentiation of the Euler equations shows that

$$v_{tt}(0) = -\nabla u_0\,\nabla q(0) - \nabla q_t(0),$$

interpolation provides the estimate

$$\begin{aligned}\sqrt{\kappa}\|v_{tt}(0)\|_{1.5} &\le \sqrt{\kappa}\|\nabla^2 u_0\,\nabla q(0)\|_{0.5} + \sqrt{\kappa}\|\nabla u_0\,\nabla^2 q(0)\|_{0.5} + \sqrt{\kappa}\|q_t(0)\|_{2.5}\\ &\le M_0 + \sqrt{\kappa}\|\mathfrak{i}_1\|_{0.5} + \sqrt{\kappa}|\mathfrak{b}_1|_2\\ &\le M_0 + \sqrt{\kappa}|\mathfrak{b}_1|_2,\end{aligned}$$

where we have used the elliptic estimate $\|q(0)\|_{2.5}\le M_0$ (from above) for both the second and third inequalities. (The remaining estimate for $|\mathfrak{b}_1|_2$ places the regularity constraints on the polynomial function M_0 in the hypothesis of the lemma.) Because $H^2(\Gamma)$ is a multiplicative algebra, the bound for $\sqrt{\kappa}|\mathfrak{b}_1|_2$ is controlled by the highest-order terms $\sqrt{\kappa}|v_t(0)|_4$ and $\sqrt{\kappa}|{u_0}_\kappa|_5 \le \sqrt{\kappa}C\|u_0\|_{5.5}$. Now,

$$\sqrt{\kappa}|v_t(0)|_4 \le \sqrt{\kappa}\|q(0)\|_{5.5} \le \sqrt{\kappa}\|\mathfrak{i}_0\|_{3.5} + \sqrt{\kappa}|\mathfrak{b}_0|_5\,,$$

and $\|\mathfrak{i}_0\|_{3.5}$ is bounded by $P(\|u_0\|_{4.5})$ while the highest-order terms in $\sqrt{\kappa}|\mathfrak{b}_0|_5$ require bounds on $\sqrt{\kappa}\|\eta_0\|_{7.5}$ and $\sqrt{\kappa}\|u_0\|_{7.5}$. With our definition of M_0, we see that

$$\kappa T\,\|P(\nabla u_0)\|^2_{L^\infty}\|v_{tt}(0)\|^2_{1.5} \le M_0$$

and hence $\int_0^T \|\sqrt{\kappa}\,\text{curl}\, v_{ttt}\|^2_{0.5} \le M_0 + C\,T\,P(\sup_{t\in[0,T]} E_\kappa(t))$.

The proof that $\int_0^T \|\sqrt{\kappa}\,\text{curl}\, v_{tt}\|^2_{1.5} \le M_0 + C\,T\,P(\sup_{t\in[0,T]} E_\kappa(t))$ is essentially identical.

The divergence estimates begin with the fundamental equation ${a_\kappa}^j_i v^i,_j = 0$. By taking one derivative of this equation and integrating by parts in time, we find that

$$\nabla\,\text{div}\,\eta = \nabla\eta^i,_j \int_0^t \partial_t {A_\kappa}^j_i + \int_0^t (\partial_t {A_\kappa}^j_i \nabla\eta^i,_j - \nabla {A_\kappa}^j_i v^i,_j\,)\,.$$

Computing the $H^{2.5}(\Omega)$-norm of this equation yields the estimate

$$\sup_{t\in[0,T]} \|\,\text{div}\,\eta(t)\|^2_{3.5} \le M_0 + C\,T\,P(\sup_{t\in[0,T]} E_\kappa(t)).$$

The divergence estimates for v, v_t, v_{tt}, $\sqrt{\kappa}v_{tt}$, and $\sqrt{\kappa}v_{ttt}$ follow the same argument as the corresponding curl estimates.

In the case that $\mathfrak{n}=3$, the estimates are found in the same way, with one minor change. Set $E_\kappa(t) = E^{3D}_\kappa(t)$. The estimates for $\text{curl}\,\eta$, which rely on Sobolev embedding, require greater regularity on v_t. The estimate (10.7) becomes

$$\sup_{t\in[0,T]} \|\,\text{curl}\,\eta\|^2_{3.5} \le T\|u_0\|^2_{4.5} + t\sup_{t'\in[0,t]} \|\int_0^t v\|^2_{3.5} + T\sup_{t\in[0,T]} \|v_t\|_3\|\eta\|^2_{4.5}\,,$$

and similarly,

$$\sup_{t\in[0,T]} \|\sqrt{\kappa}\,\mathrm{curl}\,\eta\|^2_{4.5} \le T\|\sqrt{\kappa}u_0\|^2_{5.5} + T\int_0^T \|v\|^2_{4.5} + T\sup_{t\in[0,T]}\|v_t\|_3\|\sqrt{\kappa}\eta\|^2_{5.5}\,.$$

□

11. Some geometric identities

We will usually omit writing dS_0 in our surface integrals, and for convenience we set $\sigma = 1$. Let Π^i_j denote the projection operator onto the direction normal to $\eta(\Gamma)$, defined as $\Pi^i_j = \delta^i_j - g^{\alpha\beta}\eta^i_{,\alpha}\delta_{jl}\eta^l_{,\beta}$, where g is the induced metric on $\eta(\Gamma)$ defined in (1.4). The mean curvature vector motivates us to introduce the projection operator Π. In particular, we have the important formula

(11.1)
$$\begin{aligned} -\sqrt{g}Hn\circ\eta &= \sqrt{g}\Delta_g(\eta)\\ &= \sqrt{g}\left[g^{\alpha\mu}(\delta^{ij} - g^{\nu\beta}\eta^i{,_\beta}\,\eta^j{,_\nu})\eta^j{,_{\mu\alpha}} + (g^{\alpha\beta}g^{\mu\nu} - g^{\alpha\nu}g^{\mu\beta})\eta^i{,_\beta}\,\eta^j{,_\nu}\,\eta^j{,_{\mu\alpha}}\right]\\ &= \sqrt{g}g^{\mu\alpha}\Pi^i_j\eta^j{,_{\mu\alpha}}\,, \end{aligned}$$

where the last equality follows since $(g^{\alpha\beta}g^{\mu\nu} - g^{\alpha\nu}g^{\mu\beta})\eta^i{,_\beta}\,\eta^j{,_\nu}\,\eta^j{,_{\mu\alpha}} = 0$. For a vector field F on Γ, $\Pi F = [n\cdot F]\,n$, i.e., $\Pi = n\otimes n$.

We let

$$Q(\partial\eta) = f_1(\partial\eta)/f_2(\sqrt{g}) \tag{11.2}$$

denote a generic rational function where f_1 and f_2 are smooth functions. We record for later use that that $n = \frac{a^TN}{|a^TN|} = \frac{\eta,_1\times\eta,_2}{|\eta,_1\times\eta,_2|}$ and that $|a^TN| = \sqrt{\det g}$ on Γ, as

$$\begin{aligned} |\eta,_1\times\eta,_2|^2 &= \varepsilon_{ijk}\eta^j{,_1}\,\eta^k{,_2}\;\varepsilon_{irs}\eta^r{,_1}\,\eta^s{,_2}\\ &= (\delta_{jr}\delta_{ks} - \delta_{js}\delta_{kr})\eta^k{,_1}\,\eta^j{,_s}\,\eta^r{,_1}\,\eta^s{,_2} = |\eta,_1|^2|\eta,_2|^2 - [\eta,_1\cdot\eta,_2]^2 = \det g\,, \end{aligned}$$

where ε_{ijk} denotes the permutation symbol of $(1,2,3)$. We will use the symbol Q to denote any smooth (tensor) function that can be represented as (11.2).

Remark 9. The L^∞-norm of the numerator of Q is bounded by a polynomial of the energy function, while the L^∞-norm of the denominator of Q is uniformly controlled by (3.3a). Thus, the generic constant C which appears in the following inequalities may depend on a polynomial of $\det g_0$. In particular, $\|Q(\partial\eta)\|_{L^\infty} \le C(\det g_0)\|P(\partial\eta)\|_{L^\infty}$.

For a vector field F on Γ, $F\cdot N = F\cdot n + F\cdot(N-n)$ and

$$|N-n|_{L^\infty} \le \int_0^t |n_t|_{L^\infty} = \int_0^t |Q(\partial\eta)\partial v|_{L^\infty} \le C\,t\,P(E_\kappa(t))\,,$$

the last inequality following from (3.3a). If $|\Pi F|_s \le M_0 + CP(E_\kappa(t))$, then $|F\cdot N|_s$ satisfies the same bound.

12. κ-independent estimates for the smoothed problem and existence of solutions in 2D

All of the variables in the smoothed κ-problem (4.1) implicitly depend on the parameter κ. In this section, where we study the asymptotic behavior of the solutions to (4.1) as $\kappa \to 0$, we will make this dependence explicit by placing a $\sim$ over each of the variables. We set $E_\kappa(t) = E_\kappa^{2D}(t)$.

Remark 10. The only difference between the 2D and 3D cases arises from the embedding of $\tilde{v}_t \in L^\infty(\Omega)$. In 2D, $\tilde{v}_t \in H^{2.5}(\Omega)$ is sufficient, while in 3D, we need $\tilde{v}_t \in H^3(\Omega)$.

The pressure function $\tilde{q}$ can be formulated to solve either a Dirichlet problem with boundary condition (12.5) or a Neumann problem found by taking the inner-product of the Euler equations with $\tilde{a}_\kappa^T N$. We use the latter.

Lemma 12.1 (Pressure estimates). *With $(\tilde{v}, \tilde{q})$ a solution of the κ-problem* (4.1)

$$\|\tilde{q}(t)\|_{3.5}^2 + \|\tilde{q}_t(t)\|_{2.5}^2 + \|\tilde{q}_{tt}(t)\|_1^2 \le C\, P(E_\kappa(t))\,. \tag{12.1}$$

Proof. Denoting $\tilde{a}_\kappa$ by A, we define the divergence-form elliptic operator L_A and corresponding Neumann boundary operator B_A as

$$L_A = \partial_j(\tilde{J}_\kappa^{-1} A_i^j A_i^l \partial_l)\,, \quad B_A = \tilde{J}_\kappa^{-1} A_i^j A_i^l N_j \partial_l\,.$$

For $k = 0, 1, 2$, we analyze the Neumann problems

$$L_a(\partial_t^k \tilde{q}) = f_k \ \text{ in } \ \Omega \qquad B_a(\partial_t^k \tilde{q}) = g_k \ \text{ on } \ \Gamma \tag{12.2}$$

with

$$\begin{aligned}
&f_0 = \partial_t A_i^j\, \tilde{v}^i,_j\,, && g_0 = -\tilde{v}_t \cdot \sqrt{\tilde{g}_\kappa}\tilde{n}_\kappa,\\
&f_1 = -L_{A_t}(\tilde{q}) - \partial_t^2 A_i^j\, \tilde{v}^i,_j - \partial_t A_i^j\, \tilde{v}_t^i,_j\,, && g_1 = B_{A_t}(\tilde{q}) - \tilde{v}_t \cdot (\sqrt{\tilde{g}_\kappa}\tilde{n}_\kappa)_t - \tilde{v}_{tt} \cdot \sqrt{\tilde{g}_\kappa}\tilde{n}_\kappa,\\
&f_2 = -2L_{A_t}(\tilde{q}_t) - L_{A_{tt}}(\tilde{q}) - \partial_t^3 A_i^j\, \tilde{v}^i,_j\,, && g_2 = 2B_{A_t}(\tilde{q}_t) + B_{A_{tt}}(\tilde{q}) - \tilde{v}_{ttt} \cdot \sqrt{\tilde{g}_\kappa}\tilde{n}_\kappa,\\
&\quad - 2\partial_t^2 A_i^j\, \tilde{v}_t^i,_j - \partial_t A_i^j\, \tilde{v}_{tt}^i,_j\,, && \quad - 2\tilde{v}_{tt} \cdot (\sqrt{\tilde{g}_\kappa}\tilde{n}_\kappa)_t - \tilde{v}_t \cdot (\sqrt{\tilde{g}_\kappa}\tilde{n}_\kappa)_{tt}\,.
\end{aligned}$$

For $s \ge 1$, elliptic estimates provide the inequality

$$\|\partial_t^k \tilde{q}(t)\|_s \le C_s[P(\|\eta\|_{4.5})\|f_k\|_{s-2} + |g_k|_{s-3/2} + \|\tilde{q}\|_0]\,, \tag{12.3}$$

where $\|\cdot\|_{-1}$ denotes the norm on $[H^1(\Omega)]'$. We remark that the usual H^s elliptic estimates require that coefficients have the regularity $\partial^{s-1}(A_i^l A_i^j) \in L^\infty(\Omega)$; however $\partial^{s-1}(A_i^l A_i^j) \in L^2(\Omega)$ is sufficient. See see [9] or the quasilinear estimates in [19].

As we cannot guarantee that solutions $\tilde{q}$ to the κ-problem (4.1) have zero average, we use $\|\tilde{q}\|_0 \le C\|\tilde{q}\|_1$ and the H^1 elliptic estimate for the Dirichlet problem $L_A(q) = f_0$ in Ω with $-q = \Delta_{\tilde{g}}\tilde{\eta} \cdot \tilde{n}_\kappa + \kappa\Delta_0(\tilde{v} \cdot \tilde{n}_\kappa)$ on Γ. Thus,

$$\|\tilde{q}\|_1 \le C(\|f_0\|_0 + |\Delta_{\tilde{g}}\tilde{\eta} \cdot \tilde{n}_\kappa + \kappa\Delta_0(\tilde{v} \cdot \tilde{n}_\kappa)|_{0.5}) \le C\, P(E_\kappa(t)).$$

From (10.4), it is clear that $\|f_0\|_{1.5}^2 + |g_0|_2^2 \le CP(E_\kappa(t))$; thus, from the elliptic estimate,

$$\|\tilde{q}\|_{3.5}^2 \le CP(E_\kappa(t)). \tag{12.4}$$

Next, we must show that $\|f_1\|_{0.5}^2 + |g_1|_1^2 \le C\, P(E_\kappa(t))$. But

$$f_1 \sim P(\tilde{J}_\kappa^{-1}, A, \nabla\tilde{v}_\kappa)([\nabla\tilde{v}_t]^2 + \nabla^2\tilde{q})$$

so that with (12.4), $\|f_1\|^2_{0.5} \le C\,P(E_\kappa(t))$, with the same bound for $|g_1|^2_1$, so that $\|\tilde q_t\|^2_{2.5} \le CP(E_\kappa(t))$. Using this, we find, in the same fashion, that $\|f_2\|_0 \le C\,P(E_\kappa(t))$. The normal trace theorem, read in Lagrangian variables, states that if $\tilde v_{ttt} \in L^2(\Omega)$ with $\|A^j_i \tilde v^i_{ttt},_j\|_0 \in L^2(\Omega)$, then $\tilde v_{ttt}\cdot\sqrt{\tilde g_\kappa}\tilde n_\kappa \in H^{-0.5}(\Gamma)$ with the estimate $|\tilde v_{ttt}\cdot\sqrt{\tilde g_\kappa}\tilde n_\kappa|^2_{-0.5} \le C\,P(E_\kappa(t))$. Since $\|\operatorname{Tr}(A\,\nabla\tilde v_{ttt})\|^2_0 = \|\operatorname{Tr}(3A_t\,\nabla\tilde v_{tt} + 3A_{tt}\,\nabla\tilde v_t + A_{ttt}\,\nabla\tilde v)\|^2_0 \le C\,P(E_\kappa(t))$ and using the above estimates for $\tilde q$ and $\tilde q_t$, we find that $|g_2|_{-0.5} \le C\,P(E_\kappa(t))$, thus completing the proof. $\square$

Our smoothed κ-problem (4.1) uses the boundary condition (4.1e) which we write as

$$\tilde q\,\tilde n_\kappa = \frac{\sqrt{\tilde g}}{\sqrt{\tilde g_\kappa}}\tilde H\,\tilde n\cdot\tilde n_\kappa\,\tilde n_\kappa - \kappa\Delta_0(\tilde v\cdot\tilde n_\kappa)\,\tilde n_\kappa\,, \tag{12.5}$$

where (we remind the reader) $\kappa > 0$ is the artificial viscosity,

$$\Delta_0 = \sqrt{\tilde g_\kappa}^{-1}\partial_\alpha(\sqrt{g_0}g_0^{\alpha\beta}\partial_\beta),$$

$\tilde n$ is the unit normal along the boundary $\tilde\eta(t)(\Gamma)$ and $\tilde n_\kappa$ is the unit normal along the smoothed κ-boundary $\tilde\eta_\kappa(t)(\Gamma)$.

We begin with an energy estimate for the third time-differentiated problem. Although we are doing the estimates for the 2D domain Ω, we keep the notation of the 3D problem as well as terms that only arise in 3D when differentiating the mean curvature vector. Thus, when we turn to the 3D problem in Section 14, the modifications will be trivial.

Lemma 12.2 (Energy estimates for the third time-differentiated κ-problem)**.** *For M_0 taken as in Lemma* 10.1 *and $\delta > 0$, solutions of the κ-problem* (4.1) *satisfy*

(12.6)

$$\begin{aligned}\sup_{t\in[0,T]}&\left[\|\tilde v_{ttt}\|^2_0 + |\tilde v_{tt}\cdot\tilde n|^2_1\right] + \int_0^T |\sqrt\kappa\partial_t^3\tilde v\cdot\tilde n_\kappa|^2_1\\ &\le M_0 + T\,P(\sup_{t\in[0,T]}E_\kappa(t)) + \delta\sup_{t\in[0,T]}E_\kappa(t)\\ &+ C\sup_{t\in[0,T]}[P(\|\tilde v_t\|^2_{2.5}) + P(\|\tilde v\|^2_{3.5}) + P(\|\tilde\eta\|^2_{4.5})] + CP(\|\sqrt\kappa\tilde v_{tt}\|^2_{L^2(0,T;H^{2.5}(\Omega))})\,.\end{aligned}$$

Proof. Letting $A = \tilde a_\kappa$ and testing $\partial_t^3(\tilde J_\kappa\tilde v^i_t) + \partial_t^3(A^k_i\tilde q,_k) = 0$ with $\partial_t^3\tilde v^i$ shows that

$$\begin{aligned}\int_0^T\frac12\int_\Omega\partial_t^3(\tilde J_\kappa\tilde v^i_t)\partial_t^3\tilde v^i - \int_0^T\int_\Omega\partial_t^3(A^k_i q)\ \partial_t^3\tilde v^i,_k\\ = -\int_0^T\int_\Gamma\partial_t^3(\sqrt{\tilde g_\kappa}\tilde q\tilde n_\kappa(\tilde\eta_\kappa))\cdot\partial_t^3\tilde v\ dS_0\,.\end{aligned} \tag{12.7}$$

Step 1. Boundary integral term. We rewrite the modified boundary condition (12.5) as

$$\tilde q\,\tilde n_\kappa = \frac{\sqrt g}{\sqrt{g_\kappa}}\left[\tilde H\tilde n + \tilde H\tilde n\cdot(\tilde n_\kappa - \tilde n)\,\tilde n + \tilde H\tilde n\cdot\tilde n_\kappa\,(\tilde n_\kappa - \tilde n)\right] - \kappa\Delta_0(\tilde v\cdot\tilde n_\kappa)\,\tilde n_\kappa\,. \tag{12.8}$$

We first consider the boundary integral on the right-hand side of (12.7) with only the first term on the right-hand side of (12.8):

$$
\begin{aligned}
&- \int_0^T \int_\Gamma \partial_t^3 (\sqrt{\tilde g} \tilde H \tilde n^i \circ \eta) \partial_t^3 \tilde v^i \, dS_0 \\
&= - \int_0^T \int_\Gamma \sqrt{\tilde g} \tilde g^{\alpha\beta} \Pi^i_j \partial_t^2 \tilde v^j_{,\beta} \partial_t^3 \tilde v^i_{,\alpha} - \int_0^T \int_\Gamma \sqrt{\tilde g} [\tilde g^{\mu\nu} \tilde g^{\alpha\beta} - \tilde g^{\alpha\nu} \tilde g^{\mu\beta}] \tilde\eta^j_{,\nu} \partial_t^2 \tilde v^j_{,\mu} \, \tilde\eta^i_{,\beta} \partial_t^3 \tilde v^i_{,\alpha} \\
&\qquad + \int_0^T \int_\Gamma Q^{\alpha\beta}_{ij}(\partial\tilde\eta, \partial\tilde v) \, \partial_t \tilde v^j_{,\beta} \, \partial_t^3 \tilde v^i_{,\alpha} + \int_0^T \int_\Gamma Q^\alpha_i (\partial\tilde\eta, \partial\tilde v) \, \partial_t^3 \tilde v^i_{,\alpha} \\
&=: I + II + III + IV \,.
\end{aligned} \tag{12.9}
$$

The first term I on the right-hand side of (12.9) is given by

$$
I = -\frac{1}{2} \int_\Gamma \sqrt{\tilde g} (\Pi^i_j \partial_t^2 \tilde v^j_{,\beta}) \tilde g^{\alpha\beta} (\Pi^i_k \partial_t^2 \tilde v^k_{,\alpha}) \Big]_0^T + \int_0^T \int_\Gamma Q^{\alpha\beta}_{jk} (\partial\tilde\eta, \partial\tilde v) \partial_t^2 \tilde v^k_{,\alpha} \, \partial_t^2 \tilde v^j_{,\beta} \,,
$$

where we use the notation $f]_0^T = f(T) - f(0)$. Since $\Pi^i_j \tilde v^j_{tt},_\beta = (\Pi^i_j \tilde v^j_{tt}),_\beta - \Pi^i_j,_\beta \, \tilde v^j_{tt}$ and $\Pi^i_j,_\beta = Q^{i\gamma}_{jl}(\partial\tilde\eta) \tilde\eta^l,_{\gamma\beta}$ with $Q(\partial\tilde\eta)$ defined by (11.2), for $\delta > 0$,

$$
\begin{aligned}
&- \frac{1}{2} \int_\Gamma \sqrt{\tilde g} (\Pi^i_j \partial_t^2 \tilde v^j_{,\beta}) \tilde g^{\alpha\beta} (\Pi^i_k \partial_t^2 \tilde v^k_{,\alpha}) \\
&\qquad \le -\frac{1}{2} |\Pi \tilde v_{tt}|_1^2 + \delta |\Pi \tilde v_{tt}|_1^2 + (1 + C_\delta) \left| Q^{i\alpha\beta}_{jl}(\partial\tilde\eta) \, \tilde\eta^l,_{\alpha\beta} \; \tilde v^j_{tt} \right|_0^2 ,
\end{aligned}
$$

where the constant C_δ depends inversely on δ. Since for any $t \in [0,T]$

$$
\left| \Big(Q^{i\alpha\beta}_{jl}(\partial\tilde\eta) \, \tilde\eta^l,_{\alpha\beta} \; \tilde v^j_{tt} \Big)_t \right|_0^2 (t) \le C P(E_\kappa(t)),
$$

it follows that

$$
I \le -\frac{1}{2} \sup_{t\in[0,T]} |\Pi \tilde v_{tt}|_1^2 + M_0(\delta) + \delta \sup_{t\in[0,T]} E_\kappa(t) + C \, T P(\sup_{t\in[0,T]} E_\kappa(t)) \,.
$$

The second term II requires some care (in the way in which the terms are grouped together). Letting

$$
\begin{gathered}
\mathcal{A}^1 = \begin{bmatrix} \tilde\eta_{,1} \cdot \partial_t^2 \tilde v_{,1} & \tilde\eta_{,1} \cdot \partial_t^2 \tilde v_{,2} \\ \tilde\eta_{,2} \cdot \partial_t^2 \tilde v_{,1} & \tilde\eta_{,2} \cdot \partial_t^2 \tilde v_{,2} \end{bmatrix}, \; \mathcal{A}^2 = \begin{bmatrix} \tilde v_{,1} \cdot \partial_t^2 \tilde v_{,1} & \tilde\eta_{,1} \cdot \partial_t^2 \tilde v_{,2} \\ \tilde v_{,2} \cdot \partial_t^2 \tilde v_{,1} & \tilde\eta_{,2} \cdot \partial_t^2 \tilde v_{,2} \end{bmatrix}, \\
\mathcal{A}^3 = \begin{bmatrix} \tilde\eta_{,1} \cdot \partial_t^2 \tilde v_{,1} & \tilde v_{,1} \cdot \partial_t^2 \tilde v_{,2} \\ \tilde\eta_{,2} \cdot \partial_t^2 \tilde v_{,1} & \tilde v_{,2} \cdot \partial_t^2 \tilde v_{,2} \end{bmatrix},
\end{gathered} \tag{12.10}
$$

we find that

$$
\begin{aligned}
II &= \int_0^T \int_\Gamma \det \tilde g^{-\frac{1}{2}} \, (\partial_t \det \mathcal{A}^1 - \det \mathcal{A}^2 - \det \mathcal{A}^3) \\
&= \int_0^T \int_\Gamma -(\det g^{-\frac{1}{2}})_t \, \det \mathcal{A}^1 - \det \tilde g^{-\frac{1}{2}} (\det \mathcal{A}^2 + \det \mathcal{A}^3) + \int_\Gamma \det \tilde g^{-\frac{1}{2}} \det \mathcal{A}^1 \Big]_0^T .
\end{aligned} \tag{12.11}
$$

For $\alpha = 1, 2$, let $V_\alpha = \tilde\eta,_\alpha \cdot \partial_t^2 \tilde v$; thus $V_\alpha,_\beta = \mathcal{A}^1_{\alpha\beta} + \tilde\eta^i,_{\alpha\beta}\, \tilde v^i_{tt}$ so that

$$\begin{aligned}\det \mathcal{A}^1_{\alpha\beta} &= \det(V_\alpha,_\beta - \tilde\eta^i,_{\alpha\beta}\, \tilde v^i_{tt})\\ &= \det V_\alpha,_\beta - \det(\tilde\eta^i,_{\alpha\beta}\, \tilde v^i_{tt}) + P_{ij}(\partial^2\tilde\eta)\tilde v^i_{tt}\tilde v^j_{tt} + P^\alpha_{ij}(\partial^2\tilde\eta)\tilde v^i_{tt}\tilde v^j_{tt},_\alpha\ .\end{aligned}$$

With $A = \mathrm{Cof}(\partial V)$, $\det \partial V = A^{\beta\alpha} V_\alpha,_\beta$. It follows that

$$\int_\Gamma \det \tilde g^{-\frac12} \det \partial V = -\int_\Gamma (\det \tilde g^{-\frac12}),_\beta A^{\beta\alpha} V,_\alpha\,,$$

as $A^{\beta\alpha},_\beta = 0$ since A is the cofactor matrix. Hence,

$$\int_\Gamma \det \tilde g^{-\frac12} \det \mathcal{A}_1 = \int_\Gamma P_{ij}(\partial^2\tilde\eta)\tilde v^i_{tt}\tilde v^j_{tt} + P^\alpha_{ij}(\partial^2\tilde\eta)\tilde v^i_{tt}\tilde v^j_{tt},_\alpha\ , \tag{12.12}$$

so that

$$II \le \int_0^T \int_\Gamma Q^{\alpha\beta}_{ij}(\partial\tilde\eta, \partial\tilde v)\, \partial_t^2 \tilde v^i,_\alpha\, \partial_t^2 \tilde v^j,_\beta + \left.\int_\Gamma [P_{ij}(\partial^2\tilde\eta)\tilde v^i_{tt}\tilde v^j_{tt} + P^\alpha_{ij}(\partial^2\tilde\eta)\tilde v^i_{tt}\tilde v^j_{tt},_\alpha\,]\right]_0^T .$$

By the fundamental theorem of calculus and Young's inequality, for $\delta > 0$,

$$\begin{aligned}&\int_\Gamma [P^\alpha_{ij}(\partial^2\tilde\eta)\tilde v^i_{tt}\tilde v^j_{tt},_\alpha\,](T)\\ &\quad= \int_\Gamma [P^\alpha_{ij}(\partial^2\tilde\eta)\tilde v^i_{tt}](0)\, \tilde v^j_{tt},_\alpha(T) + \int_\Gamma \int_0^T [P^\alpha_{ij}(\partial^2\tilde\eta)\tilde v^i_{tt}]_t\, dt\, \tilde v^j_{tt},_\alpha(T)\\ &\quad\le M_0(\delta) + \delta \|\tilde v_{tt}(T)\|^2_{1.5} + T\left(\int_\Gamma \sup_{t\in[0,T]} |(P^\alpha_{ij}(\partial^2\tilde\eta)\tilde v^i_{tt})_t|^2 dx\right)^{\frac12} \|\tilde v^j_{tt}(T)\|^{\frac12}_{1.5}\,.\end{aligned}$$

Since $[P^\alpha_{ij}(\partial^2\tilde\eta)\tilde v^i_{tt}]_t \in L^\infty(0,T;L^2(\Gamma))$, we conclude that

$$II \le M_0(\delta) + \delta \sup_{t\in[0,T]} E_\kappa(t) + C\,T P(\sup_{t\in[0,T]} E_\kappa(t))\,.$$

A temporal integration by parts in the third and fourth terms on the right-hand side of (12.9) yields

$$\begin{aligned}III + IV = -\int_0^T \int_\Gamma &[Q^{\alpha\beta}_{ij}(\partial\tilde\eta, \partial\tilde v)\tilde v^j_t,_\beta + Q^\alpha_i(\partial\tilde\eta,\partial\tilde v)]_t\, \tilde v^j_{tt},_\beta\\ &+ \left.\int_\Gamma [Q^{\alpha\beta}_{ij}(\partial\tilde\eta, \partial\tilde v)\tilde v^j_t,_\beta + Q^\alpha_i(\partial\tilde\eta,\partial\tilde v)]\tilde v^i_{tt},_\alpha\right]_0^T ,\end{aligned}$$

which has the same bound as term II; it follows that

$$\begin{aligned}&\int_0^T \int_\Gamma \partial_t^3(\sqrt{\tilde g}\tilde H \tilde n \circ \tilde\eta)\cdot \partial_t^3 \tilde v\\ &\quad= -\frac12 \sup_{t\in[0,T]} |\Pi \tilde v_{tt}|^2_1 + M_0(\delta) + \delta \sup_{t\in[0,T]} E_\kappa(t) + C\,T P(\sup_{t\in[0,T]} E_\kappa(t))\,.\end{aligned} \tag{12.13}$$

Remark 11. The determinant structure which appears in (12.11) is crucial in order to obtain the desired estimate. In particular, the term $\det \mathcal{A}_1$ is linear in the highest-order derivative $\partial\partial_t^2 v$ rather than quadratic (as it a priori appears).

There are three remaining boundary integral terms appearing on the right-hand side of (12.7) arising from (12.8); the terms involving κ are

(12.14)
$$-\kappa\int_0^T\left(\left[\partial_t^3(\tilde v\cdot\tilde n_\kappa),\partial_t^3\tilde v\cdot\tilde n_\kappa\right]_1+3\left[\partial_t^2(\tilde v\cdot\tilde n_\kappa),\partial_t^3\tilde v\cdot\partial_t\tilde n_\kappa\right]_1\right.$$
$$\left.+3\left[\partial_t(\tilde v\cdot\tilde n_\kappa),\partial_t^3\tilde v\cdot\partial_t^2\tilde n_\kappa\right]_1+\left[\tilde v\cdot\tilde n_\kappa,\partial_t^3\tilde v\cdot\partial_t^3\tilde n_\kappa\right]_1\right)\,.$$

The first term in (12.14) provides both the energy contribution $\int_0^T|\sqrt{\kappa}\partial_t^3\tilde v\cdot\tilde n_\kappa|_1^2$ as well as error terms. We start the analysis with the most difficult error term,

$$\kappa\int_0^T[\tilde v\cdot\partial\partial_t^3\tilde n_\kappa,\partial(\tilde n_\kappa\cdot\partial_t^3\tilde v)]_0, \tag{12.15}$$

whose highest-order contribution has an integrand (modulo L^∞ terms) of the form $\partial^2\sqrt{\kappa}\tilde v_{\kappa tt}\ \sqrt{\kappa}\partial\tilde v_{ttt}$.

With $\tilde n_\kappa=(\partial_1\tilde\eta_\kappa\times\partial_2\tilde\eta_\kappa)/\sqrt{g}=:Q(\partial\tilde\eta_\kappa)$, Q given by (11.2), the highest-order term in $\partial\partial_t^3\tilde n_\kappa$ is $Q(\partial\tilde\eta_\kappa)\partial^2\partial_t^2\tilde v_\kappa$, so that with $\mathcal{R}_1$ denoting a lower-order remainder term, and using (3.3b), we have that

$$\begin{aligned}
&-\kappa\int_0^T[\tilde v\cdot\partial\partial_t^3\tilde n_\kappa,\partial(\tilde n_\kappa\cdot\partial_t^3\tilde v)]_0\\
&\quad\le C\sup_{t\in[0,T]}|P(\tilde v,\partial\tilde\eta_\kappa)|_{L^\infty}\int_0^T|\sqrt{\kappa}\partial_t^3\tilde v\cdot\tilde n_\kappa|_1\,|\sqrt{\kappa}\partial_t^2\tilde v_\kappa|_2+\mathcal{R}_1\\
&\quad\le C\sup_{t\in[0,T]}|P(\tilde v,\partial\tilde\eta_\kappa)|_{L^\infty}|\sqrt{\kappa}\partial_t^3\tilde v\cdot\tilde n_\kappa|_{L^2(0,T;H^1(\Gamma))}|\sqrt{\kappa}\partial_t^2\tilde v_\kappa|_{L^2(0,T;H^2(\Gamma))}+\mathcal{R}_1\\
&\quad\le C_\delta\left[\sup_{t\in[0,T]}|P(\tilde v,\partial\tilde\eta_\kappa)|_{L^\infty}\|\sqrt{\kappa}\tilde v_{\kappa tt}\|_{L^2(0,T;H^{2.5}(\Omega))}\right]^2\\
&\qquad+\delta|\sqrt{\kappa}\tilde v_{ttt}\cdot\tilde n_\kappa|^2_{L^2(0,T;H^1(\Gamma))}+\mathcal{R}_1\\
&\quad\le M_0+C\,T\,P(\sup_{t\in[0,T]}E_\kappa(t))+\|\sqrt{\kappa}\tilde v_{\kappa tt}\|^4_{L^2(0,T;H^{2.5}(\Omega))}+\delta\sup_{t\in[0,T]}E_\kappa(t)\,,
\end{aligned}$$

where $\mathcal{R}_1$ also satisfies $\mathcal{R}_1\le C\,T\,P(\sup_{t\in[0,T]}E_\kappa(t))+\delta\sup_{t\in[0,T]}E_\kappa(t)$. The second term in (12.14) has a highest-order contribution with the same type of integrand, and its analysis (and bound) is identical. The third and fourth terms in (12.14) are effectively lower order by one derivative with respect to the worst case analyzed above.

Next, we estimate $\int_0^T\int_\Gamma\partial_t^3\{\tilde H\tilde n\cdot\tilde n_\kappa\,(\tilde n_\kappa-\tilde n)\}\,\partial_t^3\tilde v$. Since $\tilde n_\kappa=Q(\tilde\eta_\kappa)$ and since $|\tilde n_\kappa-\tilde n|\le\sup_\kappa|\partial Q(\partial\eta_\kappa)|\cdot|\partial\eta_\kappa-\partial\eta|$, then our assumed bounds (3.3) together with (2.6) imply that

$$|\tilde n_\kappa-\tilde n|_{L^\infty}\le C\sqrt{\kappa}\,|P(\partial\tilde\eta,\partial^2\tilde\eta)|_{L^\infty}|\tilde\eta|_{2.5}\le C\,\kappa\,P(E_\kappa(t))\,. \tag{12.16}$$

Similarly,

$$|\partial\tilde n_\kappa-\partial\tilde n|_{L^\infty}\le C\sqrt{\kappa}\,|P(\partial\tilde\eta,\partial^2\tilde\eta)|_{L^\infty}|\tilde\eta|_{3.5}\,. \tag{12.17}$$

Also by (2.6), for $k=1,2,3$,

$$|\partial_t^k\tilde n_\kappa-\partial_t^k\tilde n|_{L^\infty}\le C\sqrt{\kappa}\,|P(\partial\tilde\eta,\partial^2\tilde\eta)|_{L^\infty}\,|\partial_t^{k-1}\tilde v|_{2.5}\,, \tag{12.18}$$

and

$$|\partial_t^2 \tilde{v}_\kappa - \partial_t^2 \tilde{v}|_0 \le C\sqrt{\kappa}\,|\tilde{v}_{tt}|_{1.5}\,. \tag{12.19}$$

Taking three time derivatives of formula (11.1), we see that the highest-order term in $\partial_t^3(\sqrt{\tilde{g}}\tilde{H}\tilde{n})$ is $Q(\partial\tilde{\eta})\partial^2\tilde{v}_{tt}$. Thus, the highest-order term in the integral

$$\int_0^T \int_\Gamma \partial_t^3(\sqrt{\tilde{g}}\tilde{H}\tilde{n})^s \tilde{n}_\kappa^s\ (\tilde{n}_\kappa^r - \tilde{n}^r)\partial_t^3 \tilde{v}^r$$

is estimated using an integration by parts in space. The highest derivative count occurs when the tangential derivative is moved onto the $\tilde{v}_{ttt}$ term giving us

$$\begin{aligned}\int_0^T \int_\Gamma Q_i^s(\partial\tilde{\eta})\partial\tilde{v}_{tt}^i \tilde{n}_\kappa^s\ (\tilde{n}_\kappa^r - \tilde{n}^r)\partial\tilde{v}_{ttt}^r &\le C\int_0^T |P(\partial\tilde{\eta})|_{L^\infty}\,|\tilde{v}_{tt}|_1\,|\tilde{n}_\kappa - \tilde{n}|_{L^\infty}\,|\partial_t^3\tilde{v}|_1\\ &\le C\int_0^T |P(\partial\tilde{\eta},\partial\tilde{\eta})|_{L^\infty}\,|\tilde{\eta}|_{2.5}\,|\tilde{v}_{tt}|_1\,|\sqrt{\kappa}\tilde{v}_{ttt}|_1\\ &\le C_\delta\, T\, P(\sup_{t\in[0,T]} E_\kappa(t)) + \delta\int_0^T |\sqrt{\kappa}\tilde{v}_{ttt}|_1^2\,,\end{aligned}$$

where (12.17) is used for the second inequality. If, instead, integration by parts places the tangential derivative on $\tilde{n}_\kappa - \tilde{n}$, then (12.17) provides the same estimate for this term. The other terms are clearly lower-order.

Thanks to (12.18),

$$\begin{aligned}&\int_0^T \int_\Gamma \partial_t(\sqrt{\tilde{g}}\tilde{H}\tilde{n}^s)\partial_t^2\{\tilde{n}_\kappa^s\,(\tilde{n}_\kappa^r - \tilde{n}^r)\}\,\partial_t^3\tilde{v}^r\\ &\quad\le C\int_0^T |P(\partial\tilde{\eta},\partial^2\tilde{\eta})|_{L^\infty}\,|\partial^2\tilde{v}|_0\,|\tilde{v}_t|_{2.5}\,|\sqrt{\kappa}\tilde{v}_{ttt}|_0\\ &\quad\le C_\delta\, T\, P(\sup_{t\in[0,T]} E_\kappa(t)) + \delta \sup_{t\in[0,T]} E_\kappa(t)\,.\end{aligned}$$

We next consider the integral

$$\begin{aligned}&\int_0^T \int_\Gamma \sqrt{\tilde{g}}\tilde{H}\tilde{n}^i\partial_t^3\{\tilde{n}_\kappa^i\,(\tilde{n}_\kappa^r - \tilde{n}^r)\}\,\partial_t^3\tilde{v}^r\\ &\quad= \int_0^T \int_\Gamma \sqrt{\tilde{g}}\tilde{H}\tilde{n}\cdot\partial_t^3\tilde{n}_\kappa\ (\tilde{n}_\kappa - \tilde{n})\cdot\tilde{v}_{ttt}\\ &\qquad+ \int_0^T \int_\Gamma \sqrt{\tilde{g}}\tilde{H}\tilde{n}\cdot\tilde{n}_\kappa\ \partial_t^3(\tilde{n}_\kappa - \tilde{n})\cdot\tilde{v}_{ttt} + \mathcal{R}_2\\ &\quad=: I + II + \mathcal{R}_2\,,\end{aligned} \tag{12.20}$$

where $\mathcal{R}_2$ is a lower-order term. For term I, we use the estimate $|\tilde{n}_\kappa - \tilde{n}|_{L^\infty} \le C\,\kappa|\tilde{\eta}|_{3.5}$. One $\sqrt{\kappa}$ goes with $\partial_t^3\tilde{n}_\kappa$, and the other $\sqrt{\kappa}$ goes with $\tilde{v}_{ttt}$. Thus, $|I| \le C\,T\,P(\sup_{t\in[0,T]} E_\kappa(t)) + \delta\sup_{t\in[0,T]} E_\kappa(t)$.

To study II, we set $f = \sqrt{\tilde{g}}\tilde{H}\tilde{n}\cdot\tilde{n}_\kappa$, and consider the term $\int_0^T\int_\Gamma f\,\tilde{n}_{ttt}\cdot\tilde{v}_{ttt}$. We expand v into its normal and tangential components: set $\tau_\alpha = \tilde{\eta},_\alpha$, so that

$$\tilde{v} = v^\tau\,\tau + v^n\,\tilde{n},\ \text{ where } v^\tau\,\tau = (\tilde{v}\cdot\tau_\alpha)\,\tau_\alpha\ \text{ and } v^n = \tilde{v}\cdot\tilde{n}\,.$$

Then

$$\tilde{v}_{ttt} = v_{ttt}^\tau\tau + 3v_{tt}^\tau\tau_t + 3v_t^\tau\tau_{tt} + v^\tau\tau_{ttt} + v_{ttt}^n\tilde{n} + 3v_{tt}^n\tilde{n}_t + 3v_t^n\tilde{n}_{tt} + v^n\tilde{n}_{ttt}\,.$$

The most difficult term to estimate comes from the term $v^\tau_{ttt}\tau$, which gives the integral $\int_0^T\int_\Gamma f\,\tilde n_{ttt}\cdot\tau\, v^\tau_{ttt}$.

First, notice that $\tilde n_{ttt}\cdot\tau$ is equal to $-\tilde n\cdot\tau_{ttt}$, plus lower order terms that have at most two time derivatives on either $\tilde n$ or τ, and $\tilde n\cdot\tau_{ttt}=\tilde n\cdot\partial_\beta\tilde v_{tt}$ for $\beta=1$ or 2. Next, the κ problem states that $\tilde v^i_{ttt}=(A^k_i\tilde q,_k)_{tt}$, where we recall that

$$A=\tilde a_\kappa\,.$$

We have the formula

$$\tilde\eta^i,_\beta\,A^k_i\partial_k\tilde q_{tt}=\mathcal J^\beta\partial_\beta\tilde q_{tt}\ (\text{no sum on }\beta)\quad\text{ for }\ \beta=1,2\,,$$

where

$$\mathcal J^1=\tilde\eta^i,_1\ A^1_i\,,\qquad \mathcal J^2=\tilde\eta^i,_2\ A^2_i\,.$$

(In the case that $\kappa=0$, $\mathcal J^\beta=J=1$.) Using this, we see that the highest-order term in our integral is given by

$$\int_0^T\int_\Gamma \mathcal J f\,(\tilde n\cdot\partial_\beta\tilde v_{tt})\partial_\beta\tilde q_{tt}\,. \tag{12.21}$$

Second, write $\tilde q_{tt}$ as

$$\tilde q_{tt}=-\left[\frac{\sqrt{\tilde g}}{\sqrt{\tilde g_\kappa}}\left[\Delta_{\tilde g}(\tilde\eta)\cdot\tilde n+\Delta_{\tilde g}(\tilde\eta)\cdot(\tilde n_\kappa-\tilde n)\right]+\kappa\Delta_0(\tilde v\cdot\tilde n_\kappa)\right]_{tt}\,. \tag{12.22}$$

We begin by substituting the first term on the right-hand side of (12.22) into (12.21); the highest-order contribution comes from $\partial_\beta\partial_t^2\Delta_{\tilde g}(\tilde\eta)=Q(\partial\tilde\eta,\partial\tilde\eta_\kappa)\tilde g^{\mu\nu}\tilde n\cdot\partial_t\tilde v,_{\mu\nu\beta}$. Integrating by parts with respect to ∂_ν, the highest-order term in our integral is given by

$$\int_0^T\int_\Gamma Q(\partial\tilde\eta,\partial\tilde\eta_\kappa)\,f\,(\tilde n\cdot\tilde v_t,_{\mu\beta}\,)\,\tilde g^{\mu\nu}\,(\tilde n\cdot\tilde v_{tt},_{\nu\beta}\,)\,.$$

Letting $G^{\mu\nu}_{ij}:=Q(\partial\tilde\eta,\partial\tilde\eta_\kappa)\,f\,\tilde n_i\,\tilde n_j=Q(\partial\tilde\eta,\partial\tilde\eta_\kappa)\partial^2\eta$, integration by parts in time yields

(12.23)

$$\begin{aligned}
-\int_0^T\int_\Gamma\partial_t G^{\mu\nu}_{ij}\ \ \tilde v^j_t,_{\nu\beta}\ \tilde v^i_t,_{\mu\beta}+\left[\int_\Gamma\partial_t G^{\mu\nu}_{ij}\ \ \tilde v^j_t,_{\nu\beta}\ \tilde v^i_t,_{\mu\beta}\right]_0^T&\\
\le C\,TP(\sup_{t\in[0,T]}E_\kappa(t))+M_0+C\sup_{t\in[0,T]}|G_t|_{L^\infty}\,\|\tilde v_t\|^2_{2.5}&\\
\le M_0+C\,TP(\sup_{t\in[0,T]}E_\kappa(t))+C\sup_{t\in[0,T]}[P(\|\tilde v_t\|^2_{2.5})+P(\|\tilde v\|^2_{3.5})+P(\|\tilde\eta\|^2_{4.5})]\,.&
\end{aligned}$$

For the second term on the RHS of (12.22), the highest-order term gives the integral

$$\begin{aligned}
\int_0^T\int_\Gamma f\,Q(\partial\tilde\eta,\partial\tilde\eta_\kappa)\,(\tilde n\cdot\tilde v_{tt},_\beta\,)\,\tilde g^{\mu\nu}\Pi\tilde v_t,_{\mu\nu\beta}\cdot(\tilde n_\kappa-\tilde n)&\\
\le C\,T\,P(\sup_{t\in[0,T]}E_\kappa(t))+\delta\int_0^T\|\sqrt{\kappa}\tilde v_t\|^2_{3.5},&
\end{aligned}$$

where we used $|\tilde n_\kappa-\tilde n|_{L^\infty}<C\,\kappa|\tilde\eta|_{3.5}$ again.

For the third term on the RHS of (12.22), the highest-order term gives the integral

$$\kappa \int_0^T \int_\Gamma f\, Q(\partial\tilde\eta, \partial\tilde\eta_\kappa)\, (\tilde n \cdot \tilde\partial^2 v_{tt})\, (\tilde n_\kappa \cdot \partial^2 \tilde v_{tt}) \\ \le M_0 + C\,T\,P(\sup_{t\in[0,T]} E_\kappa(t)) + \|\sqrt{\kappa}\tilde v_{tt}\|^4_{L^2(0,T;H^{2.5}(\Omega)}\,.$$

We have thus estimated the integral $\int_0^T \int_\Gamma \partial_t^3 \{\tilde H \tilde n \cdot \tilde n_\kappa\, (\tilde n_\kappa - \tilde n)\}\, \partial_t^3 \tilde v$. The remaining integral $\int_0^T \int_\Gamma \partial_t^3 \{\tilde H \tilde n \cdot (\tilde n_\kappa - \tilde n)\, \tilde n_\kappa\}\, \partial_t^3 \tilde v$ has the same bound.

Step 2. The pressure term. We next consider the pressure term in (12.7):

$$-\int_0^T \int_\Omega \partial_t^3 (A_i^k \tilde q) \partial_t^3 \tilde v^i,_k = -\int_0^T \int_\Omega \partial_t^3 \tilde v^i,_k \left[\partial_t^3 A_i^k \tilde q + 3\partial_t^2 A_i^k\, \tilde q_t + 3\partial_t A_i^k\, \tilde q_{tt} + A_i^k \partial_t^3 \tilde q \right] \\ =: I + II + III + IV\,. \tag{12.24}$$

We record the following identities:

$$\partial_t A_i^k = \tilde J_\kappa^{-1} (A_r^s A_i^k - A_r^k A_i^s) \tilde v_\kappa^r,_s\ , \tag{12.25a}$$

$$\partial_t^2 A_i^k = \tilde J_\kappa^{-1} (A_r^s A_i^k - A_r^k A_i^s) \partial_t \tilde v_\kappa^r,_s + P_i^k (\tilde J_\kappa^{-1}, A, \nabla \tilde v_\kappa)\,, \tag{12.25b}$$

$$\partial_t^3 A_i^k = \tilde J_\kappa^{-1} (A_r^s A_i^k - A_r^k A_i^s) \partial_t^2 \tilde v_\kappa^r,_s + P_j^k (\tilde J_\kappa^{-1}, A, \nabla \tilde v_\kappa) \partial_t \tilde v_\kappa^j,_i\ . \tag{12.25c}$$

With (12.25c) and $f_{ri}^{sk} := \tilde J_\kappa^{-1} (A_r^s A_i^k - A_r^k A_i^s) \tilde q$, term I is written as

$$I = \int_0^T \int_\Omega [f_{ri}^{sk} \partial_t^3 \tilde v^i,_k\, \partial_t^2 \tilde v_\kappa^r,_s + \partial_t^3 \tilde v^i,_k\, \partial_t \tilde v_\kappa^j,_i\, P_j^k(\tilde J_\kappa^{-1}, A, \nabla \tilde v_\kappa)] \\ =: I_a + I_b\,.$$

We fix a chart θ_l in a neighborhood of the boundary Γ and let $\xi = \sqrt{\alpha_l}$, where once again, we remind the reader that $\{\alpha_i\}_{i=1}^L$ denotes the partition of unity associated to the charts $\{\theta\}_{l=1}^L$. With $\mathcal{I}_a$ denoting the restriction $I_a|_{U_l}$, where $U_l \cap \Omega = \theta_l((0,1)^3)$, and letting $\rho := \rho_{\frac{1}{\kappa}}$ and $\theta := \theta_l$, we have that

$$\mathcal{I}_a = \int_0^T \int_{(0,1)^3} f_{ri}^{sk}(\theta) \partial_t^3 \tilde v^i,_k(\theta)\, \xi\, \rho \star_h \rho \star_h \xi \partial_t^2 \tilde v^r,_s(\theta) + f(\theta) \nabla \tilde v_{ttt} G(\xi, \nabla\xi) \tilde v_{tt} \\ =: \mathcal{I}_{a1} + \mathcal{I}_{a2}\,,$$

where $G(\xi, \nabla\xi)$ is a bilinear function which arises when the gradient acts on ξ rather than $\tilde v_{tt}$. The term $\mathcal{I}_{a1}$ is the difficult term which requires forming an exact derivative, and this, in turn, requires commuting the convolution with f_{ri}^{sk}. We let

$$V(\theta) = \rho \star_h \xi \tilde v(\theta)$$

so that using the symmetry property (2.7), we see that

$$\mathcal{I}_{a1} = \int_0^T \int_{(0,1)^3} \rho \star_h [f_{ri}^{sk} \xi \partial_t^3 \tilde v^i,_k(\theta)] \partial_t^2 V^r,_s(\theta) \\ = \int_0^T \int_{(0,1)^3} \big[f_{ri}^{sk} \partial_t^3 V^i,_k(\theta)] \partial_t^2 V^r,_s(\theta) \\ + (\rho \star_h [f_{ri}^{sk} \xi \partial_t^3 \tilde v^i,_k] - f_{ri}^{sk} \rho \star_h [\xi \partial_t^3 \tilde v^i,_k])\, V_{tt}^r,_s \big] \\ =: \mathcal{I}_{a1_i} + \mathcal{I}_{a1_{ii}}\,.$$

Since f_{ri}^{sk} is symmetric with respect to i and r and k and s, we see that

$$\mathcal{I}_{a1_i} = \frac{1}{2}\int_0^T \int_{(0,1)^3} f_{ri}^{sk}(\theta)\partial_t\left[\partial_t^2 V^i,_k(\theta)\partial_t^2 V^r,_s(\theta)\right]$$
$$= -\frac{1}{2}\int_0^T \int_{(0,1)^3} \partial_t f_{ri}^{sk}(\theta)\partial_t^2 V^i,_k(\theta)\partial_t^2 V^r,_s(\theta) + \frac{1}{2}\int_{(0,1)^3} f_{ri}^{sk}(\theta)$$
$$\partial_t^2 V^i,_k(\theta)\partial_t^2 V^r,_s(\theta)\Big]_0^T .$$

We sum over our patch index $l = 1, \ldots, L$. The spacetime integral is bounded by $CTP(\sup_{t\in[0,T]} E_\kappa(t))$. For the space integral at time $t = T$, we employ the fundamental theorem of calculus:

$$\int_\Omega [f_{ri}^{sk} V_{tt}^i,_k V_{tt}^r,_s](T) = \int_\Omega V_{tt}^i,_k(T)V_{tt}^r,_s(T) f_{ri}^{sk}(0) + \int_\Omega V_{tt}^i,_k(T)V_{tt}^r,_s(T)\int_0^T \partial_t f_{ri}^{sk}$$
$$\le \|V_{tt}(T)\|_1^2\,\|\tilde{q}_0\|_2 + \|V_{tt}(T)\|_1^2\,\|\int_0^T |f_t\|_{L^\infty}$$
$$\le \|v_{tt}(T)\|_{1.5}^{4/3}\,\|v_{tt}(T)\|_0^{2/3}\,\|\tilde{q}_0\|_2 + CTP(\sup_{t\in[0,T]} E_\kappa(t))$$
$$\le \frac{\delta}{2}\|v_{tt}(T)\|_{1.5}^2 + C(\delta)\|v_{tt}(T)\|_0^2\,\|\tilde{q}_0\|_2^3 + CTP(\sup_{t\in[0,T]} E_\kappa(t))$$
$$\le \delta \sup_{t\in[0,T]} E_\kappa(t) + M_0(\delta) + CTP(\sup_{t\in[0,T]} E_\kappa(t))\,,$$

where Young's inequality has been used. For $\mathcal{I}_{a1_{ii}}$, the commutation Lemma 2.1 shows that

$$\mathcal{I}_{a1_{ii}} \le C\kappa^{\frac{1}{2}}\int_0^T \|f\|_3\,\|\sqrt{\kappa}\tilde{v}_{ttt}\|_1\,\|\tilde{v}_{tt}\|_1$$
$$\le \delta\int_0^T \|\sqrt{\kappa}\tilde{v}_{ttt}\|_1^2 + C_\delta\int_0^T \|f\|_3^2\,\|\tilde{v}_{tt}\|_1^2 .$$

Summing over $l = 1, ..., L$, we integrate by parts in time and write the term $\mathcal{I}_{a2}$ as

$$\mathcal{I}_{a2} = -\int_0^T \int_\Omega f\nabla\tilde{v}_{tt}G(\xi,\nabla\xi)(f\tilde{v}_{tt})_t + \int_\Omega f\nabla\tilde{v}_{tt}G(\xi,\nabla\xi)f\tilde{v}_{tt}\Big]_0^T .$$

This is estimated in the same way as the term $\mathcal{I}_{a1_i}$. The term I_b is handled in the identical fashion with the same bound. Thus, we have shown that

$$I \le \delta \sup_{t\in[0,T]} E_\kappa(t) + M_0(\delta) + CTP(\sup_{t\in[0,T]} E_\kappa(t)) .$$

Using (12.25b) for term II, integration by parts in time gives the identical bound as for term I. For term III, a different approach is employed; we use (12.25a) and integration by parts in space rather than time, and let $F_{ri}^{sk} := 3\tilde{J}_\kappa^{-1}(A_r^s A_i^k - A_r^k A_i^s)$ to find that

$$III = -\int_0^T \int_\Omega \tilde{v}_{ttt}^i[\tilde{v}_\kappa^r,_s F_{ri}^{sk} q_{tt}],_k + \int_0^T \int_\Gamma \tilde{v}_{ttt}^i F_{ri}^{sk} N_k q_{tt}\tilde{v}_\kappa^r,_s .$$

The Cauchy-Schwarz inequality together with the pressure estimate (12.1) give the bound $CTP(\sup_{t\in[0,T]} E_\kappa(t))$ for the first term on the right-hand side. The boundary integral term requires integration by parts in time:

$$\int_0^T \int_\Gamma \tilde v^i_{ttt} F^{sk}_{ri} N_k \tilde q_{tt} \tilde v^r_\kappa,_s = \int_\Gamma \tilde v^i_{tt} F^{sk}_{ri} N_k \tilde q_{tt} \tilde v^r_\kappa,_s \Big]_0^T - \int_0^T \int_\Gamma v^i_{tt} [F^{sk}_{ri} N_k \tilde q_{tt} \tilde v^r_\kappa,_s]_t$$
$$=: III_a + III_b\,.$$

First, note that

$$\begin{aligned}\tilde q_{tt} F^{sk}_{ri} N_k &= 3\tilde J_\kappa^{-1} A^s_r \left[(\tilde q A^k_i N_k)_{tt} - 2\tilde q_t \partial_t A^k_i N_k - \tilde q \partial_t^2 A^k_i N_k\right]\\ &\quad - 3\tilde J_\kappa^{-1} A^s_i \left[(\tilde q A^k_r N_k)_{tt} - 2\tilde q_t \partial_t A^k_r N_k - \tilde q \partial_t^2 A^k_r N_k\right]\,.\end{aligned} \tag{12.26}$$

Next, substitute the boundary condition (12.8), written as

$$\begin{aligned}-\tilde q A^k_i N_k &= \sqrt{\tilde g}\Delta_{\tilde g}(\tilde\eta^j)\,[\delta_{ji} + ((\tilde n_\kappa)_j - \tilde n_j)\tilde n_i + \tilde n_j((\tilde n_\kappa)_i - \tilde n_i)]\\ &\quad + \kappa(\sqrt{g_0} g_0^{\alpha\beta}[\tilde v\cdot \tilde n],_\beta),_\alpha \tilde n_i\,,\end{aligned} \tag{12.27}$$

into (12.26). The two bracketed terms in (12.26) are essentially the same, so it suffices to analyze just the first term. We begin by considering the term $\sqrt{\tilde g}\Delta_{\tilde g}(\tilde \eta^i)$ in (12.27).

Then III_a can be written as

$$\begin{aligned}III_a &= 3\int_\Gamma \tilde J_\kappa^{-1}\left\{\partial_t^2(\sqrt{\tilde g}\tilde g^{\alpha\beta}\tilde\eta^i,_\beta),_\alpha \tilde v^r_\kappa,_s A^s_r - 2q_t\partial_t A^k_i N_i \tilde v^r_\kappa,_s - q\partial_t^2 A^k_i N_i \tilde v^r_\kappa,_s \right\}\tilde v^i_{tt}\Big]_0^T\\ &= 3\int_\Gamma (\sqrt{\tilde g}\tilde g^{\alpha\beta}\tilde \eta^i,_\beta)_{tt}(\tilde J_\kappa^{-1}\tilde v^r_\kappa,_s A^s_r \tilde v^i_{tt}),_\alpha\\ &\qquad - 2\tilde J_\kappa^{-1} q_t \partial_t A^k_i N_i \tilde v^r_\kappa,_s \tilde v^i_{tt} - \tilde J_\kappa^{-1} q\partial_t^2 A^k_i N_i \tilde v^r_\kappa,_s \tilde v^i_{tt}\ ]_0^T\\ &\le \delta \sup_{t\in[0,T]} E_\kappa(t) + M_0(\delta) + CTP(\sup_{t\in[0,T]} E_\kappa(t))\,,\end{aligned} \tag{12.28}$$

the last inequality following from the fundamental theorem of calculus and the same argument we have used above.

In order to estimate III_b, because we do not have a trace estimate for $\partial_t^3 A$, we let $Q_r(\partial\tilde\eta_\kappa) := \sqrt{\tilde g_\kappa}(\tilde n_\kappa)_r$ and compute

$$\begin{aligned}&\partial_t Q_r = Q^\alpha_{ri}\tilde v^i_\kappa,_\alpha\,, \quad \partial_t^2 Q_r = Q^{\alpha\beta}_{rij}\tilde v^i_\kappa,_\alpha \tilde v^j_\kappa,_\beta + Q^\alpha_{ri}\partial_t \tilde v^i_\kappa,_\alpha\,,\\ &\partial_t^3 Q_r = Q^{\alpha\beta\gamma}_{rijk}\tilde v^i_\kappa,_\alpha \tilde v^j_\kappa,_\beta \tilde v^k_\kappa,_j + 3Q^{\alpha\beta}_{rij}\tilde v^i_\kappa,_\alpha \partial_t\tilde v^j_\kappa,_\beta + Q^\alpha_{ri}\partial_t^2\tilde v^i_\kappa,_\alpha\,.\end{aligned} \tag{12.29}$$

Since $\sqrt{\tilde g_\kappa}(\tilde n_\kappa)_r\tilde q_{tt} = (\sqrt{\tilde g_\kappa}(\tilde n_\kappa)_r\tilde q)_{tt} - 2\partial_t Q_r\,\tilde q_t - \partial_t^2 Q_r\,\tilde q$, it follows that

$$\begin{aligned}III_b = -3\int_0^T\int_\Gamma \tilde J_\kappa^{-1}\Big[&\tilde v^r_{tt}(\sqrt{\tilde g_\kappa}(\tilde n_\kappa)_r\tilde q)_{ttt}\tilde v^i_\kappa,_s A^s_i + \tilde v^r_{tt}(Q_r\tilde q)_{tt}(\tilde v^i_\kappa,_s A^s_i)_t\\ &-\tilde v^r_{tt}(2Q^\alpha_{rj}\tilde v^j_\kappa,_\alpha \tilde v^i_\kappa,_s A^s_i\tilde q_t + Q^{\alpha\beta}_{rlj}\tilde v^l_\kappa,_\alpha \tilde v^j_\kappa,_\beta \tilde v^i_\kappa,_s A^s_i \tilde q + Q^\alpha_{rl}\partial_t\tilde v^l_\kappa,_\alpha \tilde v^i_\kappa,_s A^s_i\tilde q)_t\Big]\,.\end{aligned} \tag{12.30}$$

Using the pressure estimates and by definition of our energy function, for $t\in(0,T)$,

$$\tilde q_{tt}(t)\in H^{0.5}(\Gamma), \quad Q_r(t), \partial_t Q_r(t)\in L^\infty(\Gamma), \quad \partial_t^3 Q_r(t)\in L^2(\Gamma),$$
$$\partial\tilde v_t(t)\in H^{1.5}(\Gamma), \quad \partial\tilde v_{tt}(t)\in L^2(\Gamma)\,.$$

Thus, all of the terms, except the first, on right-hand side of (12.30) can be easily bounded by $CTP(\sup_{t\in[0,T]} E_\kappa(t))$. Integrating by parts in space, the first term in (12.30) has the following estimate:

(12.31)

$$\begin{aligned}3\int_0^T\int_\Gamma & \sqrt{\tilde g_\kappa}\tilde g_\kappa^{\alpha\beta}\Pi^r_k \tilde v^k_{tt},_\beta (\tilde v^r_{tt}\tilde v^i_\kappa,_s J_\kappa^{-1}A^s_i),_\alpha \\ & + \sqrt{\tilde g_\kappa}(\tilde g_\kappa^{\mu\nu} g^{\alpha\beta} - g^{\alpha\nu}g^{\mu\beta})\tilde\eta^j,_\nu \tilde v^j_{tt},_\mu \tilde\eta^r,_\beta (\tilde v^r_{tt}\tilde v^i_\kappa,_s \tilde J_\kappa^{-1}A^s_i),_\alpha \\ & + [P^{\alpha\beta}_{ij}(\partial\tilde\eta,\partial\tilde v)\tilde v^j_{tt},_\beta + P^\alpha_i(\partial\tilde\eta,\partial\tilde v)](\tilde v^r_{tt}\tilde v^i_\kappa,_s \tilde J_\kappa^{-1}A^s_i),_\alpha \le CTP(\sup_{t\in[0,T]} E_\kappa(t))\,.\end{aligned}$$

The remaining three terms in the boundary condition (12.27) are now considered. The additional integrals which arise in the (12.30) are given by

$$\begin{aligned}-3\int_0^T\int_\Gamma \partial_t^3 \Big\{ & \sqrt{\tilde g}\tilde H\tilde n\cdot(\tilde n_\kappa - \tilde n)\,\tilde n_r + \sqrt{\tilde g}\tilde H\tilde n\cdot\tilde n_\kappa\,((\tilde n_\kappa)_r - \tilde n_r) \\ & + \kappa\sqrt{\tilde g_\kappa}\Delta_{\bar 0}(\tilde v\cdot\tilde n_\kappa)\tilde n_\kappa\Big\}\ \tilde v^r,_s \tilde a^s_i \tilde v^i_{tt} \\ =: {} & J_1 + J_2 + J_3\,.\end{aligned}$$

Term J_3 with the artificial viscosity provides the integral $\kappa\int_0^T[\partial_t^3(\tilde v\cdot\tilde n_\kappa), P(\nabla\tilde\eta,\nabla\tilde v)\cdot\partial_t^2\tilde v]_1$. The highest-order terms in this integral are estimated as

$$\begin{aligned}\kappa\int_0^T & \Big\{[P^{\alpha\beta}_{ij}(\partial\tilde n_\kappa,\nabla\tilde\eta,\nabla\tilde v)\,\partial_t^3\tilde v^i_{,\beta}\,,\partial_t^2\tilde v^j_{,\alpha}]_0 + [P^{\alpha\beta\gamma}_{ij}(\partial\tilde n_\kappa,\nabla\tilde\eta,\tilde v,\nabla\tilde v)\,\partial_t^2\tilde v^i_{\kappa,\beta\gamma}\,,\partial_t^2\tilde v^j_{,\alpha}]_0\Big\} \\ & \le C\sqrt\kappa\int_0^T \|P(\nabla\tilde\eta,\nabla\tilde v)\|_{L^\infty}\,|\partial_t^2\tilde v|_1\,\{|\sqrt\kappa\partial_t^3\tilde v|_1 + |\sqrt\kappa\partial_t^2\tilde v_\kappa|_2\} \\ & \le C(\delta)\int_0^T \|P(\nabla\tilde\eta,\nabla\tilde v)\|_{L^\infty}\,|\partial_t^2\tilde v|_1^2 + \kappa\delta\int_0^T \left(|\sqrt\kappa\partial_t^3\tilde v|_1^2 + |\sqrt\kappa\partial_t^2\tilde v_\kappa|_2^2\right) \\ & \le C(\delta)\,T\,P(\sup_{t\in[0,T]} E_\kappa(t)) + \delta\sup_{t\in[0,T]} E_\kappa(t)\,.\end{aligned}$$

The lower-order terms in J_3 also have the same bound. As to terms J_1 and J_2, the estimates for the terms with $(\sqrt{\tilde g}\tilde H\tilde n)_{ttt}$ are obtained exactly as in (12.31). For the terms that contain $\tilde n_{ttt}$, we use the formula (12.29) for the third time derivative of the unit normal; it immediately follows that terms J_1 and J_2 are also bounded by $\delta\sup_{t\in[0,T]} E_\kappa(t) + CTP(\sup_{t\in[0,T]} E_\kappa(t))$.

Now we need only consider the additional terms in (12.28) from the remaining three terms in the boundary condition (12.27). The only novelty is in the highest-order integral coming from integration by parts in space in the κ term:

$$\kappa\int_\gamma P^{\alpha\beta}_{ij}(T) v^i_{tt},_\alpha(T) v^j_{tt},_\beta(T)$$

where $P^{\alpha\beta}_{ij}(t)$ and $\partial_t P^{\alpha\beta}_{ij}(t)$ are both in $L^\infty(\Gamma)$ for each $t\in[0,T]$. Using the fundamental theorem of calculus and the fact that $\sqrt\kappa v_{ttt}\in L^2(0,T;H^{1.5}(\Omega))$ together with Jensen's inequality shows that this term is bounded by $M_0(\delta) + \delta\sup_{t\in[0,T]} E_\kappa(t) + CTP(\sup_{t\in[0,T]} E_\kappa(t))$.

To study term IV, we use (12.25) together with the incompressibility condition $(\tilde{v}^i,_k A_i^k)_{ttt} = 0$ to find that

$$IV = -\int_0^T \int_\Omega [(3\tilde{v}^i_{tt},_k \partial_t A_i^k + \tilde{v}^i,_k \partial_t^3 A_i^k)\tilde{q}_{ttt} + 3\tilde{v}^i_t,_k \partial_t^2 A_i^k \tilde{q}_{ttt}] =: IV_a + IV_b\,.$$

For IV_b, we integrate by parts in time:

$$IV_b = \int_0^T \int_\Omega 3(\tilde{v}^i_t,_k \partial_t^2 A_i^k)_t \tilde{q}_{tt} - \int_\Omega 3\tilde{v}^i_t,_k \partial_t^2 A_i^k \tilde{q}_{tt}\Big]_0^T\,.$$

Since $\partial_t^3 A$ is bounded in $H^{0.5}(\Omega)$, the spacetime integral is easily bounded by $CTP(\sup_{t\in[0,T]} E_\kappa(t))$; meanwhile, the remaining space integral satisfies

$$\begin{aligned} IV_{b2} &\le 3\int_\Omega [\tilde{v}^i_t,_k (0)\partial_t^2 A_i^k(0)] q_{tt}(T) + \int_\Omega \int_0^T [\tilde{v}^i_t,_k \partial_t^2 A_i^k]_t \, dt \; q_{tt}(T) + M_0 \\ &\le \delta \|q_{tt}(T)\|_0^2 + M_0(\delta) + T \sup_{t\in[0,T]} \|(\tilde{v}^i_t,_k \partial_t^2 A_i^k)_t\|_0 \, \|q_{tt}(T)\|_0 \\ &\le \delta \sup_{t\in[0,T]} E_\kappa(t) + M_0(\delta) + CTP(\sup_{t\in[0,T]} E_\kappa(t))\,. \end{aligned}$$

With $F^{sk}_{ri} := \tilde{J}_\kappa^{-1}(A^s_r A^k_i - A^k_r A^s_i)$, IV_a is written as

(12.32)

$$\begin{aligned} IV_a = &-\int_0^T\int_\Omega (3\partial_t^2 \tilde{v}^r,_s F^{sk}_{ri} \tilde{v}^i_\kappa,_k + \partial_t^2 \tilde{v}^r_\kappa,_s F^{sk}_{ri} \tilde{v}^i,_k)\tilde{q}_{ttt} + \tilde{v}^i,_k P^k_j(J_\kappa^{-1}, A, \nabla v_\kappa)\partial_t \tilde{v}^j_\kappa,_i \tilde{q}_{ttt} \\ &=: IV_{a1} + IV_{a2} + IV_{a3}\,. \end{aligned}$$

Term IV_{a3} is estimated in the same way as term IV_b. For term IV_{a2}, we integrate by parts in space to find that

$$IV_{a2} = -\int_0^T \int_\Gamma \partial_t^2 \tilde{v}^r_\kappa F^{sk}_{ri} \tilde{v}^i,_k \tilde{q}_{ttt} N_s + \int_0^T \int_\Omega \partial_t^2 \tilde{v}^r_\kappa (F^{sk}_{ri} \tilde{v}^i,_k \tilde{q}_{ttt}),_s := IV_{a2i} + IV_{a2ii}.$$

The first integral IV_{a2i} is handled identically to term III_b to give the bound $CTP(\sup_{t\in[0,T]} E_\kappa(t))$. We write the second integral as

$$IV_{a2ii} = \int_0^T \int_\Omega [\partial_t^2 \tilde{v}^r_\kappa F^{sk}_{ri} \tilde{v}^i,_k \tilde{q}_{ttt},_s + \partial_t^2 \tilde{v}^r_\kappa (F^{sk}_{ri} \tilde{v}^i,_k),_s \tilde{q}_{ttt}],$$

integrate by parts in time, and obtain

$$\begin{aligned} IV_{a2ii} = &-\int_0^T \int_\Omega [(\partial_t^2 \tilde{v}^r_\kappa F^{sk}_{ri} \tilde{v}^i,_k)_t \tilde{q}_{tt},_s + \partial_t^2 \tilde{v}^r_\kappa (F^{sk}_{ri} \tilde{v}^i,_k)_t,_s \tilde{q}_{tt}] \\ &\qquad + \int_\Omega [(\partial_t^2 \tilde{v}^r_\kappa F^{sk}_{ri} \tilde{v}^i,_k)\tilde{q}_{tt},_s + \partial_t^2 \tilde{v}^r_\kappa (F^{sk}_{ri} \tilde{v}^i,_k),_s \tilde{q}_{tt}]\Big]_0^T\,. \end{aligned}$$

Since $\partial_t^3 \tilde{v}_\kappa$ is bounded in $L^2(\Omega)$ and $(F\,\nabla v)_t$ is bounded in $L^\infty(\Omega)$, the spacetime integral is bounded by $CTP(\sup_{t\in[0,T]} E_\kappa(t))$. Next, we analyze the highest-order term in the remaining temporal boundary integral in IV_{a2}:

$$\int_\Omega [(\partial_t^2 \tilde v_\kappa^r F_{ri}^{sk} \tilde v^i,_k) \tilde q_{tt},_s](T)$$
$$= \int_\Omega (\partial_t^2 \tilde v_\kappa^r(0) F_{ri}^{sk}(0) \tilde v^i,_k(0)) \tilde q_{tt},_s(T) + \int_\Omega \int_0^T (\partial_t^2 \tilde v_\kappa^r F_{ri}^{sk} \tilde v^i,_k)_t \tilde q_{tt},_s(T)$$
$$\le M_0(\delta) + \delta \|q_{tt}(T)\|_1^2 + T \sup_{t\in[0,T]} \|(\partial_t^2 \tilde v_\kappa^r F_{ri}^{sk} \tilde v^i,_k)_t\|_0 \, \|q_{tt}(T)\|_1$$
$$\le M_0(\delta) + \delta \sup_{t\in[0,T]} E_\kappa(t) + CTP(\sup_{t\in[0,T]} E_\kappa(t)) .$$

The remaining term is analyzed in the same fashion.

Step 3. The inertia term. Finally, the inertia term in (12.7) satisfies

$$\int_0^T \int_\Omega \partial_t^3(\tilde J_\kappa \tilde v_t^i) \tilde v_{ttt}^i = \frac12 \|\tilde J_\kappa^{\frac12} \tilde v_{ttt}(T)\|_0^2 - \frac12 \|\tilde v_{ttt}(0)\|_0^2$$
$$+ \int_0^T \int_\Omega \Big[2.5 \partial_t \tilde J_\kappa |\tilde v_{ttt}|^2 + 3\partial_t^2 \tilde J_\kappa \tilde v_{tt}^i \tilde v_{ttt}^i + \partial_t^3 \tilde J_\kappa \tilde v_t^i \tilde v_{ttt}^i \Big] .$$

Since $\partial_t \tilde J_\kappa = \mathrm{Trace}(\tilde a_\kappa \, \nabla \tilde v_\kappa)$, $\partial_t^2 \tilde J_\kappa = \mathrm{Trace}(\tilde a_\kappa \, \nabla \partial_t \tilde v_\kappa) + P(\tilde J_\kappa^{-1}, \tilde a_\kappa, \nabla \tilde v_\kappa)$, and $\partial_t^2 \tilde J_\kappa = \mathrm{Trace}(\tilde a_\kappa \, \nabla \partial_t^2 \tilde v_\kappa) + P(\tilde J_\kappa^{-1}, \tilde a_\kappa, \nabla \tilde v_\kappa) \, \nabla \partial_t \tilde v_\kappa$, then using condition (3.3c), we see that

$$\sup_{t\in[0,T]} \|v_{ttt}\|_0^2 \le \|\tilde v_{ttt}(0)\|_0^2 + C\,T\,P(\sup_{t\in[0,T]} E_\kappa(t)).$$

From (12.2) evaluated at $t = 0$, we see that $\|\tilde v_{ttt}(0)\|_0^2 \le M_0$, so the lemma is proved. ◻

Lemma 12.3 (Energy estimates for the second time-differentiated κ-problem)**.** *For M_0 taken as in Lemma* 10.1 *and $\delta > 0$, solutions of the κ-problem* (4.1) *satisfy*

(12.33)
$$\sup_{t\in[0,T]} |\partial_2 \tilde v_t \cdot \tilde n|_0^2 + \int_0^T |\sqrt{\kappa} \partial^2 \tilde v_{tt} \cdot \tilde n_\kappa|_0^2$$
$$\le M_0 + T\,P(\sup_{t\in[0,T]} E_\kappa(t)) + \delta \sup_{t\in[0,T]} E_\kappa(t) + C\,P(\|\sqrt{\kappa} \tilde v_t\|^2_{L^2(0,T;H^{3.5}(\Omega))}) .$$

Proof. We let $\partial\partial_t^2$ act on (4.1b) and test with $\zeta_i^2 \partial \tilde v_{tt}$ where $\zeta_i^2 = \alpha_i$, and α_i is an element of our partition of unity. This localizes the analysis to a neighborhood of the boundary Γ where the tangential derivative is well defined. In this neighborhood, we use a normal coordinate system spanned by $(\partial_1 \eta_0, \partial_2 \eta_0, N)$.

We follow the proof of Lemma 12.2 and replace ∂_t^3 with $\partial\partial_t^2$. There are only two differences between the analysis of the second and third time-differentiated problems. The first difference can be found in the analogue of term III in (12.9), which now reads $\int_0^T \int_\Gamma P(\partial \tilde\eta, \partial \tilde v) \, \partial \tilde v_t \, \partial^2 \tilde v_{tt}$. After integration by parts in space, this term is bounded by $C\,T\,P(\sup_{t\in[0,T]} E_\kappa(t))$; however, this term requires a bound on $|\tilde v_{tt}|_1$, which requires us to study the third time-differentiated problem. (Compare this with the third time-differentiated problem wherein integration by parts in time forms an exact derivative which closes the estimate.)

The second difference is significant. Because the energy function places

$$\tilde v_{ttt} \in L^\infty(0,T;L^2(\Omega)) \text{ and } \tilde v_{tt} \in L^\infty(0,T;H^{1.5}(\Omega)),$$

there is a one-half derivative improvement that accounts for (12.33) being better than (12.6).

In particular, the analogue of term II in (12.20) is $\int_0^T\int_\Gamma \sqrt{\tilde g}\tilde H \tilde n\cdot \tilde n_\kappa\, \partial\partial_t^2(\tilde n_\kappa - \tilde n)\cdot \partial\tilde v_{tt}$, and since $|\partial\partial_t^2(\tilde n_\kappa-\tilde n)|_0 \le C\,|P(\partial\eta)|_{L^\infty}\,|\tilde v_t|_2$, then this integral is easily seen to be bounded by $C\,T\,P(\sup_{t\in[0,T]} E_\kappa(t))$. (This is in sharp contrast to the difficult analysis required at the level of the third time-differentiated problem which follows equation (12.20).)

All of the other estimates follow identically the proof of Lemma 12.2 with ∂_t^3 being replaced by $\partial\partial_t^2$. □

Lemma 12.4 (Energy estimates for the time-differentiated κ-problem)**.** *For M_0 taken as in Lemma* 10.1 *and $\delta > 0$, solutions of the κ-problem* (4.1) *satisfy*

$$\sup_{t\in[0,T]} |\partial^3\tilde v\cdot\tilde n|_0^2 + \int_0^T |\sqrt{\kappa}\partial_3\tilde v_t\cdot\tilde n_\kappa|_0^2$$

(12.34)

$$\le M_0 + T\,P(\sup_{t\in[0,T]} E_\kappa(t)) + \delta \sup_{t\in[0,T]} E_\kappa(t) + C\,P(\|\sqrt{\kappa}\tilde v\|^2_{L^2(0,T;H^{4.5}(\Omega))})\,.$$

Proof. After replacing $\partial\partial_t^2$ with $\partial^2\partial_t$, the proof is the same as the proof of Lemma 12.3. □

Lemma 12.5 (Energy estimates for the κ-problem)**.** *For M_0 taken as in Lemma* 10.1 *and $\delta > 0$, solutions of the κ-problem* (4.1) *satisfy*

(12.35)

$$\sup_{t\in[0,T]} |\partial^4\tilde\eta\cdot\tilde n|_0^2 + \int_0^T |\sqrt{\kappa}\partial_4\tilde v\cdot\tilde n_\kappa|_0^2 \le M_0 + T\,P(\sup_{t\in[0,T]} E_\kappa(t)) + \delta \sup_{t\in[0,T]} E_\kappa(t)\,.$$

Proof. Let ∂^3 act on (4.1b) and test with $\partial^3\tilde v$. All of the terms are estimated as in Lemma 12.3, except the analogue of (12.15) which reads, after replacing ∂_t^3 with ∂^3, as

$$\kappa\int_0^T [\tilde v\cdot\partial^4\tilde n_\kappa, \tilde n_\kappa\cdot\partial^4\tilde v)]_0\,.$$

Since the energy function places $\sqrt{\kappa}\tilde\eta \in L^\infty(0,T;H^{5.5}(\Omega))$, we see that this integral is bounded by $\delta\sup_{t\in[0,T]} E_\kappa(t) + C\,T\,P(\sup_{t\in[0,T]} E_\kappa(t))$. □

To the above energy estimates, we add one elliptic estimate arising from the modified boundary condition (12.8). We will make use of the following identity:

$$[\sqrt{g}\Delta_g(\eta^i)],_\gamma = [\sqrt{g}g^{\alpha\beta}\Pi^i_j\eta^j,_{\beta\gamma} + \sqrt{g}(g^{\nu\mu}g^{\alpha\beta} - g^{\alpha\nu}g^{\beta\mu})\eta^i,_\beta\,\eta^j,_\nu\,\eta^j,_{\mu\gamma}],_\alpha\,. \tag{12.36}$$

Lemma 12.6 (Elliptic estimate for $\sqrt{\kappa}\tilde\eta$)**.** *Let M_0 be given as in Lemma* 10.1*. Then for $\delta > 0$,*

$$\sup_{t\in[0,T]} |\sqrt{\kappa}\tilde\eta|_5^2(t) \le M_0(\delta) + \delta\sup_{t\in[0,T]} E_\kappa(t) + C\,T\,P(\sup_{t\in[0,T]} E_\kappa(t))\,. \tag{12.37}$$

Proof. Letting $Q^{\alpha i}_j := Q^{\alpha\beta i}_j(\partial\tilde\eta)$ denote a smooth function of $\partial\tilde\eta$, from (12.36) we see that

$$\begin{aligned}\partial^3[\sqrt{\tilde g}\Delta_{\tilde g}(\tilde\eta^i)],_\gamma &= [\sqrt{\tilde g}\tilde g^{\alpha\beta}\Pi^i_j\partial^3\tilde\eta^j,_{\beta\gamma} + \sqrt{\tilde g}(\tilde g^{\nu\mu}\tilde g^{\alpha\beta} - \tilde g^{\alpha\nu}\tilde g^{\beta\mu})\tilde\eta^i,_\beta\,\tilde\eta^j,_\nu\,\partial^3\tilde\eta^j,_{\mu\gamma}],_\alpha\\ &\quad + [\partial Q^{\alpha\beta i}_j\partial^2\tilde\eta^j,_{\beta\gamma} + \partial^2 Q^{\alpha\beta i}_j\partial\tilde\eta^j,_{\beta\gamma} + \partial^3 Q^{\alpha\beta i}_j\tilde\eta^j,_{\beta\gamma}],_\alpha\,.\end{aligned}$$

The estimate (12.37) is obtained by letting $\partial^3\partial_\gamma$ act on the modified boundary condition (12.5) and then testing this with $\zeta^2\tilde g^{\gamma\delta}\Pi^i_l\partial^3\tilde\eta^l,_\delta$, where $\zeta^2=\alpha_i$.

For convenience we drop the subscript i from Ω_i and Γ_i. (Recall that α_i denotes the partition of unity introduced in Section 2.) For the surface tension term, integration by parts with respect to ∂_α yields

$$\int_0^T [\partial^3(\sqrt{\tilde g}\tilde H\tilde n\circ\tilde\eta),_\gamma\,,\zeta^2\tilde g^{\delta\gamma}\Pi\partial^3\tilde\eta,_\delta]_0 \leq \int_0^T \left[-|\zeta\sqrt{g}^{\frac12}\Pi\partial^5\eta|_0^2 + C|F|_{L^\infty}\,|\eta|_4\,|\Pi\partial_5\eta|_0\right],$$

where $F:=P(\zeta,\partial\eta,\partial v,\partial^2\eta)$ is a polynomial of its arguments. To get this estimate we have used the fact that

$$\sqrt{\tilde g}(\tilde g^{\nu\mu}\tilde g^{\alpha\beta}-\tilde g^{\alpha\nu}\tilde g^{\beta\mu})\tilde\eta^i,_\beta\,\tilde\eta^j,_\nu\,\partial^3\tilde\eta^j,_{\mu\gamma}\,\tilde g^{\gamma\delta}\Pi^i_l\partial^3\tilde\eta^l,_{\delta\alpha}=0,$$

since $\tilde\eta^i,_\beta\,\Pi^i_l=0$. (This ensures that the error term is linear in $|\Pi\partial^5\tilde\eta|_0$ rather than quadratic.)

We next analyze the artificial viscosity term. The testing procedure gives us the integral

$$-\int_0^T \kappa[\partial^3\partial_\gamma\Delta_0(\tilde v\cdot\tilde n_\kappa)\tilde n_\kappa,\zeta^2\tilde g^{\gamma\delta}\Pi\partial^3\tilde\eta,_\delta]_0.$$

The positive term comes from $\partial^3\partial_\gamma$ acting on $\tilde v$. This gives, after integration by parts in space, the highest-order integrand $\kappa(\partial^3\tilde v,_{\alpha\gamma}\cdot\tilde n_\kappa)g^{\gamma\delta}g_0^{\alpha\beta}(\tilde n_\kappa\cdot\Pi\partial^3\tilde\eta,_{\beta\delta})$, where $\Pi=\tilde n\otimes\tilde n$. We can write this term as

$$\kappa(\partial^3\tilde v,_{\alpha\gamma}\cdot\tilde n_\kappa)g^{\gamma\delta}g_0^{\alpha\beta}(\tilde n_\kappa\cdot\partial^3\tilde\eta,_{\beta\delta})+\kappa(\partial^3\tilde v,_{\alpha\gamma}\cdot\tilde n_\kappa)g^{\gamma\delta}g_0^{\alpha\beta}(\tilde n_\kappa\cdot(\Pi-\Pi_\kappa)\partial^3\tilde\eta,_{\beta\delta}),$$

where $\Pi_\kappa=\tilde n_\kappa\otimes\tilde n_\kappa$. The first term is an exact derivative in time, and yields

$$\frac{\kappa}{2}\frac{d}{dt}|\partial^5\tilde\eta\cdot\tilde n_\kappa|^2-\kappa\partial^5\tilde\eta^i\,\partial^5\tilde\eta^j\;\tilde n^i_\kappa\partial_t\tilde n^j_\kappa.$$

The space integral of the second term is estimated by

$$C\,|F|_{L^\infty}|\sqrt{\kappa}v|_5\,|\sqrt{\kappa}\tilde\eta|_5\,|\Pi_\kappa-\Pi|_{L^\infty}$$

and

$$|\Pi_\kappa-\Pi|_{L^\infty}\leq C\,\kappa\,|\tilde\eta|_{3.5}.$$

From (12.5)

$$\kappa\partial^3\Delta_0(v\cdot\tilde n_\kappa)=-\partial^3(\sqrt{\tilde g_\kappa}^{-1}\sqrt{\tilde g}\Delta_{\tilde g}(\tilde\eta)\cdot\tilde n_\kappa)+\partial^3 q.$$

Thus,

$$C\sqrt{\kappa}|\kappa\tilde v|_5\leq\kappa|F|_{L^\infty}(|\tilde v|_3+|\sqrt{\kappa}\tilde v|_4)+(\kappa^{\frac32}+\kappa^{\frac12})|F|_{L^\infty}|\tilde\eta|_4+|F|_{L^\infty}|\sqrt{\kappa}\tilde\eta|_5+\sqrt{\kappa}|\tilde q|_3,$$

so that

$$\int_0^T\int_\Gamma\kappa(\partial^3\tilde v,_{\alpha\gamma}\cdot\tilde n_\kappa)g^{\gamma\delta}g_0^{\alpha\beta}(\tilde n_\kappa\cdot(\Pi-\Pi_\kappa)\partial^3\tilde\eta,_{\beta\delta}) \leq C\int_0^T\left[\sqrt{\kappa}|F|_{L^\infty}(|\tilde v|_3+|\sqrt{\kappa}\tilde v|_4+|\tilde\eta|_4+|\tilde q|_3)\,|\sqrt{\kappa}\tilde\eta|_5+\,|F|_{L^\infty}|\sqrt{\kappa}\tilde\eta|_5^2\right].$$

Having finished the estimates for the terms leading to the positive energy contribution, we next consider the most difficult of the error terms. This occurs when $\partial^3\partial_\gamma$ acts on $\tilde n_\kappa$, producing the integral $\int_0^T\kappa[\zeta^2\,\tilde v\cdot\partial^4\tilde n_\kappa,\tilde n_\kappa\cdot\partial^4\tilde\eta]_0$.

To analyze this term, we let $\tilde{v} = \tilde{v}^\gamma \partial_\gamma \tilde{\eta}_\kappa + \tilde{v}_n \tilde{n}_\kappa$; we remind the reader that for $\gamma = 1, 2$, $\partial_\gamma \tilde{\eta}_\kappa$ spans the tangent space of $\tilde{\eta}_\kappa(t)(\Gamma)$ at $\tilde{\eta}_\kappa(x,t)$, so that $\partial_\gamma \tilde{\eta}_\kappa$ is orthogonal to $\tilde{n}_\kappa$. It follows that $v^\gamma \partial_\gamma \tilde{\eta}_\kappa \cdot \partial^5 \tilde{n}_\kappa$ is equal to $-v^\gamma \tilde{n}_\kappa \cdot \partial^5 \partial_\gamma \tilde{\eta}_\kappa$ plus lower order terms $v^\gamma \mathcal{R}_4^\gamma(\tilde{\eta})$, which have at most only five tangential derivatives of $\tilde{\eta}_\kappa$. Note also that since $\tilde{n}_\kappa \cdot \tilde{n}_\kappa = 1$, $\tilde{n}_\kappa \cdot \partial^5 \tilde{n}_\kappa$ is a sum of lower order terms which have at most only five tangential derivatives of $\tilde{\eta}_\kappa$.

Thus,

$$\int_0^T \kappa[\zeta^2\, \tilde{v} \cdot \partial^5 \tilde{n}_\kappa, \tilde{n}_\kappa \cdot \partial^5 \tilde{\eta}]_0 = \int_0^T \kappa[\zeta^2\, \tilde{v}^\gamma \tilde{n}_\kappa \cdot \partial^5 \partial_\gamma \tilde{\eta}_\kappa + \tilde{v}^\gamma \mathcal{R}_4^\gamma(\tilde{\eta}), \tilde{n}_\kappa \cdot \partial^5 \tilde{\eta}]_0\,,$$

where the remainder term satisfies

$$\left| \int_0^T [\kappa \tilde{v}^\gamma \mathcal{R}_4^\gamma(\tilde{\eta})\,,\, \tilde{n}_\kappa \cdot \partial^5 \eta]_0 \right| \le C \int_0^T |F|_{L^\infty} \left[|\sqrt{\kappa}\tilde{v}|_4\, |\sqrt{\kappa}\tilde{\eta}|_5 + |\sqrt{\kappa}\tilde{\eta}|_5^2 \right].$$

We must form an exact derivative from the remaining highest order term

$$\int_0^T \kappa[\zeta^2\, \tilde{v}^\gamma \tilde{n}_\kappa \cdot \partial^5 \partial_\gamma \tilde{\eta}_\kappa, \tilde{n}_\kappa \cdot \partial^5 \tilde{\eta}]_0\,, \tag{12.38}$$

and this will require commuting the horizontal convolution operator, so that the $\tilde{\eta}$ on the right side of the $L^2(\Gamma)$ inner-product also has a convolution operator, and is hence converted to an $\tilde{n}_\kappa \cdot \partial^4 \rho_{\frac{1}{\kappa}} \star_h \tilde{\eta}$ term. With this accomplished, we will be able to pull-out the ∂_γ operator and form an exact derivative, which can be bounded by our energy function.

Noting that on Γ the horizontal convolution $\star_h$ restricts to the usual convolution $*$ on $\mathbb{R}^2$, we have that

$$\tilde{\eta}_\kappa = \sum_{i=1}^K \sqrt{\alpha_i} \left[\rho_{\frac{1}{\kappa}} * [\rho_{\frac{1}{\kappa}} * (\sqrt{\alpha_i}\tilde{\eta} \circ \theta_i)] \right] \circ \theta_i^{-1}.$$

For notational convenience, we set $\rho = \rho_{1/\kappa}$, $\zeta = \sqrt{\alpha_i}$, and $R = [0,1]^2 = \theta_i^{-1}(\Gamma \cap U_i)$. It follows that (12.38) can be expressed as

$$\int_0^T \kappa \sum_{i=1}^K \int_R (\tilde{v}^\gamma \tilde{n}_\kappa) \circ \theta_i \cdot \partial_\gamma \partial^5 \left[\zeta(\theta_i) \rho * \rho * (\zeta \tilde{\eta}) \circ \theta_i \right] \left(\tilde{n}_\kappa \cdot \partial^5 \tilde{\eta} \right) \circ \theta_i. \tag{12.39}$$

With $\tilde{g}_\kappa := \partial^5 \rho * \zeta(\theta_i) \tilde{\eta}(\theta_i)$, we see that

(12.40)

$$\begin{aligned} &\kappa \tilde{n}_\kappa \cdot \partial_\gamma \partial^5 \left[\zeta(\theta_i) \rho * \rho * (\zeta \tilde{\eta} \circ \theta_i) \right] \\ &\qquad = \kappa \tilde{n}_\kappa \cdot (\partial_\gamma \partial^5 \zeta(\theta_i)) \rho * \rho * (\zeta \tilde{\eta}(\theta_i)) + \kappa \zeta \tilde{n}_\kappa \cdot \rho * \partial_\gamma \tilde{g}_\kappa + \kappa \mathcal{R}_5(\tilde{\eta})\,, \end{aligned}$$

where the remainder $\mathcal{R}_5(\tilde{\eta})$ has at most five tangential derivatives on $\tilde{\eta}$. Substitution of (12.40) into (12.39) yields three terms, corresponding to the three terms on the right-hand side of (12.40). For the first term, we see that

$$\begin{aligned} &\int_0^T \kappa \sum_{i=1}^K \int_R (\partial_\gamma \partial^5 \zeta(\theta_i)) \tilde{v}^\gamma \tilde{n}_\kappa \cdot\, \rho * \rho * (\zeta \tilde{\eta}(\theta_i)\ \left(\tilde{n}_\kappa \cdot \partial^5 \tilde{\eta}(\theta_i) \right) \\ &\qquad \le C \int_0^T \|\sqrt{\kappa}\theta_i\|_{6.5} |\tilde{\eta}(\theta_i)|_{L^\infty} |\sqrt{\kappa}\tilde{\eta}|_5\,. \end{aligned}$$

The second term on the right-hand side of (12.40) gives the integral

$$\int_0^T \kappa \sum_{i=1}^K \int_R (\tilde{n}_\kappa \cdot \rho * \partial_\gamma \tilde{g}_\kappa)\ (\tilde{n}_\kappa \cdot \tilde{g}_\kappa) + \mathcal{R}_6(\tilde{\eta}),$$

where the remainder $\mathcal{R}_6(\tilde{\eta})$ is lower order containing terms which have at most four tangential derivatives on $\tilde{\eta}$ and five on $\zeta(\theta_i)$.

We fix $i \in \{1, ..., K\}$, drop the explicit composition with θ_i, set

$$\begin{aligned} \triangle_{\tilde{n}_\kappa, g_\kappa} &= \tilde{n}_\kappa \cdot \rho * \tilde{g}_\kappa - \rho * (\tilde{n}_\kappa \cdot \tilde{g}_\kappa)\,, \\ \triangle_{\tilde{n}_\kappa, \zeta\partial^5\tilde{\eta}} &= \tilde{n}_\kappa \cdot \rho * \zeta\partial^5\tilde{\eta} - \rho * (\tilde{n}_\kappa \cdot \zeta\partial^5\tilde{\eta})\,, \end{aligned}$$

and analyze the following integral:

$$\begin{aligned} &\int_R \{\zeta\tilde{n}_\kappa \cdot \rho * \tilde{g}_\kappa\}\, \{\tilde{n}_\kappa \cdot \partial^5\tilde{\eta}\} \\ &\quad = \int_R \{\rho * (\tilde{n}_\kappa \cdot \tilde{g}_\kappa)\}\, \{\tilde{n}_\kappa \cdot \zeta\partial^5\tilde{\eta}\} + \int_R \triangle_{\tilde{n}_\kappa, g_\kappa}\, \{\tilde{n}_\kappa \cdot \zeta\partial^4\tilde{\eta}\}\,, \\ &\quad = \int_R \{\tilde{n}_\kappa \cdot \tilde{g}_\kappa\}\, \{\tilde{n}_\kappa \cdot \partial^4\rho * \zeta\tilde{\eta}\} - \int_R \{\tilde{n}_\kappa \cdot \tilde{g}_\kappa\}\, \triangle_{\tilde{n}_\kappa, \zeta\partial^5\tilde{\eta}} \\ &\qquad + \int_R \triangle_{\tilde{n}_\kappa, g_\kappa}\, \{\tilde{n}_\kappa \cdot \zeta\partial^5\tilde{\eta}\} + \mathcal{R}_7(\tilde{\eta})\,, \end{aligned}$$

where the remainder $\mathcal{R}_7(\tilde{\eta})$ comes from commuting ∂^5 with the cut-off function ζ and has the same bound as $\mathcal{R}_4(\tilde{\eta})$. The first term on the right-hand side is a perfect derivative, and for the remaining terms we use Lemma 2.1 together with the estimate $\kappa|\tilde{g}_\kappa|_{0,R} \le C\|\tilde{\eta}\|_{5.5}$ to find that

$$\kappa \int_R \{\zeta\tilde{n}_\kappa \cdot \rho * \tilde{g}_\kappa\}\, \{\tilde{n}_\kappa \cdot \partial^5\tilde{\eta}\} \le C\, |F|_{L^\infty}\, |\sqrt{\kappa}\tilde{\eta}|_5^2.$$

Thus, summing over $i \in \{1, ..., K\}$,

$$\kappa \int_0^T [\zeta^2\, \tilde{v} \cdot \partial^5\tilde{n}_\kappa, \tilde{n}_\kappa \cdot \partial^5\tilde{\eta}]_0 \le C \int_0^T |F|_{L^\infty}\, (|\sqrt{\kappa}\Gamma|_6 + |\sqrt{\kappa}\tilde{v}|_4 + |\sqrt{\kappa}\tilde{\eta}|_5)\, |\sqrt{\kappa}\tilde{\eta}|_5\,,$$

where $|\sqrt{\kappa}\Gamma|_6 := \max_{i\in\{1,\dots,K\}} |\sqrt{\kappa}\theta_i|_6$.

It is easy to see that

$$\int_0^T [\partial^3\partial_\gamma(\sqrt{\tilde{g}}\Delta_{\tilde{g}}(\tilde{\eta}) \cdot (\tilde{n}_\kappa - \tilde{n})\tilde{n}_\kappa), \zeta^2\tilde{g}^{\delta\gamma}\Pi\partial^3\tilde{\eta}_{,\delta}\,] \le C \int_0^T |F|_{L^\infty}\, |\sqrt{\kappa}\tilde{\eta}|_5^2$$

with the same bound for $\int_0^T [\partial^2\partial_\gamma(\sqrt{\tilde{g}}\Delta_{\tilde{g}}(\tilde{\eta}))\cdot\tilde{n}_\kappa(\tilde{n}_\kappa - \tilde{n}), \zeta^4\tilde{g}^{\delta\gamma}\Pi\partial^2\tilde{\eta}_{,\delta}\,]$. With (3.3), we infer that

$$\begin{aligned} |\sqrt{\kappa}\partial^5\tilde{\eta} \cdot \tilde{n}_\kappa|_0^2(t) &\le M_0 + C \int_0^T |\tilde{q}|_3^2 \\ &\qquad + C \int_0^T |F|_{L^\infty}\, (|\sqrt{\kappa}\Gamma|_6 + |\sqrt{\kappa}\tilde{v}|_4 + |\sqrt{\kappa}\tilde{\eta}|_5)\, |\sqrt{\kappa}\tilde{\eta}|_5\,. \end{aligned}$$

Adding to this inequality the curl estimate (10.7) for $\sqrt{\kappa}\tilde{\eta}$ and the divergence estimate (which has the same bound as the curl estimate), and using Young's inequality to get $\int_0^T |F|_{L^\infty} |\sqrt{\kappa}\tilde{v}|_4 |\sqrt{\kappa}\tilde{\eta}|_5 \leq \delta \int_0^T |\sqrt{\kappa}\tilde{v}|_4^2 + C_\delta \int_0^T |F|_{L^\infty}^2 |\sqrt{\kappa}\tilde{\eta}|_5^2$, we see that

$$\sup_{t\in[0,T]} |\sqrt{\kappa}\tilde{\eta}|_5^2 \leq M_0 + C\,T \sup_{t\in[0,T]} \left(\|v_t\|_{2.5}^2 + \|v\|_{3.5}^2 + \|\eta\|_{4.5}^4\right)$$
$$+ C\,T \sup_{t\in[0,T]} \left(|F|_{L^\infty}^2 \left(|\sqrt{\kappa}\Gamma|_6^2 + |\sqrt{\kappa}\tilde{\eta}|_5^2\right) + \delta \int_0^T |\sqrt{\kappa}\tilde{v}|_4^2\right.,$$

from which the lemma follows. □

12.1. **Removing the additional regularity assumptions on the initial data.** At this stage, we explain how we can remove the extra regularity assumptions on the initial data, u_0 and Ω, so that the constant M_0 depends on $\|u_0\|_{4.5}$ and $|\Gamma|_{5.5}$ rather than $\sqrt{\kappa}\|u_0\|_{10.5}$ and $\sqrt{\kappa}|\Gamma|_7$ as stated in Lemma 10.1. The modification requires the following regularization of the initial data: set $u_0 = \rho_{e^{\frac{1}{\kappa}}} \star E_{\Omega_\kappa}(u(0))$, where Ω_κ is obtained by smoothing Ω via convolution with $\rho_{e^{\frac{1}{\kappa}}}$, i.e., we use $\rho_{e^{\frac{1}{\kappa}}} * \bar{\theta}_i$ as our family of charts. We make use of the fact that

$$P(\|\sqrt{\kappa}u_0\|_{10.5}) \leq CP\|u_0\|_{4.5})\,, \quad P(|\sqrt{\kappa}\Gamma|_{10}) \leq CP(|\Gamma|_{5.5})\,,$$

which follows by integration by parts of six tangential derivatives onto the mollifier $\rho_{e^{\frac{1}{\kappa}}}$; this results in the constant $C > 0$ being independent of κ.

12.2. **The limit as $\kappa \to 0$.**

Proposition 12.1. *With $M_0 = P(\|u_0\|_{4.5}, |\Gamma|_{5.5})$ a polynomial of its arguments and for $\tilde{M}_0 > M_0$,*

$$\sup_{t\in[0,T]} E_\kappa(t) \leq \tilde{M}_0\,, \tag{12.41}$$

where T depends on the data, but not on κ.

Proof. Summing the inequalities (12.6), (12.33), (12.34), (12.35), and (12.37), and using Lemma 10.1 and Proposition 5.1, we find that

$$\sup_{t\in[0,T]} E_\kappa(t) \leq M_0 + C\,T\,P(\sup_{t\in[0,T]} E_\kappa(t)) + \delta \sup_{t\in[0,T]} E_\kappa(t)\,,$$

where the polynomial P and the constant M_0 do not depend on κ. Choose $\delta < 1$. Then, from the continuity of the left-hand side in T, we may choose T sufficiently small and independent of κ, to ensure that (12.41) holds. (See [8] for a detailed account of such polynomial inequalities.) □

Proposition 12.1 provides the weak convergence as $\kappa \to 0$ of subsequences of $(\tilde{v}, \tilde{q})$ toward a limit which we denote by (v, q) in the same space. We then set $\eta = \text{Id} + \int_0^t v$, and $u = v \circ \eta^{-1}$. It is obvious that $\tilde{v}_\kappa$, arising from the double horizontal convolution by layers of $\tilde{v}$, satisfies $\tilde{v}_\kappa \to v$ in $L^2(0,T;H^{3.5}(\Omega))$, and therefore $\tilde{\eta}^\kappa \to \eta$ in $L^2(0,T;H^4(\Omega))$. It follows that $\text{div}u = 0$ in $\eta(\Omega)$ in the limit as $\kappa \to 0$ in (1.3c). Thus, the limit (v, q) is a solution to the problem (1.3), and satisfies $E_0(t) \leq \tilde{M}_0$. We then take T even small, if necessary, to ensure that (3.3) holds, which follows from the fundamental theorem of calculus.

13. A posteriori elliptic estimates

Solutions of the Euler equations gain regularity with respect to the $E_\kappa(t)$ from elliptic estimates of the boundary condition (1.3d), which we write as $\sqrt{g}Hn(\eta) = \sqrt{g}qn$.

Replacing ∂_γ with ∂_t in (12.36), we have the identities

$$\partial_t(\sqrt{g}Hn\circ\eta)^i = -[\sqrt{g}g^{\alpha\beta}\Pi^i_j v^j,_\beta + \sqrt{g}(g^{\nu\mu}g^{\alpha\beta} - g^{\alpha\nu}g^{\beta\mu})\eta^i,_\beta\,\eta^j,_\nu\,v^j,_\mu],_\alpha \tag{13.1}$$

and

(13.2)

$$\begin{aligned}&\partial_t^2(\sqrt{g}Hn\circ\eta)^i\\ &\quad = -[\sqrt{g}g^{\alpha\beta}\Pi^i_j v_t^j,_\beta + \sqrt{g}(g^{\nu\mu}g^{\alpha\beta} - g^{\alpha\nu}g^{\beta\mu})\eta^i,_\beta\,\eta^j,_\nu\,v_t^j,_\mu + Q_j^{i\alpha\beta}v^j,_\beta],_\alpha\ ,\end{aligned}$$

where $Q_j^{i\alpha\beta} = Q(\partial\eta)$ is a rational function of $\partial\eta$.

Lemma 13.1. *Taking $\tilde M_0$ as in Proposition* 12.1*, and letting $\mathcal{M}_0$ denote a polynomial function of $\tilde M_0$, for T taken sufficiently small,*

$$\sup_{t\in[0,T]}[|\Gamma(t)|_{5.5} + \|v(t)\|_{4.5} + \|v_t(t)\|_3] \le \mathcal{M}_0\,.$$

Proof. We begin with the estimate for v_t. Following the proof Lemma 12.6, we let $\partial_\gamma\partial_t^2$ act on the boundary condition (1.3d) and test with $-\zeta^2 g^{\gamma\delta}\Pi^i_k v_t^k,_\delta$, where $\zeta^2 = \alpha_i$, an element of our partition of unity. Using (13.2), we see that

$$-\int_\Gamma \partial_\gamma\partial_t^2[\sqrt{g}Hn(\eta)]\cdot\zeta^2 g^{\gamma\delta}v_t,_\delta = \int_\Gamma(\sqrt{g}qn)_{tt}\cdot[\zeta^2 g^{\gamma\delta}v_t,_\delta],_\gamma\,.$$

Using (12.41), letting $\tilde C$ denote a constant that depends on $\tilde M_0$, and summing over the partition of unity, we find that

(13.3)

$$|\partial^2 v_t\cdot n|_0^2 \le \tilde C\,[|v|_2 + |\eta|_2\,(|v_t|_1 + |v|_1) + |(q\sqrt{g}n)_{tt}|_0]\,|\partial^2 v_t\cdot n|_0 + \tilde C|v_t|_1^2|\eta|_3\,.$$

This follows since

$$\left[\sqrt{g}(g^{\nu\mu}g^{\alpha\beta} - g^{\alpha\nu}g^{\beta\mu})\eta^i,_\beta\,\eta^j,_\nu\,v_t^j,_{\mu\gamma}\right]\left[g^{\gamma\delta}\Pi^i_k v_t^k,_{\delta\alpha}\right] = 0\,,$$

while

$$\begin{aligned}\int_\Gamma&\left[\sqrt{g}(g^{\nu\mu}g^{\alpha\beta} - g^{\alpha\nu}g^{\beta\mu})\eta^i,_\beta\,\eta^j,_\nu\,v_t^j,_{\mu\gamma}\right]\left[(g^{\gamma\delta}\Pi^i_k),_\alpha\,v_t^k,_\delta\right]\\ &\le \tilde C\,|v_t|_1\,(|\partial^2 v_t\cdot n|_0 + |v_t|_1\,|\eta|_3)\,.\end{aligned}$$

Applying Young's inequality to (13.3) yields, after adjusting the constant,

$$|\partial^2 v_t\cdot n|_0^2 \le \tilde C\,[|\eta|_3^2 + |v|_2^2 + |v_t|_1^2 + |q_{tt}|_0^2]\,.$$

A similar computation shows that

$$|\partial^3 v_t\cdot n|_0^2 \le \tilde C\,[|\eta|_4^2 + |v|_3^2 + |v_t|_2^2 + |q_{tt}|_1^2]\,.$$

Thus, by interpolation

$$\sup_{t\in[0,T]}|\partial^2 v_t\cdot n|_{0.5}^2 \le \tilde C\sup_{t\in[0,T]}[|\eta|_{3.5}^2 + |v|_{2.5}^2 + |v_t|_{1.5}^2 + |q_{tt}|_{0.5}^2] \le \mathcal{M}_0\,.$$

Computing the $H^2(\Omega)$-norm of (10.9), we find that

$$\sup_{t\in[0,T]} \|\operatorname{curl} v_t\|_2 \leq \mathcal{M}_0 + \tilde{C}\,T \sup_{t\in[0,T]} \|v_t\|_3\,,$$

with the same estimate for $\sup_{t\in[0,T]} \|\operatorname{div} v_t\|_2$. Hence, for T taken sufficiently small, we infer from Proposition 5.1 that

$$\sup_{t\in[0,T]} \|v_t\|_3 \leq \mathcal{M}_0. \tag{13.4}$$

Next, we let $\partial_\gamma \partial^2 \partial_t$ act on the boundary condition (1.3d) and we test with $-\zeta^2 g^{\gamma\delta} \Pi^i_k \partial^2 v^k,_\delta$. Using (12.36), we find that

$$\sup_{t\in[0,T]} |\partial^4 v \cdot n|_0^2 \leq \tilde{C} \sup_{t\in[0,T]} [|\eta|_4^2 + |v|_3^2 + |q_t|_2^2] \leq \mathcal{M}_0\,.$$

Computing the $H^{3.5}(\Omega)$-norm of (10.8), and again taking T sufficiently small, we see that $\sup_{t\in[0,T]} \|v\|_{4.5} \leq \mathcal{M}_0$.

In order to prove our remaining estimate, we need a convenient reparameterization of $\Gamma(t)$ via a height function h in the normal bundle over Γ.

Consider the isometric immersion $\eta_0 : (\Gamma, g_0) \to (\mathbb{R}^3, \mathrm{Id})$. Let $\mathcal{B} = \Gamma \times (-\epsilon, \epsilon)$ where ϵ is chosen sufficiently small so that the map $B : \mathcal{B} \to \mathbb{R}^3 : (y,z) \mapsto y + zN(y)$ is itself an immersion, defining a tubular neighborhood of $\eta_0(\Gamma)$ in $\mathbb{R}^3$. We can choose a coordinate system $\frac{\partial}{\partial y^\alpha}$, $\alpha = 1, 2$, and $\frac{\partial}{\partial z}$.

Let $G = B^*(\mathrm{Id})$, denote the induced metric on $\mathcal{B}$, and note that $G(y,z) = G_z(y) + dz \otimes dz$, where G_z is the metric on the surface $\Gamma \times \{z\}$, and that $G_0 = g_0$. Let $h : \Gamma \to (-\epsilon, \epsilon)$ be a smooth function and consider the graph of h in $\mathcal{B}$, parameterized by $\phi : \Gamma \to \mathcal{B} : y \mapsto (y, h(y))$. The tangent space to graph(h), considered as a submanifold of $\mathcal{B}$, is spanned at a point $\phi(x)$ by the vectors

$$\phi_*(\frac{\partial}{\partial y^\alpha}) = \frac{\partial \phi}{\partial y^\alpha} = \frac{\partial}{\partial y^\alpha} + \frac{\partial h}{\partial y^\alpha}\frac{\partial}{\partial z},$$

and the normal to graph(h) is given by

$$n(y) = J_h^{-1}(y)\Big(- G^{\alpha\beta}_{h(y)} \frac{\partial h}{\partial y^\alpha}\frac{\partial}{\partial y^\beta} + \frac{\partial}{\partial z}\Big) \tag{13.5}$$

where $J_h = (1 + h,_\alpha G^{\alpha\beta}_{h(y)} h,_\beta)^{1/2}$. Therefore, twice the mean curvature H is defined to be the trace of ∇n, while

$$(\nabla n)_{ij} = G(\nabla^{\mathcal{B}}_{\frac{\partial}{\partial w^i}} n, \frac{\partial}{\partial w^j}),$$

where $\frac{\partial}{\partial w^\alpha} = \frac{\partial}{\partial y^\alpha}$ for $\alpha = 1$, 2 and $\frac{\partial}{\partial w^3} = \frac{\partial}{\partial z}$. Substituting the formula (13.5) for n, we see that

$$\begin{aligned}
(\nabla n)_{\alpha\beta} &= G\Big(\nabla^{\mathcal{B}}_{\frac{\partial}{\partial y^\alpha}}\Big[- J_h^{-1} G_h^{\gamma\delta} h,_\gamma \frac{\partial}{\partial y^\delta} + J_h^{-1}\frac{\partial}{\partial z}\Big], \frac{\partial}{\partial y^\beta}\Big)\\
&= -(G_h)_{\delta\beta}(J_h^{-1} G_h^{\gamma\delta} h,_\gamma),_\alpha + F^1_{\alpha\beta}(y, h, \partial h),\\
(\nabla n)_{33} &= G\Big(\nabla^{\mathcal{B}}_{\frac{\partial}{\partial z}}\Big[- J_h^{-1} G_h^{\gamma\delta} h,_\gamma \frac{\partial}{\partial y^\delta} + J_h^{-1}\frac{\partial}{\partial z}\Big], \frac{\partial}{\partial z}\Big)\\
&= F^2_{\alpha\beta}(y, h, \partial h)
\end{aligned}$$

for some functions $F^1_{\alpha\beta}$ and $F^2_{\alpha\beta}$, $\alpha,\beta=1,2$. Letting γ_0 denote the Christoffel symbols associated to the metric g_0 on Γ, we find that the curvature of graph(h) is given by

$$L_h(h) := H = -(J_h^{-1} G_h^{\gamma\delta} h_{,\gamma})_{,\delta} + J_h^{-1}([\gamma_0]^j_{j3} - G_h^{\gamma\delta} h_{,\gamma}[\gamma_0]^j_{j\delta})\,. \tag{13.6}$$

Note that the metric $G_h = P(h)$, and that the highest-order term is in divergence form, while the lower-order term is a polynomial in ∂h. The function h determines the *height*, and hence shape, of the surface $\Gamma(t)$ above Γ.

Given a signed height function $h:\Gamma_0\times[0,T)\to\mathbb{R}$, for each $t\in[0,T)$, define the *normal* map

$$\eta^\nu : \Gamma_0\times[0,T)\to\Gamma(t), \quad (y,t)\mapsto y + h(y,t)N(y)\,.$$

Then, there exists a unique *tangential* map $\eta^\tau:\Gamma_0\times[0,T)\to\Gamma_0$ (a diffeomorphism as long as h remains a graph) such that $\eta|_\Gamma(t)$ has the decomposition

$$\eta|_\Gamma(\cdot,t) = \eta^\nu(\cdot,t)\circ\eta^\tau(\cdot,t), \quad \eta|_\Gamma(y,t) = \eta^\tau(y,t) + h(\eta^\tau(y,t),t)N(\eta^\tau(y,t))\,.$$

The boundary condition (1.1c) can be written as $\sigma L_h(h) = q\circ(\eta^\tau)^{-1}$.

The operator L_h is a quasilinear elliptic operator; from the standard regularity theory for quasilinear elliptic operators with H^3 coefficients on a compact manifold, we have the elliptic estimate

$$|h|_{5.5} \le \tilde C\,|q\circ(\eta^\tau)^{-1}|_{3.5} \le \tilde C\,\|q\|_4\,.$$

By (12.2), we see that for all $t\in[0,T]$,

$$\|q\|_4 \le \tilde C\,\|a\|_2^2\,\|v\|_3^2 + \tilde C\,|\eta|_3\,|v_t|_2 \le \mathcal{M}_0\,,$$

the last inequality following from (13.4).

Since $\Gamma(t)$ =graph$h(t)$, this estimate shows that $\Gamma(t)$ maintains its $H^{5.5}$-class regularity for $t\le T$. □

14. κ-INDEPENDENT ESTIMATES FOR THE SMOOTHED PROBLEM AND EXISTENCE OF SOLUTIONS IN 3D

The 3D analysis of the κ-problem requires assuming that the initial data $u_0\in H^{5.5}(\Omega)$ and Γ is of class $H^{6.5}$. This is necessitated by the Sobolev embedding $\|\tilde v_t\|_{L^\infty}\le C\|\tilde v_t\|_3$.

By replacing the third-time differentiated problem with the fourth time-differentiated problem the identical analysis as in Section 12 yields

$$E_\kappa^{3D}(t) \le \tilde M_0\,,$$

where $\tilde M_0$ is a polynomial of $\|u_0\|_{5.5}$ and $|\Gamma|_{6.5}$. (In fact, our analysis in Section 12 used all of the 3D terms and notation, so no changes are required other than raising the regularity by one derivative.)

We let (v,q) again denote the limit of $(\tilde v,\tilde q)$ as $\kappa\to 0$. The identical limit process as in 2D shows that (v,q) is a solution of the Euler equations.

Having a solution (v,q) to the Euler equation, we can use the a posteriori estimates (13.1) as a priori estimates for solutions of the Euler equations. We see that

$$\sup_{t\in[0,T]}\left[E_\kappa^{2D}(t) + |\Gamma(t)|_{5.5} + \|v(t)\|_{4.5} + \|v_t(t)\|_3\right] \le \mathcal{M}_0\,,$$

where $\mathcal{M}_0$ is polynomial function of $\|u_0\|_{4.5}$ and $|\Gamma|_{5.5}$. This key point here is that the elliptic estimate for $v_t \in H^3(\Omega)$ improves the regularity given by $E^{2D}_\kappa(t)$ and allows for the required Sobolev embedding theorem to hold.

Since our initial data is a priori assumed regularized as in Subsection 12.1, we see that solutions of the Euler equations in 3D only depend on $\mathcal{M}_0$.

15. Uniqueness of solutions to (1.3)

Suppose that (η^1, v^1, q^1) and (η^2, v^2, q^2) are both solutions of (1.3) with initial data $u_0 \in H^{5.5}(\Omega$ and $\Gamma \in H^{6.5}$. Setting

$$\mathcal{E}_\eta(t) = \sum_{k=0}^{4} \|\partial_t^k \eta(t)\|^2_{5.5-k},$$

by the method of Section 12 with $\kappa = 0$, we infer that both $\mathcal{E}_{\eta^1}(t)$ and $\mathcal{E}_{\eta^2}(t)$ are bounded by a constant $\mathcal{M}_0$ depending on the data u_0 and Γ on a time interval $0 \le t \le T$ for T small enough.

Let

$$w := v^1 - v^2, \quad r := q^1 - q^2, \quad \text{and } \xi := \eta^1 - \eta^2\,.$$

Then (ξ, w, r) satisfies

$$\xi = \int_0^t w \qquad \text{in } \Omega \times (0,T]\,, \tag{15.1a}$$

$$\partial_t w^i + (a^1)^k_i\, r,_k = (a^2 - a^1)^k_i\, q^2,_k \qquad \text{in } \Omega \times (0,T]\,, \tag{15.1b}$$

$$(a^1)^j_i w^i,_j = (a^2 - a^1)^j_i {v^2}^i,_j \qquad \text{in } \Omega \times (0,T]\,, \tag{15.1c}$$

$$r n_1 = -\sigma \Pi^1 {g^1}^{\alpha\beta} \xi,_{\alpha\beta} - \sigma\sqrt{g^1}\Delta_{g^1-g^2}(\eta^2) \qquad \text{on } \Gamma \times (0,T]\,, \tag{15.1d}$$

$$(\xi, w) = (0,0) \qquad \text{on } \Omega \times \{t=0\}\,. \tag{15.1e}$$

Set

$$E(t) = \sum_{k=0}^{3} \|\partial_t^k \xi(t)\|^2_{4.5-k}.$$

We will show that $E(t) = 0$, which shows that $w = 0$. To do so, we analyze the forcing terms on the right-hand side of (15.1b) and (15.1c), as well as the term $\sigma\Delta_{g^1-g^2}(\eta^2)$ in (15.1d). We begin with the third time-differentiated problem, and study the integral $\int_0^T \int_\Omega \partial_t^3[(a^1 - a^1)\,\nabla q^2]\, w_{ttt}$. The highest-order term is

$$\int_0^t \int_\Omega (a^1 - a^2)\, \nabla q^2_{ttt} w_{ttt} \le \frac{1}{2}\mathcal{M}_0 \int_0^t \|a^1 - a^2\|^2_{L^\infty} + \frac{1}{2}\int_0^T \|w_{ttt}\|^2_0 \le C\, t\, P(E(t))\,.$$

The third space differentiated and mixed-derivative problems have forcing terms that can be similarly bounded.

The difference in pressure r satisfies, using the notation of (12.2), the following Neumann problem:

$$\begin{aligned} L_{a_1}(r) &= -\partial_t {a^1}^j_i w^i,_j + [{a^1}^j_i (a^2 - a^1)^k_i q^2,_k],_j \ \text{ in } \Omega\,, \\ B_{a_1}(r) &= -w_t \cdot \sqrt{g^1} n^1 + {a^1}^j_i (a^2 - a^1)^k_i q^2,_k N_j \text{ on } \Gamma\,. \end{aligned}$$

Since $P(\|\eta^1\|_{4.5})$ is bounded by some constant $C = P(\mathcal{M}_0)$, (12.3) provides the estimate

$$\|r\|_{3.5} \leq C[\|a^1_t\|_{1.5}\,\|w\|_{2.5} + \|a^1\|_{2.5}\,\|q^2\|_{3.5}\,\|a^1 - a^2\|_{2.5} + |\sqrt{g^1}n^1|_2\,\|w_t\|_{2.5}]\,.$$

Since $\|a^1 - a^2\|_{2.5} \leq C\|\xi\|_{3.5}$, and $\|a^1_t\|_{1.5}$, $\|a^1\|_{2.5}$, $\|q^2\|_{3.5}$, and $|\sqrt{g^1}n^1|_2$ are all bounded $\mathcal{M}_0$, we see that $\|r(t)\|_{3.5} \leq CP(E(t))$. Similar estimates for the time derivatives of r show that $\|r(t)\|_{3.5} + \|r_t(t)\|_{2.5} + \|r_{tt}(t)\|_1 \leq CP(E(t))$.

This shows that the energy estimates of Section 12 go through unchanged for equation (15.1); therefore, using (15.1e), we see that

$$\sup_{t\in[0,T]} E(t) \leq C\,T\,P(\sup_{t\in[0,T]} E(t)).$$

16. The zero surface tension case $\sigma = 0$

In this, the second part of the paper, we use our methodology to prove well-posedness of the free-surface Euler equations with $\sigma = 0$ and the Taylor sign condition (1.2) imposed, previously established by Lindblad in [13]. The main advantages of our method over the Nash-Moser approach of [13] are the significantly shorter proof and the fact that we provide directly the optimal space in which the problem is set, instead of having to separately perform an optimal energy study once a solution is known as in [6]. If one uses a Nash-Moser approach without performing the analysis of [6], then one obtains results with much higher regularity requirements than necessary, as for instance in [11] for the irrotational water-wave problem without surface tension. We also obtain lower regularity results than those given by the functional framework of [6] for the 3D case.

We will extensively make use of the horizontal convolution by layers defined in Section 3, and just as in the first part of the paper, for $v \in L^2(\Omega)$ and $\kappa \in (0,\kappa_0)$, we define the smoothed velocity v^κ by

$$v^\kappa = \sum_{i=1}^K \sqrt{\alpha_i}\big[\rho_{\frac{1}{\kappa}} \star_h [\;\rho_{\frac{1}{\kappa}} \star_h ((\sqrt{\alpha_i}v)\circ\theta_i)]\big]\circ\theta_i^{-1} + \sum_{i=K+1}^L \alpha_i v\,.$$

The horizontal convolution by layers is of crucial importance for defining an approximate problem whose asymptotic behavior will be compatible with the formal energy laws for smooth solutions of the original (unsmoothed) problem (1.1), since the regularity of the moving domain will appear as a surface integral term. In this second part of the paper, the properties of these horizontal convolutions will be featured in a more extensive way than in the surface tension case of the first part of the paper.

We remind the reader that this type of smoothing satisfies the usual properties of the standard convolution; in particular, independently of κ, we have the existence of $C > 0$ such that for any $v \in H^s(\Omega)$:

$$\|v^\kappa\|_s \leq C\;\|v\|_s, \text{ and } \;|v^\kappa|_{s-\frac{1}{2}+p} \leq C\kappa^{-p}|v|_{s-\frac{1}{2}} \text{ for } p \geq 0.$$

We will denote for any $l \in \{1,...,K\}$ the following transformed functions from v and η that will naturally arise at the variational level:

Definition 16.1.

$$v_{l\kappa} = \rho_{\frac{1}{\kappa}} \star_h (\sqrt{\alpha_l} v \circ \theta_l) \text{ in } (0,1)^3,$$
$$\eta^\kappa = \mathrm{Id} + \int_0^t v^\kappa \text{ in } \Omega,$$
$$\eta_{l\kappa} = \theta_l + \int_0^t v_{l\kappa} \text{ in } (0,1)^3.$$

Remark 12. The regularity of the moving free surface will be provided by control of each $\eta_{l\kappa}$ in a suitable norm independent of the parameter κ.

17. The smoothed κ-problem and its linear fixed-point formulation

As it turns out, the smoothed problem associated to the zero surface tension Euler equations can be found quite simply and naturally, and involves only transport-type arguments in an Eulerian framework. Also, the construction of a solution is easier if we assume more regularity on the domain and initial velocity than in Theorem 1.4. We shall therefore assume until Section 26 that Ω is of class $H^{\frac{9}{2}}$ and $u_0 \in H^{\frac{9}{2}}(\Omega)$. In Section 26, we will show how this restriction can be removed.

Letting $u = v \circ \eta^{\kappa -1}$, we consider the following sequence of approximate problems in which the transport velocity u^κ is smoothed:

$$u_t + \nabla_{u^\kappa} u + \nabla p = 0 \text{ in } \eta^\kappa(t,\Omega), \tag{17.1a}$$
$$\operatorname{div} u = 0 \text{ in } \eta^\kappa(t,\Omega), \tag{17.1b}$$
$$p = 0 \text{ on } \eta^\kappa(t,\Gamma), \tag{17.1c}$$
$$u(0) = u_0 \text{ in } \Omega. \tag{17.1d}$$

In order to solve this smoothed problem, we will use a linear problem whose fixed point will provide the desired solution. If we denote by $\bar v$ an arbitrary element of C_T defined in Section 18, and $\bar\eta^\kappa$ the corresponding Lagrangian flow defined above, then we search for w such that if $u = w \circ (\bar\eta^\kappa)^{-1}$ and $\bar u^\kappa = \bar v^\kappa \circ (\bar\eta^\kappa)^{-1}$, we have that

$$u_t + \nabla_{\bar u^\kappa} u + \nabla p = 0 \text{ in } \bar\eta^\kappa(t,\Omega), \tag{17.2a}$$
$$\operatorname{div} u = 0 \text{ in } \bar\eta^\kappa(t,\Omega), \tag{17.2b}$$
$$p = 0 \text{ on } \bar\eta^\kappa(t,\Gamma), \tag{17.2c}$$
$$u(0) = u_0 \text{ in } \Omega. \tag{17.2d}$$

A fixed point $w = \bar v$ to this problem then provides a solution to (17.1). In the following sections, $\bar v \in C_T$ is assumed given, and κ is in $(0, \kappa_0)$. κ is fixed until Section 20 where we study the asymptotic behavior of the problem (17.1) as $\kappa \to 0$.

Remark 13. Note that, for this problem, we do not add any parabolic artificial viscosity, in order to keep the transport-type structure of the Euler equations and to preserve the condition $p = 0$ on the free boundary.

18. Existence of a solution to (17.1)

18.1. A closed convex set.

Definition 18.1. For $T > 0$, we define the following closed convex set of the Hilbert space $L^2(0,T;H^{\frac{7}{2}}(\Omega))$:

$$C_T = \{v \in L^2(0,T;H^{\frac{7}{2}}(\Omega))| \sup_{[0,T]} \|v\|_{\frac{7}{2}} \le 2\|u_0\|_{\frac{7}{2}} + 1\}.$$

It is clear that C_T is nonempty (since it contains the constant in time function u_0) and is a convex, bounded and closed set of the separable Hilbert space $L^2(0,T;H^{\frac{7}{2}}(\Omega))$.

By choosing $T(\|\nabla u_0\|_{\frac{7}{2}} + 1) \le C_\Omega \epsilon_0$, condition (3.1) holds for $\eta = \mathrm{Id} + \int_0^t v$ and any $v \in C_T$ and thus (3.2) is well defined.

We then see that, by taking T smaller if necessary, we have the existence of $\kappa_1 > 0$ such that for any $\kappa \in (0,\kappa_1)$, we have the injectivity of $\eta^\kappa(t)$ on Ω for any $t \in [0,T]$, and $\nabla\eta^\kappa$ satisfies condition (3.1). We then denote $a^\kappa = [\nabla\eta^\kappa]^{-1}$, and we let $n^\kappa(\eta^\kappa(x))$ denote the exterior unit normal to $\eta^\kappa(\Omega)$ at $\eta^\kappa(x)$, with $x \in \Gamma$. We now set $\kappa_2 = \min(\kappa_0,\kappa_1)$, and assume in the following that $\kappa \in (0,\kappa_2)$.

18.2. **Existence and uniqueness for the smoothed problems (17.2) and (17.1).** Suppose that $\bar v \in C_T$ is given. Now, for $v \in C_T$ given, we define p on $\bar\eta^\kappa(t,\Omega)$ by

$$\triangle p = -\bar u^\kappa_i,_j\, u_j,_i \ \text{ in } \bar\eta^\kappa(t,\Omega), \tag{18.1a}$$

$$p = 0 \text{ on } \bar\eta^\kappa(t,\Gamma), \tag{18.1b}$$

where $u = v \circ (\bar\eta^\kappa)^{-1}$. We next define $\tilde v$ in Ω by

$$\tilde v(t) = u_0 + \int_0^t [\nabla p](t', \bar\eta^\kappa(t',\cdot))dt', \tag{18.2}$$

and we now explain why the mapping $v \to \tilde v$ has a fixed point in C_T for $T > 0$ small enough. For each $t \in [0,T]$, let $\Psi(t)$ denote the solution of $\Delta\Psi(t) = 0$ in Ω with $\Psi(t) = \bar\eta^\kappa(t)$ on Γ. For κ and T taken sufficiently small $|\bar\eta^\kappa - \mathrm{Id}|_4 << 1$ so that $\Psi(t)$ is an embedding and satisfies

$$\|\Psi(t)\|_{4.5} \le C|\bar\eta^\kappa(t)|_4\,. \tag{18.3}$$

Letting $Q(x,t) = p(\Psi(x,t),t)$ and $A(x,t) = [\nabla\Psi(x,t)]^{-1}$, (18.1) can be written as

$$\begin{aligned}[A^k_i A^j_i q,_k],_j &= -[\bar u^\kappa_i,_j\, u_j,_i](\Psi(x,t),t) \quad \text{in} \quad \Omega\,,\\ Q &= 0 \quad \text{on} \quad \Gamma\,.\end{aligned}$$

By elliptic regularity (with Sobolev class regularity on the coefficients [9]) almost everywhere in $(0,T)$ and using (18.3),

$$\|p\|_{\frac{9}{2},\bar\eta^\kappa(t,\cdot)} \le CP(|\bar\eta^\kappa|_4)\|\bar v\|_{\frac{7}{2}}\|v\|_{\frac{7}{2}}P(\|\bar\eta^\kappa\|_{\frac{7}{2}}), \tag{18.4}$$

where P denotes a generic polynomial. Now, with the definition of T and C_T, along with the properties of the convolution that allow us to state that

$$|\bar\eta^\kappa|_4 \le \frac{1}{\kappa}|\bar\eta^\kappa|_3$$

(since the derivatives involved are along the boundary, allowing our convolution by layers to smooth in these directions), this provides the following estimate:

$$\|p\|_{\frac{9}{2},\bar\eta^\kappa(t,\cdot)} \le C_\kappa P(\|\bar\eta^\kappa\|_{\frac{7}{2}})\|v\|_{\frac{7}{2}} \le C_\kappa \|v\|_{\frac{7}{2}},$$

where we have used the definition of C_T, and where C_κ denotes a generic constant depending on κ. Consequently, we get in $[0,T]$,

$$\begin{aligned} \|\tilde v(t)\|_{\frac{7}{2}} &\le \|u_0\|_{\frac{7}{2}} + \int_0^t \|p\|_{\frac{9}{2},\bar\eta^\kappa(t',\cdot)}\|\bar\eta^\kappa\|_{\frac{7}{2}} \\ &\le \|u_0\|_{\frac{7}{2}} + C_\kappa \int_0^t \|v\|_{\frac{7}{2}}\|\bar\eta^\kappa\|_{\frac{7}{2}}. \end{aligned} \tag{18.5}$$

With the definition of C_T, this yields

$$\sup_{[0,T]} \|\tilde v(t)\|_{\frac{7}{2}} \le \|u_0\|_{\frac{7}{2}} + C_\kappa T.$$

Now, for $T_\kappa \in (0,T)$ such that $T_\kappa C_\kappa \le 1$, we see that $\tilde v \in C_{T_\kappa}$, which ensures that the closed convex set C_{T_κ} is stable under the mapping $v \to \tilde v$. We could also show that this mapping is also sequentially weakly continuous in $L^2(0,T_\kappa; H^{\frac{7}{2}}(\Omega))$. Therefore, by the Tychonoff fixed point theorem, there exists a fixed point $v=\tilde v$ in C_{T_κ}. Now, to see the uniqueness of this fixed point, we see that if another fixed point $\check v$ existed, we would have by the linearity of the mapping $v \to p$ and the estimates (18.4) and (18.5) an inequality of the type

$$\|(v-\check v)(t)\|_{\frac{7}{2}} \le C\int_0^t \|v-\check v\|_{\frac{7}{2}},$$

which establishes the uniqueness of the fixed point. By construction, if we denote $u = v\circ(\bar\eta^\kappa)^{-1}$, this fixed point satisfies the equation on $(0,T_\kappa)$:

$$u_t + \bar u_i^\kappa u,_i + \nabla p = 0 \text{ in } \bar\eta^\kappa(t,\Omega).$$

Besides the definition of p in (18.1), we have

$$\operatorname{div} u_t + \bar u_i^\kappa \operatorname{div} u,_i = 0 \text{ in } \bar\eta^\kappa(t,\Omega),$$

i.e.,

$$\operatorname{div} u(t,\bar\eta^\kappa(t,x)) = \operatorname{div} u_0(x) = 0 \text{ in } \Omega.$$

This shows precisely that $u = v\circ(\bar\eta^\kappa)^{-1}$ is the unique solution of the linear system (17.2) on $(0,T_\kappa)$.

Now, we see that we again have a mapping $\bar v \to v$ from C_{T_κ} into itself, which is also sequentially weakly lower semi-continuous. It therefore also has a fixed point v_κ in C_{T_κ}, which is a solution of (17.1).

In the following we study the limit as $\kappa \to 0$ of the time of existence T_κ and of v_κ. We will also denote for the sake of conciseness v_κ, $u_\kappa = v_\kappa \circ \eta^{\kappa-1}$ and $(u_\kappa)^\kappa$ respectively by $\tilde v$, $\tilde u$ and $\tilde u^\kappa$.

19. Conventions about constants, the time of existence T_κ, and the dimension of the space

From now on, until Section 26, we shall stay in $\mathbb{R}^2$ for the sake of notational convenience. In Section 26, we shall explain the differences for the three dimensional case. In the remainder of the paper, we will denote any constant depending on $\|u_0\|_{\frac{9}{2}}$ as $N(u_0)$. So, for instance, with q_0 a solution of

$$\triangle q_0 = -u_0,^i_j\, u_0,^j_i \text{ in } \Omega, \tag{19.1a}$$

$$q_0 = 0 \text{ on } \Gamma, \tag{19.1b}$$

we have by elliptic regularity $\|q_0\|_{\frac{9}{2}} \le N(u_0)$ (since Ω is assumed in $H^{\frac{9}{2}}$ until Section 26).

We will also denote generic constants by the letter C. Moreover, we will denote

$$\|\Omega\|_s = \sum_{i=1}^{K} \|\theta_i\|_{s,(0,1)^2}.$$

Furthermore, the time $T_\kappa > 0$ will be chosen small enough so that on $[0, T_\kappa]$, we have for our solution $\tilde{v}$ given by Section 18:

$$\frac{1}{2} \le \det\nabla\tilde{\eta}^\kappa \le \frac{3}{2} \text{ in } \Omega, \tag{19.2a}$$

$$\|\tilde{\eta}\|_3 \le |\Omega| + 1, \tag{19.2b}$$

$$\|\tilde{q}\|_3 \le \|q_0\|_3 + 1 \tag{19.2c}$$

$$\|\tilde{v}\|_{\frac{5}{2}} \le \|u_0\|_{\frac{5}{2}} + 1. \tag{19.2d}$$

The right-hand sides appearing in the last three inequalities shall be denoted by a generic constant C in the estimates that we will perform. In what follows, we will prove that this can be achieved in a time independent of κ.

20. A continuous in time space energy appropriate for the asymptotic process

Definition 20.1. We choose $0 \le \xi \in \mathcal{C}^\infty(\overline{\Omega})$ such that $\text{Supp}\xi \subset \bigcap_{i=K+1}^L [\text{Supp}\alpha_i]^c$ and $\xi = 1$ in a neighborhood of Γ. We then pick $0 \le \beta \in \mathcal{D}(\Omega)$ such that $\beta = 1$ on $[\text{Supp}\xi]^c$. We then define on $[0, T_\kappa]$:

$$\begin{aligned}\tilde{E}(t) = &\sup_{[0,t]}\Big[\sum_{l=1}^{K} \|\sqrt{\alpha_l}(\theta_l)\tilde{\eta}_{l\kappa}\|_{\frac{7}{2},(0,1)^2} + \|\tilde{\eta}^\kappa\|_{\frac{7}{2}} + \|\beta\tilde{\eta}\|_{\frac{7}{2}} + \|\tilde{v}\|_3 + \|\tilde{q}\|_{\frac{7}{2}}\Big] \\ &+ \sup_{[0,t]} \sum_{l=1}^{K} \Big[\kappa\|\sqrt{\alpha_l}\tilde{v}\circ\theta_l\|_{\frac{7}{2},(0,1)^2} + \kappa^{\frac{3}{2}}\|\sqrt{\alpha_l}\tilde{v}\circ\theta_l\|_{4,(0,1)^2}\Big] + 1\,.\end{aligned} \tag{20.1}$$

Remark 14. Note the presence of κ-dependent coefficients in $\tilde{E}(t)$ that indeed arise as a necessity for our asymptotic study. The corresponding terms, without the κ, would of course not be asymptotically controlled.

Remark 15. The 1 is added to the norm to ensure that $\tilde{E} \ge 1$, which will sometimes be convenient, whereas not necessary.

Now, since from Section 18, $\tilde{v} \in \mathcal{C}^0(0, T_\kappa; H^{\frac{7}{2}}(\Omega))$ (in a way not controlled asymptotically, which does not matter for our purpose), we have $\tilde{\eta} \in \mathcal{C}^0([0, T_\kappa]; H^{\frac{7}{2}}(\Omega))$. Next with the definition (18.1), and the definition of our fixed point $\tilde{v}$, we have for $\tilde{p} = \tilde{q} \circ (\tilde{\eta}^\kappa)^{-1}$:

$$\begin{aligned}\triangle\tilde{p} &= -\tilde{u}^\kappa_i,_j\, \tilde{u}_j,_i \text{ in } \tilde{\eta}^\kappa(t, \Omega), \\ \tilde{p} &= 0 \text{ on } \tilde{\eta}^\kappa(t, \Gamma),\end{aligned}$$

which shows that $\tilde{q} \in \mathcal{C}^0([0,T_\kappa]; H^{\frac{7}{2}}(\Omega))$. Consequently, $\tilde{E}$ is a continuous function on $[0,T_\kappa]$.

We will then prove that this continuous in time energy is controlled by the same type of polynomial law as (41) of [8], which will provide a control independent of κ in a time independent of κ.

21. A commutation-type lemma

We will need the following lemma in order to later identify exact in time energy laws from terms arising from our convolution by horizontal layers:

Lemma 21.1. *Let $\delta_0 > 0$ be given. Independently of $\kappa \in (0,\delta_0)$, there exists $C > 0$ such that for any $g \in H^{\frac{1}{2}}((0,1)^2)$ and $f \in H^{\frac{5}{2}}((0,1)^2)$ such that*

$$\delta_0 < \min(dist(supp\ fg, \{1\}\times[0,1]), dist(supp\ fg, \{0\}\times[0,1])),$$

we have,

$$\begin{aligned}&\left\|\rho_{\frac{1}{\kappa}} \star_h [fg] - f\rho_{\frac{1}{\kappa}} \star_h g\right\|_{\frac{1}{2},(0,1)^2}\\ &\qquad \le C\ \|\kappa g\|_{\frac{1}{2},(0,1)^2}\|f\|_{\frac{5}{2},(0,1)^2} + C\kappa^{\frac{1}{2}}\ \|g\|_{0,(0,1)^2}\|f\|_{\frac{5}{2},(0,1)^2}.\end{aligned}$$

Proof. Let $\Delta = \rho_{\frac{1}{\kappa}} \star_h [fg] - f\ \rho_{\frac{1}{\kappa}} \star_h g$. Then, we have

$$\Delta(x) = \int_{x_1-\kappa}^{x_1+\kappa} \rho_{\frac{1}{\kappa}}(x_1-y_1)[f(y_1,x_2)-f(x_1,x_2)]\ g(y_1,x_2)\ dy_1,$$

this integral being well defined because of our condition on the support of fg. We then have, since $H^{\frac{3}{2}}$ is embedded in L^∞ in $2d$,

$$|\Delta(x)| \le C\kappa\|f\|_{\frac{5}{2},(0,1)^2}\int_{x_1-\kappa}^{x_1+\kappa} \rho_{\frac{1}{\kappa}}(x_1-y_1)\ |g(y_1,x_2)|\ dy_1,$$

showing that

$$\begin{aligned}\|\Delta\|_{0,(0,1)^2} &\le C\kappa\|f\|_{\frac{5}{2},(0,1)^2}\|\rho_{\frac{1}{\kappa}} \star_h |g|\|_{0,(0,1)^2}\\ &\le C\kappa\|f\|_{\frac{5}{2},(0,1)^2}\|g\|_{0,(0,1)^2}.\end{aligned} \tag{21.1}$$

Now, let $p \in \{1,2\}$. Then,

$$\Delta,_p = \rho_{\frac{1}{\kappa}} \star_h [fg,_p] - f\ \rho_{\frac{1}{\kappa}} \star_h g,_p + \rho_{\frac{1}{\kappa}} \star_h [f,_p g] - f,_p\ \rho_{\frac{1}{\kappa}} \star_h g.$$

The difference between the two first terms of the right-hand side of this identity can be treated in a similar fashion as (21.1), leading us to

(21.2)

$$\begin{aligned}\|\Delta,_p\|_{0,(0,1)^2} &\le C\kappa\|f\|_{\frac{5}{2},(0,1)^2}\|g\|_{1,(0,1)^2} + \|\rho_{\frac{1}{\kappa}} \star_h [f,_p g]\|_{0,(0,1)^2}\\ &\quad + \|f,_p\ \rho_{\frac{1}{\kappa}} \star_h g\|_{0,(0,1)^2}\\ &\le C\kappa\|f\|_{\frac{5}{2},(0,1)^2}\|g\|_{1,(0,1)^2} + \|f,_p g\|_{0,(0,1)^2}\\ &\quad + \|f,_p\|_{L^\infty(0,1)^2}\|\rho_{\frac{1}{\kappa}} \star_h g\|_{0,(0,1)^2}\\ &\le C\kappa\|f\|_{\frac{5}{2},(0,1)^2}\|g\|_{1,(0,1)^2} + 2\|f,_p\|_{L^\infty((0,1)^2)}\|g\|_{0,(0,1)^2}\\ &\le C\kappa\|f\|_{\frac{5}{2},(0,1)^2}\|g\|_{1,(0,1)^2} + C\|f\|_{\frac{5}{2},(0,1)^2}\|g\|_{0,(0,1)^2}.\end{aligned}$$

Consequently, we obtain by interpolation from (21.1) and (21.2):

$$\|\Delta\|_{\frac{1}{2},(0,1)^2} \leq C\kappa \|f\|_{\frac{5}{2},(0,1)^2}\|g\|_{\frac{1}{2},(0,1)^2} + C\kappa^{\frac{1}{2}}\|f\|_{\frac{5}{2},(0,1)^2}\|g\|_{0,(0,1)^2}.$$

□

We then infer the following result, whose proof follows the same patterns as the previous one:

Lemma 21.2. *Let $\delta_0 > 0$ be given. Independently of $\kappa \in (0, \delta_0)$, there exists $C > 0$ such that for any $g \in H^s((0,1)^2)$ $(s = \frac{3}{2}, \frac{5}{2})$ and for any $f \in H^{\frac{5}{2}}((0,1)^2)$ such that*

$$\delta_0 < \min(dist(supp\ fg, \{1\} \times [0,1]), dist(supp\ fg, \{0\} \times [0,1])),$$

we have

$$\begin{aligned} &\left\|\rho_{\frac{1}{\kappa}} \star_h [fg] - f\rho_{\frac{1}{\kappa}} \star_h g\right\|_{s,(0,1)^2} \\ &\qquad \leq C\ \|\kappa g\|_{s,(0,1)^2}\|f\|_{\frac{5}{2},(0,1)^2} + C\kappa^{\frac{1}{2}}\|g\|_{s-\frac{1}{2},(0,1)^2}\|f\|_{\frac{5}{2},(0,1)^2}. \end{aligned}$$

22. Asymptotic regularity of the divergence and curl of $\tilde\eta_{l\kappa}$

In this section, we state the necessary a priori controls that we have on the divergence and curl of various transformations of $\tilde v$ and $\tilde\eta$. This process has to be justified again, since the functional framework substantially differs from the case with surface tension, in this case with one time derivative on the velocity corresponding to half a space derivative.

We will base our argument on the fact that the divergence and curl of $\tilde u$ satisfy the following transport type equations:

$$D_t \mathrm{div}\tilde u = 0, \tag{22.1a}$$

$$D_t \mathrm{curl}\tilde u + \tilde u^\kappa_i,_1 \tilde u^2,_i - \tilde u^\kappa_i,_2 \tilde u^1,_i = 0. \tag{22.1b}$$

We now study the consequences of these relations on the divergence and curl of $\tilde\eta$ in the interior of Ω, and of each $\tilde\eta_{l\kappa}$ $(1 \leq l \leq N)$.

22.1. Estimate for $\mathrm{div}(\beta\tilde\eta),_s$.

From (22.1a), we then infer in Ω that $(\tilde a^\kappa)^j_i \tilde v,^i_j = 0$. Thus, for $s = 1, 2$

$$(\tilde a^\kappa)^j_i (\beta\tilde v),^i_{sj} = -\beta(\tilde a^\kappa)^j_i,_s \tilde v,^i_j + [(\tilde a^\kappa)^j_i(\beta\tilde v),^i_{sj} - \beta(\tilde a^\kappa)^j_i \tilde v,^i_{sj}],$$

and by integration in time,

$$\begin{aligned} (\tilde a^\kappa)^j_i(\beta\tilde\eta),^i_{sj}(t) = (\tilde a^\kappa)^j_i(\beta\tilde\eta),^i_{sj}(0) + \int_0^t [-\beta(\tilde a^\kappa)^j_i,_s \tilde v,^i_j + ((\tilde a^\kappa)^j_i(\beta\tilde v),^i_{sj} - \beta(\tilde a^\kappa)^j_i\tilde v,^i_{sj})] \\ + \int_0^t (\tilde a^\kappa)^j_{i\,t}(\beta\tilde\eta),^i_{sj}. \end{aligned} \tag{22.2}$$

Consequently,

$$\begin{aligned}\operatorname{div}(\beta\tilde\eta),_s(t) =&[-(\tilde a^\kappa)^j_i+\delta^j_i](\beta\tilde\eta),^i_{sj}(t)+(\tilde a^\kappa)^j_i(\beta\tilde\eta),^i_{sj}(0)+\int_0^t(\tilde a^\kappa)^j_{i\,t}(\beta\tilde\eta),^i_{sj}\\ &+\int_0^t[-\beta(\tilde a^\kappa)^j_i,_s\tilde v,^i_j+((\tilde a^\kappa)^j_i(\beta\tilde v),^i_{sj}-\beta(\tilde a^\kappa)^j_i\tilde v,^i_{sj})]\\ =&[-\int_0^t(\tilde a^\kappa)^j_{i\,t}](\beta\tilde\eta),^i_{sj}(t)+(\tilde a^\kappa)^j_i(\beta\tilde\eta),^i_{sj}(0)+\int_0^t(\tilde a^\kappa)^j_{i\,t}(\beta\tilde\eta),^i_{sj}\\ &+\int_0^t[-\beta(\tilde a^\kappa)^j_i,_s\tilde v,^i_j+((\tilde a^\kappa)^j_i(\beta\tilde v),^i_{sj}-\beta(\tilde a^\kappa)^j_i\tilde v,^i_{sj})],\end{aligned}\tag{22.3}$$

showing that

$$\begin{aligned}&\|\operatorname{div}(\beta\tilde\eta),_s(t)\|_{\frac32}\\ &\quad\le Ct\sup_{[0,t]}[\,\|\tilde a^\kappa_t\|_{\frac32}\,\|\beta\tilde\eta\|_{\frac72}]+C+Ct\sup_{[0,t]}[\,\|\tilde a^\kappa\|_{\frac52}\|\tilde v\|_{\frac52}]\le Ct\tilde E(t)+C,\end{aligned}\tag{22.4}$$

where we have used our convention stated in Section 19.

22.2. **Estimate for** $\operatorname{div}[\tilde\eta_{l\kappa},_s\circ\theta_l^{-1}\circ(\tilde\eta^\kappa)^{-1}]$**.** Since $\det\nabla\theta_l=1$, we then infer in $(0,1)^2$, with $\tilde b^\kappa_l=[\nabla\tilde\eta^\kappa\circ\theta_l]^{-1}$, that

$$(\tilde b^\kappa_l)^j_i(\tilde v\circ\theta_l),^i_j=0.$$

Therefore, as for (22.2),

$$\begin{aligned}(\tilde b^\kappa_l)^j_i((\sqrt{\alpha_l}\tilde\eta)\circ\theta_l),^i_{sj}(t)=&\,(\tilde b^\kappa_l)^j_i(\sqrt{\alpha_l}\tilde\eta\circ\theta_l),^i_{sj}(0)\\ &+\int_0^t(\tilde b^\kappa_l)^j_{i\,t}(\sqrt{\alpha_l}\tilde\eta\circ\theta_l),^i_{sj}-\int_0^t\sqrt{\alpha_l}(\theta_l)(\tilde b^\kappa_l)^j_i,_s(\tilde v\circ\theta_l),^i_j\\ &+\int_0^t[(\tilde b^\kappa_l)^j_i(\sqrt{\alpha_l}\tilde v\circ\theta_l),^i_{sj}-\sqrt{\alpha_l}(\theta_l)(\tilde b^\kappa_l)^j_i(\tilde v\circ\theta_l),^i_{sj}].\end{aligned}\tag{22.5}$$

Consequently,

$$\begin{aligned}&\rho_{\frac1\kappa}\star_h[(\tilde b^\kappa_l)^j_i((\sqrt{\alpha_l}\tilde\eta)\circ\theta_l),^i_{sj}](t)\\ &\quad=\rho_{\frac1\kappa}\star_h\left[(\tilde b^\kappa_l)^j_i(\sqrt{\alpha_l}\tilde\eta\circ\theta_l),^i_{sj}(0)\right]\\ &\quad\quad+\int_0^t\rho_{\frac1\kappa}\star_h\left[(\tilde b^\kappa_l)^j_{i\,t}(\sqrt{\alpha_l}\tilde\eta\circ\theta_l),^i_{sj}-\sqrt{\alpha_l}(\theta_l)(\tilde b^\kappa_l)^j_i,_s(\tilde v\circ\theta_l),^i_j\right]\\ &\quad\quad+\int_0^t\rho_{\frac1\kappa}\star_h\left[(\tilde b^\kappa_l)^j_i(\sqrt{\alpha_l}\tilde v\circ\theta_l),^i_{sj}-\sqrt{\alpha_l}(\theta_l)(\tilde b^\kappa_l)^j_i(\tilde v\circ\theta_l),^i_{sj}\right]\\ &=\int_0^t\rho_{\frac1\kappa}\star_h\left[(\tilde b^\kappa_l)^j_{i\,t}(\sqrt{\alpha_l}\tilde\eta\circ\theta_l),^i_{sj}\right]+R,\end{aligned}$$

with $\|R\|_{\frac{3}{2}} \le Ct\tilde{E}(t)+C$. Next, thanks to Lemma 21.2,

$$\begin{aligned}
&\big\|\rho_{\frac{1}{\kappa}}\star_h\big[(\tilde{b}_l^\kappa)_i^j(\sqrt{\alpha_l}\tilde{\eta}\circ\theta_l),_{sj}^i(t)\big]-(\tilde{b}_l^\kappa)_i^j\tilde{\eta}_{l\kappa},_{sj}^i(t)\big\|_{\frac{3}{2},(0,1)^2}\\
&\le C\|(\tilde{b}_l^\kappa)_i^j\|_{\frac{5}{2},(0,1)^2}\ \|\kappa(\sqrt{\alpha_l}\tilde{\eta}\circ\theta_l),_{sj}^i(0)+\kappa\int_0^t(\sqrt{\alpha_l}\tilde{v}\circ\theta_l),_{sj}^i\|_{\frac{3}{2},(0,1)^2}\\
&\quad+C\|(\tilde{b}_l^\kappa)_i^j\|_{\frac{5}{2},(0,1)^2}\kappa^{\frac{1}{2}}\|(\sqrt{\alpha_l}\tilde{\eta}\circ\theta_l),_{sj}^i(0)+\int_0^t(\sqrt{\alpha_l}\tilde{v}\circ\theta_l),_{sj}^i\|_{1,(0,1)^2}]\\
&\le C\kappa^{\frac{1}{2}}\tilde{E}(t)+Ct\tilde{E}(t)^2+C. \qquad (22.6)
\end{aligned}$$

By successively integrating by parts in time and using Lemma 21.2,

(22.7)

$$\begin{aligned}
&\Big\|\int_0^t\rho_{\frac{1}{\kappa}}\star_h\big[(\tilde{b}_l^\kappa)_{i\,t}^j(\sqrt{\alpha_l}\tilde{\eta}\circ\theta_l),_{sj}^i(t)\big]-\int_0^t(\tilde{b}_l^\kappa)_{i\,t}^j\rho_{\frac{1}{\kappa}}\star_h\big[(\sqrt{\alpha_l}\tilde{\eta}\circ\theta_l),_{sj}^i\big]\Big\|_{\frac{3}{2},(0,1)^2}\\
&\quad\le\Big\|\int_0^t\rho_{\frac{1}{\kappa}}\star_h\big[(\tilde{b}_l^\kappa)_i^j(\sqrt{\alpha_l}\tilde{v}\circ\theta_l),_{sj}^i(t)\big]-\int_0^t(\tilde{b}_l^\kappa)_i^j\rho_{\frac{1}{\kappa}}\star_h\big[(\sqrt{\alpha_l}\tilde{v}\circ\theta_l),_{sj}^i\big]\Big\|_{\frac{3}{2},(0,1)^2}\\
&\qquad+\Big\|\Big[\rho_{\frac{1}{\kappa}}\star_h\big[(\tilde{b}_l^\kappa)_i^j(\sqrt{\alpha_l}\tilde{\eta}\circ\theta_l),_{sj}^i(t)\big]-(\tilde{b}_l^\kappa)_i^j\rho_{\frac{1}{\kappa}}\star_h\big[(\sqrt{\alpha_l}\tilde{\eta}\circ\theta_l),_{sj}^i\big]\Big]_0^t\Big\|_{\frac{3}{2},(0,1)^2}\\
&\quad\le C\kappa^{\frac{1}{2}}\tilde{E}(t)+Ct\tilde{E}(t)^2+C.
\end{aligned}$$

Consequently, with (22.5), (22.7) and (22.6), we infer

$$\begin{aligned}
\Big\|\mathrm{div}[\tilde{\eta}_{l\kappa},_s\circ\theta_l^{-1}\circ(\tilde{\eta}^\kappa)^{-1}](\tilde{\eta}^\kappa\circ\theta_l)(t)-\int_0^t(\tilde{b}_l^\kappa)_{i\,t}^j\rho_{\frac{1}{\kappa}}\star_h\big[(\sqrt{\alpha_l}\tilde{\eta}\circ\theta_l),_{sj}^i\big]\Big\|_{\frac{3}{2},(0,1)^2}&\\
\le C\kappa^{\frac{1}{2}}\tilde{E}(t)+Ct\tilde{E}(t)^2+C&,
\end{aligned}$$

showing that

$$\big\|\mathrm{div}[\tilde{\eta}_{l\kappa},_s\circ\theta_l^{-1}\circ(\tilde{\eta}^\kappa)^{-1}]\big\|_{\frac{3}{2},\tilde{\eta}^\kappa(\theta_l((0,1)^2))}\le C\kappa^{\frac{1}{2}}\tilde{E}(t)+Ct\tilde{E}(t)^2+C. \qquad (22.8)$$

We now study the curl of the same vector fields as in the two previous subsections.

22.3. **Estimate for** $\mathrm{curl}(\beta\tilde{\eta}),_s$**.** From (22.1b), we obtain:

$$(\tilde{a}^\kappa)_1^j\tilde{v},_j^2-(\tilde{a}^\kappa)_2^j\tilde{v},_j^1=\mathrm{curl}\,\tilde{u}(0)+\int_0^t[\,-\tilde{v}^\kappa,_j^i(\tilde{a}^\kappa)_1^j\tilde{v},_k^2(\tilde{a}^\kappa)_i^k+\tilde{v}^\kappa,_j^i(\tilde{a}^\kappa)_2^j\tilde{v},_k^1(\tilde{a}^\kappa)_i^k].$$

Therefore, for $s=1,2$,

$$\begin{aligned}
(\tilde{a}^\kappa)_1^j(\beta\tilde{v}),_{sj}^2-(\tilde{a}^\kappa)_2^j(\beta\tilde{v}),_{sj}^1=&\beta\,\mathrm{curl}\,\tilde{u}(0),_s-\beta[(\tilde{a}^\kappa)_1^j,_s(\tilde{v}),_j^2-(\tilde{a}^\kappa)_2^j,_s(\tilde{v}),_j^1]\\
&+(\tilde{a}^\kappa)_1^j[(\beta\tilde{v}),_{sj}^2-\beta\tilde{v},_{sj}^2]-(\tilde{a}^\kappa)_2^j[(\beta\tilde{v}),_{sj}^1-\beta\tilde{v},_{sj}^1]\\
&+\int_0^t\beta[\,-\tilde{v}^\kappa,_j^i(\tilde{a}^\kappa)_1^j\tilde{v},_k^2(\tilde{a}^\kappa)_i^k+\tilde{v}^\kappa,_j^i(\tilde{a}^\kappa)_2^j\tilde{v},_k^1(\tilde{a}^\kappa)_i^k],_s\,,
\end{aligned}$$

which implies by integration in time that

$$\begin{aligned}
(\tilde{a}^\kappa)_1^j(\beta\tilde{\eta}),_{sj}^2-(\tilde{a}^\kappa)_2^j(\beta\tilde{\eta}),_{sj}^1=&\int_0^t[(\tilde{a}_t^\kappa)_1^j(\beta\tilde{\eta}),_{sj}^2-(\tilde{a}_t^\kappa)_2^j(\beta\tilde{\eta}),_{sj}^1]+t\beta\,\mathrm{curl}\,\tilde{u}(0),_s\\
&-\beta\int_0^t[(\tilde{a}^\kappa)_1^j,_s(\tilde{v}),_j^2-(\tilde{a}^\kappa)_2^j,_s(\tilde{v}),_j^1]+\int_0^t[f+g]\\
&+\int_0^t(\tilde{a}^\kappa)_1^j[(\beta\tilde{v}),_{sj}^2-\beta\tilde{v},_{sj}^2]-\int_0^t(\tilde{a}^\kappa)_2^j[(\beta\tilde{v}),_{sj}^1-\beta\tilde{v},_{sj}^1],
\end{aligned}$$

with $f(t')=\int_0^{t'}\beta[\,-\tilde v^\kappa,_j^i(\tilde a^\kappa)_1^j\tilde v,_k^2(\tilde a^\kappa)_i^k],_s$ and $g(t')=\int_0^{t'}\beta[\tilde v^\kappa,_j^i(\tilde a^\kappa)_2^j\tilde v,_k^1(\tilde a^\kappa)_i^k],_s$. Now, since $H^{\frac32}$ is a Banach algebra in 2d,

$$\begin{aligned}\big\|(\tilde a^\kappa)_2^j(\beta\tilde\eta),_{sj}^1-(\tilde a^\kappa)_1^j(\beta\tilde\eta),_{sj}^2\big\|_{\frac32}&\le C\int_0^t\|\tilde a_t^\kappa\|_{\frac32}\|\beta\tilde\eta\|_{\frac72}+t\|u_0\|_{\frac72}+\int_0^t\|\tilde a^\kappa\|_{\frac52}\|\tilde v\|_{\frac52}\\&\quad+\int_0^t\|f+g\|_{\frac32}\\&\le N(u_0)+Ct\tilde E(t)+\int_0^t\|f+g\|_{\frac32}.\end{aligned}\tag{22.9}$$

We now notice that

$$\begin{aligned}f(t)&=-\int_0^t\beta[\,\tilde v^\kappa,_{sj}^i(\tilde a^\kappa)_1^j\tilde v,_k^2(\tilde a^\kappa)_i^k+\tilde v^\kappa,_j^i(\tilde a^\kappa)_1^j\tilde v,_{sk}^2(\tilde a^\kappa)_i^k]\\&\quad-\int_0^t\beta\tilde v^\kappa,_j^i\tilde v,_k^2[(\tilde a^\kappa)_1^j(\tilde a^\kappa)_i^k],_s\\&=\int_0^t\beta[\,\tilde\eta^\kappa,_{sj}^i[(\tilde a^\kappa)_1^j\tilde v,_k^2(\tilde a^\kappa)_i^k]_t+\tilde\eta,_{sk}^2[\tilde v^\kappa,_j^i(\tilde a^\kappa)_1^j(\tilde a^\kappa)_i^k]_t]\\&\quad-\int_0^t\beta\tilde v^\kappa,_j^i\tilde v,_k^2[(\tilde a^\kappa)_1^j(\tilde a^\kappa)_i^k],_s\\&\quad+\big[\beta[\,\tilde\eta^\kappa,_{sj}^i(\tilde a^\kappa)_1^j\tilde v,_k^2(\tilde a^\kappa)_i^k+\tilde\eta,_{sk}^2\tilde v^\kappa,_j^i(\tilde a^\kappa)_1^j(\tilde a^\kappa)_i^k]\big]_0^t,\end{aligned}$$

which allows us to infer that

$$\begin{aligned}\|f(t)\|_{\frac32}&\le\int_0^t(\|\tilde\eta^\kappa\|_{\frac72}+\|\beta\tilde\eta\|_{\frac72}+\|\tilde\eta\|_{\frac52})[\|\tilde a_t^\kappa\|_{\frac32}\|\tilde a^\kappa\|_{\frac32}\|\tilde v\|_{\frac52}+\|\tilde a^\kappa\|_{\frac32}\|\tilde a^\kappa\|_{\frac32}\|\tilde v_t\|_{\frac52}]\\&\quad+\int_0^t\|\tilde v\|_{\frac52}^2\|\tilde a^\kappa\|_{\frac52}^2+(\|\tilde\eta^\kappa\|_{\frac72}+\|\beta\tilde\eta\|_{\frac72}+\|\tilde\eta\|_{\frac52})(t)\|\tilde a^\kappa(t)\|_{\frac32}^2\|\tilde v\|_{\frac52}+N(u_0)\\&\le Ct\tilde E(t)^2+N(u_0)+C\tilde E(t).\end{aligned}$$

Since $g(t)$ can be estimated in a similar fashion, (22.9) provides us with

$$\big\|(\tilde a^\kappa)_2^j(\beta\tilde\eta),_{sj}^1-(\tilde a^\kappa)_1^j(\beta\tilde\eta),_{sj}^2\big\|_{\frac32}\le Ct\tilde E(t)^2+N(u_0),$$

showing that

$$\begin{aligned}\big\|\operatorname{curl}(\beta\tilde\eta),_s\big\|_{\frac32}&\le\Big\|\Big[\int_0^t(\tilde a_t^\kappa)_2^j\Big](\beta\tilde\eta),_{sj}^1-\Big[\int_0^t(\tilde a_t^\kappa)_1^j\Big](\beta\tilde\eta),_{sj}^2\Big\|_{\frac32}+Ct\tilde E(t)^2+N(u_0)\\&\le Ct\tilde E(t)^2+N(u_0).\end{aligned}\tag{22.10}$$

22.4. **Estimate for** $\operatorname{curl}[\tilde\eta_{l\kappa},_s\circ\theta_l^{-1}\circ(\tilde\eta^\kappa)^{-1}]$**.** In a similar fashion as we obtained (22.8), we also have here

$$\big\|\operatorname{curl}[\tilde\eta_{l\kappa},_s\circ\theta_l^{-1}\circ(\tilde\eta^\kappa)^{-1}]\big\|_{\frac32,\tilde\eta^\kappa(\Omega)}\le Ct\tilde E(t)^2+N(u_0).\tag{22.11}$$

22.5. **Estimate for** $\kappa\,\mathrm{div}[(\sqrt{\alpha_l}\,\mathrm{div}_t \circ\theta_l),_s \circ\theta_l^{-1}\circ(\tilde\eta^\kappa)^{-1}]$. By time differentiating (22.5) twice in time, we find:

$$\begin{aligned}(\tilde b_l^\kappa)_i^j((\sqrt{\alpha_l}\tilde v_t)\circ\theta_l),_{sj}^i(t) = & -(\tilde b_l^\kappa)_{i\,tt}^j((\sqrt{\alpha_l}\tilde\eta)\circ\theta_l),_{sj}^i(t) - 2(\tilde b_l^\kappa)_{i\,t}^j((\sqrt{\alpha_l}\tilde v)\circ\theta_l),_{sj}^i(t)\\ & + [(\tilde b_l^\kappa)_{i\,t}^j(\sqrt{\alpha_l}\tilde\eta\circ\theta_l),_{sj}^i]_t - [\sqrt{\alpha_l}(\theta_l)(\tilde b_l^\kappa)_i^j,_s(\tilde v\circ\theta_l),_j^i]_t\\ & + [(\tilde b_l^\kappa)_i^j(\sqrt{\alpha_l}\tilde v\circ\theta_l),_{sj}^i - \sqrt{\alpha_l}(\theta_l)(\tilde b_l^\kappa)_i^j(\tilde v\circ\theta_l),_{sj}^i]_t. \end{aligned}\tag{22.12}$$

Therefore,

$$\begin{aligned}&\kappa\|(\tilde b_l^\kappa)_i^j((\sqrt{\alpha_l}\tilde v_t)\circ\theta_l),_{sj}^i(t)\|_{\frac32,(0,1)^2}\\ &\quad\le\; C\kappa\tilde E(t)^2 + \|(\tilde b_l^\kappa)_{i\,t}^j\|_{\frac32,(0,1)^2}\|\kappa((\sqrt{\alpha_l}\tilde v)\circ\theta_l),_{sj}^i(t)\|_{\frac32,(0,1)^2}\\ &\qquad + \|\kappa[(\tilde b_l^\kappa)_i^j,_s]_t\|_{\frac32,(0,1)^2}\|(\sqrt{\alpha_l}\tilde v\circ\theta_l),_j^i\|_{\frac32,(0,1)^2}\\ &\quad\le\; C\tilde E(t)^2 + \|\kappa[(\tilde b_l^\kappa)_i^j,_s]_t\|_{\frac32,(0,1)^2}\|(\sqrt{\alpha_l}\tilde v\circ\theta_l),_j^i\|_{\frac32,(0,1)^2}.\end{aligned}\tag{22.13}$$

Next, we for instance have

$$\begin{aligned}\kappa(\tilde b_l^\kappa,_s)_{1t}^1 = & \;\kappa\frac{\tilde v_2^\kappa\circ\theta_{l,2s}}{\det\nabla(\tilde\eta^\kappa\circ\theta_l)} - \kappa[\det(\tilde\eta^\kappa\circ\theta_l)]_{t,s}\frac{\tilde\eta_2^\kappa\circ\theta_{l,2}}{\det^2\nabla(\tilde\eta^\kappa\circ\theta_l)}\\ & - \kappa[\det(\tilde\eta^\kappa\circ\theta_l)]_t\frac{\tilde\eta_2^\kappa\circ\theta_{l,s2}}{\det^2\nabla(\tilde\eta^\kappa\circ\theta_l)} - \kappa[\det(\tilde\eta^\kappa\circ\theta_l)],_s\frac{\tilde v_2^\kappa\circ\theta_{l,2}}{\det^2\nabla(\tilde\eta^\kappa\circ\theta_l)},\end{aligned}$$

which shows that

$$\|\kappa(\tilde b_l^\kappa,_s)_t\|_{\frac32,(0,1)^2}\le C\tilde E(t),$$

and with (22.13) this implies

$$\kappa\|(\tilde b_l^\kappa)_i^j((\sqrt{\alpha_l}\tilde v_t)\circ\theta_l),_{sj}^i(t)\|_{\frac32,(0,1)^2}\le C\tilde E(t)^2,$$

and thus, still by writing $\tilde\eta^\kappa(t)=\tilde\eta^\kappa(0)+\int_0^t\tilde v^\kappa$, we finally have

$$\kappa\big\|\mathrm{div}[(\sqrt{\alpha_l}\tilde v_t\circ\theta_l),_s\circ\theta_l^{-1}\circ(\tilde\eta^\kappa)^{-1}]\big\|_{\frac32,\tilde\eta^\kappa(\Omega)}\le C\tilde E(t)^2.\tag{22.14}$$

With the same type of argument we also have the following asymptotic estimates:

22.6. **Estimate for** $\kappa\,\mathrm{curl}[(\sqrt{\alpha_l}\tilde v_t\circ\theta_l),_s\circ\theta_l^{-1}\circ(\tilde\eta^\kappa)^{-1}]$.

$$\kappa\big\|\mathrm{curl}[(\sqrt{\alpha_l}\tilde v_t\circ\theta_l),_s\circ\theta_l^{-1}\circ(\tilde\eta^\kappa)^{-1}]\big\|_{\frac32,\tilde\eta^\kappa(\Omega)}\le C\tilde E(t)^2.\tag{22.15}$$

22.7. **Estimate for** $\kappa^2\,\mathrm{div}[(\sqrt{\alpha_l}\tilde v_t\circ\theta_l),_s\circ\theta_l^{-1}\circ(\tilde\eta^\kappa)^{-1}]$.

$$\kappa^2\big\|\mathrm{div}[(\sqrt{\alpha_l}\tilde v_t\circ\theta_l),_s\circ\theta_l^{-1}\circ(\tilde\eta^\kappa)^{-1}]\big\|_{\frac52,\tilde\eta^\kappa(\Omega)}\le C\tilde E(t)^2.\tag{22.16}$$

22.8. **Estimate for** $\kappa^2\,\mathrm{curl}[(\sqrt{\alpha_l}\tilde v_t\circ\theta_l),_s\circ\theta_l^{-1}\circ(\tilde\eta^\kappa)^{-1}]$.

$$\kappa^2\big\|\mathrm{curl}[(\sqrt{\alpha_l}\tilde v_t\circ\theta_l),_s\circ\theta_l^{-1}\circ(\tilde\eta^\kappa)^{-1}]\big\|_{\frac52,\tilde\eta^\kappa(\Omega)}\le C\tilde E(t)^2.\tag{22.17}$$

Remark 16. Since we will time integrate the previous quantities, the absence of a small parameter in front of $\tilde E(t)^2$ is not problematic.

22.9. **Asymptotic control of** $\operatorname{div} \tilde{u}^\kappa(\tilde{\eta}^\kappa)$**.** We have

$$\begin{aligned}\operatorname{div}\tilde{u}^\kappa(\tilde{\eta}^\kappa) &= (\tilde{a}^\kappa)^j_k \tilde{v}^\kappa,^k_j \\ &= (\tilde{a}^\kappa)^j_k[\sqrt{\alpha_i}[\rho_{\frac{1}{\kappa}} \star_h (\tilde{v}_{i\kappa})^k](\theta_i^{-1})],_j \\ &= [(\tilde{b}^\kappa_l)^j_k[\sqrt{\alpha_l}(\theta_l)\rho_{\frac{1}{\kappa}} \star_h (\tilde{v}_{l\kappa})],^k_j](\theta_l^{-1}).\end{aligned}$$

Now, thanks to Lemma 21.2, this leads us to

$$\operatorname{div}\tilde{u}^\kappa(\tilde{\eta}^\kappa) = [\rho_{\frac{1}{\kappa}} \star_h [(\tilde{b}^\kappa_l)^j_k \sqrt{\alpha_l}(\theta_l)(\tilde{v}_{l\kappa}),^k_j]](\theta_l^{-1}) + r_1,$$

with

$$\|r_1\|_{\frac{5}{2}} \le C\|\nabla\tilde{\eta}^\kappa\|_{\frac{5}{2}}(\|\kappa\nabla\tilde{v}\|_{\frac{5}{2}} + \kappa^{\frac{1}{2}}\|\nabla\tilde{v}\|_2) \le C\tilde{E}(t)^2.$$

Next, we notice that

$$\begin{aligned}(\tilde{b}^\kappa_l)^l_k(E^{l\kappa}_t),^k_l &= (\tilde{b}^\kappa_l)^j_k \rho_{\frac{1}{\kappa}} \star_h [(\sqrt{\alpha_l}\tilde{v})(\theta_l)],^k_j \\ &= \rho_{\frac{1}{\kappa}} \star_h [\sqrt{\alpha_l}(\theta_l)(\tilde{b}^\kappa_l)^j_k \tilde{v}(\theta_l),^k_j] + r_2,\end{aligned}$$

and by virtue of Lemma 21.2,

$$\|r_2\|_{\frac{5}{2}} \le C\|\nabla\tilde{\eta}^\kappa\|_{\frac{5}{2}}(\|\kappa\nabla\tilde{v}\|_{\frac{5}{2}} + \kappa^{\frac{1}{2}}\|\nabla\tilde{v}\|_2) \le C\tilde{E}(t)^2.$$

Now, since $(\tilde{b}^\kappa_l)^l_k \tilde{v}(\theta_l),^k_l = 0$ in $(0,1)^2$, this finally provides us with

$$\|\operatorname{div}\tilde{u}^\kappa(\tilde{\eta}^\kappa)\|_{\frac{5}{2}} \le C\tilde{E}(t)^2. \tag{22.18}$$

23. Asymptotic regularity of $\kappa\tilde{v}_t$ and of $\kappa^2\tilde{v}_t$

This section is devoted to the asymptotic control of $\kappa\tilde{v}_t$ and $\kappa^2\tilde{v}_t$ in spaces smoother than the natural regularity $H^{\frac{5}{2}}(\Omega)$ for $\tilde{v}_t$, the idea still being that one degree in the power of κ allows one more degree of space regularity.

23.1. **Asymptotic control of** $\kappa\tilde{v}_t$ **in** $H^{\frac{7}{2}}(\Omega)$**.** Our starting point will be the fact that since $\tilde{q} = 0$ on Γ, we have for any $l \in \{1, ..., K\}$ on $(0,1) \times \{0\}$:

$$\tilde{v}_t \circ \theta_l + \frac{\tilde{q} \circ \theta_l,_2}{\det\nabla\tilde{\eta}^\kappa(\theta_l)} \tilde{\eta}^\kappa \circ \theta_l,^\perp_1 = 0,$$

where $x^\perp = (-x_2, x_1)$. Therefore, we have on $(0,1) \times \{0\}$:

$$\begin{aligned}(\sqrt{\alpha_l}\tilde{v}_t \circ \theta_l),_{111} \cdot \tilde{\eta}^\kappa \circ \theta_l,_1 = &-[\frac{\sqrt{\alpha_l}(\theta_l)\tilde{q} \circ \theta_l,_2}{\det\nabla\tilde{\eta}^\kappa(\theta_l)}],_{11} \tilde{\eta}^\kappa \circ \theta_l,^\perp_{11} \cdot \tilde{\eta}^\kappa \circ \theta_l,_1 \\ &- [\frac{\sqrt{\alpha_l}(\theta_l)\tilde{q} \circ \theta_l,_2}{\det\nabla\tilde{\eta}^\kappa(\theta_l)}],_1 \tilde{\eta}^\kappa \circ \theta_l,^\perp_{111} \cdot \tilde{\eta}^\kappa \circ \theta_l,_1 \\ &- \frac{\sqrt{\alpha_l}(\theta_l)\tilde{q} \circ \theta_l,_2}{\det\nabla\tilde{\eta}^\kappa(\theta_l)} \tilde{\eta}^\kappa \circ \theta_l,^\perp_{1111} \cdot \tilde{\eta}^\kappa \circ \theta_l,_1 \,,\end{aligned}$$

showing that

(23.1)

$$\kappa\|(\sqrt{\alpha_l}\tilde{v}_t \circ \theta_l),_{111} \cdot \tilde{\eta}^\kappa \circ \theta_l,_1\|_{0,\partial(0,1)^2} \le \kappa\tilde{E}(t)^2 + C\kappa\|\sqrt{\alpha_l}(\theta_l)\tilde{\eta}^\kappa \circ \theta_l,^\perp_{1111}\|_{0,\partial(0,1)^2}.$$

By definition,

$$\sqrt{\alpha_l}(\theta_l)\tilde{\eta}^\kappa \circ \theta_l,_{1111} = \sqrt{\alpha_l}(\theta_l)\sum_{i=1}^K [\sqrt{\alpha_i}(\theta_l)[\rho_{\frac{1}{\kappa}} \star_h E^{i\kappa}](\theta_i^{-1} \circ \theta_l)],_{1111} \,,$$

the sum being restricted to the indices i such that $\theta_i((0,1)^2)$ and $\theta_l((0,1)^2)$ have a nonempty intersection. We then have

(23.2)

$$\sqrt{\alpha_l}(\theta_l)\tilde{\eta}^\kappa\circ\theta_l,_{1111}=\sqrt{\alpha_l}(\theta_l)[\sum_{i=1}^K\sqrt{\alpha_i}(\theta_l)[\rho_{\frac{1}{\kappa}}\star_h E^{i\kappa}],_{i_1i_2i_3i_4}(\theta_i^{-1}\circ\theta_l)a^{i_1i_2i_3i_4}_{il,1111}+\Delta],$$

with $a^{i_1i_2i_3i_4}_{il,1111}=(\theta_i^{-1}\circ\theta_l),_1^{i_1}(\theta_i^{-1}\circ\theta_l),_1^{i_2}(\theta_i^{-1}\circ\theta_l),_1^{i_3}(\theta_i^{-1}\circ\theta_l),_1^{i_4}$ and

$$\begin{aligned}\kappa\|\sqrt{\alpha_l}(\theta_l)\Delta\|_{0,\partial(0,1)^2}&\le C\kappa\|\Omega\|_{\frac{9}{2}}\sup_i\|\rho_{\frac{1}{\kappa}}\star_h[\sqrt{\alpha_i}\tilde{\eta}\circ\theta_i]\|_{3,(0,1)^2}\\&\quad+C\kappa\sup_i\|\rho_{\frac{1}{\kappa}}\star_h[\sqrt{\alpha_i}\tilde{\eta}\circ\theta_i]\|_{\frac{7}{2},(0,1)^2}\\&\le C\kappa(\|\Omega\|_{\frac{9}{2}}+1)\tilde{E}(t)\\&\le C\kappa\tilde{E}(t)^2.\end{aligned}\tag{23.3}$$

Now, we notice that for $x_1\in(0,1)$ such that $\theta_l(x_1,0)\in\theta_i((0,1)^2)$, we necessarily have, since for all $k\in\{1,...,K\}$, $\theta_k([0,1]\times\{0\})=\partial\Omega\cap\theta_k([0,1]^2)$, that $\theta_i^{-1}\circ\theta_l(x_1,0)=(f_{il}(x_1),0)$, showing that on $(0,1)\times\{0\}$, we have $\sqrt{\alpha_l}(\theta_l)a^{i_1i_2i_3i_4}_{il,1111}=0$ except when $i_1=i_2=i_3=i_4=1$. Therefore, (23.2) can be expressed as

(23.4)

$$\sqrt{\alpha_l}(\theta_l)\tilde{\eta}^\kappa\circ\theta_l,_{1111}=\sqrt{\alpha_l}(\theta_l)[\sum_{i=1}^K\sqrt{\alpha_i}(\theta_l)[\rho_{\frac{1}{\kappa}}\star_h E^{i\kappa}],_{1111}(\theta_i^{-1}\circ\theta_l)a^{1111}_{il,1111}+\Delta].$$

Now, from the properties of our convolution by layers, we have (since the derivatives are horizontal) that

$$\kappa\|[\rho_{\frac{1}{\kappa}}\star_h E^{i\kappa}],_{1111}\|_{0,(0,1)\times\{0\}}\le C\|E^{i\kappa},_{111}\|_{0,(0,1)\times\{0\}}.\tag{23.5}$$

Thus, with (23.2), (23.3) and (23.5), we infer

$$\kappa\|\sqrt{\alpha_l}(\theta_l)\tilde{\eta}^\kappa\circ\theta_l,_{1111}\|_{0,\partial(0,1)^2}\le C\kappa\tilde{E}(t)+C\tilde{E}(t),$$

which coupled with (23.1) provides us with

$$\kappa\|(\sqrt{\alpha_l}v_t\circ\theta_l),_{111}\cdot\tilde{\eta}^\kappa\circ\theta_l,_1\|_{0,\partial(0,1)^2}\le C\tilde{E}(t)^2+C.$$

This provides us the trace estimate

$$\begin{aligned}\kappa\|(\sqrt{\alpha_l}\tilde{v}_t\circ\theta_l),_1(\theta_l^{-1}\circ(\tilde{\eta}^\kappa)^{-1})\cdot\tilde{\eta}^\kappa\circ\theta_l,_1(\theta_l^{-1}\circ(\tilde{\eta}^\kappa)^{-1})\|_{2,\partial\tilde{\eta}^\kappa(\theta_l((0,1)^2))}&\\\le C\tilde{E}(t)^2+C.&\end{aligned}\tag{23.6}$$

Consequently with the divergence and curl estimates (22.14) and (22.15) and the trace estimate (23.6), we infer by elliptic regularity:

$$\kappa\|(\sqrt{\alpha_l}\tilde{v}_t\circ\theta_l),_1(\theta_l^{-1}\circ(\tilde{\eta}^\kappa)^{-1})\|_{\frac{5}{2},\tilde{\eta}^\kappa(\theta_l((0,1)^2))}\le C\tilde{E}(t)^2+C.\tag{23.7}$$

Remark 17. It is the presence of $\|\Omega\|_{\frac{9}{2}}$ in the inequalities leading to (23.3) which explains the assumption of Ω in $H^{\frac{9}{2}}$. It is, however, not essential, as will be shown in Section 26. One way to see this is to smooth the initial domain by a convolution with the parameter κ to form Ω^κ. Then, by the properties of the convolution, $\kappa\|\Omega^\kappa\|_{\frac{9}{2}}\le C\|\Omega\|_{\frac{7}{2}}$.

23.2. **Asymptotic control of $\kappa^2\tilde{v}_t$ in $H^{\frac{9}{2}}(\Omega)$.** In a similar way as in the previous subsection, we would obtain the trace estimate:

$$\kappa^2\|(\sqrt{\alpha_l}\tilde{v}_t\circ\theta_l),_1(\theta_l^{-1}\circ(\tilde{\eta}^\kappa)^{-1})\cdot\tilde{\eta}^\kappa\circ\theta_l,_1(\theta_l^{-1}\circ(\tilde{\eta}^\kappa)^{-1})\|_{3,\partial\tilde{\eta}^\kappa(\theta_l((0,1)^2))} \\ \leq C\tilde{E}(t)^2+C,$$

which coupled with (22.16) and (22.17) provides

$$\kappa^2\|(\sqrt{\alpha_l}\tilde{v}_t\circ\theta_l),_1(\theta_l^{-1}\circ(\tilde{\eta}^\kappa)^{-1})\|_{\frac{7}{2},\tilde{\eta}^\kappa(\theta_l((0,1)^2))}\leq C\tilde{E}(t)^2+C. \tag{23.8}$$

24. Basic energy law for the control of $\tilde{v}$ and $\tilde{\eta}_{l\kappa}$ independent of κ

We will use a different type of energy than in [6], namely:

Definition 24.1.

$$H^\kappa(t)=\frac{1}{2}\sum_{l=1}^K\int_{(0,1)^2}\xi_l\circ\theta_l|(\tilde{v}\circ\theta_l),_{111}|^2,$$

where $\xi_l=\xi\ \alpha_l$, ξ being defined in Section 20.

Remark 18. The main differences with respect to the energy of [6] are in the absence in our energy of any restriction to the tangent components, allowing a more convenient set of estimates, and in a setting in Lagrangian variables.

We have

$$\begin{aligned}H_t^\kappa(t)&=\sum_{l=1}^K\int_{(0,1)^2}\xi_l\circ\theta_l\tilde{v}_t\circ\theta_l,_{111}\tilde{v}\circ\theta_l,_{111}\\&=-\sum_{l=1}^K\int_{(0,1)^2}\xi_l\circ\theta_l((\tilde{a}^\kappa)_j^k\tilde{q},_k)\circ\theta_l,_{111}\tilde{v}_j\circ\theta_l,_{111}\\&=-\sum_{l=1}^K\int_{(0,1)^2}\xi_l\circ\theta_l[(\tilde{b}_l^\kappa)_j^p\tilde{q}\circ\theta_l,_p],_{111}\tilde{v}_j\circ\theta_l,_{111},\end{aligned}$$

where $\tilde{b}_l^\kappa=[\nabla(\tilde{\eta}^\kappa\circ\theta_l)]^{-1}$. Next, we see that $H_t^\kappa=-[H_1+H_2+H_3]$, with

$$\begin{aligned}H_1(t)&=\sum_{l=1}^K\int_{(0,1)^2}\xi_l(\theta_l)(\tilde{b}_l^\kappa)_j^p,_{111}\tilde{q}\circ\theta_l,_p\tilde{v}_j\circ\theta_l,_{111},\\H_2(t)&=\sum_{l=1}^K\int_{(0,1)^2}\xi_l\circ\theta_l[(\tilde{b}_l^\kappa)_j^p\tilde{q}\circ\theta_l,_{p111}]\tilde{v}_j\circ\theta_l,_{111},\\H_3(t)&=\sum_{l=1}^K\int_{(0,1)^2}\xi_l\circ\theta_l[[(\tilde{b}_l^\kappa)_j^p\tilde{q}\circ\theta_l,_p],_{111}\\&\qquad-(\tilde{b}_l^\kappa)_j^p\tilde{q}\circ\theta_l,_{l111}-(\tilde{b}_l^\kappa)_j^p,_{111}\tilde{q}\circ\theta_l,_l]\tilde{v}_j\circ\theta_l,_{111}.\end{aligned}$$

We immediately have for the third term

$$|H_3(t)|\leq C\tilde{E}(t)^2. \tag{24.1}$$

Next, for H_2, since $(\xi_l\tilde q)\circ\theta_l=0$ on $\partial(0,1)^2$,

$$H_2(t)=-\sum_{l=1}^K\int_{(0,1)^2}\xi_l\circ\theta_l\ (\tilde b_l^\kappa)_j^p\tilde q\circ\theta_l,_{111}\,\tilde v_j\circ\theta_l,_{p111}$$
$$-\sum_{l=1}^K\int_{(0,1)^2}[\xi_l\circ\theta_l(\tilde b_l^\kappa)_j^p],_p\,\tilde q\circ\theta_l,_{111}\,\tilde v_j\circ\theta_l,_{111}\,.$$

We then notice that from the divergence condition we have $(\tilde b_l^\kappa)_j^p\ \tilde v_j\circ\theta_l,_p=0$ in $(0,1)^2$, implying

$$H_2(t)=\sum_{l=1}^K\int_{(0,1)^2}\xi_l\circ\theta_l\ \tilde q\circ\theta_l,_{111}\,(\tilde b_l^\kappa)_j^p,_{111}\,\tilde v_j\circ\theta_l,_p$$
$$+\sum_{l=1}^K\int_{(0,1)^2}\xi_l\circ\theta_l\ \tilde q\circ\theta_l,_{111}\,(\tilde b_l^\kappa)_j^p,_{11}\,\tilde v_j\circ\theta_l,_{p1}$$
$$+\sum_{l=1}^K\int_{(0,1)^2}\xi_l\circ\theta_l\ \tilde q\circ\theta_l,_{111}\,(\tilde b_l^\kappa)_j^p,_{1}\,\tilde v_j\circ\theta_l,_{p11}$$
$$-\sum_{l=1}^K\int_{(0,1)^2}[\xi_l\circ\theta_l(\tilde b_l^\kappa)_j^p],_p\,\tilde q\circ\theta_l,_{111}\,\tilde v_j\circ\theta_l,_{111}\,.$$

Now, in a way similar as (8.15), we have for any $f\in H^{\frac12}((0,1)^2)$

$$\|\xi_l\circ\theta_l f,_1\|_{H^{\frac12}((0,1)^2)'}\le C\|f\|_{H^{\frac12}((0,1)^2)},$$

since the derivative is in the horizontal direction. By applying this result to $f=(\tilde b_l^\kappa)_j^p,_{11}$ for the first integral appearing in the equality above, and by using the continuous embedding of H^1 into L^6 $(6\in(1,\infty))$, and of $H^{\frac12}$ into L^3 $(3\in(1,4))$ for the other integrals, we then obtain

$$|H_2(t)|\le C\|\tilde a^\kappa\|_{\frac52}\|\tilde q\|_{\frac72}\|\tilde v\|_3\le C\tilde E(t)^3. \tag{24.2}$$

We now come to H_1, which will require more care, and will provide us with the regularity of $\tilde\eta_{l\kappa}(\Omega)$ in $H^{\frac72}$ independent of κ. We have

$$H_1(t)=\int_{(0,1)^2}\xi_l\circ\theta_l(\tilde b_l^\kappa)_j^p,_{111}\,[\tilde q\circ\theta_l],_p\ \ \tilde v_j\circ\theta_l,_{111}$$
$$=H_{11}(t)+H_{12}(t)-\int_{(0,1)^2}\xi_l\circ\theta_l\Delta_j^p\ [\tilde q\circ\theta_l],_p\ \ \tilde v_j\circ\theta_l,_{111}\,, \tag{24.3}$$

with

$$H_{11}(t)=\int_{(0,1)^2}\xi_l\circ\theta_l\frac{(\mathrm{Cof}\nabla(\tilde\eta^\kappa\circ\theta_l))_j^p,_{111}}{\det\nabla(\tilde\eta^\kappa\circ\theta_l)}[\tilde q\circ\theta_l],_p\ \ \tilde v_j\circ\theta_l,_{111}\,,$$
$$H_{12}(t)=-\int_{(0,1)^2}\xi_l\circ\theta_l(\mathrm{Cof}\nabla(\tilde\eta^\kappa\circ\theta_l))_j^p\frac{[\det\nabla(\tilde\eta^\kappa\circ\theta_l)],_{111}}{[\det\nabla(\tilde\eta^\kappa\circ\theta_l)]^2}[\tilde q\circ\theta_l],_p\ \ \tilde v_j\circ\theta_l,_{111}\,,$$

and

$$\Delta_j^p = [\frac{(\mathrm{Cof}\nabla(\tilde\eta^\kappa\circ\theta_l))_j^p}{\det\nabla(\tilde\eta^\kappa\circ\theta_l)}],_{111} - \frac{(\mathrm{Cof}\nabla(\tilde\eta^\kappa\circ\theta_l))_j^p,_{111}}{\det\nabla(\tilde\eta^\kappa\circ\theta_l)} + (\mathrm{Cof}\nabla(\tilde\eta^\kappa\circ\theta_l))_j^p\frac{[\det\nabla(\tilde\eta^\kappa\circ\theta_l)],_{111}}{[\det\nabla(\tilde\eta^\kappa\circ\theta_l)]^2},$$

so that

$$\Big|\int_{(0,1)^2}\Delta_j^p\ [(\xi_l\tilde q)\circ\theta_l],_p\ \ \tilde v_j\circ\theta_l,_{111}\Big| \le C\|\tilde v\|_3. \tag{24.4}$$

We now turn our attention to the other terms of (24.3), and to shorten notation, we will set: $\tilde Q_l = \tilde q\circ\theta_l$. We first study the perturbation H_{12}, which would not appear if the volume preserving condition was respected by our smoothing by convolution. It turns out that we do need the double convolution by layers appearing in the definition of v^κ in order to identify time derivatives of space energies. We first notice that since θ_l does not depend on t, we have

$$(\tilde\eta^\kappa\circ\theta_l)_t = \tilde u^\kappa\circ(\tilde\eta^\kappa\circ\theta_l),$$

from which we infer in $(0,1)^2$, since θ_l is volume preserving,

$$[\det(\nabla\tilde\eta^\kappa\circ\theta_l)]_t = \mathrm{div}\tilde u^\kappa(\tilde\eta^\kappa\circ\theta_l)\ \det\nabla(\tilde\eta^\kappa\circ\theta_l). \tag{24.5}$$

24.1. **Study of H_{12}.** We have after an integration by parts in time, and the use of (24.5):

$$\int_0^t H_{12} = \sum_{i=1}^3 H_{12}^i + R_{12},$$

with

$$H_{12}^1 = \int_0^t\int_{(0,1)^2}\xi_l\circ\theta_l(\mathrm{Cof}\nabla(\tilde\eta^\kappa\circ\theta_l))_j^p \times\frac{\mathrm{div}\tilde u^\kappa(\tilde\eta^\kappa\circ\theta_l)\ [\det\nabla(\tilde\eta^\kappa\circ\theta_l)],_{111}}{[\det\nabla(\tilde\eta^\kappa\circ\theta_l)]^2}\tilde Q_l,_p\ \ \tilde\eta_j\circ\theta_l,_{111}\,,$$

$$H_{12}^2 = \int_0^t\int_{(0,1)^2}\xi_l\circ\theta_l(\mathrm{Cof}\nabla(\tilde\eta^\kappa\circ\theta_l))_j^p \times\frac{[\mathrm{div}\tilde u^\kappa(\tilde\eta^\kappa\circ\theta_l)],_{111}\ \det\nabla(\tilde\eta^\kappa\circ\theta_l)}{[\det\nabla(\tilde\eta^\kappa\circ\theta_l)]^2}\tilde Q_l,_p\ \ \tilde\eta_j\circ\theta_l,_{111}\,,$$

$$H_{12}^3 = -\int_{(0,1)^2}\xi_l\circ\theta_l(\mathrm{Cof}\nabla\tilde{(}\tilde\eta^\kappa\circ\theta_l))_j^p\frac{[\det\nabla(\tilde\eta^\kappa\circ\theta_l)],_{111}}{[\det\nabla(\tilde\eta^\kappa\circ\theta_l)]^2}\tilde Q_l,_p\ \ \tilde\eta_j\circ\theta_l,_{111}\,(t),$$

and

$$|R_{12}(t)| \le Ct\tilde E(t)^3 + C. \tag{24.6}$$

24.1.1. **Study of H_{12}^1.** For the sake of conciseness, we denote

$$A_{jl} = \xi_l\circ\theta_l(\mathrm{Cof}\nabla(\tilde\eta^\kappa\circ\theta_l))_j^p\frac{\mathrm{div}\tilde u^\kappa(\tilde\eta^\kappa\circ\theta_l)}{[\det\nabla(\tilde\eta^\kappa\circ\theta_l)]^2}[\tilde q\circ\theta_l],_p\,.$$

We then see, by expanding the third space derivative of the determinant in the integrand of H_{12}^1, that $H_{12}^1 = \sum_{i=1}^4 H_{12}^{1i} + R_{12}^1$ with the H_{12}^{1i} being estimated as H_{12}^{11}

that we make precise below and R_{12}^1 being a remainder estimated as (24.6). By definition of $\tilde{\eta}^\kappa$ we have, if we denote

$$E^{i\kappa} = \rho_{\frac{1}{\kappa}} \star_h ((\sqrt{\alpha_i}\tilde{\eta}) \circ \theta_i),$$

that

$$H_{12}^{11} = \int_0^t \int_{(0,1)^2} \big[A_{jl}(\tilde{\eta}_2^\kappa \circ \theta_l),_2 \ \big[\big[\sqrt{\alpha_i}(\theta_i)\ \rho_{\frac{1}{\kappa}} \star_h E_1^{i\kappa}\big](\theta_i^{-1} \circ \theta_l)\big],_{1111} \\ \times [\tilde{\eta}_j \circ \theta_i(\theta_i^{-1} \circ \theta_l)],_{111}\big] + R_{12}^1,$$

with $|R_{12}^1| \le Ct$ and where, because of the term $\sqrt{\alpha_i}(\theta_l)$, the only indexes i and l appearing in this sum are the ones for which $\theta_l((0,1)^2) \cap \theta_i((0,1)^2) \neq \emptyset$. Only such indexes will be considered later on when such terms arise. From our assumed regularity on Ω in $H^{\frac{9}{2}}$, we then have

$$H_{12}^{11} = \int_0^t \int_{(0,1)^2} \big[A_{jl}(\tilde{\eta}_2^\kappa \circ \theta_l),_2 \ \big[\big[\rho_{\frac{1}{\kappa}} \star_h E_1^{i\kappa}\big](\theta_i^{-1} \circ \theta_l)\big],_{1111} \\ \times [\sqrt{\alpha_i}\tilde{\eta}_j \circ \theta_i(\theta_i^{-1} \circ \theta_l)],_{111}\big] + R_{12}^{11},$$

with $|R_{12}^{11}| \le Ct\tilde{E}(t)^2$. We next have, since the charts θ_i are volume preserving,

$$H_{12}^{11} = \int_0^t \int_{\theta_i^{-1}(\theta_l(0,1)^2)} \big[[A_{jl}(\tilde{\eta}_2^\kappa \circ \theta_l),_2](\theta_l^{-1} \circ \theta_i)\ [\rho_{\frac{1}{\kappa}} \star_h E_1^{i\kappa}],_{j_1j_2j_3j_4} \\ \times c_{il,1111}^{j_1j_2j_3j_4} c_{il,111}^{i_1i_2i_3}[\sqrt{\alpha_i}\tilde{\eta}_j \circ \theta_i],_{i_1i_2i_3}\big] + R,$$

with $|R| \le Ct\tilde{E}(t)^2$ and

(24.7a)
$$c_{il,111}^{j_1j_2j_3j_4} = [(\theta_i^{-1} \circ \theta_l)^{j_1},_1 (\theta_i^{-1} \circ \theta_l)^{j_2},_1 (\theta_i^{-1} \circ \theta_l)^{j_3},_1 (\theta_i^{-1} \circ \theta_l)^{j_4},_1](\theta_l^{-1} \circ \theta_i),$$

(24.7b) $$c_{il,111}^{i_1i_2i_3} = [(\theta_i^{-1} \circ \theta_l)^{i_1},_1 (\theta_i^{-1} \circ \theta_l)^{i_2},_1 (\theta_i^{-1} \circ \theta_l)^{i_3},_1](\theta_l^{-1} \circ \theta_i).$$

Next, we notice that the term $A_{jl}(\theta_l^{-1} \circ \theta_i)$ introduces a factor $\alpha_l \circ \theta_i$ which is nonzero only if $x \in \theta_i^{-1}(\theta_l(0,1)^2)$, leading us to

$$H_{12}^{11} = \int_0^t \int_{(0,1)^2} \big[[A_{jl}(\tilde{\eta}_2^\kappa \circ \theta_l),_2](\theta_l^{-1} \circ \theta_i)\ \rho_{\frac{1}{\kappa}} \star_h E_1^{i\kappa},_{j_1j_2j_3j_4} \\ \times c_{il,1111}^{j_1j_2j_3j_4} c_{il,111}^{i_1i_2i_3}[\sqrt{\alpha_i}\tilde{\eta}_j \circ \theta_i],_{i_1i_2i_3}\big] + R,$$

where $\theta_l^{-1} \circ \theta_i$ is extended outside of $\theta_i^{-1}(\theta_l(0,1)^2)$ in any fashion. This argument of replacing an integral on a subset of $(0,1)^2$ by an integral on $(0,1)^2$ will be implicitly repeated at other places later on. Now, since ρ is even,

$$H_{12}^{11} = \int_0^t \int_{(0,1)^2} E_1^{i\kappa},_{j_1j_2j_3j_4}\, \rho_{\frac{1}{\kappa}} \star \big[[A_{jl}(\tilde{\eta}_2^\kappa \circ \theta_l),_2](\theta_l^{-1} \circ \theta_i) \\ \times c_{il,1111}^{j_1j_2j_3j_4} c_{il,111}^{i_1i_2i_3}[\sqrt{\alpha_i}\tilde{\eta}_j \circ \theta_i],_{i_1i_2i_3}\big] + R. \tag{24.8}$$

Now, let us call $f = [A_{jl}(\tilde{\eta}_2^\kappa \circ \theta_l),_2](\theta_l^{-1} \circ \theta_i) c_{il,1111}^{j_1j_2j_3j_4} c_{il,111}^{i_1i_2i_3}$ and $g = [\sqrt{\alpha_j}\tilde{\eta}_1 \circ \theta_i],_{i_1i_2i_3}$. We notice that $\|f\|_{\frac{5}{2},(0,1)^2}$ is the natural norm associated to $\tilde{E}(t)$. Here we cannot directly use Lemma 21.2 for the case where all the $j_i = 3$, since $E_{3333}^{i\kappa}$ is not necessarily in $H^{\frac{1}{2}}((0,1)^2)'$ a priori. Instead, we write

$$f(y_1,x_2) = f(x_1,x_2) + (y_1 - x_1).f,_1(x_1,x_2) + \int_{x_1}^{y_1} [f,_1(x,x_2) - f,_1(x_1,x_2)]dx,$$

which shows that on $(0,1)^2$

$$\begin{aligned}\rho_{\frac{1}{\kappa}} \star_h [fg](x_1,x_2) =& f(x_1,x_2)\rho_{\frac{1}{\kappa}} \star_h g(x_1,x_2)\\ &+ f,_1(x_1,x_2)\int_{\mathbb{R}} \rho_{\frac{1}{\kappa}}(y_1-x_1)(y_1-x_1)g(y_1,x_2)dy_1\\ &+ \int_{\mathbb{R}} \rho_{\frac{1}{\kappa}}(y_1-x_1)\int_{x_1}^{y_1}[f,_1(x,x_2)-f,_1(x_1,x_2)]dx\ g(y_1,x_2)dy_1.\end{aligned}$$

This implies

(24.9)

$$\begin{aligned}H_{12}^{11} = \int_0^t \int_{(0,1)^2} &\big[[A_{jl}(\tilde{\eta}_2^\kappa\circ\theta_l),_2](\theta_l^{-1}\circ\theta_i)\ E_1^{i\kappa},_{j_1j_2j_3j_4}\\ &\times c_{il,1111}^{j_1j_2j_3j_4} c_{il,111}^{i_1i_2i_3}\rho_{\frac{1}{\kappa}} \star_h [\sqrt{\alpha_i}(\theta_i)[\tilde{\eta}_1\circ\theta_i],_{i_1i_2i_3}]\big] + R - R_1 - R_2,\end{aligned}$$

with

$$\begin{aligned}R_1 =& \int_0^t \int_{(0,1)^2} \big[E_1^{i\kappa},_{j_1j_2j_3j_4}(x_1,x_2) f,_1(x_1,x_2)\\ &\qquad\times \int_{\mathbb{R}} \rho_{\frac{1}{\kappa}}(y_1-x_1)(y_1-x_1)g(y_1,x_2)dy_1\big]\ dx_1dx_2,\\ R_2 =& \int_0^t \int_{(0,1)^2} \big[E_1^{i\kappa},_{j_1j_2j_3j_4}(x_1,x_2)\\ &\times \int_{\mathbb{R}} \rho_{\frac{1}{\kappa}}(y_1-x_1)\int_{x_1}^{y_1}[f,_1(x,x_2)-f,_1(x_1,x_2)]dx\ g(y_1,x_2)dy_1\big]\ dx_1dx_2.\end{aligned}$$

Now, for R_2, we notice that since

$$|f,_1(x,x_2) - f,_1(x_1,x_2)| \le C|f|_{\frac{5}{2},(0,1)^2}|x-x_1|^{\frac{1}{2}} \le C|\tilde{\eta}^\kappa|_{\frac{7}{2}}|x-x_1|^{\frac{1}{2}},$$

we have

(24.10)

$$\begin{aligned}R_2 \le& C\int_0^t \|\tilde{\eta}^\kappa\|_{\frac{7}{2}} \int_{(0,1)^2} \big[|E_1^{i\kappa},_{j_1j_2j_3j_4}(x_1,x_2)| \int_{\mathbb{R}} \rho_{\frac{1}{\kappa}}(y_1-x_1)\kappa^{\frac{3}{2}}|g(y_1,x_2)|dy_1\big]dx_1dx_2\\ \le& C\int_0^t \|\tilde{\eta}^\kappa\|_{\frac{7}{2}} \int_{(0,1)^2} \big[|\kappa^{\frac{3}{2}} E_1^{i\kappa},_{j_1j_2j_3j_4}|\ \rho_{\frac{1}{\kappa}} \star_h |g|\big]\\ \le& C\int_0^t \|\tilde{\eta}^\kappa\|_{\frac{7}{2}}\kappa^{\frac{3}{2}} \big[\|E_1^{i\kappa},_{j_1j_2j_3j_4}(0)\|_{0,(0,1)^2} + \int_0^t \|(E_1^{i\kappa})_t,_{j_1j_2j_3j_4}\|_{0,(0,1)^2}\big]\|g\|_{0,(0,1)^2}\\ \le& Ct\kappa^{\frac{3}{2}}\tilde{E}(t) + Ct\tilde{E}(t)^3,\end{aligned}$$

where we have used the fact that $\|\kappa^{\frac{3}{2}}\sqrt{\alpha_i}\tilde{v}_1\circ\theta_i],_{i_1i_2i_3i_4}\|_{0,(0,1)^2}$ is contained in the definition of $\tilde{E}(t)$. We now turn our attention to R_1. We first remark that

$$\begin{aligned}f,_1 =&([A_{jl}(\tilde{\eta}_2^\kappa\circ\theta_l),_2](\theta_l^{-1}\circ\theta_i)c_{il,111}^{i_1i_2i_3}),_1 c_{il,1111}^{j_1j_2j_3j_4}\\&+\sum_{n=1}^4\big[[A_{jl}(\tilde{\eta}_2^\kappa\circ\theta_l),_2](\theta_l^{-1}\circ\theta_i)c_{il,111}^{i_1i_2i_3}[(\theta_i^{-1}\circ\theta_l),_1^{j_n}(\theta_l^{-1}\circ\theta_i)],_1\\&\qquad\times\Pi_{p\neq n}(\theta_i^{-1}\circ\theta_l),_1^{j_p}(\theta_l^{-1}\circ\theta_i)\big],\end{aligned}$$

which implies that

$$R_1=R_1^1+\sum_{n=1}^4 R_1^{i_n},$$

with

$$\begin{aligned}R_1^1=&\int_0^t\int_{(0,1)^2}c_{il,1111}^{j_1j_2j_3j_4}\big[E_1^{i\kappa},_{j_1j_2j_3j_4}(x_1,x_2)([A_{jl}(\tilde{\eta}_2^\kappa\circ\theta_l),_2](\theta_l^{-1}\circ\theta_i)c_{il,111}^{i_1i_2i_3}),_1\\&\qquad\times\int_{\mathbb{R}}\rho_{\frac{1}{\kappa}}(y_1-x_1)(y_1-x_1)g(y_1,x_2)dy_1\big]\ dx_1dx_2,\\R_1^{i_n}=&\int_0^t\int_{(0,1)^2}\big[E_1^{i\kappa},_{j_1j_2j_3j_4}(x_1,x_2)A_{jl}(\tilde{\eta}_2^\kappa\circ\theta_l),_2](\theta_l^{-1}\circ\theta_i)\\&\qquad\times[(\theta_i^{-1}\circ\theta_l),_1^{j_n}(\theta_l^{-1}\circ\theta_i)],_1\Pi_{p\neq n}(\theta_i^{-1}\circ\theta_l),_1^{j_p}(\theta_l^{-1}\circ\theta_i)\\&\qquad\times c_{il,111}^{i_1i_2i_3}\int_{\mathbb{R}}\rho_{\frac{1}{\kappa}}(y_1-x_1)(y_1-x_1)g(y_1,x_2)dy_1\big]\ dx_1dx_2.\end{aligned}$$

Let us study R_1^1. If we denote $h(x_1,x_2)=\int_{\mathbb{R}}\rho_{\frac{1}{\kappa}}(y_1-x_1)(y_1-x_1)g(y_1,x_2)dy_1$, since $(E_1^{i\kappa},_{j_2j_3j_4}\circ\theta_i^{-1}\circ\theta_l),_1(\theta_l^{-1}\circ\theta_i)=(\theta_i^{-1}\circ\theta_l),_1^{j_1}(\theta_l^{-1}\circ\theta_i)E_1^{i\kappa},_{j_1j_2j_3j_4}$,

$$\begin{aligned}R_1^1=&\int_0^t\int_{(0,1)^2}c_{il,111}^{j_2j_3j_4}\big[(E_1^{i\kappa},_{j_2j_3j_4}\circ\theta_i^{-1}\circ\theta_l),_1(\theta_l^{-1}\circ\theta_i)\\&\qquad\times([A_{jl}(\tilde{\eta}_2^\kappa\circ\theta_l),_2](\theta_l^{-1}\circ\theta_i)c_{il,111}^{i_1i_2i_3}),_1 h\big]\\=&\int_0^t\int_{(0,1)^2}\big[(E_1^{i\kappa},_{j_2j_3j_4}\circ\theta_i^{-1}\circ\theta_l),_1\\&\qquad\times[c_{il,111}^{j_2j_3j_4}([A_{jl}(\tilde{\eta}_2^\kappa\circ\theta_l),_2](\theta_l^{-1}\circ\theta_i)c_{il,111}^{i_1i_2i_3}),_1 h](\theta_i^{-1}\circ\theta_l)\big].\end{aligned}$$

Since the derivative of $(E_1^{i\kappa},_{j_2j_3j_4}\circ\theta_i^{-1}\circ\theta_l)$ is in the horizontal direction, we infer similarly as in (8.15) that

$$\begin{aligned}R_1^1\leq&\int_0^t\int_{(0,1)^2}\big[\big\|E_1^{i\kappa},_{j_2j_3j_4}\circ\theta_i^{-1}\circ\theta_l\big\|_{\frac{1}{2},(0,1)^2}\\&\qquad\times\big\|[c_{il,111}^{j_2j_3j_4}([A_{jl}(\tilde{\eta}_2^\kappa\circ\theta_l),_2](\theta_l^{-1}\circ\theta_i)c_{il,111}^{i_1i_2i_3}),_1 h](\theta_i^{-1}\circ\theta_l)\big\|_{\frac{1}{2},(0,1)^2}\big].\end{aligned}$$

Since we have by interpolation $\|h\|_{\frac{1}{2},(0,1)^2}\leq C\kappa\|g\|_{\frac{1}{2},(0,1)^2}$, we then infer

$$|R_1^1|\leq Ct\tilde{E}(t)^2. \tag{24.11}$$

In a similar fashion, for $R_1^{i_1}$ we can identify a horizontal derivative

$$\begin{aligned}&E_1^{i\kappa},_{j_1j_2j_3j_4}[(\theta_i^{-1}\circ\theta_l),_1^{j_1}(\theta_l^{-1}\circ\theta_i)],_1\Pi_{p=2}^4(\theta_i^{-1}\circ\theta_l),_1^{j_p}(\theta_l^{-1}\circ\theta_i)\\&=[(\theta_i^{-1}\circ\theta_l),_1^{j_1}(\theta_l^{-1}\circ\theta_i)],_1\left[\Pi_{p=2}^3(\theta_i^{-1}\circ\theta_l),_1^{j_p}(E_1^{i\kappa},_{j_1j_2j_3}\circ\theta_i^{-1}\circ\theta_l),_1\right](\theta_l^{-1}\circ\theta_i),\end{aligned}$$

which leads for the same reasons as for R_1^1 to $|R_1^{i_1}|\le Ct\tilde{E}(t)^2$. Since the other $R_1^{i_n}$ are similar in structure, we have

$$|R_1^{i_n}|\le Ct\tilde{E}(t)^2. \tag{24.12}$$

Consequently from (24.9), (24.10), (24.11) and (24.12), we infer

$$\begin{aligned}H_{12}^{11}=\int_0^t\int_{(0,1)^2}&\Big[[A_{jl}(\tilde{\eta}_2^\kappa\circ\theta_l),_2](\theta_l^{-1}\circ\theta_i)\ E_1^{i\kappa},_{j_1j_2j_3j_4}\\&\times c_{il,1111}^{j_1j_2j_3j_4}c_{il,111}^{i_1i_2i_3}\rho_{\frac{1}{\kappa}}\star_h[\sqrt{\alpha_i}(\theta_i)[\tilde{\eta}_1\circ\theta_i],_{i_1i_2i_3}]\Big]+r_{12}^{11},\end{aligned}$$

with

$$|r_{12}^{11}(t)|\le Ct\tilde{E}(t)^3+Ct\kappa^{\frac{3}{2}}\tilde{E}(t). \tag{24.13}$$

Since $E_1^{i\kappa},_{j_1j_2j_3j_4}c_{il,1111}^{j_1j_2j_3j_4}=c_{il,111}^{j_2j_3j_4}(E_1^{i\kappa},_{j_2j_3j_4}\circ\theta_i^{-1}\circ\theta_l),_1(\theta_l^{-1}\circ\theta_i)$, we infer as for R_1^1 that

$$|H_{12}^{11}(t)|\le Ct\sup_{j,l}\sup_{[0,t]}\|A_{jl}\|_{\frac{5}{2},(0,1)^2}\|\tilde{\eta}_{l\kappa}\|_{\frac{7}{2}}^2+|r_{12}^{11}|\le Ct\tilde{E}(t)^3+Ct\kappa^{\frac{3}{2}}\tilde{E}(t).$$

The other H_{12}^{1i} are estimated in the same fashion, leading us to

$$|H_{12}^1(t)|\le Ct\tilde{E}(t)^3+Ct\kappa^{\frac{3}{2}}\tilde{E}(t). \tag{24.14}$$

24.1.2. **Study of H_{12}^2.** Next, for H_{12}^2, we first notice from the asymptotic regularity result (22.18) on $\operatorname{div}\tilde{u}^\kappa(\tilde{\eta}^\kappa)$ that H_{12}^2 can be treated in the same fashion as H_{12}^1, leading to

$$|H_{12}^2(t)|\le Ct\tilde{E}(t)^3+Ct\kappa^{\frac{3}{2}}\tilde{E}(t). \tag{24.15}$$

24.1.3. **Study of H_{12}^3.** We simply write

$$\begin{aligned}-H_{12}^3=&\int_{(0,1)^2}\xi_l(\theta_l)(\mathrm{Cof}\nabla(\tilde{\eta}^\kappa\circ\theta_l))_j^p\frac{[\det\nabla\theta_l],_{111}}{[\det\nabla(\tilde{\eta}^\kappa\circ\theta_l)]^2}\tilde{Q}_{l},_p\ \tilde{\eta}_j\circ\theta_l,_{111}(t)+R_{12}^3\\&+\int_{(0,1)^2}\xi_l(\theta_l)(\mathrm{Cof}\nabla(\tilde{\eta}^\kappa\circ\theta_l))_j^p\frac{\int_0^t[\det\nabla(\tilde{\eta}^\kappa\circ\theta_l)],_{111}\operatorname{div}(\tilde{u}^\kappa\circ\theta_l)}{[\det\nabla(\tilde{\eta}^\kappa\circ\theta_l)]^2}\tilde{Q}_{l},_p\ \tilde{\eta}_j\circ\theta_l,_{111}(t)\\&+\int_{(0,1)^2}\xi_l(\theta_l)(\mathrm{Cof}\nabla(\tilde{\eta}^\kappa\circ\theta_l))_j^p\frac{\int_0^t\det\nabla(\tilde{\eta}^\kappa\circ\theta_l)[\operatorname{div}(\tilde{u}^\kappa\circ\theta_l)],_{111}}{[\det\nabla(\tilde{\eta}^\kappa\circ\theta_l)]^2}\tilde{Q}_{l},_p\ \tilde{\eta}_j\circ\theta_l,_{111}(t),\end{aligned}$$

with R_{12}^3 being bounded by a term similar to the right-hand side of (24.15). We also see that the first term of this equality can be estimated by a bound similar to the right-hand side of (24.15). The third term is treated in a way similar as H_{12}^1, in order to put a convolution in front of $(\tilde{\eta}_j\circ\theta_l),_{111}$. There is no difference linked to the fact that the integral from 0 to t does not apply on all terms as for H_{12}^1, since $\rho_{\frac{1}{\kappa}}$ and the θ_l do not depend on time. The fourth term follows the same treatment as H_{12}^2, leading us to

$$|H_{12}^3(t)|\le Ct\tilde{E}(t)^3+Ct\kappa^{\frac{3}{2}}\tilde{E}(t),$$

which with (24.14) and (24.15) implies

$$|H_{12}(t)| \leq Ct\tilde{E}(t)^3 + Ct\kappa^{\frac{3}{2}}\tilde{E}(t). \tag{24.16}$$

24.2. **Study of H_{11}.** As for H_{11}, we have if we still denote $E^{i\kappa} = \rho_{\frac{1}{\kappa}} \star_h ((\sqrt{\alpha_i}\tilde{\eta})\circ\theta_i)$ and ϵ^{mn} the sign of the permutation between (m,n) and $(1,2)$:

$$\begin{aligned}
H_{11}(t) &= \epsilon^{mn}\epsilon^{rs}\int_{(0,1)^2} \xi_l(\theta_l)\frac{(\tilde{\eta}^\kappa_m\circ\theta_l),_{r111}}{\det\nabla(\tilde{\eta}^\kappa\circ\theta_l)}[\tilde{q}\circ\theta_l],_s\ \ \tilde{v}_n(\theta_l),_{111}\\
&= \epsilon^{mn}\epsilon^{rs}\int_{(0,1)^2}\big[\xi_l(\theta_l)\frac{[\tilde{q}\circ\theta_i(\theta_i^{-1}\circ\theta_l)],_s}{\det\nabla(\tilde{\eta}^\kappa\circ\theta_l)}\ \big[[\sqrt{\alpha_i}(\theta_i)\rho_{\frac{1}{\kappa}}\star_h E^{i\kappa}_m](\theta_i^{-1}\circ\theta_l)\big],_{r111}\\
&\qquad\qquad \times[\tilde{v}_n\circ\theta_i(\theta_i^{-1}\circ\theta_l)],_{111}\,\big]\\
&= \epsilon^{mn}\epsilon^{rs}\int_{(0,1)^2}\big[\xi_l(\theta_i)\frac{[\tilde{q}\circ\theta_i],_{i_1}}{\det\nabla(\tilde{\eta}^\kappa\circ\theta_l)(\theta_l^{-1}\circ\theta_i)}\ \big[\rho_{\frac{1}{\kappa}}\star_h E^{i\kappa}_m\big],_{j_1j_2j_3j_4}c^{j_1j_2j_3j_4}_{il,r111}\\
&\qquad\qquad \times c^{i_1i_2i_3i_4}_{il,s111}[\sqrt{\alpha_i}\tilde{v}_n\circ\theta_i],_{i_2i_3i_4}\big]+R_{11},
\end{aligned}$$

with

$$|R_{11}(t)| \leq C\tilde{E}(t)^2$$

and

$$c^{j_1j_2j_3j_4}_{il,r111} = \big[(\theta_i^{-1}\circ\theta_l),_r^{j_1}(\theta_i^{-1}\circ\theta_l),_1^{j_2}(\theta_i^{-1}\circ\theta_l),_1^{j_3}(\theta_i^{-1}\circ\theta_l),_1^{j_4}\big](\theta_l^{-1}\circ\theta_i).$$

Therefore,

$$H_{11}=\epsilon^{mn}\epsilon^{rs}\int_{(0,1)^2}\xi_l(\theta_i)[\tilde{q}(\theta_i)],_{i_1}\rho_{\frac{1}{\kappa}}\star_h E^{i\kappa}_m,_{j_1j_2j_3j_4}[\sqrt{\alpha_i}\tilde{v}_n\circ\theta_i],_{i_2i_3i_4}h^{(ji)_{1234}}_{rs}+R_{11},$$

with

$$h^{(ji)_{1234}}_{rs} = \big[\frac{c^{j_1j_2j_3j_4}_{il,r111}c^{i_1i_2i_3i_4}_{il,s111}}{\det\nabla(\tilde{\eta}^\kappa\circ\theta_l)(\theta_l^{-1}\circ\theta_i)}\big]. \tag{24.17}$$

Similarly as in the study of H^{11}_{12} (from equations (24.8) to (24.13)) we have

$$\begin{aligned}
H_{11} &= \epsilon^{mn}\epsilon^{rs}\int_{(0,1)^2}\xi_l(\theta_i)[\tilde{q}(\theta_i)],_{i_1}h^{(ji)_{1234}}_{rs}E^{i\kappa}_m,_{j_1j_2j_3j_4}\rho_{\frac{1}{\kappa}}\star_h[\sqrt{\alpha_i}\tilde{v}_n\circ\theta_i],_{i_2i_3i_4}\\
&\quad + S_{11} + R_{11},
\end{aligned}$$

with,

$$|S_{11}| \leq Ct\tilde{E}(t)^3 + Ct\kappa^{\frac{3}{2}}\tilde{E}(t).$$

By integrating by parts in space (and using $\xi_l\tilde{q}(\theta_i) = 0$ on $\partial(0,1)^2$),

(24.18)

$$\begin{aligned}
H_{11} = -\epsilon^{mn}\epsilon^{rs}&\int_{(0,1)^2}(\xi_l\tilde{q})(\theta_i)h^{(ji)_{1234}}_{rs}\rho_{\frac{1}{\kappa}}\star_h[\sqrt{\alpha_i}\tilde{\eta}_m\circ\theta_i],_{j_1j_2j_3j_4}\\
&\qquad\times\rho_{\frac{1}{\kappa}}\star_h[\sqrt{\alpha_i}\tilde{v}_n\circ\theta_i],_{i_1i_2i_3i_4}\\
&+H^1_{11}+H^2_{11}+S_{11}+R_{11},
\end{aligned}$$

with

$$H^1_{11} = -\epsilon^{mn}\epsilon^{rs}\int_{(0,1)^2}(\xi_l\tilde{q})(\theta_i)h^{(ji)_{1234}}_{rs}\rho_{\frac{1}{\kappa}}\star_h[\sqrt{\alpha_i}\tilde{\eta}_m\circ\theta_i],_{i_1j_1j_2j_3j_4} \times \rho_{\frac{1}{\kappa}}\star_h[\sqrt{\alpha_i}\tilde{v}_n\circ\theta_i],_{i_2i_3i_4}, \tag{24.19a}$$

$$H^2_{11} = -\epsilon^{mn}\epsilon^{rs}\int_{(0,1)^2}\tilde{q}(\theta_i)[\xi_l(\theta_i)h^{(ji)_{1234}}_{rs}],_{i_1}\rho_{\frac{1}{\kappa}}\star_h[\sqrt{\alpha_i}\tilde{\eta}_m\circ\theta_i],_{j_1j_2j_3j_4} \times \rho_{\frac{1}{\kappa}}\star_h[\sqrt{\alpha_i}\tilde{v}_n\circ\theta_i],_{i_2i_3i_4}. \tag{24.19b}$$

For H^1_{11}, by taking into account the symmetric role of $\{i_2,i_3,i_4\}$ and $\{j_2,j_3,j_4\}$, we obtain

$$H^1_{11} = -\epsilon^{mn}\epsilon^{rs}\int_{(0,1)^2}(\xi_l\tilde{q})(\theta_i)h^{(ji)_{1234}}_{rs}\rho_{\frac{1}{\kappa}}\star_h[\sqrt{\alpha_i}\tilde{\eta}_m\circ\theta_i],_{i_1j_1i_2i_3i_4} \times \rho_{\frac{1}{\kappa}}\star_h[\sqrt{\alpha_i}\tilde{v}_n\circ\theta_i],_{j_2j_3j_4}.$$

Next, since $h^{(ji)_{1234}}_{rs} = h^{(ij)_{1234}}_{sr}$, this implies

$$\begin{aligned} H^1_{11} &= -\epsilon^{mn}\epsilon^{rs}\int_{(0,1)^2}(\xi_l\tilde{q})(\theta_i)h^{(ij)_{1234}}_{sr}\rho_{\frac{1}{\kappa}}\star_h[\sqrt{\alpha_i}\tilde{\eta}_m\circ\theta_i],_{i_1j_1i_2i_3i_4} \\ &\qquad\times \rho_{\frac{1}{\kappa}}\star_h[\sqrt{\alpha_i}\tilde{v}_n\circ\theta_i],_{j_2j_3j_4} \\ &= \epsilon^{mn}\epsilon^{sr}\int_{(0,1)^2}(\xi_l\tilde{q})(\theta_i)h^{(ij)_{1234}}_{sr}\rho_{\frac{1}{\kappa}}\star_h[\sqrt{\alpha_i}\tilde{\eta}_m\circ\theta_i],_{j_1i_1i_2i_3i_4} \\ &\qquad\times \rho_{\frac{1}{\kappa}}\star_h[\sqrt{\alpha_i}\tilde{v}_n\circ\theta_i],_{j_2j_3j_4}. \end{aligned}$$

Therefore, by relabeling sr as rs and ij as ji, we obtain by comparison to (24.19a):

$$H^1_{11} = -H^1_{11},$$

and thus $H^1_{11} = 0$. For H^2_{11}, we have by integrating by parts $H^2_{11} = H^{21}_{11} + H^{22}_{11}$, with

$$\begin{aligned} H^{21}_{11} &= \epsilon^{mn}\epsilon^{rs}\int_{(0,1)^2}[\tilde{q}(\theta_i)[\xi_l(\theta_i)h^{(ji)_{1234}}_{rs}],_{i_1}],_{j_1}\rho_{\frac{1}{\kappa}}\star_h[\sqrt{\alpha_i}\tilde{\eta}_m\circ\theta_i],_{j_2j_3j_4} \\ &\qquad\times \rho_{\frac{1}{\kappa}}\star_h[\sqrt{\alpha_i}\tilde{v}_n\circ\theta_i],_{i_2i_3i_4} \\ H^{22}_{11} &= \epsilon^{mn}\epsilon^{rs}\int_{(0,1)^2}\tilde{q}(\theta_i)[\xi_l(\theta_i)h^{(ji)_{1234}}_{rs}],_{i_1}\rho_{\frac{1}{\kappa}}\star_h[\sqrt{\alpha_i}\tilde{\eta}_m\circ\theta_i],_{j_2j_3j_4} \\ &\qquad\times \rho_{\frac{1}{\kappa}}\star_h[\sqrt{\alpha_i}\tilde{v}_n\circ\theta_i],_{j_1i_2i_3i_4}. \end{aligned}$$

First, for H^{21}_{11}, if we denote $E_{mn} = \epsilon^{mn}\rho_{\frac{1}{\kappa}}\star_h[\sqrt{\alpha_i}\tilde{\eta}_m\circ\theta_i],_{j_2j_3j_4}\,\rho_{\frac{1}{\kappa}}\star_h[\sqrt{\alpha_i}\tilde{\eta}_n\circ\theta_i],_{i_2i_3i_4}$, we have

$$\begin{aligned} \int_0^t H^{21}_{11} = &-\frac{1}{2}\int_0^t\epsilon^{rs}\int_{(0,1)^2}[[\tilde{q}(\theta_i)[\xi_l(\theta_i)h^{(ji)_{1234}}_{rs}],_{i_1}],_{j_1}]_t E_{mn} \\ &+\frac{1}{2}\big[\epsilon^{rs}\int_{(0,1)^2}[\tilde{q}(\theta_i)[\xi_l(\theta_i)h^{(ji)_{1234}}_{rs}],_{i_1}],_{j_1}E_{mn}\big]_0^t. \end{aligned}$$

Therefore,

$$\begin{aligned}\Big|\int_0^t H^{21}_{11}\Big| \leq & C\int_0^t |\epsilon^{rs}|\big\|[[\tilde q(\theta_i)[\xi_l(\theta_i)h^{(ji)_{1234}}_{rs}],_{i_1}],_{j_1}]_t\big\|_{\frac12,(0,1)^2}\|\tilde\eta\|^2_{\frac72}\\ &+C|\epsilon^{rs}|\big\|[\tilde q(\theta_i)[\xi_l(\theta_i)h^{(ji)_{1234}}_{rs}],_{i_1}],_{j_1}(0)\big\|_{L^\infty((0,1)^2)}\big[\|\tilde\eta\|^2_{\frac72}(t)+\|\tilde\eta\|^2_{\frac72}(0)\big]\\ &+C\int_0^t\big\|[[\tilde q(\theta_i)[\xi_l(\theta_i)h^{(ji)_{1234}}_{rs}],_{i_1}],_{j_1}]_t\big\|_{\frac12,(0,1)^2}\big[\|\tilde\eta\|^2_{\frac72}(t)+\|\tilde\eta\|^2_{\frac72}(0)\big]\,.\end{aligned}$$

With the definitions (24.17) and (22.18) for the control of the time derivative of $\det(\nabla(\tilde\eta^\kappa))$ in $H^{\frac52}(\Omega)$, we then infer

$$\Big|\int_0^t H^{21}_{11}\Big| \leq Ct\tilde E(t)^4+N(u_0). \tag{24.20}$$

Next, for H^{22}_{11}, we have by relabeling m and n

$$\begin{aligned}H^{22}_{11}=&\ \epsilon^{nm}\epsilon^{rs}\int_{(0,1)^2}\tilde q(\theta_i)[\xi_l(\theta_i)h^{(ji)_{1234}}_{rs}],_{i_1}\rho_{\frac1\kappa}\star_h[\sqrt{\alpha_i}\tilde\eta_n\circ\theta_i],_{j_2j_3j_4}\\ &\qquad\times\rho_{\frac1\kappa}\star_h[\sqrt{\alpha_i}\tilde v_m\circ\theta_i],_{j_1i_2i_3i_4}\\ =&-\epsilon^{mn}\epsilon^{rs}\int_{(0,1)^2}\tilde q(\theta_i)[\xi_l(\theta_i)h^{(ji)_{1234}}_{rs}],_{i_1}\rho_{\frac1\kappa}\star_h[\sqrt{\alpha_i}\tilde\eta_n\circ\theta_i],_{j_2j_3j_4}\\ &\qquad\times\rho_{\frac1\kappa}\star_h[\sqrt{\alpha_i}\tilde v_m\circ\theta_i],_{j_1i_2i_3i_4}\,.\end{aligned}$$

By taking into account the symmetric role of $\{i_2,i_3,i_4\}$ and $\{j_2,j_3,j_4\}$, we then obtain

$$\begin{aligned}H^{22}_{11}=&-\epsilon^{mn}\epsilon^{rs}\int_{(0,1)^2}\tilde q(\theta_i)[\xi_l(\theta_i)h^{(ji)_{1234}}_{rs}],_{i_1}\rho_{\frac1\kappa}\star_h[\sqrt{\alpha_i}\tilde\eta_n\circ\theta_i],_{i_2i_3i_4}\\ &\qquad\times\rho_{\frac1\kappa}\star_h[\sqrt{\alpha_i}\tilde v_m\circ\theta_i],_{j_1j_2j_3j_4}\,.\end{aligned} \tag{24.21}$$

Consequently, by (24.19b) and (24.21),

$$\begin{aligned}2H^2_{11}=&-\epsilon^{mn}\epsilon^{rs}\int_{(0,1)^2}\tilde q(\theta_i)[\xi_l(\theta_i)h^{(ji)_{1234}}_{rs}],_{i_1}\rho_{\frac1\kappa}\star_h[\sqrt{\alpha_i}\tilde\eta_m\circ\theta_i],_{j_1j_2j_3j_4}\\ &\qquad\times\rho_{\frac1\kappa}\star_h[\sqrt{\alpha_i}\tilde v_n\circ\theta_i],_{i_2i_3i_4}\\ &-\epsilon^{mn}\epsilon^{rs}\int_{(0,1)^2}\tilde q(\theta_i)[\xi_l(\theta_i)h^{(ji)_{1234}}_{rs}],_{i_1}\rho_{\frac1\kappa}\star_h[\sqrt{\alpha_i}\tilde\eta_n\circ\theta_i],_{i_2i_3i_4}\\ &\qquad\times\rho_{\frac1\kappa}\star_h[\sqrt{\alpha_i}\tilde v_m\circ\theta_i],_{j_1j_2j_3j_4}\\ &+H^{21}_{11}\\ =&-\epsilon^{mn}\epsilon^{rs}\int_{(0,1)^2}\tilde q(\theta_i)[\xi_l(\theta_i)h^{(ji)_{1234}}_{rs}],_{i_1}[\rho_{\frac1\kappa}\star_h[\sqrt{\alpha_i}\tilde\eta_m\circ\theta_i],_{j_1j_2j_3j_4}\\ &\qquad\times\rho_{\frac1\kappa}\star_h[\sqrt{\alpha_i}\tilde\eta_n\circ\theta_i],_{i_2i_3i_4}]_t\\ &+H^{21}_{11}.\end{aligned}$$

Therefore,

$$\int_0^t H_{11}^2 = \frac{1}{2}\epsilon^{mn}\epsilon^{rs}\int_0^t\int_{(0,1)^2}[\tilde{q}(\theta_i)[\xi_l(\theta_i)h_{rs}^{(ji)_{1234}}],_{i_1}]_t\tilde{\eta}^m_{i\kappa},_{j_1j_2j_3j_4}\tilde{\eta}^n_{i\kappa},_{i_2i_3i_4}$$
$$-\frac{1}{2}\epsilon^{mn}\epsilon^{rs}\Big[\int_{(0,1)^2}\tilde{q}(\theta_i)[\xi_l(\theta_i)h_{rs}^{(ji)_{1234}}],_{i_1}\tilde{\eta}^m_{i\kappa},_{j_1j_2j_3j_4}\tilde{\eta}^n_{i\kappa},_{i_2i_3i_4}\Big]_0^t$$
$$+\frac{1}{2}\int_0^t H_{11}^{21}.$$

Now, from (24.18) and $H_{11}^1=0$, we infer by integrating by parts in time:

$$\int_0^t H_{11} = \frac{1}{2}\epsilon^{mn}\epsilon^{rs}\int_0^t\int_{(0,1)^2}[(\xi_l\tilde{q})(\theta_i)h_{rs}^{(ji)_{1234}}]_t\tilde{\eta}^m_{i\kappa},_{j_1j_2j_3j_4}\tilde{\eta}^n_{i\kappa},_{i_1i_2i_3i_4}$$
$$-\frac{1}{2}\epsilon^{mn}\epsilon^{rs}\Big[\int_{(0,1)^2}(\xi_l\tilde{q})(\theta_i)h_{rs}^{(ji)_{1234}}\tilde{\eta}^m_{i\kappa},_{j_1j_2j_3j_4}\tilde{\eta}^n_{i\kappa},_{i_1i_2i_3i_4}\Big]_0^t$$
$$+\int_0^t[S_{11}+R_{11}+\frac{1}{2}H_{11}^2].$$

Now, we claim that the only couples (i_1,j_1) contributing to the sum above are the ones with $i_1\neq j_1$. To see that, we notice that if $i_1=j_1$, then by simply relabeling m and n, and using the symmetric role of $\{i_2,i_3,i_4\}$ and $\{j_2,j_3,j_4\}$,

$$\begin{aligned}h_{rs}^{(ji)_{1234}}\epsilon^{mn}\tilde{\eta}^m_{i\kappa},_{i_1j_2j_3j_4}\tilde{\eta}^n_{i\kappa},_{i_1i_2i_3i_4} &= h_{rs}^{(ji)_{1234}}\epsilon^{nm}\tilde{\eta}^n_{i\kappa},_{i_1j_2j_3j_4}\tilde{\eta}^m_{i\kappa},_{i_1i_2i_3i_4}\\ &= h_{rs}^{(ji)_{1234}}\epsilon^{nm}\tilde{\eta}^n_{i\kappa},_{i_1i_2i_3i_4}\tilde{\eta}^m_{i\kappa},_{j_1j_2j_3j_4},\end{aligned}$$

leading to $h_{rs}^{(ji)_{1234}}\epsilon^{mn}\tilde{\eta}^m_{i\kappa},_{i_1j_2j_3j_4}\tilde{\eta}^n_{i\kappa},_{i_1i_2i_3i_4}=0$, since $\epsilon^{mn}=-\epsilon^{nm}$. Consequently if we denote

$$d^{(ji)_{234}} = \frac{h_{rs}^{(ji)_{1234}}}{[(\theta_i^{-1}\circ\theta_l),_r^{j_1}(\theta_i^{-1}\circ\theta_l),_s^{i_1}](\theta_l^{-1}\circ\theta_i)},$$

we have

$$\int_0^t H_{11} = \frac{1}{2}\epsilon^{mn}\epsilon^{i_1j_1}\int_0^t\int_{(0,1)^2}\big[[(\xi_l\tilde{q})(\theta_i)d^{(ji)_{234}}]_t\epsilon^{rs}\epsilon^{i_1j_1}[(\theta_i^{-1}\circ\theta_l),_r^{j_1}$$
$$\times(\theta_i^{-1}\circ\theta_l),_s^{i_1}](\theta_l^{-1}\circ\theta_i)\tilde{\eta}^m_{i\kappa},_{j_1j_2j_3j_4}\tilde{\eta}^n_{i\kappa},_{i_1i_2i_3i_4}\big]$$
$$-\frac{1}{2}\epsilon^{mn}\epsilon^{i_1j_1}\Big[\int_{(0,1)^2}\big[(\xi_l\tilde{q})(\theta_i)d^{(ji)_{234}}\epsilon^{rs}\epsilon^{i_1j_1}[(\theta_i^{-1}\circ\theta_l),_r^{j_1}(\theta_i^{-1}\circ\theta_l),_s^{i_1}]$$
$$\times(\theta_l^{-1}\circ\theta_i)\tilde{\eta}^m_{i\kappa},_{j_1j_2j_3j_4}\tilde{\eta}^n_{i\kappa},_{i_1i_2i_3i_4}\big]\Big]_0^t$$
$$+\int_0^t[S_{11}+R_{11}+\frac{1}{2}H_{11}^2].$$

Now, for any fixed (i_1,j_1), we have

$$\epsilon^{rs}\epsilon^{i_1j_1}[(\theta_i^{-1}\circ\theta_l),_r^{j_1}(\theta_i^{-1}\circ\theta_l),_s^{i_1}] = -\det(\nabla(\theta_i^{-1}\circ\theta_l)) = -1,$$

leading us to

$$\begin{aligned}\int_0^t H_{11} = &-\frac{1}{2}\epsilon^{mn}\epsilon^{i_1j_1}\int_0^t\int_{(0,1)^2}[(\xi_l\tilde{q})(\theta_i)d^{(ji)_{234}}]_t\tilde{\eta}^m_{i\kappa},_{j_1j_2j_3j_4}\,\tilde{\eta}^n_{i\kappa},_{i_1i_2i_3i_4}\\ &+\frac{1}{2}\epsilon^{mn}\epsilon^{i_1j_1}\Big[\int_{(0,1)^2}(\xi_l\tilde{q})(\theta_i)d^{(ji)_{234}}\tilde{\eta}^m_{i\kappa},_{j_1j_2j_3j_4}\,\tilde{\eta}^n_{i\kappa},_{i_1i_2i_3i_4}\Big]_0^t\\ &+\int_0^t[S_{11}+R_{11}+\frac{1}{2}H^2_{11}].\end{aligned}$$

Next, by integrating by parts in space

$$\begin{aligned}\int_0^t H_{11} = &\frac{1}{2}\epsilon^{mn}\epsilon^{i_1j_1}\int_0^t\int_{(0,1)^2}[(\xi_l\tilde{q})(\theta_i)d^{(ji)_{234}}]_t,_{i_1}\,\tilde{\eta}^m_{i\kappa},_{j_1j_2j_3j_4}\,\tilde{\eta}^n_{i\kappa},_{i_2i_3i_4}\\ &-\frac{1}{2}\epsilon^{mn}\epsilon^{i_1j_1}\Big[\int_{(0,1)^2}[(\xi_l\tilde{q})(\theta_i)d^{(ji)_{234}}],_{i_1}\,\tilde{\eta}^m_{i\kappa},_{j_1j_2j_3j_4}\,\tilde{\eta}^n_{i\kappa},_{i_2i_3i_4}\Big]_0^t\\ &+\int_0^t[S_{11}+R_{11}+\frac{1}{2}H^2_{11}],\end{aligned}\tag{24.22}$$

where we have used the fact that similarly as for H^1_{11}, we have

$$0=\epsilon^{mn}\epsilon^{i_1j_1}d^{(ji)_{234}}\tilde{\eta}^m_{i\kappa},_{i_1j_1j_2j_3j_4}\,\tilde{\eta}^n_{i\kappa},_{i_2i_3i_4}\,.$$

We now come to the study of the crucial term bringing the regularity of the surface.

24.2.1. **Control of the trace of $\tilde{\eta}_{i\kappa}$ on Γ.** Let us study the second term of the right-hand side of (24.22):

$$H=-\frac{1}{2}\epsilon^{mn}\epsilon^{i_1j_1}\int_{(0,1)^2}[(\xi_l\tilde{q})(\theta_i)d^{(ji)_{234}}],_{i_1}\,\tilde{\eta}^m_{i\kappa},_{j_1j_2j_3j_4}\,\tilde{\eta}^n_{i\kappa},_{i_2i_3i_4}\,,$$

for which we have

$$\begin{aligned}H=&-\frac{1}{2}\epsilon^{mn}\epsilon^{i_1j_1}\int_{(0,1)^2}\Big[[[(\xi_l\tilde{q})(\theta_i)d^{(ji)_{234}}]\circ\theta_i^{-1}\circ(\tilde{\eta}^\kappa)^{-1}\circ\tilde{\eta}^\kappa\circ\theta_i],_{i_1}\\ &\qquad\times[\tilde{\eta}^m_{i\kappa},_{j_2j_3j_4}\circ\theta_i^{-1}\circ(\tilde{\eta}^\kappa)^{-1}\circ\tilde{\eta}^\kappa\circ\theta_i],_{j_1}\,\tilde{\eta}^n_{i\kappa},_{i_2i_3i_4}\Big]\\ =&-\frac{1}{2}\epsilon^{mn}\epsilon^{i_1j_1}\int_{(0,1)^2}\Big[[[(\xi_l\tilde{q})(\theta_i)d^{(ji)_{234}}]\circ\theta_i^{-1}\circ(\tilde{\eta}^\kappa)^{-1}],_{i'_1}\circ\tilde{\eta}^\kappa\circ\theta_i\;[\tilde{\eta}^\kappa\circ\theta_i],^{i'_1}_{i_1}\\ &\qquad\times[\tilde{\eta}^m_{i\kappa},_{j_2j_3j_4}\circ\theta_i^{-1}\circ(\tilde{\eta}^\kappa)^{-1}],_{j'_1}\circ\tilde{\eta}^\kappa\circ\theta_i\;[\tilde{\eta}^\kappa\circ\theta_i],^{j'_1}_{j_1}\,\tilde{\eta}^n_{i\kappa},_{i_2i_3i_4}\Big].\end{aligned}$$

Now, for the same reason as before, the couples (i'_1, j'_1) such that $i'_1 = j'_1$ will not contribute to the sum above, leading us to

$$\begin{aligned}
H = &-\frac{1}{2}\epsilon^{mn}\epsilon^{i'_1 j'_1}\int_{(0,1)^2}\Big[[[(\xi_l\tilde{q})(\theta_i)d^{(ji)_{234}}]\circ\theta_i^{-1}\circ(\tilde{\eta}^\kappa)^{-1}],_{i'_1}\circ\tilde{\eta}^\kappa\circ\theta_i\\
&\qquad\times\epsilon^{i_1 j_1}\epsilon^{i'_1 j'_1}[\tilde{\eta}^\kappa\circ\theta_i],_{i_1}^{i'_1}[\tilde{\eta}^\kappa\circ\theta_i],_{j_1}^{j'_1}\\
&\qquad\times[\tilde{\eta}^m_{i\kappa},_{j_2 j_3 j_4}\circ\theta_i^{-1}\circ(\tilde{\eta}^\kappa)^{-1}],_{j'_1}\circ\tilde{\eta}^\kappa\circ\theta_i\ \tilde{\eta}^n_{i\kappa},_{i_2 i_3 i_4}\Big]\\
= &-\frac{1}{2}\epsilon^{mn}\epsilon^{i'_1 j'_1}\int_{(0,1)^2}\Big[[[(\xi_l\tilde{q})(\theta_i)d^{(ji)_{234}}]\circ\theta_i^{-1}\circ(\tilde{\eta}^\kappa)^{-1}],_{i'_1}\circ\tilde{\eta}^\kappa\circ\theta_i\ \det\nabla(\tilde{\eta}^\kappa\circ\theta_i)\\
&\qquad\times[\tilde{\eta}^m_{i\kappa},_{j_2 j_3 j_4}\circ\theta_i^{-1}\circ(\tilde{\eta}^\kappa)^{-1}],_{j'_1}\circ\tilde{\eta}^\kappa\circ\theta_i\ \tilde{\eta}^n_{i\kappa},_{i_2 i_3 i_4}\Big]\\
= &-\frac{1}{2}\epsilon^{mn}\epsilon^{i'_1 j'_1}\int_{\tilde{\eta}^\kappa(\theta_i((0,1)^2))}\Big[[[(\xi_l\tilde{q})(\theta_i)d^{(ji)_{234}}]\circ\theta_i^{-1}\circ(\tilde{\eta}^\kappa)^{-1}],_{i'_1}\\
&\qquad\qquad\times[\tilde{\eta}^m_{i\kappa},_{j_2 j_3 j_4}\circ\theta_i^{-1}\circ(\tilde{\eta}^\kappa)^{-1}],_{j'_1}\ \tilde{\eta}^n_{i\kappa},_{i_2 i_3 i_4}\circ\theta_i^{-1}\circ(\tilde{\eta}^\kappa)^{-1}\Big]\\
= &\ I + J,
\end{aligned}$$

with

$$\begin{aligned}
I = &-\frac{1}{2}(\epsilon^{mn})^2\int_{\tilde{\eta}^\kappa(\theta_i((0,1)^2))}\Big[[[(\xi_l\tilde{q})(\theta_i)d^{(ji)_{234}}]\circ\theta_i^{-1}\circ(\tilde{\eta}^\kappa)^{-1}],_m\\
&\qquad\qquad\times[\tilde{\eta}^m_{i\kappa},_{j_2 j_3 j_4}\circ\theta_i^{-1}\circ(\tilde{\eta}^\kappa)^{-1}],_n\ \tilde{\eta}^n_{i\kappa},_{i_2 i_3 i_4}\circ\theta_i^{-1}\circ(\tilde{\eta}^\kappa)^{-1}\Big]\\
J = &\ \frac{1}{2}(\epsilon^{mn})^2\int_{\tilde{\eta}^\kappa(\theta_i((0,1)^2))}\Big[[[(\xi_l\tilde{q})(\theta_i)d^{(ji)_{234}}]\circ\theta_i^{-1}\circ(\tilde{\eta}^\kappa)^{-1}],_n\\
&\qquad\qquad\times[\tilde{\eta}^m_{i\kappa},_{j_2 j_3 j_4}\circ\theta_i^{-1}\circ(\tilde{\eta}^\kappa)^{-1}],_m\ \tilde{\eta}^n_{i\kappa},_{i_2 i_3 i_4}\circ\theta_i^{-1}\circ(\tilde{\eta}^\kappa)^{-1}\Big].
\end{aligned}$$

Next, we notice that

$$\begin{aligned}
J = -\frac{1}{2}\sum_n\int_{\tilde{\eta}^\kappa(\theta_i((0,1)^2))}&\Big[[[(\xi_l\tilde{q})(\theta_i)d^{(ji)_{234}}]\circ\theta_i^{-1}\circ(\tilde{\eta}^\kappa)^{-1}],_n\\
&\times[\tilde{\eta}^n_{i\kappa},_{j_2 j_3 j_4}\circ\theta_i^{-1}\circ(\tilde{\eta}^\kappa)^{-1}],_n\ \tilde{\eta}^n_{i\kappa},_{i_2 i_3 i_4}\circ\theta_i^{-1}\circ(\tilde{\eta}^\kappa)^{-1}\Big] + J_1,
\end{aligned}$$

with the perturbation term

$$\begin{aligned}
J_1 = &\ \frac{1}{2}\int_{\tilde{\eta}^\kappa(\theta_i((0,1)^2))}\Big[[[(\xi_l\tilde{q})(\theta_i)d^{(ji)_{234}}]\circ\theta_i^{-1}\circ(\tilde{\eta}^\kappa)^{-1}],_n\\
&\qquad\qquad\times\operatorname{div}[\tilde{\eta}_{i\kappa},_{j_2 j_3 j_4}\circ\theta_i^{-1}\circ(\tilde{\eta}^\kappa)^{-1}]\ \tilde{\eta}^n_{i\kappa},_{i_2 i_3 i_4}\circ\theta_i^{-1}\circ(\tilde{\eta}^\kappa)^{-1}\Big]\\
= &\ J_1^1 + J_1^2,
\end{aligned}$$

where

$$\begin{aligned}
J_1^1 = &\ \frac{1}{2}\int_{\tilde{\eta}^\kappa(\theta_i((0,1)^2))}\Big[[[(\xi_l\tilde{q})(\theta_i)c^{i_2 i_3 i_4}_{il,111}]\circ\theta_i^{-1}\circ(\tilde{\eta}^\kappa)^{-1}],_n\, c^{j_2 j_3 j_4}_{il,111}\circ\theta_i^{-1}\circ(\tilde{\eta}^\kappa)^{-1}\\
&\qquad\qquad\times\operatorname{div}[\tilde{\eta}_{i\kappa},_{j_2 j_3 j_4}\circ\theta_i^{-1}\circ(\tilde{\eta}^\kappa)^{-1}]\ \tilde{\eta}^n_{i\kappa},_{i_2 i_3 i_4}\circ\theta_i^{-1}\circ(\tilde{\eta}^\kappa)^{-1}\Big],\\
J_1^2 = &\ \frac{1}{2}\int_{\tilde{\eta}^\kappa(\theta_i((0,1)^2))}\Big[[(\xi_l\tilde{q})(\theta_i)c^{i_2 i_3 i_4}_{il,111}]\circ\theta_i^{-1}\circ(\tilde{\eta}^\kappa)^{-1}[c^{j_2 j_3 j_4}_{il,111}\circ\theta_i^{-1}\circ(\tilde{\eta}^\kappa)^{-1}],_n\\
&\qquad\qquad\times\operatorname{div}[\tilde{\eta}_{i\kappa},_{j_2 j_3 j_4}\circ\theta_i^{-1}\circ(\tilde{\eta}^\kappa)^{-1}]\ \tilde{\eta}^n_{i\kappa},_{i_2 i_3 i_4}\circ\theta_i^{-1}\circ(\tilde{\eta}^\kappa)^{-1}\Big].
\end{aligned}$$

Now, for J_1^1, let us set

$$f_{il} = [[(\xi_l \tilde{q})(\theta_i) c^{i_2 i_3 i_4}_{il,111}] \circ \theta_i^{-1} \circ (\tilde{\eta}^\kappa)^{-1}],_n [\Pi_{p=2}^3 (\theta_i^{-1} \circ \theta_l),_1^{j_p} (\theta_l^{-1} \circ \theta_i) \tilde{\eta}^n_{i\kappa},_{i_2 i_3 i_4}] \\ \times (\theta_i^{-1} \circ (\tilde{\eta}^\kappa)^{-1}),$$

so that in order to identify a horizontal derivative on the highest order term we have

$$\begin{aligned}
J_1^1 &= \frac{1}{2} \int_{\tilde{\eta}^\kappa(\theta_i((0,1)^2))} f_{il}\ (\theta_i^{-1} \circ \theta_l),_1^{j_4} (\theta_l^{-1} \circ (\tilde{\eta}^\kappa)^{-1}) \operatorname{div}[\tilde{\eta}_{i\kappa},_{j_2 j_3 j_4} \circ \theta_i^{-1} \circ (\tilde{\eta}^\kappa)^{-1}] \\
&= \frac{1}{2} \int_{\tilde{\eta}^\kappa(\theta_i((0,1)^2))} f_{il}\ (\theta_i^{-1} \circ \theta_l),_1^{j_4} (\theta_l^{-1} \circ (\tilde{\eta}^\kappa)^{-1}) (\theta_i^{-1} \circ (\tilde{\eta}^\kappa)^{-1}),_k^l \\
&\qquad \times [\tilde{\eta}_{i\kappa},^k_{j_2 j_3 j_4 l}] \circ \theta_i^{-1} \circ (\tilde{\eta}^\kappa)^{-1} \\
&= \frac{1}{2} \int_{\tilde{\eta}^\kappa(\theta_i((0,1)^2))} \big[f_{il}\ (\theta_i^{-1} \circ \theta_l),_1^{j_4} (\theta_l^{-1} \circ (\tilde{\eta}^\kappa)^{-1}) (\theta_i^{-1} \circ (\tilde{\eta}^\kappa)^{-1}),_k^l \\
&\qquad \times [\tilde{\eta}_{i\kappa},^k_{j_2 j_3 l} \circ \theta_i^{-1} \circ (\tilde{\eta}^\kappa)^{-1}],_q (\tilde{\eta}^\kappa \circ \theta_i),^q_{j_4} (\theta_i^{-1} \circ (\tilde{\eta}^\kappa)^{-1}) \big] \\
&= \frac{1}{2} \int_{\tilde{\eta}^\kappa(\theta_i((0,1)^2))} \big[f_{il}\ (\theta_i^{-1} \circ \theta_l),_1^{j_4} (\theta_l^{-1} \circ (\tilde{\eta}^\kappa)^{-1}) \\
&\qquad \times [(\theta_i^{-1} \circ (\tilde{\eta}^\kappa)^{-1}),_k^l\, \tilde{\eta}_{i\kappa},^k_{j_2 j_3 l} \circ \theta_i^{-1} \circ (\tilde{\eta}^\kappa)^{-1}],_q \\
&\qquad \times (\tilde{\eta}^\kappa \circ \theta_i),^q_{j_4} (\theta_i^{-1} \circ (\tilde{\eta}^\kappa)^{-1}) \big] + r_1^1 \\
&= \frac{1}{2} \int_{\tilde{\eta}^\kappa(\theta_i((0,1)^2))} \big[f_{il}\ [(\tilde{\eta}^\kappa \circ \theta_i),^q_{j_4} (\theta_i^{-1} \circ \theta_l)(\theta_i^{-1} \circ \theta_l),_1^{j_4}] (\theta_l^{-1} \circ (\tilde{\eta}^\kappa)^{-1}) \\
&\qquad \times \operatorname{div}[\tilde{\eta}_{i\kappa},_{j_2 j_3} \circ \theta_i^{-1} \circ (\tilde{\eta}^\kappa)^{-1}],_q \big] + r_1^1 \\
&= \frac{1}{2} \int_{\tilde{\eta}^\kappa(\theta_i((0,1)^2))} f_{il}\ (\tilde{\eta}^\kappa \circ \theta_l),_1^q (\theta_l^{-1} \circ (\tilde{\eta}^\kappa)^{-1})\ \operatorname{div}[\tilde{\eta}_{i\kappa},_{j_2 j_3} \circ \theta_i^{-1} \circ (\tilde{\eta}^\kappa)^{-1}],_q + r_1^1 \\
&= \frac{1}{2} \int_{\tilde{\eta}^\kappa(\theta_i((0,1)^2))} f_{il}\ (\operatorname{div}[\tilde{\eta}_{i\kappa},_{j_2 j_3} \circ \theta_i^{-1} \circ (\tilde{\eta}^\kappa)^{-1}] \circ \tilde{\eta}^\kappa \circ \theta_l),_1 (\theta_l^{-1} \circ (\tilde{\eta}^\kappa)^{-1}) + r_1^1
\end{aligned}$$

with $|r_1^1| \le C \|\tilde{\eta}^\kappa\|_3^2$. Now, we notice that the presence of the factor $\xi_l \circ (\tilde{\eta}^\kappa)^{-1}$ in f_{il} implies that the integrand in the integral above is zero outside of $\tilde{\eta}^\kappa(\theta_l((0,1)^2))$. Similarly, the presence of $\rho_{\frac{1}{\kappa}} \star_h [\sqrt{\alpha_i}\tilde{\eta} \circ \theta_i],_{j_2 j_3} \circ \theta_i^{-1} \circ (\tilde{\eta}^\kappa)^{-1}$ implies that $x \in \tilde{\eta}^\kappa(\theta_i((0,1)^2))$ in order for this integrand to be nonzero. Therefore,

$$J_1^1 = \frac{1}{2} \int_{(0,1)^2} f_{il}(\tilde{\eta}^\kappa \circ \theta_l)\ (\operatorname{div}[\tilde{\eta}_{i\kappa},_{j_2 j_3} \circ \theta_i^{-1} \circ (\tilde{\eta}^\kappa)^{-1}] \circ \tilde{\eta}^\kappa \circ \theta_l),_1 \det \nabla (\tilde{\eta}^\kappa \circ \theta_l) + r_1^1.$$

Now, since the derivative of $\operatorname{div}[\tilde{\eta}_{i\kappa},_{j_2 j_3} \circ \theta_i^{-1} \circ (\tilde{\eta}^\kappa)^{-1}] \circ \tilde{\eta}^\kappa \circ \theta_l$ is taken in the horizontal direction, this implies

$$\begin{aligned}
|J_1^1| &\le C \|f_{il}\|_{\frac{1}{2},(0,1)^2} \|\operatorname{div}[\tilde{\eta}_{i\kappa},_{j_2 j_3} \circ \theta_i^{-1} \circ (\tilde{\eta}^\kappa)^{-1}] \circ \tilde{\eta}^\kappa \circ \theta_l\|_{\frac{1}{2},(0,1)^2} + |r_1^1| \\
&\le C \|f_{il}\|_{\frac{1}{2},(0,1)^2} \|\operatorname{div}[\tilde{\eta}_{i\kappa},_{j_2 j_3} \circ \theta_i^{-1} \circ (\tilde{\eta}^\kappa)^{-1}]\|_{\frac{1}{2},\tilde{\eta}^\kappa(\theta_i((0,1)^2))} + C \|\mathrm{Id} + \int_0^t \tilde{v}\|_3^2.
\end{aligned}$$

Now, since we have in the same fashion as (22.8)

$$\|\operatorname{div}[\tilde{\eta}_{i\kappa},_{j_2 j_3} \circ \theta_i^{-1} \circ (\tilde{\eta}^\kappa)^{-1}]\|_{\frac{1}{2},\tilde{\eta}^\kappa(\theta_i((0,1)^2))} \le C t \tilde{E}(t)^2 + C \kappa^{\frac{1}{2}} \tilde{E}(t) + C,$$

we then have

$$|J_1^1| \le Ct\tilde{E}(t)^3 + C + C\kappa^{\frac{1}{2}}\tilde{E}(t)^2. \tag{24.23}$$

Next, for J_1^2, we notice that $[c_{il,111}^{j_1j_2j_3}\circ\theta_i^{-1}\circ(\tilde{\eta}^\kappa)^{-1}],_n$ is a sum of product, each one containing a factor $(\theta_i^{-1}\circ\theta_l),_1^{j_p}(\theta_l^{-1}\circ(\tilde{\eta}^\kappa)^{-1})$. This implies that J_1^2 can be treated in the same way as J_1^1, with the identification of a horizontal derivative on the highest order term of the integrand, leading to the same majorization. We can also treat I in a similar fashion, due to the curl estimate (similar to (22.11)):

$$\|\operatorname{curl}[\tilde{\eta}_{i\kappa},_{j_2j_3}\circ\theta_i^{-1}\circ(\tilde{\eta}^\kappa)^{-1}]\|_{\frac{1}{2},\tilde{\eta}^\kappa(\theta_i((0,1)^2))} \le Ct\tilde{E}(t)^2 + N(u_0),$$

which finally provides us with

$$\begin{aligned} H = &-\frac{1}{2}\sum_{m\neq n}\int_{\tilde{\eta}^\kappa(\theta_i((0,1)^2))}\big[[[(\xi_l\tilde{q})(\theta_i)d^{(ji)_{234}}]\circ\theta_i^{-1}\circ(\tilde{\eta}^\kappa)^{-1}],_m \\ &\qquad\qquad\times[\tilde{\eta}^n_{i\kappa},_{j_2j_3j_4}\circ\theta_i^{-1}\circ(\tilde{\eta}^\kappa)^{-1}],_m\ \tilde{\eta}^n_{i\kappa},_{i_2i_3i_4}\circ\theta_i^{-1}\circ(\tilde{\eta}^\kappa)^{-1}\big] \\ &-\frac{1}{2}\sum_{n}\int_{\tilde{\eta}^\kappa(\theta_i((0,1)^2))}\big[d^{(ji)_{234}}\circ\theta_i^{-1}\circ(\tilde{\eta}^\kappa)^{-1}(\xi_l\tilde{q})((\tilde{\eta}^\kappa)^{-1})],_n \\ &\qquad\qquad\times[\tilde{\eta}^n_{i\kappa},_{j_2j_3j_4}\circ\theta_i^{-1}\circ(\tilde{\eta}^\kappa)^{-1}],_n\ \tilde{\eta}^n_{i\kappa},_{i_2i_3i_4}\circ\theta_i^{-1}\circ(\tilde{\eta}^\kappa)^{-1}\big] + h^1 \\ = &-\frac{1}{2}\sum_{m,n}\int_{\tilde{\eta}^\kappa(\theta_i((0,1)^2))}\big[d^{(ji)_{234}}\circ\theta_i^{-1}\circ(\tilde{\eta}^\kappa)^{-1}(\xi_l\tilde{q})((\tilde{\eta}^\kappa)^{-1})],_m \\ &\qquad\qquad\times[\tilde{\eta}^n_{i\kappa},_{j_2j_3j_4}\circ\theta_i^{-1}\circ(\tilde{\eta}^\kappa)^{-1}],_m\ \tilde{\eta}^n_{i\kappa},_{i_2i_3i_4}\circ\theta_i^{-1}\circ(\tilde{\eta}^\kappa)^{-1}\big] + h^1, \end{aligned}$$

with

$$|h^1(t)| \le Ct\tilde{E}(t)^3 + N(u_0) + (C\kappa^{\frac{1}{2}}+\delta)\tilde{E}(t)^2 + C_\delta.$$

Therefore, by integrating by parts,

$$\begin{aligned} H = &-\frac{1}{4}\int_{\partial\tilde{\eta}^\kappa(\theta_i((0,1)^2))}\big[d^{(ji)_{234}}\circ\theta_i^{-1}\circ(\tilde{\eta}^\kappa)^{-1}(\xi_l\tilde{q})((\tilde{\eta}^\kappa)^{-1})],_m\tilde{n}^\kappa_m \\ &\qquad\qquad\times[\tilde{\eta}^n_{i\kappa},_{j_2j_3j_4}\circ\theta_i^{-1}\circ(\tilde{\eta}^\kappa)^{-1}]\tilde{\eta}^n_{i\kappa},_{i_2i_3i_4}\circ\theta_i^{-1}\circ(\tilde{\eta}^\kappa)^{-1}\big] + h^2, \end{aligned}$$

with

$$\begin{aligned} h^2 = &\frac{1}{4}\int_{\tilde{\eta}^\kappa(\theta_i((0,1)^2))}\big[[d^{(ji)_{234}}\circ\theta_i^{-1}\circ(\tilde{\eta}^\kappa)^{-1}(\xi\tilde{q})((\tilde{\eta}^\kappa)^{-1})],_{mm} \\ &\qquad\qquad\times[\tilde{\eta}^n_{i\kappa},_{j_2j_3j_4}\circ\theta_i^{-1}\circ(\tilde{\eta}^\kappa)^{-1}]\tilde{\eta}^n_{i\kappa},_{i_2i_3i_4}\circ\theta_i^{-1}\circ(\tilde{\eta}^\kappa)^{-1}\big] + h^1, \end{aligned}$$

so that

$$\begin{aligned} |h^2(t)| &\le [\ \|\tilde{q}\|_{\frac{7}{2}}\|\tilde{\eta}\|_3 + \|\tilde{\eta}^\kappa\|_{\frac{7}{2}}\|q\|_3]\ \|\tilde{\eta}\|_3^2 + |h^1(t)| \\ &\le (C\kappa^{\frac{1}{2}}+\delta)\tilde{E}(t)^2 + C_\delta + Ct\tilde{E}(t)^4 + N(u_0). \end{aligned}$$

Now, since $\tilde{q}=0$ and $\xi_l=\alpha_l$ on Γ, we infer

$$\begin{aligned} H = &-\frac{1}{4}\int_{\partial\tilde{\eta}^\kappa(\theta_i((0,1)^2))}\big[\alpha_l((\tilde{\eta}^\kappa)^{-1})[p_0+\int_0^t p_t],_m[N_m+\int_0^t(\tilde{n}^\kappa_m)_t] \\ &\qquad\times[d_{il}^{j_{234}}\tilde{\eta}^n_{i\kappa},_{j_2j_3j_4}]\circ\theta_i^{-1}\circ(\tilde{\eta}^\kappa)^{-1}[d_{il}^{i_{234}}\tilde{\eta}^n_{i\kappa},_{i_2i_3i_4}]\circ\theta_i^{-1}\circ(\tilde{\eta}^\kappa)^{-1}\big] + h^2, \end{aligned}$$

with $d_{il}^{k_{234}} = \dfrac{(\theta_i^{-1}\circ\theta_l),_1^{k_2}\,(\theta_i^{-1}\circ\theta_l),_1^{k_3}\,(\theta_i^{-1}\circ\theta_l),_1^{k_4}}{\sqrt{\det\nabla(\tilde\eta^\kappa\circ\theta_l)}}(\theta_l^{-1}\circ\theta_i)$. Therefore, with the initial pressure condition

$$p_0,_m N_m < -C < 0 \text{ on } \Gamma,$$

we infer

$$\begin{aligned}
-H \le & -C\int_{\partial\tilde\eta^\kappa(\theta_i((0,1)^2))} [\alpha_l(\theta_i) d_{il}^{j_{234}} \tilde\eta^n_{i\kappa},_{j_2j_3j_4}\, d_{il}^{i_{234}} \tilde\eta^n_{i\kappa},_{i_2i_3i_4}] \circ\theta_i^{-1}\circ(\tilde\eta^\kappa)^{-1} \\
& + t\tilde E(t)^2 + |h^2| \\
\le & -C\int_{\partial\tilde\eta^\kappa(\theta_i((0,1)^2))} [\alpha_i(\theta_i) d_{ii}^{j_{234}} \tilde\eta^n_{i\kappa},_{j_2j_3j_4}\, d_{ii}^{i_{234}} \tilde\eta^n_{i\kappa},_{i_2i_3i_4}] \circ\theta_i^{-1}\circ(\tilde\eta^\kappa)^{-1} \\
& + (C\kappa^{\frac12}+\delta)\tilde E(t)^2 + Ct\tilde E(t)^4 + C_\delta + N(u_0) \\
\le & -C\int_{\partial\tilde\eta^\kappa(\theta_i((0,1)^2))} [\alpha_i(\theta_i) \tilde\eta^n_{i\kappa},_{j_2j_3j_4}\, \tilde\eta^n_{i\kappa},_{i_2i_3i_4}] \circ\theta_i^{-1}\circ(\tilde\eta^\kappa)^{-1} \\
& + (C\kappa^{\frac12}+\delta)\tilde E(t)^2 + Ct\tilde E(t)^4 + C_\delta + N(u_0) \\
\le & -C\int_{\partial\tilde\eta^\kappa(\theta_i((0,1)^2))} [\alpha_i(\theta_i) \tilde\eta^n_{i\kappa},_{111}\, \tilde\eta^n_{i\kappa},_{111}] \circ\theta_i^{-1}\circ(\tilde\eta^\kappa)^{-1} \\
& + (C\kappa^{\frac12}+\delta)\tilde E(t)^2 + Ct\tilde E(t)^4 + C_\delta + N(u_0).
\end{aligned} \tag{24.24}$$

Now, it is clear that the space integral in front of the first time integral in (24.22) can be treated in a similar way, except that we do not have a control on the sign of the boundary term as in (24.24), which does not matter since a time integral is applied to it. This therefore leads us to

$$\begin{aligned}
-\int_0^t H_{11} \le & -C\int_{\partial\tilde\eta^\kappa(\theta_i((0,1)^2)} [\alpha_i(\theta_i) \tilde\eta^n_{i\kappa},_{111}\, \tilde\eta^n_{i\kappa},_{111}] \circ\theta_i^{-1}\circ(\tilde\eta^\kappa)^{-1} \\
& + (C\kappa^{\frac12}+\delta)\tilde E(t)^2 + Ct\tilde E(t)^4 + C_\delta + N(u_0),
\end{aligned}$$

which , with (24.1) and (24.2), finally gives the trace control for each $\tilde\eta_{i\kappa}$, as well as the control of v around Γ:

$$\begin{aligned}
H^\kappa(t) + C\int_{\partial\tilde\eta^\kappa(\theta_i((0,1)^2))} & |[\sqrt{\alpha_i}(\theta_i)\tilde\eta^n_{i\kappa},_{111}] \circ\theta_i^{-1}\circ(\tilde\eta^\kappa)^{-1}|^2 \\
& \le \delta\tilde E(t)^2 + C_\delta t\tilde E(t)^4 + C_\delta N(u_0).
\end{aligned} \tag{24.25}$$

24.3. **Asymptotic regularity of each $\tilde\eta_{l\kappa}$.** Consequently, we infer that for each $l\in\{1,...,K\}$, we have the trace control

$$\|[\sqrt{\alpha_l}(\theta_l)\tilde\eta_{l\kappa},_1] \circ\theta_l^{-1}\circ(\tilde\eta^\kappa)^{-1}\|^2_{2,\partial\tilde\eta^\kappa\circ\theta_l((0,1)^2)} \le \delta\tilde E(t)^2 + C_\delta t\tilde E(t)^4 + C_\delta N(u_0). \tag{24.26}$$

Consequently, with the estimates (22.8) and (22.11) on the divergence and curl, we obtain by elliptic regularity:

$$\begin{aligned}
\|[\sqrt{\alpha_l}(\theta_l) & \tilde\eta_{l\kappa},_1] \circ\theta_l^{-1}\circ(\tilde\eta^\kappa)^{-1}\|^2_{\frac52,\tilde\eta^\kappa\circ\theta_l((0,1)^2)} \\
& \le P(\|\tilde\eta^\kappa\|_{\frac52})[\delta\tilde E(t)^2 + C_\delta t\tilde E(t)^4 + C_\delta N(u_0)] \\
& \le \delta\tilde E(t)^2 + C_\delta t\tilde E(t)^4 + C_\delta N(u_0).
\end{aligned}$$

Therefore,

$$\|\sqrt{\alpha_l}(\theta_l)\tilde{\eta}_{l\kappa},_1\|^2_{\frac{5}{2},(0,1)^2} \leq \delta\tilde{E}(t)^2 + C_\delta t\tilde{E}(t)^4 + C_\delta N(u_0), \tag{24.27}$$

which implies that $\|\sqrt{\alpha_l}(\theta_l)\tilde{\eta}_{l\kappa},_{12}\|^2_{\frac{3}{2},(0,1)^2}$ and $\|\sqrt{\alpha_l}(\theta_l)\tilde{\eta}_{l\kappa},_2\|^2_{2,\partial(0,1)^2}$ are controlled by the same right-hand side as in (24.27). Consequently, with (22.8) and (22.11), we infer in the same way as we obtained (24.27) from (24.26) that

$$\|\sqrt{\alpha_l}(\theta_l)\tilde{\eta}_{l\kappa},_2\|^2_{\frac{5}{2},(0,1)^2} \leq \delta\tilde{E}(t)^2 + C_\delta t\tilde{E}(t)^4 + C_\delta N(u_0),$$

and finally that

$$\|\sqrt{\alpha_l}(\theta_l)\tilde{\eta}_{l\kappa}\|^2_{\frac{7}{2},(0,1)^2} \leq \delta\tilde{E}(t)^2 + C_\delta t\tilde{E}(t)^4 + C_\delta N(u_0). \tag{24.28}$$

24.4. **Asymptotic regularity of $\tilde{\eta}^\kappa$.** This also obviously implies that for the advected domain

$$\|\tilde{\eta}^\kappa\|^2_{\frac{7}{2},\Omega^K} \leq \delta\tilde{E}(t)^2 + C_\delta t\tilde{E}(t)^4 + C_\delta N(u_0), \tag{24.29}$$

where $\Omega^K = \Omega \bigcap_{i=K+1}^{L} (\text{supp}\alpha_i)^c$.

From the divergence and curl estimates (22.4) and (22.10), we then infer that

$$\|\beta\tilde{\eta}\circ\theta_l\|^2_{\frac{7}{2},(0,1)^2} \leq \delta\tilde{E}(t)^2 + C_\delta t\tilde{E}(t)^4 + C_\delta N(u_0), \tag{24.30}$$

which with (24.28) provides

$$\|\tilde{\eta}^\kappa\|^2_{\frac{7}{2}} \leq \delta\tilde{E}(t)^2 + C_\delta t\tilde{E}(t)^4 + C_\delta N(u_0, \kappa\|\Omega\|_{\frac{9}{2}}). \tag{24.31}$$

24.5. **Asymptotic regularity of $\tilde{v}$.** The relation (24.25) provides us with the asymptotic regularity of $\tilde{v}$ near $\partial\Omega$. For the interior regularity, we notice that if we time-differentiate the analog of (22.5) for the cut-off $\beta \in \mathcal{D}(\Omega)$, we obtain

$$\left\|\text{div}((\beta\tilde{v}\circ\theta_l),_s \circ\theta_l^{-1}\circ(\tilde{\eta}^\kappa)^{-1})\right\|_{1,\tilde{\eta}^\kappa(\Omega)} \leq C. \tag{24.32}$$

Similarly, we also have

$$\left\|\text{curl}((\beta\tilde{v}\circ\theta_l),_s \circ\theta_l^{-1}\circ(\tilde{\eta}^\kappa)^{-1})\right\|_{2,\tilde{\eta}^\kappa(\Omega)} \leq C. \tag{24.33}$$

From (24.32) and (24.33), elliptic regularity yields

$$\|\beta\tilde{v}\circ\theta_l\|_{2,(0,1)^2} \leq C,$$

which, together with (24.25), provides

$$\|\tilde{v}\|^2_3 \leq \delta\tilde{E}(t)^2 + C_\delta t\tilde{E}(t)^4 + C_\delta N(u_0). \tag{24.34}$$

24.6. **Asymptotic regularity of $\tilde{q}$.** From the elliptic system

$$(\tilde{a}^\kappa)^j_i[(\tilde{a}^\kappa)^k_i\tilde{q},_k],_j = -(\tilde{u}^\kappa,^j_i\,\tilde{u},^i_j)(\tilde{\eta}^\kappa) \text{ in } \Omega,$$
$$\tilde{q} = 0 \text{ on } \Gamma,$$

we then infer on $(0, T_\kappa)$ (ensuring that (19.2) is satisfied):

$$\|\tilde{q}\|^2_{\frac{7}{2}} \leq C\|\tilde{\eta}^\kappa\|^2_{\frac{7}{2}} \leq \delta\tilde{E}(t)^2 + C_\delta t\tilde{E}(t)^4 + C_\delta N(u_0). \tag{24.35}$$

24.7. **Asymptotic regularity of $\kappa\sqrt{\alpha_l}\tilde{v}\circ\theta_l$.** From (23.7), we have

$$[\kappa\|\sqrt{\alpha_l}\tilde{v}\circ\theta_l\|_{\frac{7}{2},(0,1)^2}]^2 \leq [\kappa\|u_0\|_{\frac{7}{2}}]^2 + \delta\tilde{E}(t)^2 + C_\delta t\tilde{E}(t)^4 + C_\delta N(u_0). \tag{24.36}$$

24.8. **Asymptotic regularity of $\kappa^{\frac{3}{2}}\sqrt{\alpha_l}\tilde{v}\circ\theta_l$.** From (23.8), we have

$$[\kappa^2\|\sqrt{\alpha_l}\tilde{v}\circ\theta_l\|_{\frac{9}{2},(0,1)^2}]^2 \leq [\kappa^2\|u_0\|_{\frac{9}{2}}]^2 + C_\delta t\tilde{E}(t)^4 + C_\delta N(u_0),$$

which by interpolation leads to

$$[\kappa^{\frac{3}{2}}\|\sqrt{\alpha_l}\tilde{v}\circ\theta_l\|_{4,(0,1)^2}]^2 \leq [\kappa^{\frac{3}{2}}\|u_0\|_4]^2 + \delta\tilde{E}(t)^2 + C_\delta t\tilde{E}(t)^4 + C_\delta N(u_0). \tag{24.37}$$

24.9. **Asymptotic regularity of $\tilde{q}_t$.** From the elliptic system

$$\begin{aligned}&(\tilde{a}^\kappa)_i^j[(\tilde{a}^\kappa)_i^k\tilde{q}_{t},_k],_j = -[\tilde{u}^\kappa,_i^j\,\tilde{u},_j^i\,(\tilde{\eta}^\kappa)]_t - [(\tilde{a}^\kappa)_i^j[(\tilde{a}^\kappa)_i^k]_t\tilde{q},_k],_j \text{ in } \Omega,\\ &\tilde{q} = 0 \text{ on } \Gamma,\end{aligned}$$

we then infer on $(0,T_\kappa)$

$$\|\tilde{q}_t\|_3^2 \leq \delta\tilde{E}(t)^2 + C_\delta t\tilde{E}(t)^4 + C_\delta N(u_0). \tag{24.38}$$

24.10. **Asymptotic regularity of $\tilde{v}_t$.** Since $\tilde{v}_t^i = -(\tilde{a}^\kappa)_i^j q,_j$, we then infer on $(0,T_\kappa)$

$$\|\tilde{v}_t\|_{\frac{5}{2}}^2 \leq C(\|\tilde{\eta}^\kappa\|_{\frac{7}{2}} + \|\tilde{q}\|_{\frac{7}{2}})^2 \leq \delta\tilde{E}(t)^2 + C_\delta t\tilde{E}(t)^4 + C_\delta N(u_0). \tag{24.39}$$

25. Time of existence independent of κ and solution to the limit problem

By (24.31), (24.35), (24.34), (24.30), (24.28), (24.36), (24.37) we then infer the control on $(0,T_\kappa)$:

$$\tilde{E}(t)^2 \leq \delta\tilde{E}(t)^2 + C_\delta t\tilde{E}(t)^4 + C_\delta N(u_0),$$

which for a choice of δ_0 small enough provides us with

$$\tilde{E}(t)^2 \leq C_{\delta_0}N(u_0) + C_{\delta_0}t\tilde{E}(t)^4.$$

Similarly as in Section 9 of [8], this provides us with a time of existence $T_\kappa = T_1$ independent of κ and an estimate on $(0,T_1)$ independent of κ of the type

$$\tilde{E}(t)^2 \leq N_0(u_0),$$

as long as the conditions (19.2) hold. Now, since $\|\tilde{\eta}(t)\|_3 \leq \|\mathrm{Id}\|_3 + \displaystyle\int_0^t \|\tilde{v}\|_3$, we see that condition (19.2b) will be satisfied for $t \leq \dfrac{1}{N_0(u_0)}$. The other conditions in (19.2) are satisfied with similar arguments ((24.38) and (24.39) are used for (19.2c) and (19.2d)). This leads us to a time of existence $T_2 > 0$ independent of κ for which we have the estimate on $(0,T)$

$$\tilde{E}(t)^2 \leq N_0(u_0),$$

which provides by weak convergence the existence of a solution (v,q) of (1.1), with $\sigma = 0$, on $(0,T)$.

26. Optimal regularity

In this section, we assume that Ω is of class $H^{\frac{7}{2}}$ in $\mathbb{R}^3$, that $u_0 \in H^3(\Omega)$, and that the pressure condition is satisfied. We denote by $N(u_0)$ a generic constant depending on $\|u_0\|_3$. With these requirements, we will only get the $H^{\frac{7}{2}}$ regularity of the moving domain $\eta(\Omega)$ and not of the mapping η.

Due to the fact that $H^{\frac{3}{2}}$ is not continuously embedded in L^∞ in the case that Ω is three-dimensional, we cannot directly study the integral terms as in Section 24 as we did for the two-dimensional case. Instead, we are forced to also regularize the initial domain, by a standard convolution, with a parameter $\epsilon > 0$ fixed independently of κ, on the charts defining it locally, so that the initial regularized domain $\Omega_\epsilon = \tilde{\Omega}$ obtained in this fashion is of class C^∞. The regularized initial velocity, by a standard convolution, will be denoted $u_0(\epsilon)$. We then start at Section 18 in the same way except that the regularity of the functional framework is increased by one degree for each quantity. This leaves us with the existence of a solution to (17.1) on $(0, T_{\kappa,\epsilon})$, with initial domain Ω_ϵ and initial velocity $u_0(\epsilon)$. We then perform the same asymptotic analysis as $\kappa \to 0$ as we did in Sections 19 to 24, in this new framework. We then see that the problematic term is now updated to one which can be treated directly by the Sobolev embedding of H^2 into L^∞ in 3d. This leads us to the existence of a solution to a system similar to (1.1) (with $\sigma = 0$) with initial domain Ω_ϵ on $(0, T_\epsilon)$, with $\eta_\epsilon \in L^\infty(0, T_\epsilon; H^{\frac{9}{2}}(\Omega_\epsilon))$, with initial domain Ω_ϵ and initial velocity $u_0(\epsilon)$.

We then study hereafter the asymptotic behavior of this solution and of T_ϵ as $\epsilon \to 0$. This will be less problematic than in Section 24 since the convolutions by layers with the parameter κ do not appear in the problem (1.1) with smoothed initial data and domain. We will denote the dependence on ϵ this time by a tilde, $\tilde{v}$ standing here for v_ϵ for instance, and prove that as $\epsilon \to 0$, the time of existence and norms of $\tilde{v}$ are $\epsilon-$independent, which leads to the existence of a solution with optimal regularity on the initial data, as stated in Theorem 1.4.

Our functional framework will be different than in Sections 19 to 24. Our continuous in time energy will be:

Definition 26.1.

$$\tilde{H}(t) = \sup_{[0,t]} \left[|\tilde{n}|_{2,\tilde{\Gamma}} + \|\tilde{v}\|_3 + \|\tilde{v}_t\|_{\frac{5}{2}} + \|\tilde{q}\|_3 + \|\tilde{v}_{tt}\|_2 \right] + 1, \tag{26.1}$$

where $\tilde{n}$ denotes the unit exterior normal to $\tilde{\eta}(\Omega)$.

Our condition on T_ϵ will be that on $(0, T_\epsilon)$,

$$\frac{1}{2} \leq \det \nabla \tilde{\eta} \leq \frac{3}{2} \text{ in } \tilde{\Omega}, \tag{26.2a}$$

$$\|\tilde{\eta}\|_3 \leq |\Omega| + 1, \quad \|\tilde{q}\|_3 \leq \|q_0\|_3 + 1, \quad \|\tilde{v}\|_{\frac{5}{2}} \leq \|u_0\|_{\frac{5}{2}} + 1, \tag{26.2b}$$

$$\|\tilde{v}_t\|_2 \leq \|w_1\|_2 + 1, \tag{26.2c}$$

$$\forall l \in \{1, ..., K\}, \; \left| \tilde{\eta} \circ \theta_{l,1} \times \tilde{\eta} \circ \theta_{l,2} \right| \geq \frac{1}{2} \left| \theta_{l,1} \times \theta_{l,2} \right| \text{ on } (0,1)^2 \times \{0\}, \tag{26.2d}$$

where $w_1 = -\nabla q_0 \in H^{\frac{5}{2}}(\Omega)$. We will use a more straightforward approach than in Section 24, which is enabled by the fact that we have $\tilde{a}$ instead of the convolution by layers $\tilde{a}^\kappa$ in our equation, by defining the following energy:

Definition 26.2.

$$E^\epsilon(t)=\sum_{l=1}^K\int_{(0,1)^3}\xi\circ\theta_l|D^2(\tilde v_{tt}\circ\theta_l)|^2,$$

where D^2f stands for any second space derivative in a horizontal direction, i.e., $f,_{\alpha_1\alpha_2}$, where $\alpha_i\in\{1,2\}$. Summation over all horizontal derivatives is taken in the expression for E^κ.

Remark 19. We also note that this energy is associated with the second time-differentiated problem; we thus avoid the use of the curl relation (22.11) for $\tilde\eta$, which necessitates the supplementary condition $\operatorname{curl} u_0\in H^{\frac52}(\Omega)$ (which we do not have here).

With $\tilde b_l=[\nabla(\tilde\eta\circ\theta_l)]^{-1}$, we have: $E^\epsilon_t=\sum_{i=1}^9 E_i$, with

$$E_1(t)=-\sum_{l=1}^K\int_{(0,1)^3}\xi(\theta_l)D^2([(\tilde b_l)^k_j]_{tt})(\tilde q\circ\theta_l),_k D^2(\tilde v_{tt}\circ\theta_l)^j,$$

$$E_2(t)=-2\sum_{l=1}^K\int_{(0,1)^3}\xi(\theta_l)D([(\tilde b_l)^k_j]_{tt})D(\tilde q\circ\theta_l),_k D^2(\tilde v_{tt}\circ\theta_l)^j,$$

$$E_3(t)=-\sum_{l=1}^K\int_{(0,1)^3}\xi(\theta_l)[(\tilde b_l)^k_j]_{tt}D^2(\tilde q\circ\theta_l),_k D^2(\tilde v_{tt}\circ\theta_l)^j,$$

$$E_4(t)=-4\sum_{l=1}^K\int_{(0,1)^3}\xi(\theta_l)D[(\tilde b_l)^k_j]_t D(\tilde q_t\circ\theta_l),_k D^2(\tilde v_{tt}\circ\theta_l)^j,$$

$$E_5(t)=-2\sum_{l=1}^K\int_{(0,1)^3}\xi(\theta_l)D^2[(\tilde b_l)^k_j]_t(\tilde q_t\circ\theta_l),_k D^2(\tilde v_{tt}\circ\theta_l)^j,$$

$$E_6(t)=-2\sum_{l=1}^K\int_{(0,1)^3}\xi(\theta_l)[(\tilde b_l)^k_j]_t D^2(\tilde q_t\circ\theta_l),_k D^2(\tilde v_{tt}\circ\theta_l)^j,$$

$$E_7(t)=-\sum_{l=1}^K\int_{(0,1)^3}\xi(\theta_l)D^2(\tilde b)^k_j(\tilde q_{tt}\circ\theta_l),_k D^2(\tilde v_{tt}\circ\theta_l)^j,$$

$$E_8(t)=-2\sum_{l=1}^K\int_{(0,1)^3}\xi(\theta_l)D(\tilde b)^k_j D(\tilde q_{tt}\circ\theta_l),_k D^2(\tilde v_{tt}\circ\theta_l)^j,$$

$$E_9(t)=-\sum_{l=1}^K\int_{(0,1)^3}\xi(\theta_l)(\tilde b)^k_j D^2(\tilde q_{tt}\circ\theta_l),_k D^2(\tilde v_{tt}\circ\theta_l)^j.$$

26.1. **Estimate for $\tilde q_t$, $\tilde q_{tt}$ and $\tilde q_{ttt}$.** From the elliptic system

$$\tilde a^j_i(\tilde a^k_i\tilde q_t,_k),_j=-[\tilde a^j_i(\tilde a^k_i]_t\tilde q,_k),_j-[\tilde a^k_j\tilde v^i,_k\tilde a^l_i\tilde v^j,_l]_t \text{ in } \tilde\Omega,$$
$$\tilde q_t=0 \text{ on } \partial\tilde\Omega,$$

we infer

$$\|\tilde q_t\|_3\le C[\ \|\tilde v\|_3+\|\tilde q\|_3+\|\tilde\eta\|_3+\|\tilde v_t\|_2]\le C\tilde H(t). \tag{26.3}$$

For similar reasons, we also have

$$\|\tilde{q}_{tt}\|_{\frac{5}{2}} \leq C\tilde{H}(t), \tag{26.4a}$$

$$\|\tilde{q}_{ttt}\|_2 \leq C\tilde{H}(t). \tag{26.4b}$$

26.2. **Estimate for E_2, E_4, E_5, E_6, E_8.** Thanks to the embedding of H^1 into L^6 and $H^{\frac{1}{2}}$ into L^3 we first immediately have

(26.5a)

$$|E_2(t)| \leq C\|\tilde{a}_{tt}\|_1\|\tilde{q}\|_{\frac{5}{2}}\|\tilde{v}_{tt}\|_2 \leq C[\ \|\tilde{v}_t\|_2\|\tilde{\eta}\|_3 + \|v\|_3^2]\ \|\tilde{q}\|_3\|\tilde{v}_{tt}\|_2 \leq C\tilde{H}(t)^4,$$

(26.5b)

$$|E_4(t)| \leq C\|\tilde{a}_t\|_2\|\tilde{q}_t\|_{\frac{5}{2}}\|\tilde{v}_{tt}\|_2 \leq C\tilde{H}(t)^3,$$

(26.5c)

$$|E_5(t)| \leq C\|\tilde{v}\|_3\|\tilde{q}_t\|_3\|\tilde{v}_{tt}\|_2 \leq C\tilde{H}(t)^3,$$

(26.5d)

$$|E_6(t)| \leq C\|\tilde{v}\|_3\|\tilde{q}_t\|_3\|\tilde{v}_{tt}\|_2 \leq C\tilde{H}(t)^3,$$

(26.5e)

$$|E_8(t)| \leq C\|\tilde{a}\|_2\|\tilde{q}_{tt}\|_{\frac{5}{2}}\|\tilde{v}_{tt}\|_2 \leq C\tilde{H}(t)^3,$$

where we have used (26.3) for (26.5c), (26.5d), and (26.4a) for (26.5e).

26.3. **Estimate for E_3.** By integrating by parts, and using $[(\tilde{b})^k_j],_k = 0$, we obtain $E_3 = E_3^1 + E_3^2$, with

$$E_3^1 = \sum_{l=1}^K \int_{(0,1)^3} \xi(\theta_l)[(\tilde{b})^k_j]_{tt} D^2(\tilde{q}\circ\theta_l) D^2(\tilde{v}_{tt}\circ\theta_l),^j_k\,,$$

$$E_3^2 = \sum_{l=1}^K \int_{(0,1)^3} \xi(\theta_l),_k\,[(\tilde{b})^k_j],_{tt}\, D^2(\tilde{q}\circ\theta_l) D^2(\tilde{v}_{tt}\circ\theta_l)^j.$$

We first have

$$|E_3^2(t)| \leq C\|\tilde{a}_{tt}\|_1\|\tilde{q}\|_3\|\tilde{v}_{tt}\|_2 \leq C\tilde{H}(t)^4.$$

Next, $E_3^1 = \sum_{i=1}^3 E_3^{1i}$, with

$$E_3^{11} = -\sum_{l=1}^K \int_{(0,1)^3} \xi(\theta_l) D[(\tilde{b})^k_j]_{tt} D^2(\tilde{q}\circ\theta_l) D(\tilde{v}_{tt}\circ\theta_l),^j_k\,,$$

$$E_3^{12} = -\sum_{l=1}^K \int_{(0,1)^3} \xi(\theta_l)[(\tilde{b})^k_j]_{tt} D^3(\tilde{q}\circ\theta_l) D(\tilde{v}_{tt}\circ\theta_l),^j_k\,,$$

$$E_3^{13} = -\sum_{l=1}^K \int_{(0,1)^3} D(\xi(\theta_l))[(\tilde{b})^k_j]_{tt} D^2(\tilde{q}\circ\theta_l) D(\tilde{v}_{tt}\circ\theta_l),^j_k\,.$$

We obviously have $|E_3^{13}(t)|\le C\tilde H(t)^4$. Next, we have by integrating by parts in time:

$$\int_0^t E_3^{12}=\sum_{l=1}^K\int_0^t\int_{(0,1)^3}\xi(\theta_l)\big([(\tilde b)^k_j]_{tt}D^3(\tilde q_t\circ\theta_l)+[(\tilde b)^k_j]_{ttt}D^3(\tilde q\circ\theta_l)\big)D(\tilde v_t\circ\theta_l),^j_k$$
$$+\big[\sum_{l=1}^K\int_{(0,1)^3}\xi(\theta_l)[(\tilde b)^k_j]_{tt}D^3(\tilde q\circ\theta_l)D(\tilde v_t\circ\theta_l),^j_k\big]_0^t,$$

showing, with the continuous embedding of H^1 into L^6 and of $H^{\frac12}$ into L^3:

(26.6)

$$\begin{aligned}\Big|\int_0^t E_3^{12}\Big|&\le Ct\sup_{[0,t]}[\|\tilde a_{tt}\|_1\|\tilde q_t\|_3\|\tilde v_t\|_{\frac52}+\|\tilde a_{ttt}\|_1\|\tilde q\|_3\|\tilde v_t\|_{\frac52}]\\&\quad+\big[\sum_{l=1}^K\int_{(0,1)^3}\xi(\theta_l)D(\tilde v_t\circ\theta_l),^j_k\,\big[[(\tilde b)^k_j]_{tt}(0)D^3(\tilde q\circ\theta_l)(0)\\&\quad+\int_0^{\cdot}[[(\tilde b)^k_j]_{tt}D^3(\tilde q\circ\theta_l)]_t\big]\big]_0^t\\&\le Ct\sup_{[0,t]}[\|\tilde a_{tt}\|_1\|\tilde q_t\|_3\|\tilde v_t\|_{\frac52}+\|\tilde a_{ttt}\|_1\|\tilde q\|_3\|\tilde v_t\|_{\frac52}]\\&\quad+\|\tilde v_t|_{\frac52}\|\tilde q(0)\|_3\|\tilde b_{tt}(0)\|_1+\|\tilde v_t|_{\frac52}t\sup_{[0,t]}[\|\tilde q_t\|_3\|\tilde b_{tt}\|_1+\|\tilde q\|_3\|\tilde b_{ttt}\|_1]+N(u_0)\\&\le C\delta\tilde H(t)^2+t\tilde H(t)^4+C_\delta N(u_0),\end{aligned}$$

for any $\delta>0$. For the remaining term E_3^{11},

$$|E_3^{11}|\le C\|\tilde a_{tt}\|_{\frac32}\|\tilde q\|_3\|\tilde v_{tt}\|_2\le C\tilde H(t)^4.$$

Consequently, we have

$$\Big|\int_0^t E_3\Big|\le C\delta\tilde H(t)^2+t\tilde H(t)^4+C_\delta N(u_0). \tag{26.7}$$

26.4. **Estimate for E_7.** By integrating by parts, $E_7=E_7^1+E_7^2$, with

$$E_7^1=\sum_{l=1}^K\int_{(0,1)^3}\xi(\theta_l)D^2(\tilde b)^k_j(\tilde q_{tt}\circ\theta_l)D^2(\tilde v_{tt}\circ\theta_l),^j_k\,,$$
$$E_7^2=\sum_{l=1}^K\int_{(0,1)^3}\xi(\theta_l),_k\,D^2(\tilde b)^k_j(\tilde q_{tt}\circ\theta_l)D^2(\tilde v_{tt}\circ\theta_l)^j.$$

We first have

$$|E_7^2(t)|\le C\|\tilde a\|_2\|\tilde q_{tt}\|_2\|\tilde v_{tt}\|_2\le C\tilde H(t)^4.$$

Next, we notice by integrating by parts in time and space that

$$\int_0^t E_7^1=\sum_{l=1}^K\int_0^t\int_{(0,1)^3}[D^2(\tilde b_t)^k_j(\xi\tilde q_{tt}\circ\theta_l),_k+D^2(\tilde b)^k_j(\xi\tilde q_{ttt}\circ\theta_l),_k]D^2(\tilde v_t\circ\theta_l)^j$$
$$+\big[\sum_{l=1}^K\int_{(0,1)^3}\xi(\theta_l)D^2(\tilde b)^k_j(\xi\tilde q_{tt}\circ\theta_l),_k\,D^2(\tilde v_t\circ\theta_l)^j\big]_0^t.$$

In the same fashion as we obtained (26.6), we then infer

(26.8)

$$
\begin{aligned}
\Big|\int_0^t E_7^1\Big| &\le Ct\sup_{[0,t]}[\|\tilde b_t\|_2\|\tilde q_{tt}\|_2\|\tilde v_t\|_{\frac52}+\|\tilde b\|_2\|\tilde q_{ttt}\|_2\|\tilde v_t\|_{\frac52}]\\
&\quad+\|\tilde v_t\|_{\frac52}\|\tilde q_{tt}(0)\|_2\|\tilde b(0)\|_2+\|\tilde v_t\|_{\frac52}t\sup_{[0,t]}[\|\tilde q_{ttt}\|_2\|\tilde b\|_2+\|\tilde q_{tt}\|_2\|\tilde b_t\|_2]+N(u_0)\\
&\le C\delta\tilde H(t)^2+t\tilde H(t)^4+C_\delta N(u_0),
\end{aligned}
$$

where we have used (26.4b) for $\tilde q_{ttt}$. Consequently, we have

$$
\Big|\int_0^t E_7\Big|\le C\delta\tilde H(t)^2+t\tilde H(t)^4+C_\delta N(u_0). \tag{26.9}
$$

26.5. **Estimate for E_9.** We notice by integrating by parts in space that

$$
\begin{aligned}
E_9(t)&=\sum_{l=1}^K\int_{(0,1)^3}\xi(\theta_l)[(\tilde b_l)_j^k]D^2(\tilde q_{tt}\circ\theta_l)D^2(\tilde v_{tt}\circ\theta_l),_k^j\\
&\quad+\sum_{l=1}^K\int_{(0,1)^3}\xi(\theta_l),_k[(\tilde b_l)_j^k]D^2(\tilde q_{tt}\circ\theta_l)D^2(\tilde v_{tt}\circ\theta_l)^j.
\end{aligned}
$$

Next by the divergence condition,

$$
\begin{aligned}
E_9(t)&=-\sum_{l=1}^K\int_{(0,1)^3}\xi(\theta_l)D^2[(\tilde b_l)_j^k]D^2(\tilde q_{tt}\circ\theta_l)(\tilde v_{tt}\circ\theta_l),_k^j\\
&\quad-2\sum_{l=1}^K\int_{(0,1)^3}\xi(\theta_l)D[(\tilde b_l)_j^k]D^2(\tilde q_{tt}\circ\theta_l)D(\tilde v_{tt}\circ\theta_l),_k^j\\
&\quad+\sum_{l=1}^K\int_{(0,1)^3}\xi(\theta_l),_k[(\tilde b_l)_j^k]D^2(\tilde q_{tt}\circ\theta_l)D^2(\tilde v_{tt}\circ\theta_l)^j,
\end{aligned}
$$

showing that

$$
|E_9(t)|\le C\|\tilde b\|_2\|\tilde q_{tt}\|_{\frac52}\|\tilde v_{tt}\|_2\le C\tilde H(t)^4. \tag{26.10}
$$

26.6. **Estimate for E_1.** If ϵ^{ijk} denotes the sign of the permutation between $\{i,j,k\}$ and $\{1,2,3\}$, if i,j,k are distinct, and is set to zero otherwise, we obtain

$$
E_1=E_1^1+E_1^2,
$$

with

$$
\begin{aligned}
E_1^1&=\sum_{l=1}^K\int_{(0,1)^3}\xi(\theta_l),_k\tilde q\circ\theta_l D^2[(\tilde b_l)_j^k]_{tt}D^2(\tilde v_{tt}\circ\theta_l)^j,\\
E_1^2&=\frac12\sum_{l=1}^K\epsilon^{mnj}\epsilon^{pqk}\int_{(0,1)^3}\xi\tilde q(\theta_l)D^2[\tilde\eta\circ\theta_l,_p^m\tilde\eta\circ\theta_l,_q^n]_{tt}D^2(\tilde v_{tt}\circ\theta_l),_k^j.
\end{aligned}
$$

26.6.1. **Estimate of E_1^1.** Now, for E_1^1, since

$$D_t\triangle\tilde{u}+\tilde{u},_j^i\,\tilde{u},_{ij}-\nabla[\tilde{u},_i^j\,\tilde{u},_j^i]=0,$$

we obtain in $\tilde{\Omega}$ that

$$\tilde{a}_i^j(\tilde{a}_i^k\tilde{v},_k),_j(t)=\triangle\tilde{u}(0)+\int_0^t[\tilde{a}_n^m(\tilde{a}_i^k\tilde{v},_k^j\,\tilde{a}_j^l\tilde{v},_l^i),_m]_{n=1}^3-\int_0^t\tilde{a}_j^k\tilde{v},_k^i\,\tilde{a}_j^m(\tilde{a}_i^l\tilde{v},_l),_m,$$

and thus,

$$\tilde{a}_i^j(\tilde{a}_i^k\tilde{v}_{t},_k),_j=-[\tilde{a}_i^j(\tilde{a}_i^k]_t\tilde{v},_k),_j+[\bar{a}_n^m(\tilde{a}_i^k\tilde{v},_k^j\,\tilde{a}_j^l\tilde{v},_l^i),_m]_{n=1}^3-\tilde{a}_j^k\tilde{v},_k^i\,\tilde{a}_j^m(\tilde{a}_i^l\tilde{v},_l),_m. \tag{26.11}$$

By elliptic regularity in the interior of $\tilde{\Omega}$, we infer that for any ω whose closure is contained in $\tilde{\Omega}$,

$$\|\tilde{v}_t\|_{3,\omega}\le C_\omega[\|\bar{a}_i^j(\bar{a}_i^k\tilde{v}_t,_k),_j\|_{1,\tilde{\Omega}}+\|\tilde{v}_t\|_2]\le C_\omega\tilde{H}(t).$$

With this estimate and the condition $\xi\circ\theta_l,_k=0$ in a neighborhood of $(0,1)^2\times\{0\}$, we then obtain

$$\begin{aligned}|E_1^1|&\le C\|\tilde{q}\|_2\big[C\tilde{H}(t)\|\tilde{\eta}\|_3+\|\tilde{v}\|_3^2\big]\|\tilde{v}_{tt}\|_2\\&\le C(\tilde{H}(t)^3+1).\end{aligned} \tag{26.12}$$

26.6.2. **Estimate for E_1^2 and the trace regularity.** We now study E_1^2, which will be the term bringing the asymptotic regularity of the moving domain $\tilde{\eta}(\tilde{\Omega})$. We have that

$$E_1^2=\sum_{l=1}^3E_1^{2l},$$

with

$$\begin{aligned}E_1^{21}&=\sum_{l=1}^K\epsilon^{mnj}\epsilon^{pqk}\int_{(0,1)^3}\xi\tilde{q}(\theta_l)D^2[\tilde{v}\circ\theta_l,_p^m\,\tilde{v}\circ\theta_l,_q^n]D^2(\tilde{v}_{tt}\circ\theta_l),_k^j,\\E_1^{22}&=2\sum_{l=1}^K\epsilon^{mnj}\epsilon^{pqk}\int_{(0,1)^3}\xi\tilde{q}(\theta_l)[D(\tilde{v}_t\circ\theta_l),_p^m\,D(\tilde{\eta}\circ\theta_l),_q^n]D^2(\tilde{v}_{tt}\circ\theta_l),_k^j,\\E_1^{23}&=\sum_{l=1}^K\frac{1}{2}\epsilon^{mni}\epsilon^{pqj}\int_{(0,1)^3}\xi(\theta_l)[D^2(\tilde{\eta}_{tt}\circ\theta_l),_p^m\,D^2(\tilde{\eta}_{tt}\circ\theta_l),_j^i]_t\tilde{q}\circ\theta_l(\tilde{\eta}\circ\theta_l),_q^n.\end{aligned}$$

We first notice that

$$\begin{aligned}|E_1^{21}|+|E_1^{22}|&\le C\|\tilde{q}\|_3\|\tilde{v}\|_3^2\|\tilde{v}_{tt}\|_2+C\|\tilde{q}\|_3\|\tilde{v}_t\|_{\frac{5}{2}}\|\tilde{\eta}\|_3\|\tilde{v}_{tt}\|_2\\&\le C\tilde{H}(t)^4.\end{aligned} \tag{26.13}$$

Now, for the remaining term E_1^{23}, an integration by parts in time provides

$$\int_0^tE_1^{23}=\frac{1}{2}E_1^{231}+\frac{1}{2}[E_1^{232}]_0^t,$$

with

$$E_1^{231} = -\sum_{l=1}^K \epsilon^{mni}\epsilon^{pqj}\int_0^t\int_{(0,1)^3}\xi(\theta_l)D^2(\tilde v_t\circ\theta_l),_p^m D^2(\tilde v_t\circ\theta_l),_j^i (\tilde q\circ\theta_l\ (\tilde\eta\circ\theta_l),_q^n)_t,$$

$$E_1^{232} = \epsilon^{mni}\epsilon^{pqj}\int_{(0,1)^3}\xi(\theta_l)D^2(\tilde v_t\circ\theta_l),_p^m D^2(\tilde v_t\circ\theta_l),_j^i\,\tilde q\circ\theta_l\ (\tilde\eta\circ\theta_l),_q^n .$$

First, for the perturbation term E_1^{231}, by integrating by parts in space (and using $\tilde q=0$ on Γ):

$$E_1^{231} = \int_0^t\sum_{l=1}^K \epsilon^{mni}\epsilon^{pqj}\int_{(0,1)^3}\xi(\theta_l)D^2(\tilde v_t\circ\theta_l),_{pj}^m D^2(\tilde v_t\circ\theta_l)^i(\tilde q\circ\theta_l\ (\tilde\eta\circ\theta_l),_q^n)_t$$
$$+\int_0^t\sum_{l=1}^K \epsilon^{mni}\epsilon^{pqj}\int_{(0,1)^3} D^2(\tilde v_t\circ\theta_l),_p^m D^2(\tilde v_t\circ\theta_l)^i(\xi\tilde q\circ\theta_l\ (\tilde\eta\circ\theta_l),_q^n)_{t,j} .$$

For the first integral, we notice that for any f, g smooth,

$$\epsilon^{mni}\epsilon^{pqj}f,_{pj}^m f^i g,_q^n = \epsilon^{mni}\epsilon^{jqp}f,_{jp}^m f^i g,_q^n ,$$

and, since $\epsilon^{pqj}=-\epsilon^{jqp}$, this quantity equals zero, leading to

$$E_1^{231} = \int_0^t\sum_{l=1}^K \epsilon^{mni}\epsilon^{pqj}\int_{(0,1)^3} D^2(\tilde v_t\circ\theta_l),_p^m D^2(\tilde v_t\circ\theta_l)^i(\xi\tilde q\circ\theta_l(\tilde\eta\circ\theta_l),_q^n)_{t,j} .$$

Consequently, as in (8.15) since the derivatives in D^2 are horizontal,

$$|E_1^{231}| \le C\int_0^t \|\tilde v_t\|_{\frac52}^2[\ \|\nabla(\tilde q_t\nabla\eta)\|_1 + \|\nabla(\tilde q\nabla\tilde v)\|_1]$$
$$\le CtH(t)^4. \tag{26.14}$$

Now for E_1^{232}, we will introduce the notation

$$V(l)=\tilde v\circ\theta_l \text{ and } E(l)=\tilde\eta\circ\theta_l.$$

We then have after a change of variables made in order to get vector fields whose divergence and curl are controlled:

$$E_1^{232} = \sum_{l=1}^K \epsilon^{mni}\epsilon^{pqj}\int_{(0,1)^3}\big[\xi\tilde q\circ\theta_l\ (D^2V_t(l)\circ E(l)^{-1}),_{p_1}^m (E(l))E(l),_p^{p_1}$$
$$\times (D^2V_t(l)\circ E(l)^{-1}),_{j_1}^i (E(l))E(l),_j^{j_1}\ (E(l)\circ E(l)^{-1}),_{q_1}^n E(l),_q^{q_1}\big].$$

Now, we notice that any triplet (i_1,j_1,q_1) such that $\mathrm{Card}\{i_1,j_1,q_1\}<3$ will not contribute to this sum. For instance, if $j_1=q_1$, we notice that by relabeling j and q,

$$\epsilon^{pqj}E(l),_p^{p_1}E(l),_j^{j_1}E(l),_q^{j_1} = \epsilon^{pjq}E(l),_p^{p_1}E(l),_q^{j_1}E(l),_j^{j_1} ,$$

where $j_1=q_1$ is fixed in the sum above. Now, since $\epsilon^{pqj}=-\epsilon^{pjq}$, this shows that

$$\epsilon^{pqj}E(l),_p^{p_1}E(l),_j^{j_1}E(l),_q^{j_1}=0.$$

By a similar argument,

$$\epsilon^{pqj}E(l),_p^{p_1}E(l),_j^{p_1}E(l),_q^{q_1}=0.$$

Consequently, only the triplets where $\text{Card}\{i_1, j_1, q_1\} = 3$ contribute to E_1^{232}, showing that

$$E_1^{232} = \sum_{l=1}^{K} \epsilon^{mni}\epsilon^{p_1q_1j_1}\int_{(0,1)^3} \big[\xi\tilde{q}\circ\theta_l\ \epsilon^{pqj}\epsilon^{p_1q_1j_1} E(l),_p^{p_1} E(l),_j^{j_1} E(l),_q^{q_1} \\ \times (D^2V_t(l)\circ E(l)^{-1}),_{p_1}^m (E(l))(D^2V_t(l)\circ E(l)^{-1}),_{j_1}^i (E(l))\delta_{q_1}^n\big].$$

Since for each given (p_1, j_1, q_1) we have $\epsilon^{pqj}\epsilon^{p_1q_1j_1} E(l),_p^{p_1} E(l),_j^{j_1} E(l),_q^{q_1} = \det\nabla E(l) = 1$, we then infer

$$E_1^{232} = \sum_{l=1}^{K} \epsilon^{mni}\epsilon^{pqj}\int_{\tilde{\eta}(\tilde{\Omega})} \big[\xi\tilde{q}\circ\tilde{\eta}^{-1}(D^2V_t(l)\circ E(l)^{-1}),_p^m (D^2V_t(l)\circ E(l)^{-1}),_j^i \delta_q^n\big].$$

By integrating by parts in space, we get by using $\tilde{q} = 0$ on $\tilde{\Gamma}$:

$$E_1^{232} = -\sum_{l=1}^{K} \epsilon^{mni}\epsilon^{pqj}\int_{\tilde{\eta}(\tilde{\Omega})} \big[\xi\tilde{q}\circ\tilde{\eta}^{-1}(D^2V_t(l)\circ E(l)^{-1}),_{pj}^m (D^2V_t(l)\circ E(l)^{-1})^i\delta_q^n\big] \\ -\sum_{l=1}^{K} \epsilon^{mni}\epsilon^{pqj}\int_{\tilde{\eta}(\tilde{\Omega})} \big[(\xi\tilde{q}\circ\tilde{\eta}^{-1}),_j (D^2V_t(l)\circ E(l)^{-1}),_p^m (D^2V_t(l)\circ E(l)^{-1})^i\delta_q^n\big].$$

Next, since for any f smooth,

$$\epsilon^{mni}\epsilon^{pqj} f,_{pj}^m \delta_q^n = 0,$$

we infer

$$E_1^{232} = -\sum_{l=1}^{K} \epsilon^{mni}\epsilon^{pqj}\int_{\tilde{\eta}(\tilde{\Omega})} (\xi\tilde{q}\circ\tilde{\eta}^{-1}),_j (D^2V_t(l)\circ E(l)^{-1}),_p^m (D^2V_t(l)\circ E(l)^{-1})^i\delta_q^n.$$

Now, we notice that for $\delta_n^q\epsilon^{mni}\epsilon^{pqj} \neq 0$, if $p = i$, then necessarily $j = m$. Similarly, if $p \neq i$, then since $p \neq n$, necessarily, $p = m$, and thus $i = j$. Therefore,

$$E_1^{232} = -\sum_{l=1}^{K} \epsilon^{mni}\epsilon^{inm}\int_{\tilde{\eta}(\tilde{\Omega})} (D^2V_t(l)\circ E(l)^{-1}),_i^m (D^2V_t(l)\circ E(l)^{-1})^i\ (\xi\tilde{q}\circ\tilde{\eta}^{-1}),_m \\ -\sum_{l=1}^{K} \epsilon^{mni}\epsilon^{mni}\int_{\tilde{\eta}(\tilde{\Omega})} (D^2V_t(l)\circ E(l)^{-1}),_m^m (D^2V_t(l)\circ E(l)^{-1})^i\ (\xi\tilde{q}\circ\tilde{\eta}^{-1}),_i\,.$$

Now, in the same way as we obtained the divergence and curl estimates (24.32) and (24.33), we have the same type of estimates for $V_t(l)$ leading us to

$$\|\sqrt{\xi}(\theta_l)\ [(D^2V_t(l)\circ E(l)^{-1}),_i^m - (D^2V_t(l)\circ E(l)^{-1}),_m^i]\|_{H^{\frac12}(\tilde{\eta}(\tilde{\Omega}))'} \le C, \\ \|\sqrt{\xi}(\theta_l)\text{div}(D^2V_t(l)\circ E(l)^{-1})\|_{H^{\frac12}(\tilde{\eta}(\tilde{\Omega}))'} \le C.$$

The fact that D^2 contains horizontal derivatives once again played a crucial role in these estimates. This implies

$$
\begin{aligned}
E_1^{232} &= \sum_{l=1}^{K} \sum_{i\neq m} \int_{\tilde\eta(\tilde\Omega)} (D^2 V_t(l)\circ E(l)^{-1}),_m^i\, (D^2 V_t(l)\circ E(l)^{-1})^i\ (\xi\tilde q\circ\tilde\eta^{-1}),_m \\
&\quad + \sum_{l=1}^{K} \int_{\tilde\eta(\tilde\Omega)} (D^2 V_t(l)\circ E(l)^{-1}),_i^i\, (D^2 V_t(l)\circ E(l)^{-1})^i\ (\xi\tilde q\circ\tilde\eta^{-1}),_i + R_1 \\
&= \sum_{l=1}^{K} \int_{\tilde\eta(\tilde\Omega)} (D^2 V_t(l)\circ E(l)^{-1}),_m^i\, (D^2 V_t(l)\circ E(l)^{-1})^i\ (\xi\tilde q\circ\tilde\eta^{-1}),_m + R_1,
\end{aligned}
$$

with $|R_1(t)|\le Ct\tilde H(t)$. Consequently,

$$
\begin{aligned}
E_1^{232} &= -\frac12 \sum_{l=1}^{K} \int_{\tilde\eta(\tilde\Omega)} |D^2 V_t(l)\circ E(l)^{-1}|^2 \triangle(\xi\tilde q\circ\tilde\eta^{-1}) \\
&\quad + \frac12 \sum_{l=1}^{K} \int_{\partial\tilde\eta(\tilde\Omega)} |D^2 V_t(l)\circ E(l)^{-1}|^2 (\xi\tilde q\circ\tilde\eta^{-1}),_m \tilde n_m + R_1.
\end{aligned}
$$

If we write $\tilde q=\tilde q(0)+\int_0^t \tilde q_t$, $\tilde n = N+\int_0^t \tilde n_t$, and use the fact that $\xi=1$ on $\tilde\Gamma$, we then get

$$
E_1^{232} = \frac12 \sum_{l=1}^{K} \int_{\partial\tilde\eta(\tilde\Omega)} |D^2 V_t(l)\circ E(l)^{-1}|^2\ \tilde q(0),_m \tilde N_m + R_2, \tag{26.15}
$$

with $|R_2(t)|\le \delta\tilde H(t)^2 + C_\delta t\tilde H(t)^4 + C_\delta N(u_0)$. Together with (26.5), (26.7), (26.9), (26.10), (26.12), (26.13) and (26.14), this provides us on $[0,T_\kappa]$ with

$$
\begin{aligned}
\sum_{l=1}^{K} \int_{(0,1)^3} \xi(\theta_l)|D^2(\tilde v_{tt}\circ\theta_l)|^2 + \frac12 \sum_{l=1}^{K} \int_{\partial\tilde\eta(\tilde\Omega)} |D^2 V_t(l)\circ E(l)^{-1}|^2\ (-\tilde q(0),_m \tilde N_m) \\
\le \delta\tilde H(t)^2 + C_\delta t\tilde H(t)^4 + C_\delta N(u_0).
\end{aligned} \tag{26.16}
$$

Similarly as in Section 24, from the pressure condition (1.2), this provides us with an estimate of the type

$$
\|\tilde v_{tt}\|_2^2 + \|\tilde v_t\|_{\frac52}^2 \le \delta\tilde H(t)^2 + C_\delta t\tilde H(t)^4 + C_\delta N(u_0). \tag{26.17}
$$

The control

$$
\|\tilde q\|_3^2 \le \delta\tilde H(t)^2 + C_\delta t\tilde H(t)^4 + C_\delta N(u_0) \tag{26.18}
$$

is then easy to achieve by elliptic regularity on the pressure system.

Next, we see that (26.17) implies that for any $l\in\{1,\dots,K\}$, $\tilde v_t\circ\theta_l$, and thus $(\tilde q\circ\theta_l),_3\, \tilde\eta\circ\theta_l,_1 \times \tilde\eta\circ\theta_l,_2$, are controlled in $H^2((0,1)^2\times\{0\})$ by the right-hand side of (26.17). This implies the same control on $(0,T_\epsilon)$ for $\dfrac{\tilde q\circ(\theta_l),_3\,\tilde\eta\circ\theta_l,_1\times\tilde\eta\circ\theta_l,_2}{|\tilde q\circ(\theta_l),_3\,\tilde\eta\circ\theta_l,_1\times\tilde\eta\circ\theta_l,_2|}$ in $H^2((0,1)^2\times\{0\})$, *i.e.*, that

$$
|\tilde n|_{2,\tilde\Gamma}^2 \le \delta\tilde H(t)^2 + C_\delta t\tilde H(t)^4 + C_\delta N(u_0), \tag{26.19}
$$

which brings the $H^{\frac72}$ regularity of the *domain* $\tilde\eta(\tilde\Omega)$.

Now, for $\tilde{v}$, we notice that from the identity on $(0,1)^2\times\{0\}$:

$$V_{tt}(l)+\tilde{q}\circ\theta_{l,3}\,V(l),_1\times E(l),_2+\tilde{q}\circ\theta_{l,3}\,E(l),_1\times V(l),_2+\tilde{q}_t\circ\theta_{l,3}\,E(l),_1\times E(l),_2=0,$$

we infer by taking the scalar product of the above vector by $E(l),_1$ that

$$|V(l),_1\cdot\tilde{n}|^2_{\frac{3}{2},(0,1)^2\times\{0\}}\le\delta\tilde{H}(t)^2+C_\delta t\tilde{H}(t)^4+C_\delta N(u_0),$$

which by divergence and curl relations for $\xi(\theta_l)V(l),_1\,(E(l)^{-1})$ similar in spirit to the ones in Section 22, leads to

$$\|\xi(\theta_l)V(l),_1\|^2_{2,(0,1)^3}\le\delta\tilde{H}(t)^2+C_\delta t\tilde{H}(t)^4+C_\delta N(u_0).$$

In a similar fashion,

$$\|\xi(\theta_l)V(l),_2\|^2_{2,(0,1)^3}\le\delta\tilde{H}(t)^2+C_\delta t\tilde{H}(t)^4+C_\delta N(u_0).$$

Now, with divergence and curl relations for $\xi(\theta_l)V(l)(E(l)^{-1})$ similar in spirit to the ones in Section 22, this leads to

$$\|\xi\tilde{v}\|^2_3\le\delta\tilde{H}(t)^2+C_\delta t\tilde{H}(t)^4+C_\delta N(u_0),$$

and consequently, with the control of the divergence and curl of $\tilde{v}$ inside $\tilde{\Omega}$ as in Section 22, we get

$$\|\tilde{v}\|^2_3\le\delta\tilde{H}(t)^2+C_\delta t\tilde{H}(t)^4+C_\delta N(u_0). \tag{26.20}$$

Now, with the estimates (26.17), (26.18), (26.19) and (26.20), we then get similarly as in Section 25 the existence of a time $T>0$ independent of ϵ such that on $(0,T)$ the estimates (26.2) hold, and such that we have $\tilde{H}(t)\le N(u_0)$ on $(0,T)$ for any $\epsilon>0$ small enough. Therefore, we have a solution to the problem with optimal regularity on the initial data and domain as the weak limit as $\epsilon\to 0$.

27. Uniqueness

Let (v,q) and $(\bar{v},\bar{q})$ be solutions of (1.2) on $[0,T]$. We denote $\delta v=v-\bar{v}$ and $\delta q=q-\bar{q}$. We then introduce the energy

$$f(t)=\sum_{l=1}^K\int_{(0,1)^d}\xi(\theta_l)|D^2(v_{tt}\circ\theta_l-\bar{v}_{tt}\circ\theta_l)|^2,$$

where D^2v stands for any second order horizontal space derivative $v,_{i_1i_2}$. By proceeding in the same way as in the previous section, and using the fact that the divergence and curl of δv have a transport type structure as well, we obtain an energy inequality similar to (26.17), without the presence of $N(u_0)$ (since $\delta v(0)=0$). This establishes uniqueness of solutions.

Acknowledgments

The authors were supported by the National Science Foundation under grant NSF DMS-0313370 and NSF EAR-0327799. We thank the referee for the major time and effort expended on the careful reading of our manuscript.

References

[1] R.A. Adams, Sobolev Spaces, Academic Press, 1978. MR0450957 (56:9247)
[2] D.M Ambrose, Well-posedness of vortex sheets with surface tension, SIAM J. Math. Anal. **35** (2003), no. 1, 211-244. MR2001473 (2005g:76006)
[3] D.M. Ambrose and N. Masmoudi, The zero surface tension limit of two-dimensional water waves, Comm. Pure Appl. Math., **58** (2005), 1287–1315. MR2162781 (2006d:35189)
[4] J.T. Beale, T. Hou and J. Lowengrub, Growth rates for the linearized motion of fluid interfaces away from equilibrium, Comm. Pure Appl. Math., **46**, (1993) 1269–1301. MR1231428 (95c:76016)
[5] W. Craig, An existence theory for water waves and the Boussinesq and Korteweg-de Vries scaling limits, Comm. Partial Differential Equations, **10** (1985), no. 8, 787-1003. MR795808 (87f:35210)
[6] D. Christodoulou and H. Lindblad, On the motion of the free surface of a liquid, Comm. Pure Appl. Math., **53** (2000), 1536–1602. MR1780703 (2002c:76025b)
[7] D. Coutand, S. Shkoller, On the motion of an elastic solid inside of an incompressible viscous fluid, Arch. Rational Mech. Anal., **176**, (2005), 25–102. MR2185858 (2006g:74045)
[8] D. Coutand, S. Shkoller, The interaction between quasilinear elastodynamics and the Navier-Stokes equations, Arch. Rational Mech. Anal., **179**, (2006), 303–352. MR2208319 (2006j:74023)
[9] S. Ebenfeld, L^2-regularity theory of linear strongly elliptic Dirichlet systems of order $2m$ with minimal regularity in the coefficients, Quart. Appl. Math., **60** (2002), 547–576. MR1914441 (2003d:35071)
[10] D. Ebin, The equations of motion of a perfect fluid with free boundary are not well posed, Comm. Part. Diff. Eq., **10**, (1987), 1175–1201. MR886344 (88g:58029)
[11] D. Lannes, Well-posedness of the water-waves equations, J. Amer. Math. Soc., **18**, (2005) 605–654. MR2138139 (2005m:76021)
[12] H. Lindblad, Well-posedness for the linearized motion of an incompressible liquid with free surface boundary, Comm. Pure Appl. Math., **56**, (2003), 153–197. MR1934619 (2003i:35234)
[13] H. Lindblad, Well-posedness for the motion of an incompressible liquid with free surface boundary, Annals of Math., **162**, (2005), 109–194. MR2178961 (2006g:35293)
[14] V.I. Nalimov, The Cauchy-Poisson Problem (in Russian), Dynamika Splosh. Sredy, **18** (1974),104–210. MR0609882 (58:29458)
[15] J. Shatah, C. Zeng, Geometry and a priori estimates for free boundary problems of the Euler's equation, preprint, (2006).
[16] B. Schweizer, On the three-dimensional Euler equations with a free boundary subject to surface tension, Ann. Inst. H. Poincaré Anal. Non Linéaire, **22** (2005), 753–781. MR2172858 (2006g:35222)
[17] V.A. Solonnikov, Solvability of the problem of evolution of a viscous incompressible fluid bounded by a free surface on a finite time interval, St. Petersburg Math. J., **3** (1992), 189–220. MR1120848 (92h:35245)
[18] V.A. Solonnikov and V.E.Scadilov, On a boundary value problem for a stationary system of Navier-Stokes equations, Proc. Steklov Inst. Math., **125**, (1973), 186–199. MR0364910 (51:1164)
[19] M. Taylor, Partial Differential Equations, Vol. I-III, Springer, (1996). MR1395148 (98b:35002b); MR1395149 (98b:35003); MR1477408 (98k:35001)
[20] S. Wu, Well-posedness in Sobolev spaces of the full water wave problem in 2-D, Invent. Math., **130** (1997), 39–72. MR1471885 (98m:35167)

[21] S. Wu, Well-posedness in Sobolev spaces of the full water wave problem in 3-D, J. Amer. Math. Soc., **12** (1999), 445–495. MR1641609 (2001m:76019)

[22] H. Yosihara, Gravity Waves on the Free Surface of an Incompressible Perfect Fluid, Publ. RIMS Kyoto Univ., **18** (1982), 49–96. MR660822 (83k:76017)

Department of Mathematics, University of California at Davis, Davis, California 95616

E-mail address: `coutand@math.ucdavis.edu`

Department of Mathematics, University of California at Davis, Davis, California 95616

E-mail address: `shkoller@math.ucdavis.edu`