Smooth Global Lagrangian Flow for the 2D Euler and Second-Grade Fluid Equations

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Abstract—We present a very simple proof of the global existence of a $C^1$ Lagrangian flow map for the 2D Euler and second-grade fluid equations (on a compact Riemannian manifold with boundary) which has $C^1$ dependence on initial data $u_0$ in the class of $H^s$ divergence-free vector fields for $s > 2$. © 2001 Elsevier Science Ltd. All rights reserved.

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1. INCOMPRESSIBLE EULER EQUATIONS

Let $(M, g)$ be a $C^\infty$ compact oriented Riemannian 2-manifold with smooth boundary $\partial M$, let $\nabla$ denote the Levi-Civita covariant derivative, and let $\mu$ denote the Riemannian volume form. The incompressible Euler equations are given by

$$\begin{align*}
\frac{\partial}{\partial t} u + \nabla u &= -\nabla p, \\
\text{div } u &= 0, \quad u(0) = u_0, \quad g(u, n) = 0, \quad \text{on } \partial M,
\end{align*}$$

(1.1)

where $p(t, x)$ is the pressure function, determined (modulo constants) by solving the Neumann problem $-\Delta p = \text{div } \nabla u$ with boundary condition $g(\nabla p, n) = S_n(u)$, $S_n$ denoting the second fundamental form of $\partial M$.

The now standard global existence result for two-dimensional classical solutions states that for initial data $u_0 \in \chi^s := \{v \in H^s(TM) \mid \text{div } u = 0, \ g(u, n) = 0\}$, $s > 2$, the solution $u$ is in $C^0(\mathbb{R}, \chi^s)$ and has $C^0$ dependence on $u_0$ (see, for example, [1]). Equation (1.1) gives the Eulerian or spatial representation of the dynamics of the fluid. The Lagrangian representation which is in terms of the volume-preserving fluid particle motion or flow map $\eta(t, x)$ is obtained by solving

$$\begin{align*}
\frac{\partial}{\partial t} \eta(t, x) &= u(t, \eta(t, x)), \\
\eta(0, x) &= x.
\end{align*}$$

(1.2)
This is an ordinary differential equation on the infinite-dimensional volume-preserving diffeomorphism group $\mathcal{D}_\mu$, the set of $H^s$ class bijective maps of $M$ into itself with $H^s$ inverses which leave $\partial M$ invariant. Ebin and Marsden [2] proved that $\mathcal{D}_\mu$ is a $C^\infty$ manifold whenever $s > 2$. They also showed that for an interval $I$, whenever $u \in C^s(I, \chi^s)$ and $s > 3$, there exists a unique solution $\eta \in C^1(I, \mathcal{D}_\mu)$ to (1.2). Thus, for $s > 3$ the existence of a global $C^1$ flow map immediately follows from the fact that $u$ remains bounded in $H^s$ for all time. It is often essential, however, for the Euler flow to depend smoothly on the initial data; in the case of vortex methods, for example, Hald in Assumption 3 of [3] requires this as a necessary condition to establish convergence.

**Theorem 1.1.** For $u_0 \in \chi^s$, $s > 2$, there exists a unique global solution to (1.3) which is in $C^\infty(\mathbb{R}, T\mathcal{D}_\mu)$ and has $C^\infty$ dependence on $u_0$.

**Proof.** The smoothness of the flow map follows by considering the Lagrangian version of (1.1) given by

$$\frac{D}{dt} \partial_t \eta(t, x) = -\text{grad} \, p(t, \eta(t, x)), \quad \det T\eta(t, x) = 1,$$

$$\partial_t \eta(0, x) = u_0(x),$$

$$\eta(0, x) = x,$$

where $T\eta(t, x)$ denotes the tangent map of $\eta$ (which in local coordinates is given by the $2 \times 2$ matrix of partial derivatives $(\eta_{ij})$, and where $\frac{D}{dt}$ is the covariant derivative along the curve $t \mapsto \eta(t, x)$ (which in Euclidean space is the usual partial time derivative). Since

$$\text{grad} \, p \circ \eta = \text{grad} \, \Delta^{-1} [\text{Tr}(\nabla u \cdot \nabla u) + \text{Ric}(u, u)] \circ \eta,$$

where Ric is the Ricci curvature of $M$, and since $S_n$ is $C^\infty$ and $H^{s-1}(TM)$ forms a multiplicative algebra whenever $s > 2$, we see that the linear operator $u \mapsto \text{grad} \, \Delta^{-1}[\text{Tr}(\nabla u \cdot \nabla u) + \text{Ric}(u, u)]$ maps $H^s$ back into $H^s$. Denote by $f : T\mathcal{D}_\mu \rightarrow T(T\mathcal{D}_\mu)$ the vector field

$$(\eta, \partial_t \eta) \mapsto \text{grad} \, \Delta^{-1} [\text{Tr}(\nabla u \cdot \nabla u) + \text{Ric}(u, u)] \circ \eta.$$

Then,

$$f (\eta, \partial_t \eta) = \text{grad}_\eta \Delta^{-1} [\text{Tr}(\nabla_\eta \partial_t \eta \cdot \nabla_\eta \partial_t \eta) + \text{Ric}_\eta (\partial_t \eta, \partial_t \eta)],$$

where $\text{grad}_\eta g = [\text{grad}(g \circ \eta^{-1})] \circ \eta$ for all $g \in H^s(M)$, $\text{div}_\eta X_\eta = [\text{div}(X_\eta \circ \eta^{-1})] \circ \eta$, and $\nabla_\eta (X_\eta) = [\nabla(X_\eta \circ \eta^{-1})] \circ \eta$ for all $X_\eta \in T\mathcal{D}_\mu$, $\Delta_\eta = \text{div}_\eta \circ \text{grad}_\eta$, and $\text{Ric}_\eta = \text{Ric} \circ \eta$. It follows from Lemmas 4-6 in [4] and Appendix A in [2] that $f$ is a $C^\infty$ vector field. Thus, (1.3) is an ordinary differential equation on the tangent bundle $T\mathcal{D}_\mu$ governed by a $C^\infty$ vector field on $T\mathcal{D}_\mu$; it immediately follows from the fundamental theorem of ordinary differential equations on Hilbert manifolds, that (1.3) has a unique $C^\infty$ solution on finite time intervals which depends smoothly on the initial velocity field $u_0$, i.e., there exists a unique solution $\partial_t \eta \in C^\infty((-T, T), T\mathcal{D}_\mu)$ with $C^\infty$ dependence on initial data $u_0$, where $T$ depends only on $||u_0||_{H^s}$.

When $s > 3$, this interval can be extended globally to $\mathbb{R}$ by virtue of $\eta$ remaining in $\mathcal{D}_\mu$. Unfortunately, the global existence and uniquess of a $C^\infty$ flow map $\eta(t, x)$ does not follow for initial data $u_0 \in \chi^s$ for $s \in (2, 3)$, so we provide a simple argument to fill this gap. We must show that $\eta$ can be continued in $\mathcal{D}_\mu$. It suffices to prove that $T\eta$ and $T\eta^{-1}$ are both bounded in $H^{s-1}$. This is easily achieved using energy estimates. We have that

$$\frac{D}{dt} T\eta = \nabla \partial_t \eta = \nabla u \cdot T\eta$$

and

$$\frac{D}{dt} T\eta^{-1} = -T\eta^{-1} \cdot \nabla \partial_t \eta \cdot T\eta^{-1} = -T\eta^{-1} \cdot \nabla u.$$
Computing the $H^{s-1}$ norm of $T\eta$ and $T\eta^{-1}$, respectively, we obtain
\[ \frac{1}{2} \frac{d}{dt} \| T\eta \|_{H^{s-1}} = \langle D^{s-1} (\nabla u \cdot T\eta), D^{s-1} T\eta \rangle_{L^2} \]
and
\[ \frac{1}{2} \frac{d}{dt} \| T\eta^{-1} \|_{H^{s-1}} = \langle D^{s-1} (T\eta^{-1} \cdot \nabla u), D^{s-1} T\eta^{-1} \rangle_{L^2} . \]
It is easy to estimate
\[ \langle D^{s-1} (\nabla u \cdot T\eta), D^{s-1} T\eta \rangle_{L^2} \leq C \left( \| \nabla u \|_{L^\infty} \| T\eta \|_{H^{s-1}} \| \nabla u \|_{H^{s-1}} + \| T\eta \|_{L^\infty} \| T\eta \|_{H^{s-1}} \right) \]
\[ \leq C \left( \| \nabla u \|_{L^\infty} \| T\eta \|_{H^{s-1}}^2 + \| u \|_{H^s} \| T\eta \|_{H^{s-1}}^2 \right) , \]
where the first inequality is due to Cauchy-Schwartz and Moser’s inequalities and the second is the Sobolev embedding theorem. Similarly,
\[ \langle D^{s-1} (-T\eta^{-1} \cdot \nabla u), D^{s-1} T\eta^{-1} \rangle_{L^2} \leq C \left( \| \nabla u \|_{L^\infty} \| T\eta^{-1} \|_{H^{s-1}}^2 + \| u \|_{H^s} \| T\eta^{-1} \|_{H^{s-1}}^2 \right) . \]
Since the solution $u$ to (1.1) is in $\chi^s$ for all $t$, we have that $\| u \|_{H^s}$ is bounded for all $t$. Because the vorticity $\omega = \nabla \times u$ is in $L^\infty$, we have by Lemma 2.4 in [1, Chapter 17] that $\| \nabla u \|_{L^\infty} \leq C(1 + \log \| u \|_{H^s})$; hence, $\| \nabla u \|_{L^\infty}$ is bounded for $t$. It then follows that $\eta$ and $\eta^{-1}$ are in $D^s_{\mu}$ for all time.

## 2. SECOND-GRADE FLUID EQUATIONS

In this section, we establish the global existence of a $C^\infty$ Lagrangian flow map for the second-grade fluids equations, also known as the Lagrangian averaged Euler or Euler-$\alpha$ equations when $\nu = 0$, which has $C^\infty$ dependence on initial data. These equations are given on $(M, g)$ by
\[ \partial_t (1 - \alpha \Delta_r) u - \nu \Delta_r u + \nabla u \cdot (1 - \alpha \Delta_r) u - \alpha (\nabla u)^T \cdot \Delta_r u = -\text{grad} p, \]
\[ \text{div} u = 0, \quad u(0) = u_0, \quad u = 0, \quad \text{on } \partial M, \]
\[ \alpha > 0, \quad \nu \geq 0, \quad \Delta_r = -(d\delta + \delta d) + 2 \text{Ric}, \]
(see [4]), and were first derived in 1955 by Rivlin and Ericksen [5] in Euclidean space ($\text{Ric} = 0$) as a first-order correction to the Navier-Stokes equations. In Euclidean space, the operator $\Delta_r$ is just the component-wise Laplacian, and the equation may be written as
\[ \partial_t (1 - \alpha \Delta) u - \nu \Delta u + \nabla (1 - \alpha \Delta) u \times u = -\text{grad} p. \]

For convenience, we set $\alpha = 1$. We define the unbounded, self-adjoint operator $(1 - \mathcal{L}) = (1 - 2 \text{Def}^* \text{Def})$ on $L^2(TM)$ with domain $H^2(TM) \cap H^1_0(TM)$. The operator $\text{Def}^*$ is the formal adjoint of $\text{Def}$ with respect to $L^2$; $2 \text{Def}^* \text{Def} u = -(\Delta + \text{grad} \text{div} + 2 \text{Ric})u$ so that $2 \text{Def}^* \text{Def} u = -(\Delta + 2 \text{Ric})u$ if $\text{div} u = 0$. We let $\mathcal{D}_{\mu,D}^s$ denote the subgroup of $\mathcal{D}_{\mu,\partial M}^s$ whose elements restrict to the identity on the boundary $\partial M$. $\mathcal{D}_{\mu,D}^s$ is a $C^\infty$ manifold (see [2,4]). Define $\chi^s_D := \{ u \in \chi^s | u = 0 \}$ on $\partial M$.

The following is Proposition 5 in [4].

**Proposition 2.1.** For $s > 2$, let $\eta(t)$ be a curve in $\mathcal{D}_{\mu,D}^s$, and set $u(t) = \partial_t \eta \circ \eta(t)^{-1}$. Then $u$ is a solution of the initial-boundary value problem (2.1) with Dirichlet boundary conditions $u = 0$ on $\partial M$ if and only if
\[ \mathcal{D}_\eta \circ \left[ \frac{\nabla \eta}{dt} + [-\nu (1 - \mathcal{L})^{-1} \Delta_r u + \mathcal{U}(u) + \mathcal{R}(u)] \circ \eta \right] = 0, \quad \text{Det} T\eta(t, x) = 1, \]
\[ \partial_t \eta(0, x) = u_0(x), \]
\[ \eta(0, x) = x, \]
(2.2)
where
\[
U(u) = (1 - \mathcal{L})^{-1} \{ \text{div} [\nabla u \cdot \nabla u^t + \nabla u \cdot \nabla u - \nabla u^t \cdot \nabla u] + \text{grad } \text{Tr} (\nabla u \cdot \nabla u) \},
\]
\[
\mathcal{R}(u) = (1 - \mathcal{L})^{-1} \{ \text{Tr} [\nabla (R(u, \cdot) u) + R(u, \cdot) \nabla u + R(\nabla u, \cdot) u] 
+ \text{grad } \text{Ric} (u, u) - (\nabla \text{Ric}) \cdot u + \nabla u^t \cdot \text{Ric}(u) \},
\]
and \( \mathcal{P}_\eta : T_\eta \mathcal{D}_{\mu}^s \rightarrow T_\eta \mathcal{D}_{\mu, D}^s \) is the Stokes projector defined by
\[
\mathcal{P}_\eta : T_\eta \mathcal{D}_{\mu, D}^s \rightarrow T_\eta \mathcal{D}_{\mu, D}^s,
\]
\[
\mathcal{P}_\eta (X_\eta) = [\mathcal{P}_\epsilon (X_\eta \circ \eta^{-1})] \circ \eta,
\]
and where \( \mathcal{P}_\epsilon (F) = v, v \) being the unique solution of the Stokes problem
\[
(1 - \mathcal{L}) v + \text{grad } p = (1 - \mathcal{L}) F,
\]
\[
\text{div } v = 0,
\]
\[
v = 0, \quad \text{on } \partial M.
\]

Equation (2.2) is an ordinary differential equation for the Lagrangian flow. Notice again that \( H^{s-1}, s > 2 \), forms a multiplicative algebra, so that both \( \mathcal{U} \) and \( \mathcal{R} \) map \( H^s \) into \( H^s \).

**Theorem 2.1.** For \( u_0 \in \chi_\mathcal{D}^s, s > 2 \), and \( \nu \geq 0 \), there exists a unique global solution to (2.2) which is in \( C^\infty(\mathbb{R}, \mathcal{T} \mathcal{D}_{\mu}^s) \) and has \( C^\infty \) dependence on \( u_0 \).

We note that one cannot prove the statement of this theorem from an analysis of (2.1) alone (see [6,7], and references therein).

**Proof.** The ordinary differential equation (2.2) can be written as \( \partial_t \eta = S(\eta, \partial_t \eta) \) (see in [4, p. 23]). Remarkably, \( S : \mathcal{T} \mathcal{D}_{\mu, D}^s \rightarrow \mathcal{T} \mathcal{T} \mathcal{D}_{\mu, D}^s \) is a \( C^\infty \) vector field, and [4, Theorem 2] provides the existence of a unique short-time solution to (2.2) in \( C^\infty((-T, T), \mathcal{T} \mathcal{D}_{\mu, D}^s) \) which depends smoothly on \( u_0 \), and where \( T \) only depends on \( \|u_0\|_{H^s} \).

Thus, it suffices to prove that the solution curve \( \eta \) does not leave \( \mathcal{D}_{\mu, D}^s \). Following the proof of Theorem 1.1, and using the fact that the solution \( u(t, x) \) to (2.1) remains in \( H^s \) for all time [6,7], it suffices to prove that \( \nabla u \) is bounded in \( L^\infty \).

Letting \( q = \text{curl}(1 - \alpha \Delta_\epsilon) u \) denote the potential vorticity, and computing the curl of (2.1), we obtain the 2D vorticity form as
\[
\partial_t q + g (\text{grad } q, u) = \nu \text{curl } u.
\]

It follows that for all \( \nu \geq 0, q(t, x) \) is bounded in \( L^2 \) (conserved when \( \nu = 0 \)), and therefore, by standard elliptic estimates \( \nabla u(t, x) \) is bounded in \( H^2 \), and hence, in \( L^\infty \).

As a consequence of Theorem 2.1 being independent of viscosity, we immediately obtain the following.

**Corollary 2.1.** Let \( \eta^\nu(t, x) \) denote the Lagrangian flow solving (2.2) for \( \nu > 0 \), so that \( u^\nu = \partial_t \eta^\nu \circ \eta^\nu^{-1} \) solves (2.1). Then for \( u_0 \in \chi_\mathcal{D}^s, s > 2 \), the viscous solution \( \eta^\nu \in C^\infty(\mathbb{R}, \mathcal{T} \mathcal{D}_{\mu}^s) \) converges regularly (in \( H^s \)) to the inviscid solution \( \eta^0 \in C^\infty(\mathbb{R}, \mathcal{T} \mathcal{D}_{\mu}^s) \). Consequently, \( u^\nu \rightarrow u^0 \) in \( H^s \) on infinite-time intervals.

This gives an improvement of Busuioc’s result in [8] in two ways:

1. we are able to prove the regular limit of zero viscosity on manifolds with boundary, and
2. in the Lagrangian framework, we are able to get \( C^\infty \) in time solutions.
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