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Well-posedness of the full Ericksen–Leslie model of nematic liquid crystals

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Abstract. The Ericksen–Leslie model of nematic liquid crystals is a coupled system between the Navier–Stokes and the Ginzburg–Landau equations. We show here the local well-posedness for this problem for any initial data regular enough, by a fixed point approach relying on some weak continuity properties in a suitable functional setting. By showing the existence of an appropriate local Lyapunov functional, we also give sufficient conditions for the global existence of the solution, and some stability conditions. © 2001 Académie des sciences/Éditions scientifiques et médicales Elsevier SAS

Sur le caratère bien posé du modèle de cristaux liquides nématiques d'Ericksen–Leslie

Résumé. *Le modèle de cristaux liquides nématiques d'Ericksen–Leslie est un système couplant les équations de Navier–Stokes et de Ginzburg–Landau. Par une approche de point fixe basée sur des propriétés de continuité faible, nous montrons l'existence d'une unique solution, en temps petit, pour toute donnée initiale suffisamment régulière. En montrant l'existence d'une fonctionnelle de Lyapunov locale appropriée, nous donnons également des conditions suffisantes d'existence globale de la solution, ainsi que des propriétés de stabilité.* © 2001 Académie des sciences/Éditions scientifiques et médicales Elsevier SAS

Version française abrégée

Dans cette Note, on analyse le caractère bien posé du modèle de cristal liquide nématique d'Ericksen–Leslie formulé dans [4–6].

Une version simplifiée de ce modèle a été introduite par F.H. Lin dans [7] puis analysée récemment par F.H. Lin et C. Liu (*voir* [8,9]) par une méthode de Galerkin modifiée et par S. Shkoller (*voir* [10]) par une méthode de contraction en temps petit suivie d'estimations a priori sur des énergies appropriées. Les différentes méthodes proposées par ces auteurs ne permettent pas d'établir un résultat d'existence locale pour le modèle d'Ericksen–Leslie, essentiellement en raison de l'absence de lois énergétiques appropriées pour ce modèle et d'un phénomène de perte de régularité au niveau de la partie linéaire du système.

Note présentée par Philippe G. CIARLET.

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Le modèle d'Ericksen–Leslie est un système d'évolution couplant les équations de Navier–Stokes et de Ginzburg–Landau, où l'inconnue est d'une part la vitesse $u(t,x)$ du fluide et d'autre part le vecteur directeur $d(t,x)$ représentant le paramètre d'orientation du cristal liquide, situé dans un ouvert régulier Ω de $\mathbb{R}^n$ ($n=3$ ou 2).

Dans ce qui suit, on suppose donné un élément $h \in \mathrm{H}^{5/2}(\Omega,\mathbb{R}^n)$ tel que $|h|=1$ sur $\partial\Omega$. L'inconnue (u,d) est solution du système d'évolution :

$$u_t + \nabla_u u = -\operatorname{grad} p + \nu\Delta u - \lambda \operatorname{Div}\bigl(\nabla d^{\mathrm{T}} \cdot \nabla d\bigr), \tag{1a}$$

$$\operatorname{div} u(t,x) = 0, \tag{1b}$$

$$d_t + \nabla_u d - \nabla_d u = \gamma\left(\Delta d - \frac{1}{\varepsilon^2}\bigl(|d|^2-1\bigr)d\right), \tag{1c}$$

$$u=0 \quad \text{sur } \partial\Omega, \quad d=h \quad \text{sur } \partial\Omega, \tag{1d}$$

$$u(0,\cdot)=u_0, \quad d(0,\cdot)=d_0. \tag{1e}$$

Dans ce qui précède u_0 et d_0 sont respectivement la vitesse et le champ directeur initiaux, et satisfont à $u_0|_{\partial\Omega}=0$ et $d_0|_{\partial\Omega}=h$. En ce qui concerne les coefficients, $\nu>0$ représente la viscosité du fluide, $\lambda>0$ est une constante d'élasticité, $\varepsilon>0$ est un coefficient de pénalisation par rapport à la contrainte unitaire et $\gamma>0$ est un coefficient de relaxation par rapport au temps.

Dans ce qui suit Ω est supposé être de classe $\mathcal{C}^3$, et la donnée initiale (u_0,d_0) dans $V^2 \times \mathrm{H}^3(\Omega;\mathbb{R}^n)$, où $V^2=\bigl\{u\in \mathrm{H}^2(\Omega;\mathbb{R}^n); \operatorname{div} u=0,\, u=0 \text{ sur } \partial\Omega\bigr\}$.

Notre résultat essentiel concerne le caractère localement bien posé du modèle d'Ericksen–Leslie :

THÉORÈME 1. – *Il existe* $T>0$ (*dépendant de* $\|u_0\|_{\mathrm{H}^2}$ *et de* $\|d_0\|_{\mathrm{H}^3}$) *tel que le système* (1) *admet une unique solution* (u,d) *dans* $\mathrm{L}^2((0,T);V^2)\times \mathrm{L}^2((0,T);\mathrm{H}^3(\Omega;\mathbb{R}^n))$.

La solution mise en évidence est de plus dans $\mathcal{C}([0,T];V^2)\times\mathcal{C}((0,T];\mathrm{H}^3(\Omega;\mathbb{R}^n))$. *Si en outre la donnée initiale satisfait la condition de compatibilité* $\gamma\Delta d_0+\nabla_{d_0}u_0=0$ *sur* $\partial\Omega$*, alors le vecteur orientation* d *est dans* $\mathcal{C}([0,T];\mathrm{H}^3(\Omega;\mathbb{R}^n))$.

En outre, pour tout $0<t<T$, $(u(t,\cdot),d(t,\cdot))$ *dépend de manière continue de* (u_0,d_0) *dans* $V^2\times \mathrm{H}^3(\Omega;\mathbb{R}^n)$.

Au sujet de ce théorème, la question la plus délicate réside dans l'existence locale qui ne peut s'obtenir par la méthode de Galerkin modifiée de [8] ou par la méthode de contraction de [10] concernant le problème simplifié. Notre preuve réside dans une approche de compacité et de continuité faible par le théorème de point fixe de Tychonoff (*voir* [2] et la version anglaise).

Concernant la question de l'existence globale en temps, dans le cas particulier où l'orientation prescrite sur la frontière est une constante, on peut mettre en évidence une fonctionnelle de Lyapunov locale appropriée et obtenir le résultat suivant :

THÉORÈME 2. – *On suppose la condition frontière* $h=e$ *sur* $\partial\Omega$*, où* e *est un vecteur donné de* $\mathbb{R}^n$*. Il existe un voisinage* Θ *de* $(0,e)$ *dans* $V^2\times\mathrm{H}^3(\Omega;\mathbb{R}^n)$ *tel que, pour toute donnée initiale* $(u_0,d_0)\in\Theta$*, la solution mise en évidence par le théorème précédent est définie sur* $[0,+\infty[$ *avec les propriétés de régularité précédentes pour tout* $T>0$*, et reste dans un voisinage de* $(0,e)$ *dans* $V^2\times\mathrm{H}^3(\Omega;\mathbb{R}^n)$.

Au sujet de la stabilité du système, avec une hypothèse supplémentaire de petitesse sur la constante d'élasticité par rapport au produit de la viscosité et de la constante de relaxation, on obtient en outre :

THÉORÈME 3. – *Avec les hypothèses du théorème* 2*, si l'on suppose en outre* $\lambda<\nu\gamma$*, alors la solution* $(u(t,\cdot),d(t,\cdot))$ *tend vers* $(0,e)$ *dans* $V^2\times\mathrm{H}^3(\Omega;\mathbb{R}^n)$ *lorsque* t *tend vers* $+\infty$.

1. Introduction

In this Note, we establish the well-posedness of the Ericksen–Leslie model of nematic liquid crystals formulated in [4–6].

A simplified version of the Ericksen–Leslie model was introduced by Lin in [7] and recently analyzed by Lin and Liu [8,9] who used a modified Galerkin approach, and by Shkoller [10] who relied on a contraction mapping argument coupled with appropriate energy estimates. The simplified model ignores the stretching of the director field induced by the straining of the fluid, and yields a maximum principle for the Ginzburg–Landau heat flow. When this streching term is kept in the analysis, a priori energy laws, present in the simplified model, cease to hold; thus, the different methods used in [8,9] and [10] cannot be used for a local existence theory for the *full* Ericksen–Leslie model, primarily because of a loss of regularity created by the stretching of the director field. The full model may be found in (1.1)–(1.15) of [9]; herein, we shall analyze the simplest form of this model which retains the streching of the director field and as well as the main mathematical features of the problem.

The Ericksen–Leslie model is a system of PDEs coupling the Navier–Stokes and Ginzburg–Landau equations, where the unknown is the couple consisting of the time-dependent divergence-free velocity field $u(t,x)$ of the fluid and of the director field $d(t,x)$ representing the orientation parameter of the liquid crystal, contained inside an open, bounded, $\mathcal{C}^3$ domain $\Omega \subset \mathbb{R}^n$ with $n=2$ or 3. In [2], we show that this system arises quite naturally from a certain variational principle.

We shall assume that boundary data for the director field d is given by $h \in \mathrm{H}^{5/2}(\Omega,\mathbb{R}^n)$ such that $|h|=1$ on $\partial\Omega$. The evolution of the unknown (u,d) is governed by the following system:

$$u_t + \nabla_u u = -\operatorname{grad} p + \nu\Delta u - \lambda \operatorname{Div}\big(\nabla d^{\mathrm{T}} \cdot \nabla d\big), \tag{1.1a}$$

$$\operatorname{div} u(t,x) = 0, \tag{1.1b}$$

$$d_t + \nabla_u d - \nabla_d u = \gamma\left(\Delta d - \frac{1}{\varepsilon^2}\big(|d|^2-1\big)d\right), \tag{1.1c}$$

$$u=0 \quad \text{on } \partial\Omega, \qquad d=h \quad \text{on } \partial\Omega, \tag{1.1d}$$

$$u(0,\cdot)=u_0, \qquad d(0,\cdot)=d_0, \tag{1.1e}$$

where u_0 and d_0 are, respectively, the initial velocity and director fields, satisfying $u_0|_{\partial\Omega}=0$ and $d_0|_{\partial\Omega}=h$. Concerning the coefficients, $\nu>0$ represents the viscosity of the fluid, $\lambda>0$ is an elasticity constant, $\varepsilon>0$ is a penalization parameter with respect to the unitary constraint, and $\gamma>0$ is a relaxation-time constant.

In the following,

$$P: \mathrm{L}^2\big(\Omega;\mathbb{R}^n\big) \to H = \big\{u \in \mathrm{L}^2\big(\Omega;\mathbb{R}^n\big);\ \operatorname{div} u = 0,\ u\cdot n = 0 \text{ on } \partial\Omega\big\}$$

denotes the Leray projector onto divergence-free vector fields, and $A=-P\Delta$ the Stokes operator. We also assume that the initial data (u_0,d_0) belongs to $V^2 \times \mathrm{H}^3(\Omega;\mathbb{R}^n)$, where

$$V^2 = \big\{u \in \mathrm{H}^2\big(\Omega;\mathbb{R}^n\big);\ \operatorname{div} u = 0,\ u = 0 \text{ on } \partial\Omega\big\}.$$

Our main result establishes the local well-posedness of the Ericksen–Leslie problem for any regular enough initial data:

THEOREM 1. – *There exists* $T>0$ (*depending on* $\|u_0\|_{\mathrm{H}^2}$ *and on* $\|d_0\|_{\mathrm{H}^3}$) *such that the system* (1.1) *admits a unique solution* (u,d) *in*

$$\mathrm{L}^2\big((0,T);V^2\times \mathrm{H}^3\big(\Omega,\mathbb{R}^n\big)\big) \cap \mathcal{C}\big([0,T];V^2\big) \times \mathcal{C}\big((0,T];\mathrm{H}^3\big(\Omega;\mathbb{R}^n\big)\big).$$

Moreover, if we denote by $(\tilde{u},\tilde{d})$ *the solution associated to the initial data* $(\tilde{u}_0,\tilde{d}_0)$ *located in a neighborhood of* (u_0,d_0)*, then for any* $t\in(0,T)$*,* $(\tilde{u}(t,\cdot),\tilde{d}(t,\cdot))\to(u(t,\cdot),d(t,\cdot))$ *in* $V^2\times \mathrm{H}^3(\Omega,\mathbb{R}^n)$ *as* $(\tilde{u}_0,\tilde{d}_0)\to(u_0,d_0)$ *in* $V^2\times \mathrm{H}^3(\Omega,\mathbb{R}^n)$*.*

Furthermore, if the initial data satisfies the compatibility condition $\gamma\Delta d_0+\nabla_{d_0}u_0=0$ *on* $\partial\Omega$*, then the director field* d *belongs to* $\mathcal{C}([0,T];\mathrm{H}^3(\Omega;\mathbb{R}^n))$*.*

We also establish a small-data result based upon a certain local Lyapunov functional.

THEOREM 2. – *Suppose that* $d|_{\partial\Omega}=e$*, where* e *is a given unit vector on* $\mathbb{R}^n$*. Then, there exists a neighborhood* Θ *of* $(0,e)$ *in* $V^2\times\mathrm{H}^3(\Omega;\mathbb{R}^n)$ *such that for any initial data* $(u_0,d_0)\in\Theta$*, the solution given by the preceding theorem is defined on* $[0,+\infty[$ *with the same regularity properties for any* $T>0$*, Moreover, the solution stays in a neighborhood of* $(0,e)$ *in* $V^2\times\mathrm{H}^3(\Omega;\mathbb{R}^n)$*.*

With a supplementary assumption on the coefficients, we can establish the stability of the system about the stationary solution $(0,e)$.

THEOREM 3. – *Suppose the hypotheses of Theorem* 2*, and additionally assume that* $\lambda<\nu\gamma$*. Then the solution* $(u(t,\cdot),d(t,\cdot))\to(0,e)$ *in* $V^2\times\mathrm{H}^3(\Omega;\mathbb{R}^n)$ *as* $t\to+\infty$*.*

Remark 1. – In the statement of Theorems 2 and 3, the constant boundary condition e can be replaced by a non-constant boundary condition h, provided h is sufficiently close to e in $\mathrm{H}^{7/2}(\partial\Omega;\mathbb{R}^n)$.

2. Sketch of the proofs

We shall use the Tychonoff fixed-point theorem (*see*, for instance, [3]) to establish the existence result. This states that for a reflexive Banach space X, and $C\subset X$ a closed, convex, bounded subset, if $F:C\to C$ is weakly sequentially continuous into X, then F has at least one fixed-point.

We now introduce the functional framework which will enable us to prove the well-posedness for the Ericksen–Leslie model.

DEFINITION 4. – For a given time $T>0$ we define the following Hilbert space:

$$X_T=\big\{(u,d)\in\mathrm{L}^2\big((0,T);V^2\big)\times\mathrm{L}^2\big((0,T);\mathrm{H}^3(\Omega;\mathbb{R}^n)\big)\;\big|\; (u_t,d_t)\in\mathrm{L}^2\big((0,T);\mathrm{L}^2\big(\Omega;\mathbb{R}^n\big)\big)\times\mathrm{L}^2\big((0,T);\mathrm{H}^1\big(\Omega;\mathbb{R}^n\big)\big)\big\}$$

endowed with its natural scalar product:

$$\forall\big((u,d),(v,e)\big)\in X_T^2,\quad \big((u,d),(v,e)\big)=\int_0^{\mathrm{T}}\big\{(u,v)_{\mathrm{H}^2}+(d,e)_{\mathrm{H}^3}+(u_t,v_t)_{\mathrm{L}^2}+(d_t,e_t)_{\mathrm{H}^1}\big\}\,\mathrm{d}t.$$

Remark 2. – While it may appear that the above condition on u_t and d_t is not necessary for a natural functional framework for these equations, it appears that the constraint we impose is indeed required to use the Tychonoff fixed-point procedure.

We next define C_T as being the set of elements (u,d) of X_T satisfying the initial condition:

$$\big(u(0,\cdot),d(0,\cdot)\big)=(u_0,d_0),$$

and the following six inequalities for any $t\in[0,T]$:

$$\nu\int_0^t\big\|Au(\tau,\cdot)\big\|_{\mathrm{L}^2}^2\,\mathrm{d}\tau+\big\|\nabla u(t,\cdot)\big\|_{\mathrm{L}^2}^2\leqslant c_1,\quad \int_0^t\big\|u_t(\tau,\cdot)\big\|_{\mathrm{L}^2}^2\,\mathrm{d}\tau\leqslant c_2\,t^{1/4},$$

$$\big\|\nabla d(t,\cdot)\big\|_{\mathrm{L}^2}^2 \leqslant c_3, \quad \gamma\int_0^t \big\|\nabla d_t(\tau,\cdot)\big\|_{\mathrm{L}^2}^2\,\mathrm{d}\tau + \frac{1}{2}\big\|d_t(t,\cdot)\big\|_{\mathrm{L}^2}^2 \leqslant c_4, \tag{2.1}$$
$$\int_0^t \big\|\nabla\Delta d(\tau,\cdot)\big\|_{\mathrm{L}^2}^2\,\mathrm{d}\tau \leqslant c_5, \quad \big\|\Delta d(t,\cdot)\big\|_{\mathrm{L}^2}^2 \leqslant c_6,$$

where the $(c_i)_{1\leqslant i\leqslant 6}$ are positive constants depending on $\|u_0\|_{\mathrm{H}^1}$, $\|d_0\|_{\mathrm{H}^2}$ and the coefficients of the system (*see* [2]). It is readily seen that C_T is a non-empty closed, convex and bounded set in the Hilbert space X_T. From classical regularity results for linear evolution equations (*see*, for instance, [11]), we define the mapping $F : C_T \to X_T$ which associates to a given element $(u,d)\in C_T$ the unique element $(U,D)\in X_T$ such that:

$$U_t + \nu AU = -P\,\nabla_u u - \lambda\,P\,\mathrm{Div}\big(\nabla d^{\mathrm{T}}\cdot\nabla d\big), \tag{2.2a}$$
$$\mathrm{div}\,U(t,x) = 0, \tag{2.2b}$$
$$D_t - \gamma\,\Delta D = -\nabla_u d + \nabla_d u - \frac{\gamma}{\varepsilon^2}\big(|d|^2-1\big)d, \tag{2.2c}$$
$$U = 0 \quad \text{on } \partial\Omega, \quad D = h \quad \text{on } \partial\Omega, \tag{2.2d}$$
$$U(0,\cdot) = u_0, \quad D(0,\cdot) = d_0. \tag{2.2e}$$

LEMMA 1. – *The mapping F is weakly sequentially continuous from C_T into X_T.*

Proof. – Given a sequence $(u_n,d_n)_{n\in\mathbb{N}}$ of elements of C_T weakly convergent towards a given element $(u,d)\in C_T$, we establish by compactness arguments that for any subsequence $F(u_\sigma(n),d_\sigma(n))_{n\in\mathbb{N}}$ there exists an extracted subsequence $F(u_{\sigma'(n)},d_{\sigma'(n)})_{n\in\mathbb{N}}$ weakly convergent in X_T towards a limit satisfying the system of equations (2.2) associated to (u,d). Hence we can conclude that $F(u_n,d_n)_{n\in\mathbb{N}}$ is weakly convergent in X_T towards $F(u,d)$. □

We next have the following crucial result concerning the range of F.

LEMMA 2. – *There exists $T>0$ (depending on $\|u_0\|_{\mathrm{H}^2}$ and $\|d_0\|_{\mathrm{H}^3}$) such that F maps C_T into itself.*

Proof. – In order to establish that (U,D) satisfies the set of inequalities (2.1), we essentially use standard parabolic methods to get those six different estimates (*see* [2]). We mention here that the estimate with the constant c_4 is the most delicate to establish. □

Proof of Theorem 1. – From Lemmas 1 and 2, we infer from the Tychonoff fixed point theorem that the mapping F admits a fixed point (which is not necessarily unique) in C_T, which precisely proves the local existence in Theorem 1. The regularity result is a consequence of standard parabolic regularity results (*see*, for instance, [11]). The uniqueness is established from a Gronwall type inequality on the director field, which is obtained from the use of another differential inequality (involving both unknowns and arising from the Navier–Stokes equation) into an inequality arising from the Ginzburg–Landau equation. The regularity of the solution is also used in the obtainment of those inequalities. Continuity with respect to the initial data is obtained from compactness arguments following from the structure of the set C_T (the bounds and time existence depending on the norm of the initial data in $V^2\times\mathrm{H}^3(\Omega;\mathbb{R}^n)$), and from parabolic type convergence results. □

Concerning the proofs for Theorems 2 and 3, we refer to [2]. Here, we will just mention that the proof for the global existence result relies mainly upon the identification of some appropriate neighborhood Θ of $(0,e)$ in $V^2\times\mathrm{H}^3(\Omega;\mathbb{R}^n)$ such that for each initial data in Θ, the functional

$$f(t) = \frac{40}{\gamma\nu}\big\|u_t(t,\cdot)\big\|_{\mathrm{L}^2}^2 + \big\|\nabla d_t(t,\cdot)\big\|_{\mathrm{L}^2}^2,$$

is decreasing on $]0,+\infty[$. The stability theorem is a consequence of the preceding result and of some energy estimates which hold in the neighborhood Θ, with the assumption made on the coefficients (*see* [2]).

Remark 3. – A loss of regularity phenomenon appears with the linear part of the problem (*see* [2]). As such, one might conclude that a Nash–Moser iteration scheme (*see*, for instance, [3]) should be successful for proving the local (in time) existence for initial data in a neighborhood of (O, d_{GL}), where d_{GL} is a solution to the stationary Ginzburg–Landau equations. However, our approach presents the advantage of establishing the local well-posedness for *any* initial data, and also requires less regularity than a Nash–Moser procedure would.

Remark 4. – A natural question which arises now concerns the global well-posedness of the problem for initial data in a neighborhood of (O, d_{GL}), where d_{GL} is a minimizing solution to the stationary Ginzburg–Landau equations. The properties of such solutions (*see*, for instance, [1]) have not allowed us to reach this conclusion for small values of ε.

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References

[1] Bethuel F., Brezis H., Hélein F., Ginzburg–Landau Vortices, Birkhäuser, 1994.
[2] Coutand D., Shkoller S., Article in preparation (2001).
[3] Deimling K., Nonlinear Functional Analysis, Springer-Verlag, 1985.
[4] Ericksen J., Conservation laws for liquid crystals, Trans. Soc. Rheol. 5 (1961) 22–34.
[5] Ericksen J., Equilibrium theory for liquid crystals, in: G. Brown (Ed.), Advances in Liquid Crystals, Vol. 2, Academic Press, New York, 1975, pp. 233–398.
[6] Leslie F.M., Theory of flow phenomena in liquid crystals, in: G. Brown (Ed.), Advances in Liquid Crystals, Vol. 4, Academic Press, New York, 1979, pp. 1–81.
[7] Lin F.-H., Nonlinear theory of defects in nematic liquid crystals: phase transition and flow phenomena, Comm. Pure Appl. Math. 42 (1989) 789–814.
[8] Lin F.-H., Liu C., Nonparabolic dissipative systems modeling the flow of liquid crystals, Comm. Pure Appl. Math. 48 (1995) 501–537.
[9] Lin F.-H., Liu C., Existence of solutions for the Ericksen–Leslie system, Arch. Rat. Mech. Anal. 154 (2000) 135–156.
[10] Shkoller S., Well-posedness and global attractors for liquid crystals on Riemannian manifolds, Comm. Partial Differ. Eq. (2001) (to appear).
[11] Temam R., Infinite-Dimensional Dynamical Systems in Mechanics and Physics, Springer-Verlag, 1988.