On the Motion of the Free Surface of a Liquid

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1 Introduction

We consider Euler's equations

$$(1.1)\qquad \big(\partial_t + v^k\partial_k\big)v_j = -\partial_j p\,, \quad j = 1,\dots,n \ \text{ in } \mathcal{D}\,, \ \text{ where } \partial_i = \frac{\partial}{\partial x^i}\,,$$

describing the motion of a perfect incompressible fluid in vacuum:

$$(1.2)\qquad \operatorname{div} v = \partial_k v^k = 0 \quad \text{in } \mathcal{D}$$

where $v = (v_1,\dots,v_n)$ and $\mathcal{D} \subset [0,T]\times\mathbb{R}^n$ are to be determined. Here $v^k = \delta^{ki}v_i = v_k$, and we have used the summation convention that repeated upper and lower indices are summed over. Given a simply connected bounded domain $\mathcal{D}_0 \subset \mathbb{R}^n$ and initial data v_0 satisfying the constraint $\operatorname{div} v_0 = 0$, we want to find a set $\mathcal{D} \subset [0,T]\times\mathbb{R}^n$ and a vector field v solving (1.1) and (1.2) and satisfying the initial conditions

$$(1.3)\qquad \begin{cases} \{x : (0,x)\in\mathcal{D}\} = \mathcal{D}_0 \\ v = v_0 \quad \text{on } \{0\}\times\mathcal{D}_0\,. \end{cases}$$

Let $\mathcal{D}_t = \{x\in\mathbb{R}^n : (t,x)\in\mathcal{D}\}$. We also require the boundary conditions on the free boundary $\partial\mathcal{D}_t$,

$$(1.4)\qquad \begin{cases} p = 0 & \text{on } \partial\mathcal{D}_t \\ v_{\mathcal{N}} = \kappa & \text{on } \partial\mathcal{D}_t \end{cases}$$

for each t, where $\mathcal{N}$ is the exterior unit normal to $\partial\mathcal{D}_t$, $v_{\mathcal{N}} = \mathcal{N}^i v_i$, and κ is the normal velocity of $\partial\mathcal{D}_t$. The second condition can also be expressed as $(\partial_t + v^k\partial_k)|_{\partial\mathcal{D}} \in T(\partial\mathcal{D})$. We will prove a priori bounds for the initial value problem (1.1)–(1.4) in Sobolev spaces under the assumption

$$(1.5)\qquad \nabla_{\mathcal{N}} p \le -\varepsilon < 0 \quad \text{on } \partial\mathcal{D}_t \ \text{ where } \nabla_{\mathcal{N}} = \mathcal{N}^i\partial_{x^i}\,.$$

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(1.5) is a natural *physical condition* since the pressure p has to be positive in the interior of the fluid. It is essential for well-posedness in Sobolev spaces. Taking the divergence of (1.1),

$$-\triangle p = (\partial_j v^k)\partial_k v^j \ \text{ in } \mathcal{D}_t\,, \qquad p = 0 \text{ on } \partial\mathcal{D}_t\,. \tag{1.6}$$

In the irrotational case (1.5) always holds, as shown in [6, 16, 17]. Then $(\operatorname{curl} v)_{ij} = \partial_i v^j - \partial_j v^i = 0$ so $\triangle p < 0$ and hence $p > 0$ and (1.5) holds by the strong maximum principle (see [11]).

The incompressible perfect fluid is to be thought of as an idealization of a liquid. For small bodies like water drops, surface tension should help to hold the liquid together; for a large dense body like a star, its own gravity should play a role. Here we neglect the influence of such forces. Instead it is the incompressibility condition that prevents the body from expanding, and it is the fact that the pressure is positive that prevents the body from breaking up in the interior. Let us also point out that from a physical point of view one can alternatively think of the pressure as being a small positive constant on the boundary instead of vanishing. The aim of this paper is to show that we have a priori bounds for the free boundary problem (1.1)–(1.5) in any number of space dimensions. What makes this problem difficult is that the regularity of the boundary enters to highest order. Roughly speaking, the velocity tells the boundary where to move, and the boundary is the zero set of the pressure that determines the acceleration.

It is generally possible to prove local existence for analytic data for a free interface between two fluids with the same normal component of the velocity; see [2] and [13] for the irrotational case. However, this type of problem might be subject to instability in Sobolev norms. The classical examples are Rayleigh-Taylor instability, which occurs in a local linear analysis when a heavier fluid lies above a lighter fluid in a gravitational field, and Kelvin-Helmholtz instability, which occurs when the tangential velocities of the two fluids along the interface are different; see, e.g., [1]. In our case it is the first kind of instability that we must exclude. No gravitational fields are present in our problem; however, a uniform exterior gravitational field would not make a difference because it can be transformed away by going to an accelerated frame. It is condition (1.5) that excludes the possibility of this kind of instability. In fact, without taking into account the sign condition (1.5), the problem is actually ill-posed in Sobolev spaces; see [8].

Some existence results in Sobolev spaces are known in the irrotational case for the closely related water wave problem that describes the motion of the surface of the ocean under the influence of Earth's gravity. In that problem, the gravitational field can be considered as uniform, and as we remarked above, this problem reduces to our problem by going to an accelerated frame. The domain $\mathcal{D}_t$ is unbounded for the water wave problem coinciding with a half space in the case of still water. Nalimov [12] and Yosihara [18] proved local existence in Sobolev spaces in two space dimensions for initial conditions sufficiently close to still water. Beale, Hou, and Lowengrab [3] have given an argument to show that that problem is linearly well posed in a weak sense in Sobolev spaces if a condition is assumed that can be shown to be equivalent to (1.5). The condition (1.5) prevents the Rayleigh-Taylor instability from occurring when the water wave turns over. Recently Wu [16, 17] proved local existence in general in two and three dimensions for the water wave problem. Wu showed that (1.5) holds for an unbounded domain in the irrotational case. More importantly, Wu [17] is the first existence result in three space dimensions in Sobolev spaces; going from two to three dimensions required introduction of new techniques.

The method of proof in the above papers relies heavily on the assumption that the velocity is curl-free and hence satisfies Laplace's equation in the interior. This makes it possible to reduce the problem to one involving the boundary alone. In this reduction the Dirichlet-to-Neumann map enters, and it is estimated in fractional Sobolev spaces on the boundary. In the general case, with nonvanishing curl, no existence results in Sobolev spaces are known. However, recently Ebin [9] announced a local existence result for the same equations but with the boundary condition containing surface tension, which makes the problem more regular.

We prove a priori bounds in the case of nonvanishing vorticity in any number of space dimensions. We also show that the Sobolev norms remain bounded essentially as long as (1.5) holds, the second fundamental form of the free surface is bounded, and the first-order derivatives of the velocity are bounded. The proof works with lower regularity assumptions on initial data. This is partly due to the fact that our result is in terms of norms in the Eulerian space coordinates and the second fundamental form of the free surface. The norms are hence independent of a parametrization of the boundary, so we do not have to be concerned with the possibility of a parametrization becoming singular. On the other hand, it is more difficult to put up an iteration in this approach. However, existence will follow from analogous estimates and existence in the presence of surface tension, reducing to the estimates presented here in the limit of vanishing surface tension. Let us also point out that an existence result even for infinitely differentiable data together with the a priori bounds here imply existence and continuation for low regularity data. This is in particular true in the irrotational case where existence is known.

Our approach is quite elementary and geometric in nature. We use a new type of energy that controls the geometry of the free surface. The energy has a boundary part and an interior part; this fact allows us to avoid the use of fractional Sobolev spaces on the boundary. The boundary part controls the norms of the second fundamental form of the free surface, whereas the interior part controls the norms of the velocity and hence the pressure. We show that the time derivative of the energy is controlled by the energy. A crucial point is that the time derivative of the interior part will, after integrating by parts, contribute a boundary term that exactly cancels the leading-order term in the time derivative of the boundary integral. The equations look ill-posed at first sight, but if one differentiates them, one gets a well-posed system for higher-order derivatives of the velocity and the pressure.

Our energy contains the components of this higher-order system. In the interior it contains most components and on the boundary only the tangential components. Due to the fact that the pressure vanishes on the boundary, the tangential components of this higher-order system are more regular. Another crucial point is then to estimate the projection of a tensor to the tangent space of the boundary, which involves the second fundamental form.

Let us first introduce *Lagrangian coordinates*. In these coordinates the boundary is fixed. Let Ω be a domain in $\mathbb{R}^n$, and let $f_0 : \Omega \to \mathcal{D}_0$ be a diffeomorphism that is volume preserving, $\det(\partial f_0/\partial y) = 1$. Assume that $v(t,x)$ and $p(t,x)$, $(t,x) \in \mathcal{D}$, are given satisfying (1.1)–(1.4). The Lagrangian coordinates $x = x(t,y) = f_t(y)$ are given by solving

$$\frac{dx}{dt} = v(t, x(t,y)), \quad x(0,y) = f_0(y), \; y \in \Omega. \tag{1.7}$$

Then $f_t : \Omega \to \mathcal{D}_t$ is a volume-preserving diffeomorphism, since $\operatorname{div} v = 0$, and the boundary becomes fixed in the new y-coordinates. Let us introduce the notation

$$D_t = \frac{\partial}{\partial t}\Big|_{y=\text{constant}} = \frac{\partial}{\partial t}\Big|_{x=\text{constant}} + v^k \frac{\partial}{\partial x^k} \tag{1.8}$$

for the material derivative and

$$\partial_i = \frac{\partial}{\partial x^i} = \frac{\partial y^a}{\partial x^i}\frac{\partial}{\partial y^a}. \tag{1.9}$$

Sometimes it is convenient to work in the Eulerian coordinates (t,x), and sometimes it is easier to work in the Lagrangian coordinates (t,y). In the Lagrangian picture the partial derivative with respect to the time coordinate has more direct significance than the partial derivative with respect to the time coordinate in the Eulerian picture. However, this is not true for the partial derivatives with respect to the space coordinates. The notion of space derivative that plays a more significant role in the Lagrangian picture is that of covariant differentiation with respect to the metric $g_{ab}(t,y) = \delta_{ij}\,\partial x^i/\partial y^a\, \partial x^j/\partial y^b$, the pullback by f_t of the Eulerian metric δ_{ij} on $\mathcal{D}_t \subset \mathbb{R}^n$. The covariant space derivatives of the Lagrangian picture are simply and directly related to the partial derivatives with respect to the Cartesian space coordinates of the Eulerian picture. We will work mostly in the Lagrangian coordinates in this paper. However, our statements are coordinate independent, and to simplify the exposition we will present the results in the Eulerian picture in the introduction.

In the notation of (1.8) and (1.9), Euler's equations (1.1) become

$$D_t v_i = -\partial_i p. \tag{1.10}$$

Note that the commutator satisfies

$$[D_t, \partial_i] = -(\partial_i v^k)\partial_k. \tag{1.11}$$

By (1.10) we obtain the second-order equation for the velocity

$$D_t^2 v_i - (\partial_k p)\partial_i v^k = -\partial_i D_t p\,. \tag{1.12}$$

Our estimates make use of (1.12) restricted to the boundary together with the boundary condition

$$p = 0 \quad \text{on } \partial\mathcal{D}_t \implies D_t p = 0 \quad \text{on } \partial\mathcal{D}_t\,. \tag{1.13}$$

In the interior we will make use of the equation obtained by taking the curl of (1.10), using (1.11),

$$D_t(\operatorname{curl} v)_{ij} = -(\partial_i v^k)(\operatorname{curl} v)_{kj} + (\partial_j v^k)(\operatorname{curl} v)_{ki} \tag{1.14}$$

together with

$$\operatorname{div} v = 0 \quad \text{in } \mathcal{D}_t\,. \tag{1.15}$$

If we take take the divergence of (1.10) and (1.12), respectively, by using (1.11) and (1.15), we get the elliptic equations

$$\triangle p = -(\partial_i v^\ell)\partial_\ell v^i \qquad \text{in } \mathcal{D}_t\,, \quad p = 0 \ \textit{on}\ \partial\mathcal{D}_t\,, \tag{1.16}$$

$$\triangle D_t p = (\partial_k p)\triangle v^k + G(\partial v, \partial^2 p) \quad \text{in } \mathcal{D}_t\,, \quad D_t p = 0 \text{ on } \partial\mathcal{D}_t\,, \tag{1.17}$$

where $G(\partial v, \partial^2 p) = 4\delta^{ij}(\partial_i v^k)\partial_j\partial_k p + 2(\partial_i v^j)(\partial_j v^k)\partial_k v^i$. Equation (1.16) gains regularity; neglecting the problem with the boundary regularity, one derivative of v in the interior gives two derivatives of p, which gives a gain of one time derivative of v in (1.10). If $\operatorname{curl} v = 0$, then $\triangle v = 0$, so then the equation for $D_t p$ is as good as the equation for p.

To see the importance of the condition $\nabla_{\mathcal{N}} p \le -\varepsilon < 0$, let us look at a simplified linear model problem: Since $p = D_t p = 0$ on $\partial\mathcal{D}_t$, it follows that $\partial_i p = N_i \nabla_{\mathcal{N}} p$ and $\partial_i D_t p = N_i \nabla_{\mathcal{N}} D_t p$ there, so by (1.12)

$$D_t^2 v_i - (\nabla_{\mathcal{N}} p) N^k \partial_i v_k = -(\nabla_{\mathcal{N}} D_t p) N_i \quad \text{on } \partial\mathcal{D}_t\,. \tag{1.18}$$

We linearize by taking $\mathcal{D}_t = \Omega$ and $x(t,y) = y$ independently of t. In the irrotational case, $\mathcal{N}^k \partial_i v_k = \mathcal{N}^k \partial_k v_i = \nabla_{\mathcal{N}} v_i$ and $\triangle v_i = \delta^{jk}\partial_j\partial_k v_i = \delta^{jk}\partial_i\partial_j v_k = \partial_i \operatorname{div} v = 0$. Let us therefore consider the equations

$$D_t^2 v_i + \nu^{-1}\nabla_{\mathcal{N}} v_i = F_i \ \text{ on } \partial\Omega\,, \quad \triangle v_i = 0 \ \text{ in } \Omega\,, \tag{1.19}$$

for a vector field v on Ω depending on t, where ν and F_i are given functions on Ω and $D_t = \partial_t$. To simplify further, let us assume that $\nu^{-1} = \varepsilon$ is constant, $F = 0$, and Ω is the unit disc in $\mathbb{R}^2$. Then the solutions of $\triangle v = 0$ are given in polar coordinates by $v(t,r,\theta) = \sum c_k(t) r^{|k|} e^{ik\theta}$. The boundary condition in (1.19) implies that $c_k''(t) + \varepsilon|k|c_k(t) = 0$, with solutions $c_k(t) = c_k^+ e^{t\lambda_k} + c_k^- e^{-t\lambda_k}$, $\lambda_k = \sqrt{-\varepsilon|k|}$, so the high frequencies remain bounded for $t > 0$ if $\varepsilon > 0$, but they are exponentially increasing if $\varepsilon < 0$. Note that if data are analytic, i.e., $c_k^\pm = o(e^{-\delta|k|})$, $\delta > 0$, then the solution exists independently of the sign condition. The model problem is related to Enbin's counterexample. By linearizing around a rigid rotation $v = (x_2, -x_1)$, he gets an equation for the variation similar to (1.19) with $\nu^{-1} = -\nabla_N p = -1$. (1.19) is also up to terms of lower order the equation Wu [17] uses. Furthermore, a similar model problem shows up in [6] when one studies the equation for the derivatives of the velocity (1.24)–(1.25).

The model problem also suggests a candidate for an energy:

$$E(t) = \int_{\Omega} |\partial v|^2 \, dx + \int_{\partial\Omega} |D_t v|^2 \nu \, dS\,, \quad \nu > 0\,. \tag{1.20}$$

If we differentiate below the integral sign and integrate by parts, we get a bound for the energy:

$$\begin{aligned} \frac{dE}{dt} &= 2\int_{\Omega} \partial v \partial D_t v \, dx + 2\int_{\partial\Omega} D_t v \, D_t^2 v \, \nu \, dS + \int_{\partial\Omega} |D_t v|^2 D_t \nu \, dS \\ &= -2\int_{\Omega} \triangle v \, D_t v \, dx + 2\int_{\partial\Omega} D_t v (D_t^2 v + \nu^{-1} \nabla_{\mathcal{N}} v) \nu \, dS \\ &\quad + \int_{\partial\Omega} |D_t v|^2 D_t \nu \, dS \\ &\leq 2\|F\|_{L^2(\partial\Omega, \nu dS)} E^{1/2} + \|\nu^{-1} D_t \nu\|_{L^\infty(\partial\Omega)} E\,. \end{aligned} \tag{1.21}$$

An easy modification gives (1.21) with an extra term $2\|D_t \omega\|_{L^2(\Omega)} E^{1/2}$ also for a divergence-free vector field, $\operatorname{div} v = 0$, with $\operatorname{curl} v = \omega$ satisfying $D_t^2 v_i + \nu^{-1} N^k \partial_i v_k = F_i$ on the boundary. This estimate, however, is not by itself good enough to deal with (1.12), since we cannot expect a bound for $\|\partial D_t p\|_{L^2(\partial\Omega)}$ from a bound for $\|\partial v\|_{L^2(\Omega)}$ due to the loss of regularity in (1.17) in the irrotational case. One derivative of v in the interior gives only one derivative of $D_t p$ in the interior, and restricting to the boundary we lose half a derivative.

An additional idea is required that has to do with exploiting our special boundary conditions $D_t p = 0$. If we modify our energy so it contains only tangential components on and close to the boundary, then only the projection onto the tangential components of (1.12) on the boundary will occur in the energy estimate, and the tangential components of $\partial D_t p$ vanish. The components we lose control over in the energy can then be gotten back by elliptic estimates. Although the pressure and the regularity of the boundary did not enter in the above simplified model, they will enter once we go to higher-order energies, which are needed to close the argument. We will now develop these higher-order energies.

One can think of (1.10) and (1.12) as a system of equations for v and $\dot{v} = D_t v = -\partial p$:

$$D_t v_i = -\partial_i p\,, \tag{1.22}$$

$$D_t \partial_i p + (\partial_k p) \partial_i v^k = \partial_i D_t p\,. \tag{1.23}$$

To see better what goes on, let us differentiate once more with respect to the spatial coordinates

$$D_t \partial_i v_j = -\partial_i \partial_j p - (\partial_i v^k)\partial_k v_j \,, \tag{1.24}$$

$$D_t \partial_i \partial_j p + (\partial_k p)\partial_i \partial_j v^k = \partial_i \partial_j D_t p - (\partial_i v^k)\partial_k \partial_j p - (\partial_j v^k)\partial_k \partial_i p \,, \tag{1.25}$$

where we used (1.11).

We want to project (1.25) to the tangent space of the boundary. The orthogonal projection Π to the tangent space of the boundary of a $(0, r)$ tensor α is defined to be the projection of each component along the normal:

$$(\Pi\alpha)_{i_1\cdots i_r} = \Pi_{i_1}^{j_1} \cdots \Pi_{i_r}^{j_r} \alpha_{j_1\cdots j_r} \quad \text{where } \Pi_i^j = \delta_i^j - \mathcal{N}_i \mathcal{N}^j \,. \tag{1.26}$$

Let $\overline{\partial}_i = \Pi_i^j \partial_j$ be a tangential derivative. If $q = 0$ on $\partial \mathcal{D}_t$, it follows that $\overline{\partial}_i q = 0$ there and

$$(\Pi \partial^2 q)_{ij} = \theta_{ij} \nabla_{\mathcal{N}} q \quad \text{where } \theta_{ij} = \overline{\partial}_i N_j \tag{1.27}$$

is the second fundamental form of $\partial \mathcal{D}_t$. In fact,

$$\begin{aligned} 0 = \overline{\partial}_i \overline{\partial}_j q = \Pi_i^{i'} \partial_{i'} \Pi_j^{j'} \partial_{j'} q = \Pi_i^{i'} \Pi_j^{j'} \partial_{i'} \partial_{j'} q - (\overline{\partial}_i N_j) \mathcal{N}^k \partial_k q - N_j (\overline{\partial}_i \mathcal{N}^k) \partial_k q \\ = (\Pi \partial^2 q)_{ij} - \theta_{ij} \nabla_{\mathcal{N}} q \end{aligned}$$

since $N_k \overline{\partial}_i \mathcal{N}^k = \overline{\partial}_i (N_k \mathcal{N}^k)/2 = 0$.

Our energy for the second-order equation (1.25) will be a modification of (1.20) that contains only the tangential components $\Pi \partial D_t v = -\Pi \partial^2 p$ on the boundary and $(\tilde{\Pi} \partial^2) v$ in the interior, where $\tilde{\Pi}$ is an extension of the projection to the interior. Taking the time derivative of this energy and integrating by parts as in (1.21), we get a boundary term that involves the projection of (1.25). Because $\Pi \partial^2 D_t p = \theta \nabla_{\mathcal{N}} D_t p$, this can be controlled by one less derivative $\partial D_t p$. The energy together with elliptic estimates controls two derivatives of v in the interior, so (1.17) gives us two derivatives of $D_t p$ in the interior and hence one derivative on the boundary. In our discussion so far we have neglected the problem of boundary regularity, which comes in to highest order. However, our energy also controls the second fundamental form. By (1.27) and $|\nabla_{\mathcal{N}} p| \geq \varepsilon > 0$, the boundary part of the energy, $|\Pi \partial^2 p|^2 \geq |\theta|^2 |\nabla_{\mathcal{N}} p|^2 \geq |\theta|^2 \varepsilon^2$, gives an estimate for the second fundamental form θ.

The energies we propose are of the form

$$\begin{aligned} E_r(t) = \int_{\mathcal{D}_t} \delta^{mn} Q(\partial^r v_m, \partial^r v_n)\, dx + \int_{\mathcal{D}_t} |\partial^{r-1} \operatorname{curl} v|^2\, dx \\ + \int_{\partial \mathcal{D}_t} Q(\partial^r p, \partial^r p) \nu\, dS \,, \end{aligned} \tag{1.28}$$

where $\nu = (-\nabla_{\mathcal{N}} p)^{-1}$. Here Q is a positive definite quadratic form which, when restricted to the boundary, is the inner product of the tangential components

$Q(\alpha,\beta) = \langle \Pi\alpha, \Pi\beta\rangle$, and in the interior $Q(\alpha,\alpha)$ increases to the norm $|\alpha|^2$. To be more specific, we define

$$Q(\alpha,\beta) = q^{i_1 j_1}\cdots q^{i_r j_r}\alpha_{i_1\cdots i_r}\beta_{j_1\cdots j_r} \tag{1.29}$$

where

$$q^{ij} = \delta^{ij} - \eta(d)^2\mathcal{N}^i\mathcal{N}^j\,, \quad d(x) = \operatorname{dist}(x,\partial\mathcal{D}_t)\,, \quad \mathcal{N}^i = -\delta^{ij}\partial_j d\,. \tag{1.30}$$

Here η is a smooth cutoff function satisfying $0 \le \eta(d) \le 1$, $\eta(d) = 1$ when $d < d_0/4$ and $\eta(d) = 0$ when $d > d_0/2$. d_0 is a fixed number that is smaller than the injectivity radius of the normal exponential map ι_0, defined to be the largest number ι_0 such that the map

$$\partial\mathcal{D}_t \times (-\iota_0,\iota_0) \to \{x\in\mathbb{R}^n : \operatorname{dist}(x,\partial\mathcal{D}_t) < \iota_0\} \tag{1.31}$$

given by

$$(\overline{x},\iota) \to x = \overline{x} + \iota\mathcal{N}(\overline{x})$$

is an injection. These energies, in fact, control all components of $\partial^r v$, $\partial^r p$, and $\partial^{r-2}\theta$; see (1.41)–(1.42).

We prove an energy estimate implying that the energies are bounded as long as certain a priori assumptions are true. More specifically, we prove that there are continuous functions C_r such that

$$\left|\frac{dE_r(t)}{dt}\right| \le C_r\left(K, \frac{1}{\varepsilon}, L, M, \operatorname{Vol}\mathcal{D}_t, E^*_{r-1}(t)\right)E^*_r(t)\,, \tag{1.32}$$

$$\text{where } E^*_r(t) = \sum_{s=0}^{r} E_s(t)\,,$$

if $0 \le r \le 4$ or $r \ge n/2 + 3/2$, provided that

$$|\theta| \le K,\ \frac{1}{\iota_0} \le K\,, \quad \text{on } \partial\mathcal{D}_t\,, \tag{1.33}$$

$$-\nabla_{\mathcal{N}} p \ge \varepsilon > 0 \qquad \text{on } \partial\mathcal{D}_t\,, \tag{1.34}$$

$$|\partial^2 p| + |\nabla_{\mathcal{N}} D_t p| \le L \qquad \text{on } \partial\mathcal{D}_t\,, \tag{1.35}$$

$$|\partial v| + |\partial p| \le M \qquad \text{in } \mathcal{D}_t\,. \tag{1.36}$$

The bounds (1.33) give us control of the geometry of the free surface $\partial\mathcal{D}$. A bound for the second fundamental form θ gives a bound for the curvature of $\partial\mathcal{D}_t$, and a lower bound for the injectivity radius of the normal exponential map ι_0 measures how far off the surface is from self-intersecting.

Now, the lowest-order energy and the volume are in fact conserved:

$$E_0(t) = \int_{\mathcal{D}_t}\delta^{mn}v_m v_n\,dx = E_0(0)\,, \qquad \operatorname{Vol}\mathcal{D}_t = \int_{\mathcal{D}_t} dx = \operatorname{Vol}\mathcal{D}_0\,. \tag{1.37}$$

Recursively it follows from (1.32) and (1.37):

THEOREM 1.1 *Let $n \le 7$. Then there are continuous functions $\mathcal{F}_r$, $r = 0, 1, \dots,$ with $\mathcal{F}_r|_{t=0} = 1$ such that any smooth solution of the free boundary problem for Euler's equations* (1.1)–(1.5) *for $0 \le t \le T$ that satisfy the a priori assumptions* (1.33)–(1.36) *also satisfy the energy bound*

$$E_r^*(t) \le \mathcal{F}_r\left(t, K, \frac{1}{\varepsilon}, L, M, E_{r-1}^*(0), \mathrm{Vol}\,\mathcal{D}_0\right) E_r^*(0)\,, \quad 0 \le t \le T\,. \tag{1.38}$$

Most of the a priori bounds (1.33)–(1.36) can be obtained from the energy through (1.41) and (1.42) below using Sobolev's lemma if $r > (n-1)/2 + 2$. However, the lower bounds for ε and ι_0 cannot be obtained in this way; instead one has to try to get evolution equations for these.

Let $K(0)$ and $\varepsilon(0)$ be the minimum and maximum values, respectively, such that (1.33) and (1.34) hold when $t = 0$.

THEOREM 1.2 *Let r_0 be the smallest integer such that $r_0 > n/2 + 3/2$. Then there are continuous functions $\mathcal{T}_r > 0$, $r = r_0, r_0 + 1, \dots,$ such that if*

$$T \le \mathcal{T}_r\left(K(0), \frac{1}{\varepsilon(0)}, E_{r_0}^*(0), \mathrm{Vol}\,\mathcal{D}_0\right), \tag{1.39}$$

then any smooth solution of the free boundary problem for Euler's equations (1.1)–(1.5) *for $0 \le t \le T$ satisfies*

$$E_r^*(t) \le 2E_r^*(0)\,, \quad 0 \le t \le T\,. \tag{1.40}$$

Remark. The restriction $n \le 7$ in Theorem 1.1, i.e., the restriction for (1.32) to hold, is just a result of the proof becoming simpler in this case. The assumption that $\mathrm{Vol}\,\mathcal{D}_0 < \infty$ is just used to get an L^2 estimate for p, so it can be omitted if we add $\int p^2\,dx$ to the energy. We need only a lower bound for the interior radius of injectivity of the normal exponential map in (1.31) for the energy estimates to hold. The bound for the exterior one is to prevent the surface from self-intersecting.

Let us first point out that since $\mathrm{div}\, v = 0$ and $-\triangle p = (\partial_i v^k)\partial_k v^i$, one can use *elliptic estimates* to control all components of $\partial^r v$ and $\partial^r p$ from the tangential components $\Pi \partial^r p$ in the energy:

$$\|\partial^r v\|^2_{L^2(\mathcal{D}_t)} + \|\partial^{r-1} v\|^2_{L^2(\partial\mathcal{D}_t)} + \|\partial^r p\|^2_{L^2(\partial\mathcal{D}_t)} + \|\partial^r p\|^2_{L^2(\mathcal{D}_t)} \le C(K, M, \mathrm{Vol}\,\mathcal{D}_0) E_r^*\,. \tag{1.41}$$

A bound for the energy also implies a bound for the second fundamental form of the free boundary

$$\|\overline{\partial}^{r-2}\theta\|^2_{L^2(\partial\mathcal{D}_t)} \le C\left(K, L, M, \frac{1}{\varepsilon}, E_{r-1}^*, \mathrm{Vol}\,\mathcal{D}_t\right) E_r^* \tag{1.42}$$

that controls the regularity. In fact, we prove higher-order versions of the projection formula (1.27):

$$\Pi \partial^r q = (\overline{\partial}^{r-2}\theta)\nabla_{\mathcal{N}} q + O(\partial^{r-1} q) + O(\overline{\partial}^{r-3}\theta) \quad \text{if } q = 0 \text{ on } \partial\mathcal{D}_t\,. \tag{1.43}$$

Since $|\nabla_{\mathcal{N}} p| \geq \varepsilon > 0$, it follows from (1.43) that

$$|\overline{\partial}^{r-2}\theta| \leq C|\Pi\partial^r p| + O(\partial^{r-1}p) + O(\overline{\partial}^{r-3}\theta)\,,$$

where the lower-order terms can be bounded using (1.41) and (1.42) for smaller r, so (1.42) follows inductively.

Once we have the bound (1.42) for the second fundamental form, we can get estimates for any solution of the Dirichlet problem. In particular, since $D_t p$ satisfies the elliptic equation (1.17), we get

$$\|\Pi\partial^r D_t p\|^2_{L^2(\partial\mathcal{D}_t)} + \|\partial^{r-1} D_t p\|^2_{L^2(\partial\mathcal{D}_t)} \leq C\left(K, L, M, \frac{1}{\varepsilon}, E^*_{r-1}, \mathrm{Vol}\,\mathcal{D}_t\right)E^*_r\,. \tag{1.44}$$

This follows from the elliptic estimates, used to prove (1.41), and (1.43) applied to $D_t p$, where $\overline{\partial}^{r-2}\theta$ is now bounded by (1.42) and $\partial^{r-1}D_t p$ is lower order. $\Pi\partial^r D_t p$ shows up in the energy estimate when we take the time derivative of the boundary part of the energy $\Pi\partial^r p$. Although a bound for the energy implies bounds for all components of $\partial^r p$, we cannot bound the time derivative of the nontangential components on the boundary in the case of nonvanishing curl, since the elliptic estimates give control of only the tangential components $\Pi\partial^r D_t p$ in (1.44) because of the term with $\triangle v$ in (1.17).

Let us now outline the proof of Theorems 1.1 and 1.2. First, we explain the proof of the energy estimate (1.32), which uses integration by parts as in the model problem. Then we give the main elliptic estimates and the projection formula used in proving (1.41)–(1.44). Finally, we discuss how to control the geometry of the free surface and the a priori bounds (1.33)–(1.36), the time evolution of ι_0 and ε, and other geometric quantities that control the Sobolev constants that are needed for Theorem 1.2.

1.1 Energy Estimates (Sections 5 and 7)

We will now outline the proof of the energy estimate (1.32). In order to take the time derivative of the energy (1.28), we make use of the fact that if f is an arbitrary function on $\overline{\mathcal{D}}_t$ depending on t, then

$$\frac{d}{dt}\int_{\mathcal{D}_t} f\,dx = \int_{\mathcal{D}_t} D_t f\,dx \quad \text{and} \quad \frac{d}{dt}\int_{\partial\mathcal{D}_t} f\,dS = \int_{\partial\mathcal{D}_t} \big(D_t f - (\nabla_{\mathcal{N}} v_{\mathcal{N}})f\big)\,dS$$

since $\operatorname{div} v = 0$ (this can be seen, e.g., using the Lagrangian coordinates). We have

$$\frac{dE_r}{dt} = \int_{\mathcal{D}_t} D_t\big(\delta^{mn}Q(\partial^r v_m, \partial^r v_n) + |\partial^{r-1}\operatorname{curl} v|^2\big)dx + \int_{\partial\mathcal{D}_t} D_t\big(Q(\partial^r p, \partial^r p)\nu\big) - Q(\partial^r p, \partial^r p)\nu\nabla_{\mathcal{N}} v_{\mathcal{N}}\,dS \tag{1.45}$$

The derivatives of the coefficients of Q and the measures can bounded by the constants in (1.33)–(1.36):

$$|D_t q^{ij}| \leq CM\,, \quad |\partial q^{ij}| \leq CK\,, \quad |\nabla_{\mathcal{N}} v_{\mathcal{N}}| \leq CM\,; \tag{1.46}$$

see Section 3. The time derivative of the higher-order tensors $\partial^r v$ and $\partial^r p$ can be obtained from (1.22) and (1.23) by repeated use of (1.11),

$$D_t \partial^r v_n = -\partial^r \partial_n p + \sum_{0\leq s\leq r-1} c_{sr}(\partial^{s+1} v)\cdot \partial^{r-s} v_n\,, \tag{1.47}$$

$$D_t \partial^r p + (\partial_k p)\partial^r v^k = \partial^r D_t p + \sum_{0\leq s\leq r-2} d_{sr}(\partial^{s+1} v)\cdot \partial^{r-s} p\,, \tag{1.48}$$

where the symmetrized dot product is defined in Lemma 2.4. Now

$$\|(\partial^{s+1} v)\cdot \partial^{r-s} v\|_{L^2(\mathcal{D}_t)} \leq C(K)\|\partial v\|_{L^\infty(\mathcal{D}_t)} \sum_{s\leq r} \|\partial^s v\|_{L^2(\mathcal{D}_t)}\,, \quad 0\leq s\leq r-1\,. \tag{1.49}$$

This is clear for $s = 0, r-1$, and follows in general by interpolation. Hence by (1.45)–(1.48) and (1.41),

$$\frac{dE_r}{dt} = -2\int_{\mathcal{D}_t} \delta^{mn} Q\big(\partial^r v_m, \partial_n \partial^r p\big)dx + 2\int_{\partial\mathcal{D}_t} Q\big(\partial^r p, D_t\partial^r p\big)\nu\, dS + \text{lower-order terms}\,, \tag{1.50}$$

where "lower-order term" means something that is controlled by the energy E_r^* and by K, L, M, and $1/\varepsilon$ so it can be bounded by the right-hand side of (1.32).

If we integrate by parts in the first term, we get

$$\frac{dE_r}{dt} = 2\int_{\mathcal{D}_t} \delta^{mn} Q\big(\partial^r \partial_n v_m, \partial^r p\big)dx + 2\int_{\partial\mathcal{D}_t} Q\big(\partial^r p, D_t\partial^r p - \nu^{-1}\mathcal{N}_m \partial^r v^m\big)\nu\, dS + \text{lower-order term.} \tag{1.51}$$

The first term vanishes since $\operatorname{div} v = 0$. Since $-\nu^{-1}\mathcal{N}_m = \partial_m p$, the second is the inner product of $\Pi\partial^r p$ and

$$\Pi\big(D_t\partial^r p + (\partial_m p)\partial^r v^m\big) = \Pi(\partial^r D_t p) + \sum_{0\leq s\leq r-2} d_{sr}\Pi\big((\partial^{s+1} v)\cdot \partial^{r-s} p\big) \tag{1.52}$$

by (1.48). Here $\Pi\partial^r D_t p$ is under control by (1.44), and we really need to use the projection since in the case of nonvanishing curl we cannot control all components of $\partial^r D_t p$ on the boundary. The other terms in (1.52) are bounded by the a priori assumptions times (1.41). This is clear for $s = 0$, $r = 2$, but dealing with the intermediate terms is the most involved part of the manuscript. This is because the interpolation has to be done on the boundary and the expression involves nontangential components. Note that if $0 \le r \le 2$, then the boundary terms simplify and the lower-order terms are easily bounded by (1.32). The boundary terms vanish if $r = 0, 1$, and if $r = 2$ then $Q(\partial^2 p, \partial^2 p) = |\Pi \partial^2 p|^2 = |\theta|^2 |\nabla_N p|^2$, where $|\nabla_N p| \ge \varepsilon > 0$ and $Q(\partial^2 D_t p, \partial^2 D_t p) = |\theta|^2 |\nabla_N D_t p|^2$.

1.2 Elliptic Estimates Using the Energy Bound (Section 5)

The bound (1.41) follows from

$$|\partial^r v|^2 \le C\big(\delta^{mn} Q(\partial^r v_n, \partial^r v_m) + |\partial^{r-1} \operatorname{div} v|^2 + |\partial^{r-1} \operatorname{curl} v|^2\big) \tag{1.53}$$

$$\|\partial^r p\|^2_{L^2(\partial \mathcal{D}_t)} + \|\partial^r p\|^2_{L^2(\mathcal{D}_t)} \le C(K, \operatorname{Vol} \mathcal{D}_t) \sum_{s \le r} \big(\|\Pi \partial^s p\|^2_{L^2(\partial \mathcal{D}_t)} + \|\partial^{s-1} \triangle p\|^2_{L^2(\mathcal{D}_t)} \big) . \tag{1.54}$$

In fact, using that the measure in the boundary part of the energy $\ge \|\nabla_{\mathcal{N}} p\|^{-1}_{L^\infty} dS$, we get from (1.16) and (1.49), respectively,

$$\begin{aligned} \|\Pi \partial^r p\|^2_{L^2(\partial \mathcal{D}_t)} &\le \|\partial p\|_{L^\infty(\partial \mathcal{D}_t)} E_r \quad \text{and} \\ \|\partial^{r-1} \triangle p\|^2_{L^2(\mathcal{D}_t)} &\le C \|\partial v\|^2_{L^\infty(\mathcal{D}_t)} E_r . \end{aligned} \tag{1.55}$$

(1.53) follows because $\operatorname{curl} v$ is the antisymmetric part of ∂v, so only the symmetric part of $\partial^r v$ needs to be estimated; moreover, the first term in the right contains one normal component while, since $\mathcal{N}^m \mathcal{N}^n \partial_m v_n = -q^{mn} \partial_m v_n + \delta^{mn} \partial_m v_n$, two normal components can be expressed in terms of tangential components and the divergence. (1.54) follows inductively from the following inequalities:

$$\|\partial^r p\|^2_{L^2(\partial \mathcal{D}_t)} \le C \|\Pi \partial^r p\|^2_{L^2(\partial \mathcal{D}_t)} + C\big(\|\partial^{r-1} \triangle p\|_{L^2(\mathcal{D}_t)} + K \|\partial^r p\|_{L^2(\mathcal{D}_t)}\big) \|\partial^r p\|_{L^2(\mathcal{D}_t)} , \tag{1.56}$$

$$\|\partial^r p\|^2_{L^2(\mathcal{D}_t)} \le \|\partial^r p\|_{L^2(\partial \mathcal{D}_t)} \|\partial^{r-1} p\|_{L^2(\partial \mathcal{D}_t)} + \|\partial^{r-2} \triangle p\|^2_{L^2(\mathcal{D}_t)} , \tag{1.57}$$

$$\|p\|_{L^2(\mathcal{D}_t)} \le C (\operatorname{Vol} \mathcal{D}_t)^{1/n} \|\triangle p\|_{L^2(\mathcal{D}_t)} \quad \text{if } p = 0 \text{ on } \partial \mathcal{D}_t. \tag{1.58}$$

Estimate (1.56) follows from repeated use of the fact that the square of the normal derivative minus the square of the tangential one behaves better on the boundary: Let $\tilde{Q}$ be any quadratic form acting on $(0, r)$ tensors constructed from δ^{ij} and q^{ij}, and let $\tilde{\mathcal{N}} = \eta(d) \mathcal{N}$ be an extension of the normal to the interior; see (1.30). Let $T_{ij} = 2\tilde{Q}(\partial_i \alpha, \partial_j \alpha) - \delta_{ij} \delta^{mn} \tilde{Q}(\partial_m \alpha, \partial_n \alpha)$. Then $\partial_i T^i_j = 2\tilde{Q}(\triangle \alpha, \partial_j \alpha) + 2\delta^{im} (\partial_i \tilde{Q})(\partial_m \alpha, \partial_j \alpha) - \delta^{mn} (\partial_j \tilde{Q})(\partial_m \alpha, \partial_n \alpha)$, so

$$\left| \int_{\partial \mathcal{D}_t} (\mathcal{N}^i \mathcal{N}^j - q^{ij}) \tilde{Q}(\partial_i \alpha, \partial_j \alpha) dS \right| = \left| \int_{\partial \mathcal{D}_t} \mathcal{N}^i \mathcal{N}^j T_{ij} \, dS \right|$$

$$= \left| \int_{\mathcal{D}_t} \partial_i \big(\tilde{\mathcal{N}}^j T^i_j\big) dx \right|$$
$$\leq \int_{\mathcal{D}_t} 2|\triangle\alpha||\partial\alpha| + CK|\partial\alpha|^2 \, dx$$

by the divergence theorem. (1.57) results from integration by parts twice. (1.58) is the product of applying the Faber-Krahns theorem; see [14].

1.3 The Projection Formula and Estimate for the Second Fundamental Form (Section 4)

We prove an estimate for the projection: If $q = 0$ on $\partial\mathcal{D}_t$, then for $m = 0, 1$ and $0 \leq r \leq 4$ or $r \geq (n-1)/2 + 2$,

$$\|\Pi\partial^r q - (\nabla_{\mathcal{N}} q)\overline{\partial}^{r-2}\theta\|_{L^2} \tag{1.59}$$
$$\leq \varepsilon \|\nabla_{\mathcal{N}} q\|_{L^\infty} \|\overline{\partial}^{r-2}\theta\|_{L^2} + C_\varepsilon \|\theta\|_{L^\infty} \|\partial^{r-1} q\|_{L^2}$$
$$+ C\big(\|\theta\|_{L^\infty}\big) \Bigg(\|\theta\|_{L^\infty} + \sum_{s \leq r-2-m} \|\overline{\partial}^s \theta\|_{L^2} \Bigg) \sum_{s \leq r-2+m} \|\partial^s q\|_{L^2}$$

for any $\varepsilon > 0$, where $L^p = L^p(\partial\mathcal{D}_t)$ and θ is the second fundamental form. The bound (1.42) for the second fundamental form θ follows from (1.41) and (1.59) by using the a priori bound $|\nabla_{\mathcal{N}} p| \geq \varepsilon \|\nabla_{\mathcal{N}} p\|_{L^\infty}/2$.

Let us now briefly discuss the proof of (1.59). In Section 4 we derive a formula for the projection:

$$\Pi\partial^r q = \tag{1.60}$$
$$\overline{\partial}^r q + \nabla_{\mathcal{N}} q \overline{\partial}^{r-2}\theta + \sum_{\ell=1}^{r-2} \binom{r}{\ell} (\overline{\partial}^{r-2-\ell}\theta) \tilde{\otimes} (\overline{\partial}^\ell \nabla_{\mathcal{N}} q)$$
$$+ \sum_{\substack{r_0 + r_1 + \cdots + r_k + \ell = r-k \\ k-\ell=m=0 \bmod 2,\ k \geq \ell \geq 0,\ k \geq 2}} a_{r_0 \cdots r_k \ell m} C^m \big(\overline{\partial}^{r_1}\theta \tilde{\otimes} \cdots \tilde{\otimes} \overline{\partial}^{r_k}\theta \tilde{\otimes} \overline{\partial}^{r_0} \nabla^\ell_{\mathcal{N}} q\big),$$

where $\theta = \partial\mathcal{N}$ is the second fundamental form, $\tilde{\otimes}$ stands for some partial symmetrization of the tensor product, and C^m stands for contraction over m pairs of indices; see Section 4. Note that in (1.60) the total number of derivatives decreases by 1 as the number of factors of θ increases by 1. Therefore, since we have assumed that we have control of $\|\theta\|_{L^\infty}$, the terms on the second row will be lower order. (1.60) follows by expressing tangential derivatives of normal derivatives as projections onto tangential and normal components. The general form of the terms in (1.60) follows from the fact that the projections are defined in terms of the normal, and each time a derivative falls on the normal we get a factor of θ and at the same time the total number of derivatives decreases by 1.

One way to obtain the leading-order terms is to expand q in the distance to the boundary $d(x) = \operatorname{dist}(x, \partial\mathcal{D}_t)$. To highest order $\Pi\partial^r q \sim \overline{\partial}^r q$. To calculate the next terms, let us assume that $q = 0$ on $\partial\mathcal{D}_t$. Then $q/d = \nabla_{\mathcal{N}} q$ on $\partial\mathcal{D}_t$, and since $d = \Pi d = 0$ and $\theta = \nabla d$ on $\partial\mathcal{D}_t$, we have

$$\begin{aligned}\Pi\partial^r q &= \Pi\partial^r\left(d\frac{q}{d}\right)\\ &= \sum_{\ell=0}^{r-2}\binom{r}{\ell}\Pi(\partial^{r-2-\ell}\theta)\widetilde{\otimes}\Pi\partial^{\ell}\left(\frac{q}{d}\right)\\ &= \sum_{\ell=0}^{r-2}\binom{r}{\ell}(\overline{\partial}^{r-2-\ell}\theta)\widetilde{\otimes}(\overline{\partial}^{\ell}\nabla_{\mathcal{N}} q) + \text{lower-order terms},\end{aligned} \tag{1.61}$$

where "lower-order terms" means terms that contain at least one more factor of θ. In the appendix we give interpolation inequalities to deal with the products on the first row of (1.60)

$$\begin{aligned}&\big\| |\overline{\partial}^{\ell}\nabla_{\mathcal{N}} q|\,|\overline{\partial}^{r-2-\ell}\theta|\big\|_{L^2(\partial\mathcal{D}_t)} \leq \\ &\quad \varepsilon\|\nabla_{\mathcal{N}} q\|_{L^\infty(\partial\mathcal{D}_t)}\|\overline{\partial}^{r-2}\theta\|_{L^2(\partial\mathcal{D}_t)} + C_\varepsilon\|\theta\|_{L^\infty(\partial\mathcal{D}_t)}\|\overline{\partial}^{r-2}\nabla_{\mathcal{N}} q\|_{L^2(\partial\mathcal{D}_t)}\,.\end{aligned} \tag{1.62}$$

The lower-order terms on the second row of (1.60) are estimated by interpolation and Sobolev's lemma.

1.4 Elliptic Estimates Using the Bound for the Second Fundamental Form (Section 5)

If $q = 0$ on $\partial\mathcal{D}_t$ and $0 \leq r \leq 4$ or $r \geq (n-1)/2 + 2$, then we obtain the following estimate from (1.59) and (1.54):

$$\begin{aligned}\|\partial^{r-1}q\|_{L^2(\partial\mathcal{D}_t)} &\leq C\big(K, \operatorname{Vol}\mathcal{D}_t, \|\theta\|_{L^2(\partial\mathcal{D}_t)}, \dots, \|\overline{\partial}^{r-3}\theta\|_{L^2(\partial\mathcal{D}_t)}\big)\\ &\qquad\left(\|\nabla_{\mathcal{N}} q\|_{L^\infty(\partial\mathcal{D}_t)} + \sum_{s\leq r-2}\|\nabla^s\triangle q\|_{L^2(\mathcal{D}_t)}\right).\end{aligned} \tag{1.63}$$

If, in addition, $r > (n-1)/2+2$, then it follows from (1.59), (1.54), and Sobolev's lemma that

$$\begin{aligned}&\|\partial^{r-1}q\|_{L^2(\partial\mathcal{D}_t)} + \|\partial q\|_{L^\infty(\partial\mathcal{D}_t)} \leq \\ &\quad C\big(K, \operatorname{Vol}\mathcal{D}_t, \|\theta\|_{L^2(\partial\mathcal{D}_t)}, \dots, \|\overline{\partial}^{r-3}\theta\|_{L^2(\partial\mathcal{D}_t)}\big)\sum_{s\leq r-2}\|\nabla^s\triangle q\|_{L^2(\mathcal{D}_t)}\,.\end{aligned} \tag{1.64}$$

(1.63) together with (1.42) now gives a bound for $\|\partial^{s-1}D_t p\|_{L^2(\partial\mathcal{D}_t)}$ for $s \leq r$, since, by (1.17), $\|\partial^{s-2}\triangle D_t p\|_{L^2(\mathcal{D}_t)} = \|O(\partial^s p) + O(\partial^s v)\|_{L^2(\mathcal{D}_t)}$ is bounded by

(1.41) for $s \le r$ and since $\|\nabla_{\mathcal{N}} D_t p\|_{L^\infty(\partial\mathcal{D}_t)}$ is bounded by the a priori assumptions. The bound for $\|\partial^{s-1} D_t p\|_{L^2(\partial\mathcal{D}_t)}$ for $s \le r$ together with (1.59) and (1.42) gives (1.44). This suffices to prove the energy estimate. However, in order to prove Theorem 1.2, we also need to get back bounds for the a priori assumptions, which is where (1.64) will be used.

1.5 Bounds for the Geometry and the A Priori Assumptions (Sections 3 and 7)

We need to control the Sobolev constants for the surface and the derivatives of the coefficients of the quadratic form Q. These are easily controlled by an upper bound for the second fundamental form θ and a lower bound for the injectivity radius of the normal exponential map ι_0. This proves Theorem 1.1. To prove Theorem 1.2, we also need to control the time evolution of the a priori assumptions (1.33)–(1.36). However, there is a difficulty with (1.33) because we do not have an evolution equation for ι_0 and the evolution equation for θ loses regularity, so we have to control these in an indirect way. It turns out that in order to control the Sobolev constants for the interior as well as for the boundary (see Lemma A.4 and Lemma A.2, respectively), the constant in the elliptic estimate (1.41), and the constant in the interpolation inequality (1.49), it suffices to have an upper bound $1/\iota_1 \le K_1$ instead of (1.33), where $\iota_1 = \iota_1(\varepsilon_1)$ is defined to be the largest number such that

$$|\mathcal{N}(\overline{x}_1) - \mathcal{N}(\overline{x}_2)| \le \varepsilon_1 \quad \text{whenever } |\overline{x}_1 - \overline{x}_2| \le \iota_1 \text{ and } \overline{x}_1, \overline{x}_2 \in \partial\mathcal{D}_t \tag{1.65}$$

for some fixed number $0 < \varepsilon_1 < 2$.

To prove this, one makes a partition of unity into neighborhoods where (1.65) holds. An upper bound for θ and a lower bound for ι_1 then implies a lower bound for ι_0:

$$\iota_0 \ge \min\left(\frac{\iota_1}{2}, \frac{1}{\|\theta\|_{L^\infty}}\right). \tag{1.66}$$

In fact, suppose that $x^* = \overline{x} - \iota_0 N(\overline{x})$, $\overline{x} \in \partial\mathcal{D}_t$, is a point in $\mathcal{D}_t$ such that the interior normal exponential map of $\partial\mathcal{D}_t$ fails to be injective just beyond x^* along the normal line $\lambda \to \overline{x} - \lambda N(\overline{x})$, while $\operatorname{dist}(x^*, \partial\mathcal{D}_t) = \iota_0$, the injectivity radius. Then either x^* is a focal point, i.e., θ has an eigenvalue $1/\iota_0$, or the line $\lambda \to \overline{x} - \lambda N(\overline{x})$ is contained in $\mathcal{D}_t$ for all $\lambda \in (0, 2\iota_0)$ and intersects $\partial\mathcal{D}_t$ normally at $\lambda = 2\iota_0$, in which case (1.65) cannot be true for the two endpoints. Since a similar argument holds for the exterior normal exponential map, (1.66) follows.

The bounds (1.35) and (1.36) are easily controlled by the energy using (1.41), where K can be replaced by $K_1 \ge 1/\iota_1$, and Sobolev's lemma if $r \ge r_0 > n/2 + 3/2$: By Sobolev's lemma (Lemma A.4) and (1.53),

$$\|v\|^2_{L^\infty(\mathcal{D}_t)} + \|\partial v\|^2_{L^\infty(\mathcal{D}_t)} \le C(K_1)\sum_{s=0}^{r_0} \|\partial^s v\|^2_{L^2(\mathcal{D}_t)} \le C(K_1) E^*_{r_0}\,. \tag{1.67}$$

The proof of the fact that we can replace K by K_1 in (1.54), however, requires some work; see Lemma 5.7. By (1.54), (1.55) (note that p enters quadratically in the left and linearly in the right), (1.67), and Sobolev's lemma (Lemma A.4 and Lemma A.2),

$$\|\partial p\|^2_{L^\infty(\mathcal{D}_t)} + \|\partial^2 p\|^2_{L^\infty(\partial\mathcal{D}_t)} \le C\big(K_1, \operatorname{Vol}\mathcal{D}_0, E^*_{r_0}\big)\,. \tag{1.68}$$

Since the evolution equation for θ loses regularity, and since the L^2 estimate for θ depends on the L^∞ estimate, we will control it in an indirect way. By (1.27) and (1.68),

$$\begin{aligned}\|\theta\|_{L^\infty} &\le \mathcal{E}\|\Pi\partial^2 p\|_{L^\infty(\partial\mathcal{D}_t)}\\ &\le \mathcal{E}\,\|\partial^2 p\|_{L^\infty(\partial\mathcal{D}_t)} \le C\big(K_1, \operatorname{Vol}\mathcal{D}_0, \mathcal{E}, E^*_{r_0}\big)\,,\end{aligned} \tag{1.69}$$

$$\text{where } \mathcal{E}(t) = \|(\nabla_{\mathcal{N}} p(t,\cdot))^{-1}\|_{L^\infty(\partial\mathcal{D}_t)}\,. \tag{1.70}$$

The estimate for $\|\nabla_{\mathcal{N}} D_t p\|_{L^\infty(\partial\mathcal{D}_t)}$ follows from (1.64).

It remains to control the evolution of K_1 and $\mathcal{E}$. The bound for K_1 follows since we can control the time evolution of the boundary in the Lagrangian coordinates $x(t, y)$ and of the normal $\mathcal{N}(x(t, y))$

$$D_t x = v \quad\text{and}\quad D_t \mathcal{N}_i = -(\overline{\partial}_i v_k)\mathcal{N}^k\,, \tag{1.71}$$

where the right-hand sides are bounded by (1.67). We also have evolution equations for $\mathcal{E}$ and E_r,

$$\left|\frac{d\mathcal{E}}{dt}\right| \le \|\nabla_{\mathcal{N}} D_t p\|_{L^\infty}\mathcal{E}^2 \le C\big(K_1, \mathcal{E}, E^*_{r_0}, \operatorname{Vol}\mathcal{D}_0\big), \tag{1.72}$$

$$\left|\frac{dE_r}{dt}\right| \le C\big(K_1, \mathcal{E}, E^*_{\max(r_0, r-1)}, \operatorname{Vol}\mathcal{D}_0\big)E^*_r\,. \tag{1.73}$$

Assuming (1.65), the energy bound (1.40), and the bound $\mathcal{E}(t) \le 2\mathcal{E}(0)$, integration of (1.71)–(1.73) gives back slightly better bounds if $t \le \mathcal{T}(K_1(0), \mathcal{E}(0), E^*_{r_0}(0), \operatorname{Vol}(\mathcal{D}_0))$ is sufficiently small, so Theorem 1.2 follows. In fact, integrating (1.71) by using (1.67), we see that the change in $\mathcal{N}$ and x are under control if $t \le \mathcal{T}$ is small. Hence we get back the bound (1.65) if it is true with $\varepsilon_1/2$ and $2\iota_1$ initially.

2 Transformation of the Free Boundary to a Fixed Boundary: Lagrangian Coordinates, the Metric, and Covariant Differentiation in the Interior

Assume that we are given a velocity vector field $v(t, x)$ defined in a set $\mathcal{D} \subset [0, T] \times \mathbb{R}^n$ such that the boundary of $\mathcal{D}_t = \{x : (t, x) \in \mathcal{D}\}$ moves with the velocity, i.e., $(1, v) \in T(\partial\mathcal{D})$. We will now introduce Lagrangian or comoving coordinates, i.e., coordinates that are constant along the integral curves of the velocity vector field so that the boundary becomes fixed in these coordinates.

Let $x = f_t(y)$ be the change of variables given by

$$\frac{dx}{dt} = v(t, x(t, y)), \quad x(0, y) = f_0(y) \text{ if } (t, y) \in [0, T] \times \Omega. \tag{2.1}$$

Initially, when $t = 0$, we can start with either the Euclidean coordinates in $\Omega = \mathcal{D}_0$ or some other coordinates $f_0 : \Omega \to \mathcal{D}_0$ where f_0 is a diffeomorphism in which the domain Ω becomes simple. For each t we will then have a change of coordinates $f_t : \Omega \to \mathcal{D}_t = \{x : (t, x) \in \mathcal{D}\}$, taking $y \to x(t, y)$. The Euclidean metric δ_{ij} in $\mathcal{D}_t$ then induces a metric

$$g_{ab}(t, y) = \delta_{ij} \frac{\partial x^i}{\partial y^a} \frac{\partial x^j}{\partial y^b} \tag{2.2}$$

in Ω for each fixed t. We will use covariant differentiation in Ω with respect to the metric $g_{ab}(t, y)$, since it corresponds to differentiation in $\mathcal{D}_t$ under the change of coordinates $\Omega \ni y \to x(t, y) \in \mathcal{D}_t$, and we will work in both coordinate systems. This also avoids possible singularities in the change of coordinates. We will denote covariant differentiation in the y_a-coordinates by ∇_a, $a = 0, \dots, n$, and differentiation in the x_i-coordinates by ∂_i, $i = 1, \dots, n$.

Covariant Differentiation. The covariant differentiation of a $(0, r)$ tensor $k(t, y)$ is the $(0, r + 1)$ tensor given by

$$\nabla_a k_{a_1 \cdots a_r} = \frac{\partial k_{a_1 \cdots a_r}}{\partial y^a} - \Gamma^d_{a a_1} k_{d \cdots a_r} - \cdots - \Gamma^d_{a a_r} k_{a_1 \cdots d}, \tag{2.3}$$

where the Christoffel symbols Γ^d_{ab} are given by

$$\Gamma^c_{ab} = \frac{g^{cd}}{2} \left(\frac{\partial g_{bd}}{\partial y^a} + \frac{\partial g_{ad}}{\partial y^b} - \frac{\partial g_{ab}}{\partial y^d} \right) = \frac{\partial y^c}{\partial x^i} \frac{\partial^2 x^i}{\partial y^a \partial y^b}, \tag{2.4}$$

where g^{cd} is the inverse of g_{ab}. If $w(t, x)$ is the $(0, r)$ tensor expressed in the x-coordinates, then the same tensor $k(t, y)$ expressed in the y-coordinates is given by

$$k_{a_1 \cdots a_r}(t, y) = \frac{\partial x^{i_1}}{\partial y^{a_1}} \cdots \frac{\partial x^{i_r}}{\partial y^{a_r}} w_{i_1 \cdots i_r}(t, x), \quad x = x(t, y), \tag{2.5}$$

and by the transformation properties for tensors,

$$\nabla_a k_{a_1 \cdots a_r} = \frac{\partial x^i}{\partial y^a} \frac{\partial x^{i_1}}{\partial y^{a_1}} \cdots \frac{\partial x^{i_r}}{\partial y^{a_r}} \frac{\partial w_{i_1 \cdots i_r}}{\partial x^i}. \tag{2.6}$$

Covariant differentiation is constructed so the norms of tensors are invariant under changes of coordinates,

$$g^{a_1 b_1} \cdots g^{a_r b_r} k_{a_1 \cdots a_r} k_{b_1 \cdots b_r} = \delta^{i_1 j_j} \cdots \delta^{i_r j_r} w_{i_1 \cdots i_r} w_{j_1 \cdots j_r}. \tag{2.7}$$

Furthermore, expressed in the y-coordinates,

$$\partial_i = \frac{\partial}{\partial x^i} = \frac{\partial y^a}{\partial x^i} \frac{\partial}{\partial y^a}. \tag{2.8}$$

Since the curvature vanishes in the x-coordinates, it must do so in the y-coordinates, and hence

$$[\nabla_a, \nabla_b] = 0\,. \tag{2.9}$$

Let us introduce the notation $k_{a\cdots}{}^{b}{}_{\cdots c} = g^{bd} k_{a\cdots d\cdots c}$, and recall that covariant differentiation commutes with lowering and rising indices: $g^{ce}\nabla_a k_{b\cdots e\cdots d} = \nabla_a g^{ce} k_{b\cdots e\cdots d}$. Let us also introduce a notation for the material derivative:

$$D_t = \frac{\partial}{\partial t}\Big|_{y=\text{const}} = \frac{\partial}{\partial t}\Big|_{x=\text{const}} + v^k \frac{\partial}{\partial x^k}\,. \tag{2.10}$$

In this section, indices $a, b, c, \ldots,$ will refer to quantities in the y-coordinates, and indices $i, j, k, \ldots,$ will refer to quantities in the x-coordinates.

It is now important to be able to compute time derivatives of the change of coordinates and commutators between time derivatives and space derivatives.

LEMMA 2.1 *Let* $x = f_t(y)$ *be the change of variables given by* (2.1), *and let* g_{ab} *be the metric given by* (2.2). *Let* $v_i = \delta_{ij} v^j = v^i$, *and set*

$$\begin{aligned} u_a(t, y) &= v_i(t, x)\partial x^i/\partial y^a\,, & u^a &= g^{ab} u_b\,, \\ h_{ab} &= D_t g_{ab}\,, & h^{ab} &= g^{ac} g^{bd} h_{cd}\,. \end{aligned} \tag{2.11}$$

Then

$$D_t \frac{\partial x^i}{\partial y^a} = \frac{\partial x^k}{\partial y^a}\frac{\partial v_i}{\partial x^k}\,, \qquad D_t \frac{\partial y^a}{\partial x^i} = -\frac{\partial y^a}{\partial x^k}\frac{\partial v_k}{\partial x^i}\,, \tag{2.12}$$

$$D_t g_{ab} = \nabla_a u_b + \nabla_b u_a\,, \quad D_t g^{ab} = -h^{ab}\,, \quad D_t d\mu_g = \frac{g^{ab} h_{ab}\, d\mu_g}{2}\,, \tag{2.13}$$

$$D_t \Gamma^c_{ab} = \nabla_a \nabla_b u^c\,, \tag{2.14}$$

where $d\mu_g$ *is the Riemannian volume element on* Ω *in the metric* g.

PROOF: We have

$$D_t \frac{\partial x^i}{\partial y^a} = \frac{\partial D_t x^i}{\partial y^a} = \frac{\partial v_i}{\partial y^a} = \frac{\partial x^k}{\partial y^a}\frac{\partial v_i}{\partial x^k}\,,$$

which proves the first part of (2.12). Furthermore,

$$0 = D_t\left(\frac{\partial y^b}{\partial x^i}\frac{\partial x^j}{\partial y^b}\right) = \left(D_t \frac{\partial y^b}{\partial x^i}\right)\frac{\partial x^j}{\partial y^b} + \frac{\partial y^b}{\partial x^i} D_t \frac{\partial x^j}{\partial y^b}\,.$$

Multiplying by $\partial y^a/\partial x^j$ and using the first part of (2.12) now gives the second part. To prove the first part of (2.13), we note that that by (2.2) $D_t g_{ab}$ is the sum over i of

$$\begin{aligned} D_t\left(\frac{\partial x^i}{\partial y^a}\frac{\partial x^i}{\partial y^b}\right) &= \left(D_t \frac{\partial x^i}{\partial y^a}\right)\frac{\partial x^i}{\partial y^b} + \frac{\partial x^i}{\partial y^a}\left(D_t \frac{\partial x^i}{\partial y^b}\right) \\ &= \frac{\partial x^k}{\partial y^a}\frac{\partial v_i}{\partial x^k}\frac{\partial x^i}{\partial y^b} + \frac{\partial x^i}{\partial y^a}\frac{\partial x^k}{\partial y^b}\frac{\partial v_i}{\partial x^k} = \nabla_a u_b + \nabla_b u_a \end{aligned}$$

by (2.6). The second part of (2.13) follows from the first since $0 = D_t(g^{ab}g_{bc}) = D_t(g^{ab})g_{bc} + g^{ab}D_t(g_{bc})$, so $D_t g^{ak} = -g^{ck}g^{ab}D_t g_{bc}$. The last part of (2.13) follows since in local coordinates $d\mu_g = \sqrt{\det g}\,dy$ and $D_t \det g = \det g\, g^{ab}D_t g_{ab}$. It follows from (2.4) and (2.13) that

$$D_t\Gamma^c_{ab} = \frac{g^{cd}}{2}(\nabla_a D_t g_{bd} + \nabla_b D_t g_{ad} - \nabla_d D_t g_{ab}) = g^{cd}\nabla_a\nabla_b u_d\,.$$

□

LEMMA 2.2 *Let $w_{i_1\cdots i_r}(t,x)$ be an arbitrary $(0,r)$ tensor, and let*

$$k_{a_1\cdots a_r}(t,y) = w_{i_1\cdots i_r}(t,x)\frac{\partial x^{i_1}}{\partial y^{a_1}}\cdots\frac{\partial x^{i_r}}{\partial y^{a_r}} \quad \text{where } x = f(t,y)\,. \tag{2.15}$$

Let $D_t = \partial_t\big|_{y=\text{constant}}$ and $v^\ell(t,x) = \partial_t f^\ell(t,y)$. Then

$$\begin{aligned} D_t k_{a_1\cdots a_r} &= \Big(D_t w_{i_1\cdots i_r} + w_{\ell\cdots i_r}\frac{\partial v^\ell}{\partial x^{i_1}} + \cdots + w_{i_1\cdots \ell}\frac{\partial v^\ell}{\partial x^{i_r}}\Big)\frac{\partial x^{i_1}}{\partial y^{a_1}}\cdots\frac{\partial x^{i_r}}{\partial y^{a_r}} \\ &= \big(\partial_t\big|_{x=\text{const}} w_{i_1\cdots i_r} + (\mathcal{L}_v w)_{i_1\cdots i_r}\big)\frac{\partial x^{i_1}}{\partial y^{a_1}}\cdots\frac{\partial x^{i_r}}{\partial y^{a_r}} \end{aligned} \tag{2.16}$$

and $\mathcal{L}_v$ is the Lie derivative.

PROOF: Note that if the tensor and the velocity depend only on t through x, then this would just be the definition of the Lie derivative. Now

$$\begin{aligned} &\frac{\partial}{\partial t}\bigg|_{y=\text{const}} w_{i_1\cdots i_r}(t,x)\frac{\partial x^{i_1}}{\partial y^{a_1}}\cdots\frac{\partial x^{i_r}}{\partial y^{a_r}} \\ &\quad= \bigg(\partial_t\bigg|_{x=\text{const}} w_{i_1\cdots i_r}(t,x) + (\partial_\ell w_{i_1\cdots i_r})(t,x)\frac{\partial x^\ell}{\partial t}\bigg)\frac{\partial x^{i_1}}{\partial y^{a_1}}\cdots\frac{\partial x^{i_r}}{\partial y^{a_r}} \\ &\qquad + w_{i_1\cdots i_r}(t,x)\frac{\partial^2 x^{i_1}}{\partial t\partial y^{a_1}}\cdots\frac{\partial x^{i_r}}{\partial y^{a_r}} + \cdots + w_{i_1\cdots i_r}(t,x)\frac{\partial x^{i_1}}{\partial y^{a_1}}\cdots\frac{\partial^2 x^{i_r}}{\partial t\partial y^{a_r}}\,. \end{aligned}$$

Since $v^\ell(t,x) = \partial x^\ell/\partial t$, we see that

$$\begin{aligned} w_{i_1\cdots i_r}(t,x)\frac{\partial^2 x^{i_1}}{\partial t\partial y^{a_1}}\cdots\frac{\partial x^{i_r}}{\partial y^{a_r}} &= w_{i_1\cdots i_r}(t,x)\frac{\partial v^{i_1}}{\partial y^{a_1}}\cdots\frac{\partial x^{i_r}}{\partial y^{a_r}} \\ &= w_{\ell\cdots a_r}(t,x)\frac{\partial v^\ell}{\partial x^{i_1}}\frac{\partial x^{i_1}}{\partial y^{a_1}}\cdots\frac{\partial x^{i_r}}{\partial y^{a_r}}\,, \end{aligned}$$

and similarly for the other terms. This proves (2.16), since by definition

$$(\mathcal{L}_v w)_{i_1\cdots i_r} = v^\ell(\partial_\ell w_{i_1\cdots i_r}) + w_{\ell\cdots i_r}\frac{\partial v^\ell}{\partial x^{i_1}} + \cdots + w_{i_1\cdots\ell}\frac{\partial v^\ell}{\partial x^{i_r}}\,.$$

□

We will now calculate commutators between the material derivative D_t and space derivatives ∂_i in Lemma 2.3 and covariant derivatives ∇_a in Lemma 2.4. In order to calculate commutators between D_t and higher-order derivatives $\partial_{i_1}\cdots\partial_{i_r}$ or $\nabla_{a_1}\cdots\nabla_{a_r}$, we will introduce some notation incorporating that these commutators are symmetric under permutations of the indices $(i_1,\dots,i_r)$ and $(a_1,\dots,a_r)$, respectively. Let $(\partial^r)_{i_1\cdots i_r} = \partial^r_{i_1\cdots i_r} = \partial_{i_1}\cdots\partial_{i_r}$ and $(\nabla^r)_{a_1\cdots a_r} = \nabla^r_{a_1\cdots a_r} = \nabla_{a_1}\cdots\nabla_{a_r}$. In particular, it is convenient to introduce the symmetric dot product in (2.19) and (2.24):

LEMMA 2.3 *Let ∂_i be given by* (2.8). *Then*

$$[D_t\partial_i] = -(\partial_i v^k)\partial_k\,. \tag{2.17}$$

Furthermore,

$$[D_t,\partial^r] = \sum_{s=0}^{r-1} -\binom{r}{s+1}(\partial^{1+s}v)\cdot\partial^{r-s}\,, \tag{2.18}$$

where the symmetric dot product is defined to be in components

$$\big((\partial^{1+s}v)\cdot\partial^s\big)_{i_1\cdots i_r} = \frac{1}{r!}\sum_{\sigma\in\Sigma_r}\big(\partial^{1+s}_{i_{\sigma_1}\cdots i_{\sigma_{1+s}}}v^k\big)\partial^s_{ki_{\sigma_{s+2}}\cdots i_{\sigma_r}}\,. \tag{2.19}$$

PROOF: The proof of (2.17) follows from (2.8) and (2.12). In the notation of (2.18), we can write (2.17) as

$$[D_t,\partial] = -(\partial v)\cdot\partial\,.$$

Using this repeatedly, we obtain

$$\begin{aligned}[D_t,\partial^r] = \sum_{\ell=0}^{r}\partial^\ell[D_t,\partial]\partial^{r-\ell-1} &= -\sum_{\ell=0}^{r-1}\partial^\ell(\partial v)\cdot\partial^{r-\ell}\\ &= -\sum_{\ell=0}^{r-1}\sum_{s=0}^{\ell}\binom{\ell}{s}(\partial^{1+s}v)\cdot\partial^{r-s}\,.\end{aligned}$$

Since $\sum_{\ell=s}^{r-1}\binom{\ell}{s} = \binom{r}{s+1}$, this proves (2.18). □

LEMMA 2.4 *Let $T_{a_1\cdots a_r}$ be a $(0,r)$ tensor. We have*

$$[D_t,\nabla_a]T_{a_1\cdots a_r} = -(\nabla_{a_1}\nabla_a u^d)T_{da_2\cdots a_r} - \cdots - (\nabla_{a_r}\nabla_a u^d)T_{a_1\cdots a_{r-1}d}\,. \tag{2.20}$$

If $\triangle = g^{cd}\nabla_c\nabla_d$ and q is a function, we have

$$[D_t,g^{ab}\nabla_a]T_b = -h^{ab}\nabla_a T_b - (\triangle u^e)T_e\,, \tag{2.21}$$

$$[D_t,\triangle]q = -h^{ab}\nabla_a\nabla_b q - (\triangle u^e)\nabla_e q\,. \tag{2.22}$$

Furthermore,

$$[D_t,\nabla^r]q = \sum_{s=1}^{r-1} -\binom{r}{s+1}(\nabla^{s+1}u)\cdot\nabla^{r-s}q\,, \tag{2.23}$$

where the symmetric dot product is defined to be in components

$$\big((\nabla^{s+1}u)\cdot\nabla^{r-s}q\big)_{a_1\cdots a_r} = \frac{1}{r!}\sum_{\sigma\in\Sigma_r}\big(\nabla^{s+2}_{a_{\sigma_1}\cdots a_{\sigma_{s+1}}}u^d\big)\nabla^{r-s}_{da_{\sigma_{s+3}}\cdots a_{\sigma_r}}q\,. \tag{2.24}$$

PROOF: (2.20) is a consequence of (2.13) since in components the covariant derivative is given by $\nabla_a T_{a_1\cdots a_r} = \partial T_{a_1\cdots a_r}/\partial y^a - \Gamma^d_{a_1 a} T_{da_2\cdots a_3} - \cdots - \Gamma^d_{a_r a} T_{a_1\cdots a_{r-1}d}$. Now

$$\big[D_t, g^{ab}\nabla_a\big]T_b = (D_t g^{ab})\nabla_a T_b + g^{ab}\big[D_t,\nabla_a\big]T_b\,,$$

and (2.21) follows from (2.12) and (2.20). (2.22) follows from (2.21) applied to $T_b = \nabla_b\psi$, since $D_t\nabla_b q = \partial_t\partial q(t,y)/\partial y^b = \nabla_b D_t q$.

In the notation of (2.24), we have by (2.20)

$$\big[D_t,\nabla\big]\nabla^s q = -s(\nabla^2 u)\cdot\nabla^s q\,. \tag{2.25}$$

Using this repeatedly, we get

$$\begin{aligned}\big[D_t,\nabla^r\big]q &= \sum_{\ell=0}^{r-1}\nabla^\ell\big[D_t,\nabla\big]\nabla^{r-\ell-1}q\\ &= -\sum_{\ell=0}^{r-1}\nabla^\ell(r-\ell-1)\big(\nabla^2 u\big)\cdot\nabla^{r-\ell-1}q\\ &= -\sum_{\ell=0}^{r-2}\sum_{s=0}^{\ell}(r-\ell-1)\binom{\ell}{s}\big(\nabla^{s+2}u\big)\cdot\nabla^{r-s-1}q\,.\end{aligned}$$

Since $\sum_{\ell=s}^{r-2}(r-\ell-1)\binom{\ell}{s} = \binom{r}{s+2}$, this proves (2.23). □

Notice that the difference between (2.18) and (2.23) is that in (2.23) the term with $s=0$ is absent, which is the advantage of going to covariant differentiation.

3 The Geometry and Regularity of the Boundary: The Second Fundamental Form and Extension of the Normal to the Interior

In this section we will deal with the geometry and regularity of the boundary. The regularity is measured by the regularity of the normal, in particular by the first space derivative, i.e., the second fundamental form. We also need to control how far off the boundary is from self-intersecting since we want to foliate the domain close the boundary into surfaces that do not self-intersect. This can be achieved by the level sets of the distance function to the boundary. This gives an extension of the normal to the interior, which we need to prove our estimates. The size of the neighborhood in which the level sets are well-defined and smooth determines the size of the derivatives of our extension of the normal to a vector field defined everywhere in the interior. We also want to control the time evolution of the boundary, which can be measured by the time derivative of the normal in the Lagrangian coordinates.

We will use both the Eulerian coordinates and the Lagrangian coordinates. When we calculate time derivatives, it is of course most convenient to do so in the Lagrangian coordinates, whereas the Eulerian coordinates are more convenient to use when we measure how the surface lies in space, since we want to be able to compare the normal at different points. In this section we will also define the projection of a tensor to the boundary that we will use to define covariant differentiation on the boundary. The projection will play an important role in our estimates, and we will discuss it in detail in Section 4.

DEFINITION 3.1 Let N^a denote be the unit normal to $\partial\Omega$,

$$g_{ab}N^aN^b = 1\,, \quad g_{ab}N^aT^b = 0 \ \text{ if } T \in T(\partial\Omega)\,, \tag{3.1}$$

and let $N_a = g_{ab}N^b$ denote the unit conormal, $g^{ab}N_aN_b = 1$. The induced metric γ on the tangent space to the boundary $T(\partial\Omega)$ extended to be 0 on the orthogonal complement in $T(\Omega)$ is then given by

$$\gamma_{ab} = g_{ab} - N_aN_b\,, \quad \gamma^{ab} = g^{ab} - N^aN^b\,. \tag{3.2}$$

The orthogonal projection of an (r, s) tensor S to the boundary is given by

$$(\Pi S)^{a_1\cdots a_r}_{b_1\cdots b_s} = \gamma^{a_1}_{c_1}\cdots\gamma^{a_s}_{c_s}\gamma^{d_1}_{b_1}\cdots\gamma^{d_s}_{b_s}S^{c_1\cdots c_r}_{d_1\cdots d_s}\,, \tag{3.3}$$

where

$$\gamma^c_a = \delta^c_a - N_aN^c \quad \text{and} \quad \gamma^a_c = \delta^a_c - N^aN_c\,. \tag{3.4}$$

Covariant differentiation on the boundary $\overline{\nabla}$ is given by

$$\overline{\nabla}S = \Pi\nabla S\,. \tag{3.5}$$

The second fundamental form of the boundary is given by

$$\theta_{ab} = (\Pi\nabla N)_{ab} = \gamma^c_a\nabla_cN_b\,. \tag{3.6}$$

Note first that $\overline{\nabla}$ is invariantly defined since the projection and the covariant derivative are. Note also that $\overline{\nabla}$ indeed corresponds to the intrinsic covariant derivative $\not\nabla$ of the boundary:

LEMMA 3.2 *Suppose that the coordinates are chosen so that locally the boundary is given by* $\partial\Omega = \{y : y^n = 0\}$ *and parameterized by* $(y^1, \dots, y^{n-1})$. *Let* $\not\nabla$ *denote covariant differentiation on* $\partial\Omega$. *Then*

$$\overline{\nabla}^aT^b = \begin{cases} \not\nabla^aT^b & \text{for } a, b = 1, \dots, n-1 \\ 0 & \text{for } a = n \text{ or } b = n \end{cases} \quad \text{if } T^n = 0\,. \tag{3.7}$$

PROOF: The conormal is $N_a = \delta_{an}/\sqrt{g^{nn}}$, and the normal is $N^a = g^{ac}N_c = g^{an}/\sqrt{g^{nn}}$. The induced metric is given by $\gamma_{ab} = g_{ab}$ for $a, b = 1, \dots, n-1$, and its inverse is given by $\gamma^{ab} = g^{ab} - N^a N^b$ for $a, b = 1, \dots, n-1$. Note also that

$$\begin{cases} \gamma_a^n = \gamma_a^n = \gamma^{na} = \gamma^{an} = 0 & \text{when } a \le n \\ \gamma_a^b = \gamma_a^b = \delta_a^b & \text{when } a < n \\ \gamma_n^b = \gamma_n^b = -g^{bn}/g^{nn} & \text{when } b < n\,. \end{cases}$$

Let us at this point use the notation $\nabla^a = g^{ab}\nabla_b$, $\overline{\nabla}^a = g^{ab}\overline{\nabla}_b$, and $\not{\nabla}^a = \gamma^{ab}\not{\nabla}_b$, where the last sum is only over $b = 1, \dots, n-1$. To prove (3.7), we first note that $\overline{\nabla}^a T^b = \gamma_{a'}^a \gamma_{b'}^b \nabla^{a'} T^{b'} = \gamma_{a'}^a \nabla^{a'} T^b = 0$ when $i = n$ or $b = n$ since $\gamma_{a'}^n = 0$. On the other hand, if $1 \le a, b \le n-1$, then

$$\overline{\nabla}^a T^b = \gamma_{a'}^a g^{a'a''}\left(\frac{\partial T^b}{\partial y^{a''}} + g^{bb''}\Gamma_{a''b''c}T^c\right) = \gamma^{aa'}\frac{\partial T^b}{\partial y^{a'}} + \gamma^{aa'}\gamma^{bb'}\Gamma_{a'b'c}T^c\,,$$

and if $1 \le a, b, c \le n-1$, then

$$\Gamma_{abc} = \frac{1}{2}\left(\frac{\partial g_{bc}}{\partial y^a} + \frac{\partial g_{ac}}{\partial y^b} - \frac{\partial g_{ab}}{\partial y^c}\right) = \frac{1}{2}\left(\frac{\partial \gamma_{bc}}{\partial y^a} + \frac{\partial \gamma_{ac}}{\partial y^b} - \frac{\partial \gamma_{ab}}{\partial y^c}\right) = \not{\Gamma}_{abc}$$

gives the intrinsic connection, so (3.7) follows. □

It follows that any invariant quantities formed from either side of (3.7) have to be equal. If the coordinates are chosen so $y^n = 0$ on $\partial\Omega$, then the curvature of $\partial\Omega$ is related to the second fundamental form by Gauss equations

$$\overline{R}_{cab}^{\,d} = \theta_{ac}\theta_b^d - \theta_{bc}\theta_a^d\,. \tag{3.8}$$

Recall also that if T is tangential,

$$\left[\overline{\nabla}_a, \overline{\nabla}_b\right]T_{a_1\cdots a_r} = -\overline{R}_{cab}^{\,a_1}T_{c\cdots a_r} - \cdots - \overline{R}_{cab}^{\,a_r}T_{a_1\cdots c}\,. \tag{3.9}$$

We also need to extend the normal to a neighborhood of the boundary. The exact extension of the normal to the interior is not so important at this point. Basically we want to have control of the supremum norm of the time and space derivatives of the normal in the interior. One way to define an extension of the normal in the interior is to consider a foliation of Ω close to $\partial\Omega$,

$$S_\lambda = \{y \in \Omega : d(t, y) = \lambda\}\,, \quad d > 0 \text{ in } \Omega\,, \quad d = 0 \text{ on } \partial\Omega\,. \tag{3.10}$$

The unit conormal to S_λ is then given by

$$N_a = \frac{\partial_a d}{\sqrt{g^{bc}\partial_b d\,\partial_c d}}\,. \tag{3.11}$$

It is natural to take $d(t, y) = \text{dist}_g(y, \partial\Omega)$ to be the geodesic distance to the boundary, which is the same as the Euclidean distance in the x-variables. If d is the geodesic distance in the metric g, then the conormal is $N_a = \nabla_a d$ and $\theta = \nabla N = \nabla^2 d = \Pi\nabla^2 d$, and the normal derivative of the normal vanishes $\nabla_N N = 0$. Since $\theta = \Pi\nabla^2 d = \nabla^2 d$, it follows that $\overline{\nabla}\theta = \Pi\nabla\Pi\nabla^2 d = \Pi\nabla^3 d$ is symmetric as well. $\overline{\nabla}^2\theta$, however, is not symmetric, but the antisymmetric part is lower order. By Gauss equations (3.8)–(3.9),

$$\overline{\nabla}_a\overline{\nabla}_b\theta_{cd} - \overline{\nabla}_b\overline{\nabla}_a\theta_{cd} = \left[\overline{\nabla}_a, \overline{\nabla}_b\right]\theta_{cd} = -\overline{R}^{e}_{cab}\theta_{ed} - \overline{R}^{e}_{dab}\theta_{de}\,. \tag{3.12}$$

Furthermore, since $N \cdot N = 1$, we get $N \cdot \nabla^2 N + (\nabla N)\cdot(\nabla N) = 0$; in other words,

$$\nabla_N \theta_{ab} = -\theta_a^c\theta_{cb}\,, \tag{3.13}$$

so the second fundamental forms for the surfaces S_λ for small λ are as regular as for $\partial\Omega$. We will discuss this and the regularity of the extension of the normal to the interior further in Lemma 3.10.

Let us now discuss two definitions to control the geometry and regularity of the boundary. Let us express our surface in the x-variables $\partial\mathcal{D}_t \subset \mathbb{R}^n$ using the metric there.

DEFINITION 3.3 Let $\mathcal{N}(\overline{x})$ be the outward unit normal to $\partial\mathcal{D}_t$ at $\overline{x} \in \partial\mathcal{D}_t$. Let $\operatorname{dist}(x_1, x_2) = |x_1 - x_2|$ denote the Euclidean distance in $\mathbb{R}^n$, and for $\overline{x}_1, \overline{x}_2 \in \partial\mathcal{D}_t$ let $\operatorname{dist}_{\partial\mathcal{D}_t}(\overline{x}_1, \overline{x}_2)$ denote the geodesic distance on the boundary. Let $\operatorname{dist}(x, \partial\mathcal{D}_t)$ be the Euclidean distance from x to the boundary.

DEFINITION 3.4 Let ι_0 be the injectivity radius of the normal exponential map of $\partial\mathcal{D}_t$, i.e., the largest number such that the map

$$\begin{gathered}\partial\mathcal{D}_t \times (-\iota_0, \iota_0) \to \{x \in \mathbb{R}^n : \operatorname{dist}(x, \partial\mathcal{D}_t) < \iota_0\}\\ \text{given by } (\overline{x}, \iota) \to x = \overline{x} + \iota\mathcal{N}(\overline{x})\end{gathered} \tag{3.14}$$

is an injection.

Note that $\iota_0 \geq 1/|\theta|_{L^\infty(\partial\mathcal{D}_t)}$, for along the normal line from $\overline{x} \in \partial\mathcal{D}_t$, the first focal point is at a distance $1/|\theta(\overline{x})|$, where $|\theta(\overline{x})| = \sup_{|v|=1}|\theta(\overline{x})\cdot v|$ is the greatest eigenvalue in magnitude. Instead of using the injectivity radius ι_0, we can use a radius ι_1 that, in conjunction with a bound for the second fundamental form, is comparable. The radius ι_1 works equally well for controlling the Sobolev constants, and it is easier to control the time evolution off.

DEFINITION 3.5 Let $0 < \varepsilon_1 < 2$ be a fixed number, and let $\iota_1 = \iota_1(\varepsilon_1)$ the largest number such that

$$|\mathcal{N}(\overline{x}_1) - \mathcal{N}(\overline{x}_2)| \leq \varepsilon_1 \quad \text{whenever } |\overline{x}_1 - \overline{x}_2| \leq \iota_1\,,\ \overline{x}_1, \overline{x}_2 \in \partial\mathcal{D}_t\,. \tag{3.15}$$

Remark. Note that Definition 3.5 also says that the intersection $\partial\mathcal{D}_t \cap B(\iota_1, \overline{x}_0)$ of the surface with an open ball of radius ι_1 centered at any point $\overline{x}_0 \in \partial\mathcal{D}_t$ is connected, and it can be written as a graph over the plane orthogonal to the normal $\mathcal{N}(\overline{x}_0)$ at the center $\overline{x}_0$. In fact, we claim that the line segment in $B(\iota_1, \overline{x}_0)$ along the exterior normal $\mathcal{N}(\overline{x}_0)$ from any point $\overline{x}_1$ in the same component of $\partial\mathcal{D}_t$ as $\overline{x}_0$ is completely contained in the complement $\complement\mathcal{D}_t$ (and the line segment in the opposite direction is completely contained in $\mathcal{D}_t$). In fact, if not, then there would be a point $\overline{x}_2 \in \partial\mathcal{D}_t$ where it would enter the region $\mathcal{D}_t$ again, and at that point the exterior normal $\mathcal{N}(\overline{x}_2)$ would have to make an angle of at least $\pi/2$ with $\mathcal{N}(\overline{x}_0)$, contradicting the condition in Definition 3.5.

LEMMA 3.6 *Suppose that* $|\theta| \le K$, *and let* ι_0 *and* ι_1 *be as in Definitions* 3.4 *and* 3.5. *Then*

$$\iota_0 \ge \min\left(\frac{\iota_1}{2}, \frac{1}{K}\right) \quad \text{and} \quad \iota_1 \ge \min\left(2\iota_0, \frac{\varepsilon_1}{K}\right). \tag{3.16}$$

PROOF: Let

$$\iota_3 = \min_{\operatorname{dist}_{\partial\mathcal{D}_t}(\overline{x},\overline{z}) \ge \pi/K} |\overline{x} - \overline{z}|\,.$$

We claim that

$$\iota_0 = \frac{\iota_3}{2} \ge \frac{\iota_1}{2} \quad \text{if } \min\left(\iota_0, \frac{\iota_3}{2}\right) \le \frac{1}{K}\,.$$

By Definition 3.4 there are $\overline{x_1} \ne \overline{x}_2$ on the boundary such that

$$\overline{x}_1 + aN(\overline{x}_1) = \overline{x}_2 + bN(\overline{x}_2) \quad \text{for some } |a| \le \iota_0,\ |b| \le \iota_0\,.$$

If $\iota_0 < 1/K$, then by Lemma 3.7 $\operatorname{dist}_{\partial\mathcal{D}_t}(\overline{x_1}, \overline{x}_2) \ge \pi/K$, and hence

$$\iota_3 = \min_{\operatorname{dist}_{\partial\mathcal{D}_t}(\overline{x},\overline{z}) \ge \pi/K} |\overline{x} - \overline{z}| \le |\overline{x}_1 - \overline{x}_2| \le 2\iota_0 < \frac{2}{K}\,.$$

If $\iota_3 < 2/K$, it follows from Lemma 3.7 that the minima above are attained at some, possibly different, $(\overline{x}_3, \overline{x}_4) \in \partial\mathcal{D}_t \times \partial\mathcal{D}_t$ with $\operatorname{dist}_{\partial\mathcal{D}_t}(\overline{x}_3, \overline{x}_4) > \pi/K$. Hence $\partial\mathcal{D}_t \times \partial\mathcal{D}_t \ni (\overline{x}, \overline{z}) \to |\overline{x} - \overline{z}|$ has a local minimum at $(\overline{x}_3, \overline{x}_4)$, so the normals $\mathcal{N}(\overline{x}_3)$ and $\mathcal{N}(\overline{x}_4)$ are parallel to the line between $\overline{x}_3$ and $\overline{x}_4$. From this it follows that $\iota_0 \le \iota_3/2$, and it also contradicts the condition in Definition 3.5 so we conclude that $\iota_3 = |\overline{x}_3 - \overline{x}_4| > \iota_1$. This proves the first part of (3.16), and the second part follows in a similar way; if $\operatorname{dist}_{\partial\mathcal{D}_t}(\overline{x_1}, \overline{x}_2) \le \pi/K$, then by Lemma 3.7

$$\begin{aligned} |\mathcal{N}(\overline{x}_1) - \mathcal{N}(\overline{x}_2)| &\le 2\sin\left(K \operatorname{dist}_{\partial\mathcal{D}_t} \frac{\overline{x_1}, \overline{x}_2}{2}\right) \\ &\le K \operatorname{dist}_{\partial\mathcal{D}_t}(\overline{x_1}, \overline{x}_2) \le K\pi \frac{|\overline{x_1} - \overline{x}_2|}{2} \le \varepsilon_1\,, \end{aligned}$$

if $|\overline{x_1} - \overline{x}_2| \le \varepsilon_1 2/K\pi$. If, on the other hand, $\operatorname{dist}_{\partial\mathcal{D}_t}(\overline{x_1}, \overline{x}_2) > \pi/K$, then $|\overline{x}_1 - \overline{x}_2| \ge \iota_3$, and if $\iota_3 < 2/K$, then $\iota_3 = 2\iota_0$ so $|\overline{x}_1 - \overline{x}_2| \ge \min(2/K, 2\iota_0)$. □

LEMMA 3.7 *Suppose that* $|\theta| \le K$ *and* $0 < \operatorname{dist}_{\partial\mathcal{D}_t}(\overline{x}_1, \overline{x}_2) < \pi/K$. *Then*

$$\overline{x}_1 + a\mathcal{N}(\overline{x}_1) \ne \overline{x}_2 + b\mathcal{N}(\overline{x}_2) \quad \text{for } |a| \le \frac{1}{K},\ |b| \le \frac{1}{K}\,. \tag{3.17}$$

Furthermore, if $|\theta| \le K$ *and* $\operatorname{dist}_{\partial\mathcal{D}_t}(\overline{x}_1, \overline{x}_2) \le \pi/K$, *then*

$$|\overline{x}_1 - \overline{x}_2| \ge \frac{2\operatorname{dist}_{\partial\mathcal{D}_t}(\overline{x}_1, \overline{x}_2)}{\pi} \quad \text{and} \tag{3.18}$$

$$\mathcal{N}(\overline{x}_1) \cdot \mathcal{N}(\overline{x}_2) \ge \cos\left(K \operatorname{dist}_{\partial\mathcal{D}_t}(\overline{x}_1, \overline{x}_2)\right).$$

PROOF: Let $\alpha(s)$ be a geodesic in $\partial\Omega$ parameterized by arc length, $|\dot\alpha(s)| = 1$, with $\alpha(s_i) = \overline{x}_i$. Let $s_0 = (s - 1 + s_2)/2$. To simplify notation, we assume that $s_0 = 0$ and $\alpha(0) = 0$ and set $\dot\alpha(0) = \mathcal{T}$. Let $\mathcal{N}(s)$ be the normal to $\alpha(s)$, and $k(s) = \theta(\dot\alpha(s), \dot\alpha(s))$ be the (normal) curvature of $\alpha(s)$, i.e., $\ddot\alpha(s) = \pm k(s)\mathcal{N}(s)$. We will show that $\mathcal{T} \cdot (\alpha(s) + a\mathcal{N}(s)) > 0$ for $|a| < K$ and that $\mathcal{T} \cdot \alpha(s) \geq \sin(Ks)/K$ provided that $0 < s < \pi/2K$. Since the same result is true in the negative direction, this would prove the lemma.

Let $\phi(s)$ be the angle that $\dot\alpha(s)$ makes with $\mathcal{T}$; i.e., $\dot\alpha(s) \cdot \mathcal{T} = \cos\phi(s)$. Then $|\dot\phi(s)| \leq K$ so $0 \leq \phi(s) \leq Ks$. Let $x(s) = \alpha(s) \cdot \mathcal{T}$ and $r(s) = |\alpha(s) - \mathcal{T}(\alpha(s) \cdot \mathcal{T})|$. Then $\dot x(s) = \cos\phi(s) \geq \cos(Ks)$ and $|\dot r(s)| \leq \sin\phi(s) \leq \sin(Ks)$. Hence $x(s) \geq \sin(Ks)/K$ and $r(s) \leq (1 - \cos(Ks))/K$. Furthermore, $\mathcal{T} \cdot \mathcal{N}(s) \geq \cos(\phi(s) + \pi/2) = -\sin\phi(s) \geq -\sin(Ks)$, which proves the lemma. □

Note that it follows from the remark after Definition 3.5 that in a neighborhood of $\overline{x}_0 \in \partial\mathcal{D}_t$, we can write the boundary as a graph. We can now make a partition of unity into coordinate neighborhoods where this is true, which will be used to control the Sobolev constants:

LEMMA 3.8 *Suppose that $\mathcal{D}_t \subset \mathbb{R}^n$ with the boundary satisfying the condition in Definition* 3.5 *with $\iota_1 \geq 1/K_1$. Then there are $\chi_i \in C_0^\infty(\mathbb{R}^n)$, $i = 1, 2, \dots,$ such that*

$$\sum_p \chi_i = 1\,, \quad \sum_p |\partial^\alpha \chi_i| \leq C_\alpha K_1^{|\alpha|}\,, \quad \operatorname{diam}(\operatorname{supp}(\chi_i)) \leq \frac{1}{K_1}\,, \tag{3.19}$$

and for each $x \in \mathbb{R}^n$ there are at most 32^n i such that $\chi_i(x) \neq 0$. Furthermore, either $\operatorname{supp}(\chi_i) \cap \partial\mathcal{D}_t$ *is empty or is part of a graph contained in $\partial\mathcal{D}_t$, which after a rotation is given by*

$$\begin{gathered} x^n = f_i(x')\,, \quad (x', x^n) \in \mathbb{R}^n\,, \quad |x' - x_i'| \leq \iota_1\,, \\ |\partial f_i| \leq \varepsilon_1\,, \quad x_i \in \partial\mathcal{D}_t\,, \quad \mathcal{N}(x_i) = (0, \dots, 0, 1)\,. \end{gathered} \tag{3.20}$$

PROOF: Let $B(r, x)$ denote the ball of radius r centered at x. Let $\rho_1 = \iota_1/16$, and let $\{B(2\rho_1, x_i)\}$ be a cover of $\mathbb{R}^n$ such that $\{B(\rho_1, x_i)\}$ are disjoint. We define

$$\chi_i(x) = \frac{\chi(|x - x_i|/4\iota_1)}{\sum_i \chi(|x - x_i|/4\iota_1)}$$

where $\chi \in C_0^\infty$ satisfy $0 \leq \chi \leq 1$, $\chi(s) = 1$ when $s \leq 0$ and $\chi(s) = 0$ when $s \geq 2$. The number of disjoint balls of radius ρ_1 that can be contained in a ball of radius $16\rho_1$ is 16^n. Since $\operatorname{supp}(\chi_i)$ is contained in a ball of radius $8\rho_1$, this proves that for each $x \in \mathbb{R}^n$ there are at most 16^n i such that $\chi_i(x) \neq 0$. □

We will now estimate first-order derivatives of the extension of the normal to the interior. In Lemma 3.9 we estimate the time derivatives on the boundary. It is now natural to work in the Lagrangian coordinates. In Lemma 3.10 we estimate the geodesic extension of the normal to the interior in a neighborhood of the boundary.

LEMMA 3.9 *Let N be the unit normal to $\partial\Omega$, and let $h_{ab} = D_t g_{ab}/2$. On $[0, T] \times \partial\Omega$ we have*

$$D_t N_a = h_{NN} N_a\,, \quad D_t N^c = -2h^c_d N^d + h_{NN} N^c\,, \tag{3.21}$$

$$D_t \gamma^{ab} = -2\gamma^{ac}\gamma^{bd} h_{cd}\,. \tag{3.22}$$

The volume element on $\partial\Omega$ satisfies

$$D_t d\mu_\gamma = (\operatorname{tr} h - h_{NN}) d\mu_\gamma = (\operatorname{tr}\theta u \cdot N + \gamma^{ab}\overline{\nabla}_a \overline{u}_b) d\mu_\gamma\,. \tag{3.23}$$

PROOF: Since the right-hand sides of (3.21) restricted to $[0, T] \times \partial\Omega$ are independent of the extension of the normal to the interior, we may choose the foliation

$$N_a = \frac{\partial_a u}{\sqrt{g^{cd}\partial_c u \partial_d u}} \quad \text{where } \partial\Omega = \{y : u(y) = 0\}, \quad u < 0 \text{ in } \Omega\,.$$

Then

$$D_t N_a = -\frac{1}{2} N_a (D_t g^{cd}) N_c N_d = h_{NN} N_a$$

and

$$D_t N^a = D_t g^{ad} N_d = (D_t g^{ad}) N_d + g^{ad} D_t N_d = -2h^{ad} N_d + h_{NN} N^a\,,$$

which proves (3.21). (3.22) follows from

$$\begin{aligned} D_t \gamma^{ab} &= D_t (g^{ab} - N^a N^b) \\ &= D_t g^{ab} - (D_t N^a) N^b - N^a D_t N^b \\ &= -2h^{ab} + 2h^a_c N^c N^b + 2h^b_d N^d N^a - 2h_{kl} N^k N^l N^a N^b \\ &= (\delta^a_d - N^a N_d)(\delta^b_d - N^b N_d) h^{cd} = -2\gamma^a_d \gamma^b_d h^{cd}\,. \end{aligned}$$

Introducing coordinates, we have $d\mu_g = \sqrt{\det g}\, dy$ and $D_t \sqrt{\det g} = \sqrt{\det g} \operatorname{tr} h$. Now $d\mu_\gamma = \sqrt{\det g}(\sum N_n^2)^{-1/2} dS$, where dS is the Euclidean surface measure, and $D_t (\sum N_n^2)^{-1/2} = -(\frac{1}{2})(\sum N_n^2)^{-3/2} \sum 2N_n D_t N_n$. But $D_t N_n = h_{NN} N_n$, which proves that $D_t d\mu_\gamma = (\operatorname{tr} h - h_{NN}) d\mu_\gamma$. Now $\operatorname{tr} h - h_{NN} = \gamma^{ab}\nabla_a v_b = \gamma^{ab}\overline{\nabla}_a (N_b v \cdot N) + \gamma^{ab}\overline{\nabla}_a \overline{v}_b$. □

We will now extend the normal to a vector field defined and regular everywhere in the interior such that when $d(t, y) \le \iota_0/4$, it is the normal to the sets $\{y : (t, y) = d_0\}$, and in the interior it drops off to 0.

LEMMA 3.10 *Let ι_0 be as in Definition* 3.4*, and let $d(y) = \operatorname{dist}_g(y, \partial\Omega)$ be the geodesic distance in the metric g from y to $\partial\Omega$. Then the conormal $n = \nabla d$ to the sets $S_a = \partial\{y \in \Omega : d(y) = a\}$ satisfies*

$$|\nabla n| \le 2|\theta|_{L^\infty(\partial\Omega)} \quad \text{and} \quad |D_t n(t, y)| \le 6|h|_{L^\infty(\Omega)} \quad \text{when } d(y) < \frac{\iota_0}{2}\,. \tag{3.24}$$

PROOF: Now since $n \cdot n = 1$, it follows that $n \cdot \nabla n = 0$ and hence $(\nabla n) \cdot \nabla n + n \cdot \nabla^2 n = 0$; since $\theta = \nabla n$, we get $\nabla_N \theta = -\theta \cdot \theta$. It follows that $|\nabla_N |\theta|| \le |\theta|^2$. If $d(y) = \text{dist}_g(y, \partial\Omega) < \iota_0$, then there is a unique $\overline{y} \in \partial\Omega$ such that $d(y, \overline{y}) = \text{dist}_g(y, \partial\Omega)$. Hence we can introduce d and $\overline{y}$ as new variables so that $y = y(d, \overline{y})$. In these coordinates $\nabla_N = \partial/\partial d$, so with $f(d) = |\theta(d, \overline{y})|$ we get the inequality $|f'(d)| \le f(d)^2$ for each fixed $\overline{y}$. It's easy to see that $f(d) \le 2f(0)$ if $2df(0) \le 1$, and hence $|\theta(d, \overline{y})| \le 2|\theta|_{L^\infty(\partial\Omega)}$ if $2d|\theta|_{L^\infty(\partial\Omega)} \le 1$, which proves the first part of (3.24). We claim that

$$\nabla_N D_t d = h_{NN}\,, \quad \nabla_N \dot{n} + \theta \cdot \dot{n} = \theta \cdot h \cdot n \ \text{ if } \dot{n} = D_t n - h \cdot n\,. \tag{3.25}$$

In fact, since $g^{ab} N_a N_b = 1$, we have

$$0 = 2g^{ab} N_a D_t N_b + (D_t g^{ab}) N_a N_b = 2\nabla_N D_t d - 2h^{ab} N_a N_b\,,$$

and the first equation in (3.25) follows. Since

$$\begin{aligned}\nabla_c h_{NN} = \nabla_c(N^a N^b h_{ab}) &= N^a N^b \nabla_c h_{ab} + h_{ab} \nabla_c (N^a N^b)\\ &= N^a N^b \nabla_a h_{cb} + h_{ab}(N^b \theta_c^a + N^a \theta_c^b)\,,\end{aligned}$$

by differentiating the first equation in (3.25) we get

$$\begin{aligned}\nabla_N D_t N_c + \theta_c^e D_t N_e = \nabla_c N^e \nabla_e D_t d &= \nabla_c h_{NN}\\ &= \nabla_N (h_{cb} N^b) + \theta_c^e h_{eb} N^b + \theta_c^b N^a h_{ab}\,.\end{aligned}$$

With $\dot{n}_c = D_t N_c - h_{cb} N^b$, we get $\nabla_N \dot{n}_c + \theta_c^e \dot{n}_e = \theta_c^b N^a h_{ab}$, which proves the second part of (3.25),

$$|\nabla_N |\dot{n}|| \le |\theta||\dot{n}| + |\theta||h| \le K|\dot{n}| + K|h| \quad \text{if } K = 2|\theta|_{L^\infty(\partial\Omega)}\,. \tag{3.26}$$

Thus using the coordinates $y = y(d, \overline{y})$, we get

$$\begin{aligned}|\dot{n}(t, y)| &\le e^{d(t,y)K} |\dot{n}(t, \overline{y})| + \int_0^{d(t,y)} e^{(d(t,y)-s)K} K|h| ds\\ &\le e^{d(t,y)K} \big(|\dot{n}(t, \overline{y})| + Kd(t, y)|h|_{L^\infty(\Omega)}\big)\,,\end{aligned}$$

where $\overline{y} \in \partial\Omega$ satisfies $d(t, y) = \text{dist}_g(y, \overline{y})$. Since $Kd_0 \le \frac{1}{2}$, we get $|\dot{n}(t, y)| \le 2|\dot{n}(t, \overline{y})| + |h|_{L^\infty(\Omega)}$ when $d(t, y) \le d_0$. Since $D_t n(t, \overline{y}) = h_{NN}(t, \overline{y}) n(t, \overline{y})$ and $\dot{n} = D_t n - h \cdot n$, we get $|D_t n(t, y)| \le 6|h|_{L^\infty(\Omega)}$. ☐

LEMMA 3.11 *Let ι_0 be the reduced injectivity radius of the normal exponential map of $\partial\Omega$, and let d_0 be a fixed number such that $\iota_0/16 \le d_0 \le \iota_0/2$. Let $\eta \in C^\infty(\mathbb{R})$ be such that $\eta(s) = 1$ when $|s| \le 1/2$, $\eta(s) = 0$ when $|s| \ge 3/4$, $0 \le \eta(s) \le 1$, and $|\eta'(s)| \le 4$. Then the pseudo-Riemannian metric γ given by*

$$\gamma_{ab} = g_{ab} - \tilde{n}_a \tilde{n}_b\,, \quad \gamma^{ab} = g^{ab} - N^a N^b\,, \quad N^a = g^{ab} \tilde{n}_a\,, \tag{3.27}$$

$$\textit{where } \tilde{n}_c = \eta\left(\frac{d}{d_0}\right) \nabla_c d$$

satisfies

$$|\nabla\gamma|_{L^\infty(\Omega)} \le 256\left(|\theta|_{L^\infty(\partial\Omega)} + \frac{1}{\iota_0}\right) \quad \textit{and} \quad |D_t\gamma(t,y)| \le 64|h|_{L^\infty(\Omega)}\,. \tag{3.28}$$

PROOF: We have $\nabla_c\tilde{n}_a = -\eta(d/d_0)\nabla_c N_a - \eta'(d/d_0)N_a N_c/d_0$, which in view of (3.27) proves that $|\nabla\tilde{\gamma}| \le 2|\nabla n| + 16/d_0$, so the first inequality in (3.28) follows. Since $\gamma_{ab} = g_{ab} - \tilde{n}_a\tilde{n}_b$, where $\tilde{n}_b = \eta(d/d_0)N_b$, we have $D_t\tilde{n}_b = \eta(d/d_0)D_t N_b + \eta'(d/d_0)N_b D_t d/d_0$. Integrating the first equation in (3.25) gives $|D_t d(t,y)| \le |h_{NN}|_{L^\infty(\Omega)}d(t,y)$, and since $d/d_0 \le 1$ in the support of $\eta(d/d_0)$, this proves the second part of (3.28). □

Note that in a neighborhood of $\partial\Omega$, $\tilde{\gamma}$ is just the induced metric on the surfaces $S_\lambda = \{y \in \mathbb{R}^n : d(y,\partial\Omega) = \lambda\}$, and in the interior $\tilde{\gamma}$ is just the interior metric g.

4 Estimates for the Projection of a Tensor to the Tangent Space of the Boundary

DEFINITION 4.1 Let N be the unit normal to $\partial\Omega$, and let $\nabla_N = N^j\nabla_j$ be the normal derivative. Let $d(t,y) = \mathrm{dist}_g(y,\partial\Omega)$ be the geodesic distance from y to $\partial\Omega$, and let $N_i = \nabla_i d$ be the geodesic extension of the normal to the interior. Let $\theta_{ij} = \nabla_i N_j = \nabla_i\nabla_j d$ be the second fundamental form of $\partial\Omega$. Let $\gamma_i^j = \delta_i^j - N_i N^j$, and if $I = (i_1,\dots,i_r)$ and $J = (j_1,\dots,j_r)$ are multi-indices of length $|I| = r$, set $\gamma_I^J = \gamma_{i_1}^{j_1}\dots\gamma_{i_r}^{j_r}$ and $N^I = N^{i_1}\cdots N^{i_r}$. If β is a $(0,r)$ tensor in Ω, define the projection $\Pi\beta$ to a tensor on $\partial\Omega$ to be $(\Pi\beta)_I = \gamma_I^J\beta_J$. Let $\overline{\nabla}\beta = \Pi\nabla\beta$ denote the tangential covariant derivative. This is the intrinsic covariant derivative of $\partial\Omega$ if β is already tangential to $\partial\Omega$, i.e., if $\beta_{i_1\cdots i_k\cdots i_r}N^k = 0$, $k = 1,\dots,r$; see Lemma 3.2. Furthermore, let ∇^r and $\overline{\nabla}^r$ be the operators that in components are given by $\nabla_I^r = \nabla_{i_1}\cdots\nabla_{i_r}$ and $\overline{\nabla}_I^r = \overline{\nabla}_{i_1}\cdots\overline{\nabla}_{i_r}$, respectively.

DEFINITION 4.2 Let α be a $(0,s)$ tensor and β a $(0,r)$ tensor. We will let $\alpha\tilde{\otimes}\beta$ denote some partial symmetrization of the tensor product $\alpha\otimes\beta$, i.e., a sum over some subset of the permutations of the indices divided by the number of permutations in that subset. In each situation there is of course a specific subset, but in our estimates it does not matter which one, so to simplify the exposition we do not write out the exact permutations. Similarly, we let $\alpha\,\tilde{\cdot}\,\beta$ denote a partial symmetrization of the dot product $\alpha\cdot\beta$, which in turn is defined to be a contraction of the last index of α with the first index of β: $(\alpha\cdot\beta)_{i_1\cdots i_{r+s-2}} = g^{ij}\alpha_{i_1\cdots i_{s-1}i}\beta_{j i_s\cdots i_{r+s-2}}$.

The simple observation that will help us is that if $q = 0$ on $\partial\Omega$, then the projection of the tensor $\nabla^2 q$ to the boundary will only contain first-order derivatives of q and will contain all components of the second fundamental form. In fact,

$$\Pi\nabla^2 q = \overline{\nabla}^2 q + \theta\nabla_N q\,, \tag{4.1}$$

where the tangential derivatives $\overline{\nabla}^2 q = 0$ on the boundary. To prove (4.1) we note that

$$\gamma_j^k \nabla_i \gamma_k^l = -\gamma_j^k \nabla_i (N_k N^l) = -\gamma_j^k \theta_{ik} N^l - \gamma_j^k N_k \theta_i^l = -\theta_{ij} N^l \,, \tag{4.2}$$

so

$$\begin{aligned} \overline{\nabla}_i \overline{\nabla}_j q = \gamma_i^{i'} \gamma_j^{j'} \nabla_{i'} \gamma_{j'}^{j''} \nabla_{j''} q &= \gamma_i^{i'} \gamma_j^{j'} \gamma_{j'}^{j''} \nabla_{i'} \nabla_{j''} q + \gamma_i^{i'} \gamma_j^{j'} (\nabla_{i'} \gamma_{j'}^{j''}) \nabla_{j''} q \\ &= \gamma_i^{i'} \gamma_j^{j'} \nabla_{i'} \nabla_{j'} q - \theta_{ij} \nabla_N q \,. \end{aligned} \tag{4.3}$$

We now want to find a higher-order version of (4.1). One way to understand why there should be such a formula if $q = 0$ on $\partial\Omega$ is to expand q in a Taylor series in the geodesic distance d from the boundary. If $q = 0$ on $\partial\Omega$, then $q/d \sim \nabla_N q$ is a well-defined function in a neighborhood of $\partial\Omega$, and hence we can write

$$\Pi \nabla^r q = \Pi \nabla^r \left(d \frac{q}{d} \right) = \sum_{s=0}^{r} \binom{r}{s} \Pi (\nabla^{r-s} d) \widetilde{\otimes} \Pi \nabla^s \left(\frac{q}{d} \right).$$

Since, however, $d = \Pi \nabla d = 0$ on $\partial\Omega$ and $\nabla^2 d = \theta$, we obtain

$$\Pi \nabla^r q = \sum_{s=0}^{r-2} \binom{r}{s} \Pi (\nabla^{r-2-s} \theta) \widetilde{\otimes} \Pi \nabla^s \left(\frac{q}{d} \right). \tag{4.4}$$

PROPOSITION 4.3 *On $\partial\Omega$ we have*

$$\begin{aligned} &\left| (\Pi \nabla^r) q - \overline{\nabla}^r q - \nabla_N q \, \overline{\nabla}^{r-2} \theta - \sum_{s=1}^{r-2} \binom{r}{s} (\overline{\nabla}^{r-2-s} \theta) \widetilde{\otimes} (\overline{\nabla}^s \nabla_N q) \right| \leq \\ &\qquad C \sum_{\substack{r_0 + r_1 + \cdots + r_k + \ell = r - k \\ k - \ell = 0 \mod 2, \; k \geq \ell \geq 0, \; k \geq 2}} |\overline{\nabla}^{r_1} \theta| \cdots |\overline{\nabla}^{r_k} \theta| |\overline{\nabla}^{r_0} \nabla_N^\ell q| \end{aligned} \tag{4.5}$$

and

$$|\overline{\nabla}^{r_0} \nabla_N^\ell q| \leq C \sum_{\tilde{r}_0 + \tilde{r}_1 + \cdots + \tilde{r}_k = r_0 + \ell - k} |\overline{\nabla}^{\tilde{r}_1} \theta| \cdots |\overline{\nabla}^{\tilde{r}_k} \theta| |\nabla^{\tilde{r}_0} q| \tag{4.6}$$

$$|\nabla^{r_0} q| \leq C \sum_{\tilde{r}_0 + \ell + \tilde{r}_1 + \cdots + \tilde{r}_k = r_0 - k} |\overline{\nabla}^{\tilde{r}_1} \theta| \cdots |\overline{\nabla}^{\tilde{r}_k} \theta| |\overline{\nabla}^{\tilde{r}_0} \nabla_N^\ell q| \,, \tag{4.7}$$

where the sums are over all positive integers $r_i \geq 0$, $\tilde{r}_i \geq 0$, and $k, \ell \geq 0$.

PROPOSITION 4.4 *We have*

$$\begin{aligned} &(\Pi \nabla^r)_J q = \\ &\qquad \sum_{r_0 + r_1 + \cdots + r_k + \ell = r - k} c_{k\ell J I_0 \cdots I_k}(g) (\overline{\nabla}^{r_1} \theta)_{I_1} \otimes \cdots \otimes (\overline{\nabla}^{r_k} \theta)_{I_k} \otimes \overline{\nabla}^{r_0}_{I_0} \big(\nabla_N^\ell q \big) \,, \end{aligned} \tag{4.8}$$

where the sum is over positive integers $k, \ell, m \geq 0$, $k - \ell = 2m \geq 0$, $r_i \geq 0$, and all permutations $(I_0, I_1, \ldots, I_k)$ of $(J, i_1, \ldots, i_{2m})$. Here

$$c_{klJ I_0 \cdots I_k}(g) = d_{k\ell m J I_0 \cdots I_k} g^{i_1 i_2} \cdots g^{i_{2m-1} i_{2m}} \tag{4.9}$$

denotes contractions over m indices. Furthermore,

$$(\Pi\nabla^r)q = \widetilde{\nabla}^r q + \sum_{s=0}^{r-2}\binom{r}{s}(\overline{\nabla}^{r-2-s}\theta)\widetilde{\otimes}(\overline{\nabla}^s\nabla_N q) + F\,, \tag{4.10}$$

where F is of the form in the right-hand side of (4.8) *but with $k \geq 2$ in the sum.*

Remark. Propositions 4.1 and 4.2 apply to the function q being replaced by the $(0, s)$ tensor α as well if the projections and tangential and normal derivatives are correctly interpreted: Only the first r indices should be projected. This will be explained later in this section; see Proposition 4.11.

The proof of Propositions 4.3 and 4.4 consists of turning projections onto the tangential and normal components into tangential derivatives of normal derivatives. The basic idea is that any derivative ∇^r of order r can be expressed as a sum of combinations of tangential derivatives $\overline{\nabla}$ and normal derivatives ∇_N of total order at most $s \leq r$, and similarly any combination of normal and tangential derivatives of total order r can be expressed as a sum of derivatives ∇^s for $s \leq r$. Since the coefficients of both the normal derivative and of the projection involved in the tangential derivative are made up out of the normal, it follows that the coefficients in expressing a derivative ∇^r in terms of normal ∇_N and tangential $\overline{\nabla}$ derivatives will consist of derivatives of the normal, i.e., derivatives of the second fundamental form θ. Whenever a derivative in, say (4.5)–(4.8), falls on the normal, it produces a new factor θ. At the same time, the total number of derivatives involved has gone down by 1, so the total number of derivatives in expressions (4.5)–(4.8) goes down by 1 for each new factor of θ. This simple observation will be used to prove (4.6), (4.7), and (4.8). The more detailed information in (4.5) and (4.10) formally follows from (4.4) and the above argument.

The key to turn tangential and normal components into tangential derivatives of normal components is Lemma 4.5 below. In Lemma 4.6 it is then expressed in a form that is more directly adapted to the situation in Propositions 4.3 and 4.4.

LEMMA 4.5 *Suppose that S is a $(0, r+\ell+s)$ tensor that is symmetric with respect to the first $r+\ell$ indices. Let*

$$S^{r,\ell}_{i_1\cdots i_{r+s}} = (\Pi^{r,\ell}S)_{i_1\cdots i_{r+s}} = \gamma_{i_1}^{j_1}\cdots\gamma_{i_r}^{j_r}N^{j_{r+1}}\cdots N^{j_{r+\ell}}S_{j_1\cdots j_{r+\ell}i_{r+1}\cdots i_{r+s}} \tag{4.11}$$

be the projection of the first indices onto r tangential and ℓ normal components. Then

$$\Pi^{r+1,0}\nabla\Pi^{r,\ell}S = \Pi^{r+1,\ell}\nabla S - r\,\theta\widetilde{\otimes}\Pi^{r-1,\ell+1}S + \ell\,\theta\cdot\Pi^{r+1,\ell-1}S \tag{4.12}$$

where

$$\big(\theta\widetilde{\otimes}\Pi^{r-1,\ell+1}S\big)_{i_0 i_1\cdots i_r i_{r+1}\cdots i_{r+s}} = \frac{1}{r}\sum_{p=1}^{r}\theta_{i_0 i_p}(\Pi^{r-1,\ell+1}S)_{I_p i_{r+1}\cdots i_{r+s}}\,, \tag{4.13}$$

$$\big(\theta\cdot\Pi^{r+1,\ell-1}S\big)_{i_0 i_1\cdots i_r i_{r+1}\cdots i_{r+s}} = \theta_{i_0}^{j}(\Pi^{r+1,\ell-1}S)_{j\,i_1\cdots i_r i_{r+1}\cdots i_{r+s}}\,, \tag{4.14}$$

where $I_p = (i_1, \dots, i_{p-1}, i_{p+1}, \dots, i_r)$.

PROOF OF LEMMA 4.5: To simplify notation, we assume that $s = 0$. Now

$$S^{r,\ell}_{i_1\cdots i_r} = \gamma_I^J N^{J'} S_{JJ'} = \gamma_{i_1}^{j_1} \cdots \gamma_{i_r}^{j_r} N^{j_{r+1}} \cdots N^{j_{r+\ell}} S_{j_1\cdots j_{r+\ell}} ,$$

where $I = (i_1, \dots, i_r)$ and $J = (j_1, \dots, j_r)$ are multi-indices of length r, and $J' = (j_{r+1}, \dots, j_{r+\ell})$ is a multi-index of length ℓ. Now

$$\begin{aligned}\overline{\nabla}_{i_0} S^{r,\ell}_{i_1\cdots i_r} &= \gamma_{i_0}^{j_0} \gamma_I^L \nabla_{j_0} (\gamma_L^J N^{J'} S_{JJ'}) \\ &= \gamma_{i_0}^{j_0} \gamma_I^J N^{J'} \nabla_{j_0} S_{JJ'} + \gamma_{i_0}^{j_0} \gamma_I^L (\nabla_{j_0} \gamma_L^J) N^{J'} S_{JJ'} + \gamma_{i_0}^{j_0} \gamma_I^J (\nabla_{j_0} N^{J'}) S_{JJ'} .\end{aligned}$$

By (4.2)

$$\gamma_I^L \nabla_{i_0} \gamma_L^J = -\sum_{p=1}^{r} \theta_{i_0 i_p} \gamma_{I_p}^{J_p} N^{j_p} ,$$

where $I_p = (i_1, \dots, i_{p-1}, i_{p+1}, \dots, i_r)$ and $J_p = (j_1, \dots, j_{p-1}, j_{p+1}, \dots, j_r)$. Furthermore,

$$\nabla_{i_0} N^{J'} = \sum_{p=r+1}^{r+\ell} \theta_{i_0}^{j_p'} N^{J_p'} ,$$

where $J_p' = (j_{r+1}, \dots, j_{p-1}, j_{p+1}, \dots, j_{r+\ell})$. If we now assume that S is symmetric, the notation simplifies a bit and we obtain the lemma. □

Now we want to apply Lemma 4.5 to $S = \nabla^{r+\ell} q$. Since in geodesic coordinates $\nabla_N N = 0$, it follows that

(4.15) $$[\nabla_N, \Pi] = 0 , \quad \nabla_N^\ell = N^{i_1} \cdots N^{i_\ell} \nabla_{i_1} \cdots \nabla_{i_\ell} .$$

LEMMA 4.6 *Let*

(4.16) $$S^{r,\ell} = \Pi^{r,\ell} \nabla^{r+\ell} q = \nabla_N^\ell \Pi \nabla^r q .$$

Then

(4.17) $$S^{r+1,\ell} = \overline{\nabla} S^{r,\ell} + r\theta \widetilde{\otimes} S^{r-1,\ell+1} - \ell\theta \cdot S^{r+1,\ell-1} .$$

Furthermore,

(4.18) $$S^{r,\ell} - \overline{\nabla}^r S^{0,\ell} = \sum_{k=0}^{r-2} \overline{\nabla}^{r-2-k} \big((k+1)\theta \widetilde{\otimes} S^{k,\ell+1} - \ell\theta \cdot S^{k+2,\ell-1} \big)$$

and

(4.19) $$\begin{aligned} S^{r,\ell} - \overline{\nabla}^r S^{0,\ell} = &\sum_{m=0}^{r-2} \binom{r}{m} (\overline{\nabla}^m \theta) \widetilde{\otimes} \overline{\nabla}^{r-2-m} S^{0,\ell+1} \\ &- \sum_{m=0}^{r-2} \ell \binom{r-1}{m} (\overline{\nabla}^m \theta) \tilde{\cdot} \overline{\nabla}^{r-m} S^{0,\ell-1} \end{aligned}$$

$$+\sum a_{r_1r_2k}(\overline{\nabla}^{r_1}\theta)\widetilde{\otimes}(\overline{\nabla}^{r_2}\theta)\widetilde{\otimes}\overline{\nabla}^{r-4-r_1-r_2-k}S^{k,\ell+2}$$

$$+(\ell+1)\sum b_{r_1r_2k}(\overline{\nabla}^{r_1}\theta)\widetilde{\otimes}(\overline{\nabla}^{r_2}\theta)\,\tilde{\cdot}\,\overline{\nabla}^{r-2-r_1-r_2-k}S^{k,\ell}$$

$$+\ell\sum c_{r_1r_2k}(\overline{\nabla}^{r_1}\theta)\,\tilde{\cdot}\,(\overline{\nabla}^{r_2}\theta)\widetilde{\otimes}\overline{\nabla}^{r-2-r_1-r_2-k}S^{k,\ell}$$

$$+\ell(\ell-1)\sum d_{r_1r_2k}(\overline{\nabla}^{r_1}\theta)\,\tilde{\cdot}\,(\overline{\nabla}^{r_2}\theta)\,\tilde{\cdot}\,\overline{\nabla}^{r-r_1-r_2-k}S^{k,\ell-2}\,,$$

where the sums are over all integers $r_1, r_2, k \geq 0$ *such that all exponents of differentiation also are* ≥ 0.

PROOF: (4.17) follows from (4.12). Now by repeated use of (4.17)

$$\begin{aligned}
S^{r,\ell} &= \overline{\nabla}S^{r-1,\ell}+(r-1)\theta\widetilde{\otimes}S^{r-2,\ell+1}-\ell S^{r,\ell-1}\\
&= \overline{\nabla}\big(\overline{\nabla}S^{r-2,\ell}+(r-2)\theta\widetilde{\otimes}S^{r-3,\ell+1}-\ell\theta\cdot S^{r-1,\ell-1}\big)\\
&\quad +(r-1)\theta\widetilde{\otimes}S^{r-2,\ell+1}-\ell\,\theta\cdot S^{r,\ell-1}\\
&= \cdots = \overline{\nabla}^r S^{0,\ell}+\sum_{k=0}^{r-2}\overline{\nabla}^{r-2-k}\big((k+1)\theta\widetilde{\otimes}S^{k,\ell+1}-\ell\theta\cdot S^{k+2,\ell-1}\big)\,,
\end{aligned}$$

which proves (4.18). To proceed further, we must use (4.18) twice. In the right-hand side of (4.18) we use (4.18) to write $S^{k,\ell+1}$ as $\overline{\nabla}^k S^{k,\ell+1}$ plus terms involving one factor of θ, and write $S^{k+2,\ell-1}$ as $\overline{\nabla}^{k+2}S^{0,\ell-1}$ plus terms involving one factor of θ.

Let us first calculate the term involving one factor of θ. By Leibniz' rule we have

$$\begin{aligned}
&\sum_{k=0}^{r-2}\overline{\nabla}^{r-2-k}\big((k+1)\theta\widetilde{\otimes}\overline{\nabla}^k S^{0,\ell+1}-\ell\theta\cdot\overline{\nabla}^k S^{0,\ell-1}\big)\\
&\quad=\sum_{k=0}^{r-2}\sum_{m=0}^{r-2-k}\binom{r-2-k}{m}(k+1)(\overline{\nabla}^m\theta)\widetilde{\otimes}\overline{\nabla}^{r-2-m}S^{0,\ell+1}\\
&\qquad-\ell\sum_{k=0}^{r-2}\sum_{m=0}^{r-2-k}\binom{r-2-k}{m}(\overline{\nabla}^m\theta)\,\tilde{\cdot}\,\overline{\nabla}^{r-2-m}S^{0,\ell-1}\\
&\quad=\sum_{m=0}^{r-2}\binom{r}{m}(\overline{\nabla}^m\theta)\widetilde{\otimes}\overline{\nabla}^{r-2-m}S^{0,\ell+1}-\ell\sum_{m=0}^{r-2}\binom{r-1}{m}(\overline{\nabla}^m\theta)\,\tilde{\cdot}\,\overline{\nabla}^{r-m}S^{0,\ell-1}\,,
\end{aligned}$$

since $\sum_{k=0}^{r-2-m}(k+1)\binom{r-2-k}{m}=\binom{r}{m}$ and $\sum_{k=0}^{r-2-m}\binom{r-2-k}{m}=\binom{r-1}{m}$. This explains the terms involving one factor of θ in the first row of (4.19). Using (4.18) and Leibniz' rule, it is easy to see that the term involving two factors of θ has to be of the form in (4.19). □

PROOF OF PROPOSITIONS 4.3 AND 4.4: The proof is just an application of Lemma 4.6: (4.5) follows from (4.8). (4.8) follows by induction from (4.17), noticing that the total order of the tensor goes down by 1 for each new factor of θ. (4.10) follows from (4.19). (4.6) and (4.7) follow from the same argument. □

Using (4.17) and (4.18), one can show that

$$\Pi\nabla^2 q=\overline{\nabla}^2 q+\theta\nabla_N q\,, \tag{4.20}$$

$$\Pi\nabla^3 q=\overline{\nabla}^3 q-2\theta\tilde{\otimes}(\theta\tilde{\cdot}\overline{\nabla}q)+(\overline{\nabla}\theta)\nabla_N q+3\theta\tilde{\otimes}\overline{\nabla}\nabla_N q\,, \tag{4.21}$$

$$\begin{aligned}\Pi\nabla^4 q&=\overline{\nabla}^4 q-\theta\tilde{\otimes}\big(5(\overline{\nabla}\theta)\tilde{\cdot}\overline{\nabla}q+8\theta\tilde{\cdot}\overline{\nabla}^2 q\big)-2(\overline{\nabla}\theta)\tilde{\otimes}(\theta\tilde{\cdot}\overline{\nabla}q)\\&\quad+(\overline{\nabla}^2\theta)\nabla_N q+4(\overline{\nabla}\theta)\tilde{\otimes}\overline{\nabla}\nabla_N q+6\theta\tilde{\otimes}\overline{\nabla}^2\nabla_N q\\&\quad-3\theta\tilde{\otimes}(\theta\tilde{\cdot}\theta)\nabla_N q+3\theta\tilde{\otimes}\theta\nabla_N^2 q\,.\end{aligned} \tag{4.22}$$

Since $\nabla_N=N\cdot\nabla$, $\Pi\nabla N=\nabla N=\theta$, $\Pi\nabla^2 N=\Pi\nabla\theta=\Pi\nabla\Pi\theta=\overline{\nabla}\theta$, and $\nabla_N\theta=\Pi N\cdot\nabla^2 q=-\Pi(\nabla N)\cdot(\nabla N)=-\theta\cdot\theta$ (see (4.51)), we get

$$\overline{\nabla}q=\Pi\nabla q\,, \tag{4.23}$$

$$\overline{\nabla}\nabla_N q=\Pi N\cdot\nabla^2 q+\theta\cdot\nabla q\,, \tag{4.24}$$

$$\nabla_N^2 q=N\cdot(N\cdot\nabla^2 q)\,, \tag{4.25}$$

$$\overline{\nabla}^2 q=\Pi\nabla^2 q-\theta N\cdot\nabla q\,, \tag{4.26}$$

$$\begin{aligned}\overline{\nabla}^2\nabla_N q&=\Pi N\cdot\nabla^3 q+2\theta\tilde{\cdot}\Pi\nabla^2 q\\&\quad+(\overline{\nabla}\theta)\cdot\Pi\nabla q-\theta\cdot\theta N\cdot\nabla q-\theta N\cdot(N\cdot\nabla^2 q)\,,\end{aligned} \tag{4.27}$$

where in (4.27) we used that $\overline{\nabla}^2\nabla_N q=\Pi\nabla^2\nabla_N q-\theta\nabla_N^2 q$.

PROPOSITION 4.7 *Suppose that $q=0$ on $\partial\Omega$ and $0\le r\le 4$ or $r\ge(n-1)/2+2$. Let $L^p=L^p(\partial\Omega)$, and suppose that $\iota_1\ge 1/K_1$, where ι_1 is as in Definition 3.4. Then for $m=0,1$ and any $\varepsilon>0$, we have*

$$\begin{aligned}&\big\|\Pi\nabla^r q-(\nabla_N q)\overline{\nabla}^{r-2}\theta\big\|_{L^2}\\&\le\varepsilon\|\nabla_N q\|_{L^\infty}\|\overline{\nabla}^{r-2}\theta\|_{L^2}+C(1/\varepsilon)\sum_{k=1}^{r-1}\|\theta\|_{L^\infty}^k\|\nabla^{r-k}q\|_{L^2}\\&\quad+C\big(K_1,1/\varepsilon,\|\theta\|_{L^\infty}\big)\Bigg(\|\theta\|_{L^\infty}+\sum_{0\le s\le r-2-m}\|\overline{\nabla}^s\theta\|_{L^2}\Bigg)\\&\quad\sum_{0\le s\le r-2+m}\|\nabla^s q\|_{L^2}\end{aligned} \tag{4.28}$$

where the second line drops out if $r \le 4$.

PROOF OF PROPOSITION 4.7 FOR $r \le 4$: We want to prove (4.28) for $r = 4$, since the proof for $r \le 3$ is simpler and follows in the same way. By (4.22) we have, if $q = 0$ on $\partial\Omega$,

$$\Pi\nabla^4 q = \big(\overline{\nabla}^2\theta\big)\nabla_N q + 4\big(\overline{\nabla}\theta\big)\widetilde{\otimes}\overline{\nabla}\nabla_N q + 6\theta\widetilde{\otimes}\overline{\nabla}^2\nabla_N q \\ - 3\theta\widetilde{\otimes}(\theta\cdot\theta)\nabla_N q + 3\theta\widetilde{\otimes}\theta\nabla_N^2 q\,.$$

The only problematic term can be controlled by Lemma A.1 (here $L^p = L^p(\partial\Omega)$):

$$\begin{aligned}\big\| |\overline{\nabla}\theta||\overline{\nabla}\nabla_N q|\big\|_{L^2} &\le \|\overline{\nabla}\theta\|_{L^4}\|\overline{\nabla}\nabla_N q\|_{L^4} \\ &\le C\|\theta\|_{L^\infty}^{1/2}\|\overline{\nabla}^2\theta\|_{L^2}^{1/2}\|\nabla_N q\|_{L^\infty}^{1/2}\|\overline{\nabla}^2\nabla_N q\|_{L^2}^{1/2} \\ &\le C2^{-1}\varepsilon\|\nabla_N q\|_{L^\infty}\|\overline{\nabla}^2\theta\|_{L^2} + C2^{-1}\varepsilon^{-1}\|\theta\|_{L^\infty}\|\overline{\nabla}^2\nabla_N q\|_{L^2} \\ &\quad \text{for any } \varepsilon > 0.\end{aligned}$$

By (4.27), since $\Pi\nabla q = 0$ on $\partial\Omega$,

$$\|\overline{\nabla}^2\nabla_N q\|_{L^2} \le \|\nabla^3 q\|_{L^2} + 3\|\theta\|_{L^\infty}\|\nabla^2 q\|_{L^2} + \|\theta\|_{L^\infty}^2\|\nabla q\|_{L^2}\,.$$

□

The basic inequalities that we will use on the boundary for the proof of Theorem 4.5 in general can be summarized in the following:

LEMMA 4.8 *Let* $L^p = L^p(\partial\Omega)$ *and let* $t = r - 2$. *Then if* $t - m \ge s$,

$$\|\overline{\nabla}^s\alpha\|_{L^{2t/(s+m)}} \le C\|\alpha\|_{L^{2t/m}}^{1-s/(t-m)}\|\overline{\nabla}^{t-m}\alpha\|_{L^2}^{s/(t-m)}, \quad m \ge 0,\ t-m \ge s\,, \tag{4.29}$$

$$\|\nabla^s\alpha\|_{L^{2t/(s-m)}} \le \\ C(K_1)\sum_{\ell=s}^{t+m}\|\nabla^\ell\alpha\|_{L^2} \quad \textit{if } t \ge \frac{n-1}{2},\ s-m\ge 0,\ t+m \ge s\,, \tag{4.30}$$

where K_1 *is a constant such that* $\iota_1 \ge 1/K_1$ *and* ι_1 *is as in Definition* 3.4. *Furthermore,*

$$\big\||\overline{\nabla}^{r_1}\theta|\cdots|\overline{\nabla}^{r_k}\theta|\big\|_{L^p} \le C\|\theta\|_{L^\infty}^{k-1}\|\overline{\nabla}^{r_1+\cdots+r_k}\theta\|_{L^p}\,, \tag{4.31}$$

$$\|\overline{\nabla}^s\theta\|_{L^{2t/(s+m)}} \le C\|\theta\|_{L^{2t/m}}^{1-(s+m)/t}\|\overline{\nabla}^{t-m}\theta\|_{L^2}^{(s+m)/t}, \quad m \ge 0\,. \tag{4.32}$$

Furthermore, we have for every $\varepsilon > 0$ *if* $1 \le s \le t$

$$\big\||\overline{\nabla}^{t-s}\theta||\overline{\nabla}^s\nabla_N q|\big\|_{L^2} \le \\ \varepsilon\|\nabla_N q\|_{L^\infty}\|\overline{\nabla}^t\theta\|_{L^2} + C\varepsilon^{-(t-s)/s}\|\theta\|_{L^\infty}\|\overline{\nabla}^t\nabla_N q\|_{L^2}\,, \tag{4.33}$$

and if $0 \le m \le s \le t - m$,

$$\big\| |\overline{\nabla}^{t-s}\theta||\nabla^s q| \big\|_{L^2} \le \|\overline{\nabla}^{t-s}\theta\|_{L^{2t/(t-s+m)}} \|\nabla^s q\|_{L^{2t/(s-m)}} \le C(K_1)\|\theta\|_{L^{2t/m}}^{1-(t-s+m)/t} \|\overline{\nabla}^{t-m}\theta\|_{L^2}^{(t-s+m)/t} \sum_{\ell=s}^{t+m} \|\nabla^\ell q\|_{L^2} . \tag{4.34}$$

PROOF OF LEMMA 4.8: Equations (4.29) and (4.31)–(4.33) are just the interpolation inequality (A.4) in Lemma A.1. For the proof of (4.31), one first uses Hölder's inequality. (4.30), on the other hand, is a special case of Sobolev's lemma, Lemma A.2, which by the remark after the lemma holds with the covariant differentiation of the interior restricted to the boundary. By Hölder's inequality and (4.29) with $m = 0$:

$$\begin{aligned} \big\| |\overline{\nabla}^{t-s}\theta||\overline{\nabla}^s \nabla_N q| \big\|_{L^2} &\le \|\nabla^{t-s}\theta\|_{L^{2t/(t-s)}} \|\overline{\nabla}^s \nabla_N q\|_{L^{2t/s}} \\ &\le C\|\theta\|_{L^\infty}^{s/t} \|\overline{\nabla}^t\theta\|_{L^2}^{1-s/t} \|\nabla_N q\|_{L^\infty}^{1-s/t} \|\overline{\nabla}^t \nabla_N q\|_{L^2}^{s/t} \\ &\le \varepsilon \|\nabla_N q\|_{L^\infty} \|\overline{\nabla}^t\theta\|_{L^2} + C\varepsilon^{-(t-s)/s} \|\theta\|_{L^\infty} \|\overline{\nabla}^t \nabla_N q\|_{L^2} \\ &\quad \text{for any } \varepsilon > 0, \end{aligned}$$

which proves (4.33). (4.34) follows from Hölder's inequality and (4.30) applied to $\alpha = q$ and (4.32). □

PROOF OF PROPOSITION 4.7 IN THE CASE $r \ge 5$: The proof is an application of Proposition 4.3 and Lemma 4.8. Since $q = 0$, the term $\overline{\nabla}^r q = 0$ on the left of (4.5) and the terms on the right with $\ell = 0$ vanishes as well so $\ell \ge 2$ in the right sum. Each term in the sum on the left of (4.5) can be estimated using (4.33). Then we can use (4.6) to estimate $\|\theta\|_{L^\infty} \|\overline{\nabla}^{r-2}\nabla_N q\|_{L^2}$ by $\|\theta\|_{L^\infty}\|\nabla^{r-1} q\|_{L^2}$ plus a sum of terms of the form

$$\|\theta\|_{L^\infty} \big\| |\overline{\nabla}^{r_2}\theta| \cdots |\overline{\nabla}^{r_k}\theta||\nabla^{r_0} q| \big\|_{L^2} , \qquad r_0 + r_2 + \cdots + r_k = r - k,\ k \ge 2 . \tag{4.35}$$

Similarly, if we use (4.6), we can estimate the terms in the right of (4.5) (the second line of (4.5)) by

$$\big\| |\overline{\nabla}^{r_1}\theta| \cdots |\overline{\nabla}^{r_k}\theta||\nabla^{r_0} q| \big\|_{L^2} , \quad r_0 + r_1 + \cdots + r_k = r - k,\ k \ge 2 . \tag{4.36}$$

Now a typical term looks like

$$\|\theta\|_{L^\infty} \big\| |\overline{\nabla}^{r-2-s}\theta||\nabla^s q| \big\|_{L^2} ,$$

which can be estimated by (4.34) with $m = 0, 1$. The general term is not much harder: Using Hölder's inequality and (4.31), we see that we must estimate

$$\|\theta\|_{L^\infty}^{k-1} \|\overline{\nabla}^{r'}\theta\|_{L^p} \|\nabla^{r_0} q\|_{L^{p'}} , \quad r_0 + r' = r - k,\ k \ge 2 , \tag{4.37}$$

for some $1/p + 1/p' = 1/2$, which are to be determined. If $r' = 0$, then we can take $p = \infty$, so we may assume that $r' \geq 1$. Similarly, we may assume that $r_0 \geq 2$, since if $r_0 = 1$, we can take $p' = \infty$. We pick

$$p = \frac{2(r-2)}{r'+m}, \quad p' = \frac{2(r-2)}{r-2-r'-m}, \tag{4.38}$$

and use (4.34) with $m = 0, 1$. □

Note that Propositions 4.3 and 4.4 apply to q being replaced by the $(0, t)$ tensor α as well if the projections and tangential and normal derivatives are correctly interpreted. Only the first r indices should be projected; i.e., all indices referring to θ should be projected as well as the ones referring to differentiation of α, but the ones referring to α itself should not. So we should replace $\Pi\nabla^r$ by $\Pi^{r,0}\nabla^r$, and we should replace $\overline{\nabla}^r$ when applied to α by $\overline{\overline{\nabla}}^r = \Pi^{r,0}\nabla\Pi^{r-1,0}\nabla\cdots\Pi^{2,0}\nabla\Pi^{1,0}\nabla$. (One should keep the old definition of $\overline{\nabla}^r\theta$, since all these indices are projected over.) In components, this means the following:

DEFINITION 4.9 Let

$$\begin{aligned}(\Pi^{r,0}\nabla^r)_{i_1\cdots i_r}\alpha_{i_{r+1}\cdots i_{+t}} &= \gamma_{i_1}^{j_1}\cdots\gamma_{i_r}^{j_r}\nabla_{j_1}\cdots\nabla_{j_r}\alpha_{i_{r+1}\cdots i_{r+t}}, \\ \nabla_N\alpha_{i_1\cdots i_t} &= N^k\nabla_k\alpha_{i_1\cdots i_t},\end{aligned} \tag{4.39}$$

and

$$\begin{aligned}&\left(\overline{\overline{\nabla}}^r\right)_{i_1\cdots i_r}\alpha_{i_{r+1}\cdots i_{r+t}} = \\ &\quad \gamma_{i_1}^{j_1}\cdots\gamma_{i_r}^{j_r}\nabla_{j_1}\Big(\gamma_{j_2}^{k_2}\cdots\gamma_{j_r}^{k_r}\nabla_{k_2}\Big(\cdots\gamma_{m_{r-2}}^{N_{r-2}}\gamma_{m_{r-1}}^{N_{r-1}}\gamma_{m_r}^{N_r}\nabla_{N_{r-2}} \\ &\qquad \Big(\gamma_{N_{r-1}}^{o_{r-1}}\gamma_{N_r}^{o_r}\nabla_{o_{r-1}}\big(\gamma_{o_r}^{p_r}\nabla_{p_r}\alpha_{i_{r+1}\cdots i_{r+t}}\big)\Big)\cdots\Big)\Big).\end{aligned} \tag{4.40}$$

In fact, with this modification the proofs of Lemmas 4.5 and 4.6 go through. Also, the interpolation inequality in Lemma A.1 remains true. One just has to modify the proof to work with mixed tangential and full inner products:

$$\langle\alpha,\beta\rangle_{\gamma g} = \gamma^{i_1 j_1}\cdots\gamma^{i_s j_s}g^{i_{s+1}j_{s+1}}\cdots g^{i_{s+t}j_{s+t}}\alpha_{i_1\cdots i_s i_{s+1}\cdots i_{s+t}}\beta_{j_1\cdots j_s j_{s+1}\cdots j_{s+t}}. \tag{4.41}$$

Hence we obtain the following version of the interpolation inequality:

LEMMA 4.10 *Suppose that α is a $(0, t)$ tensor, and let $\overline{\overline{\nabla}}^s$ be defined as in* (4.40). *Then if $s \leq r-2$*

$$\left\|\overline{\overline{\nabla}}^s\alpha\right\|_{L^{2(r-2)/s}} \leq C\|\alpha\|_{L^\infty}^{1-s/(r-2)}\left\|\overline{\overline{\nabla}}^{r-2}\alpha\right\|_{L^2}^{s/(r-2)}. \tag{4.42}$$

In order to deal with some lower-order terms, the following is useful:

PROPOSITION 4.11 *Suppose that* α *is a* $(0,\mu)$ *tensor, and let* $\Pi^{s,0}\nabla^s$ *and* $\overline{\overline{\nabla}}^s$ *be defined as in* (4.39) *and* (4.40). *Let* $t = r-2$. *Then*

$$|(\Pi^{s,0}\nabla^s)\alpha - \overline{\overline{\nabla}}^s\alpha| \le C \sum_{\substack{r_0+r_1+\cdots+r_k=s-k\\ k\ge 1,\ r_0\ge 1}} |\overline{\nabla}^{r_1}\theta|\cdots|\overline{\nabla}^{r_k}\theta||\nabla^{r_0}\alpha|\,. \tag{4.43}$$

Here $\overline{\overline{\nabla}}^r\alpha$ *is defined by projecting over only the first* r *components as in* (4.40), *whereas* $\overline{\nabla}^r\theta$ *is defined as before by projecting over all* $r+2$ *components. If* $s\le t$

$$\begin{aligned}\|(\Pi^{s,0}\nabla^s)\alpha\|_{L^{2t/s}} \le{}& \\ &C\|\alpha\|_{L^\infty}^{1-s/t}\|\overline{\overline{\nabla}}^t\alpha\|_{L^2}^{s/t}\\ &+C(K_1)(1+\|\theta\|_{L^\infty})^s\big(\|\theta\|_{L^\infty}+\|\overline{\nabla}^t\theta\|_{L^2}\big)^{s/t}\sum_{\ell=0}^{t-1}\|\nabla^\ell\alpha\|_{L^2}\,,\end{aligned} \tag{4.44}$$

where K_1 *is a constant such that* $\iota_1 \ge 1/K_1$ *and* ι_1 *is as in Definition* 3.4. *Furthermore,*

$$\begin{aligned}\|\overline{\overline{\nabla}}^t\alpha\|_{L^2} \le{}& \\ &C\|\nabla^t\alpha\|_{L^2}+C(K_1)(1+\|\theta\|_{L^\infty})^t\big(\|\theta\|_{L^\infty}+\|\overline{\nabla}^t\theta\|_{L^2}\big)\sum_{\ell=0}^{t-1}\|\nabla^\ell\alpha\|_{L^2}\end{aligned} \tag{4.45}$$

and

$$\begin{aligned}&\Big\||(\Pi^{s,0}\nabla^s)\alpha||(\Pi^{t-s,0}\nabla^{t-s})\beta|\Big\|_{L^2}\\ &\quad\le \Big\|(\Pi^{s,0}\nabla^s)\alpha\Big\|_{L^{2t/s}}\Big\|(\Pi^{t-s,0}\nabla^{t-s})\beta\Big\|_{L^{2t/(t-s)}}\\ &\quad\le C(K_1)\left(\|\alpha\|_{L^\infty}+\sum_{\ell=0}^{t-1}\|\nabla^\ell\alpha\|_{L^2}\right)\|\nabla^t\beta\|_{L^2}\\ &\qquad+C(K_1)\left(\|\beta\|_{L^\infty}+\sum_{\ell=0}^{t-1}\|\nabla^\ell\beta\|_{L^2}\right)\|\nabla^t\alpha\|_{L^2}\\ &\qquad+C(K_1)(1+\|\theta\|_{L^\infty})^t\big(\|\theta\|_{L^\infty}+\|\overline{\nabla}^t\theta\|_{L^2}\big)\\ &\qquad\quad\left(\|\alpha\|_{L^\infty}+\sum_{\ell=0}^{t-1}\|\nabla^\ell\alpha\|_{L^2}\right)\left(\|\beta\|_{L^\infty}+\sum_{\ell=0}^{t-1}\|\nabla^\ell\beta\|_{L^2}\right).\end{aligned} \tag{4.46}$$

PROOF: (4.43) follows from Lemma 4.5. And if $r' = r_1+\cdots+r_k$, $r'+r_0 = s-k$, then by Hölder's inequality, (4.32) with $m=0$ and (4.30) with $m=-k$, respectively,

$$\big\| |\overline{\nabla}^{r_1}\theta| \cdots |\overline{\nabla}^{r_k}\theta||\nabla^{r_0}\alpha| \big\|_{L^{2t/s}} \tag{4.47}$$

$$\le C\|\theta\|_{L^\infty}^{k-1} \|\overline{\nabla}^{r'}\theta\|_{L^{2t/r'}} \|\nabla^{r_0}\alpha\|_{L^{2t/(r_0+k)}}$$

$$\le C(K_1)\|\theta\|_{L^\infty}^{k-r'/t}\|\overline{\nabla}^t\theta\|_{L^2}^{r'/t}\sum_{\ell=r_0}^{t-k}\|\nabla^\ell\alpha\|_{L^2}$$

$$\le C(K_1)\big(1+\|\theta\|_{L^\infty}\big)^s\big(\|\theta\|_{L^\infty}+\|\overline{\nabla}^t\theta\|_{L^2}\big)^{s/t}\sum_{\ell=r_0}^{t-k}\|\nabla^\ell\alpha\|_{L^2}\,.$$

If $s = t$ this proves (4.45). (4.44) follows from (4.43), (4.42), and (4.47). (4.46) follows from (4.44), (4.45), and our usual convexity inequality $a^{s/t}b^{1-s/t} \le a + b$. □

Let us now derive some properties of the projection. Since $g^{ij} = \gamma^{ij} + N^i N^j$, we have

$$\Pi(S\cdot R) = \Pi(S)\cdot\Pi(R) + \Pi(S\cdot N)\widetilde{\otimes}\Pi(N\cdot R)\,. \tag{4.48}$$

Note also that

$$[\nabla_N, \Pi]S = 0\,, \qquad [\overline{\nabla}, \Pi]S = 0\,, \tag{4.49}$$
$$[\nabla_N, \nabla]S = -\theta\cdot\nabla S\,, \quad [\nabla_N, \overline{\nabla}]S = -\theta\cdot\overline{\nabla}S\,,$$

where we have used that $[\nabla_N, \overline{\nabla}] = [\nabla_N, \Pi\nabla\Pi] = \Pi[\nabla_N, \nabla]\Pi$. Since $N\cdot\overline{\nabla}^k\theta = 0$, we get

$$[\nabla_N, \overline{\nabla}^r]S = \sum_{\ell=0}^{r-1}\overline{\nabla}^\ell[\nabla_N, \overline{\nabla}]\overline{\nabla}^{r-1-\ell}S = -\sum_{k=0}^{r-1}\binom{r}{k+1}(\overline{\nabla}^k\theta)\cdot\overline{\nabla}^{r-k}S\,, \tag{4.50}$$

where we used that $\sum_{\ell=0}^{r-1}\binom{\ell}{k} = \binom{r}{k+1}$ and $\overline{\nabla}\big((\Pi R)\cdot\Pi S\big) = (\overline{\nabla}\Pi R)\cdot\Pi S + (\Pi R)\cdot\overline{\nabla}\Pi S$. Furthermore, $0 = \nabla^2(N\cdot N) = 2N\cdot\nabla^2 N + 2(\nabla N)\cdot\nabla N$ and thus $\nabla_N\theta = -\theta\cdot\theta$, so (4.50) applied to $S=\theta$ gives

$$\nabla_N\overline{\nabla}^r\theta = -\sum_{k=0}^{r}\left(\binom{r}{k+1}+\binom{r}{k}\right)(\overline{\nabla}^k\theta)\cdot\overline{\nabla}^{r-k}\theta\,. \tag{4.51}$$

5 Elliptic Estimates and Energy Estimates for the Boundary Problem

Most of the results here will be stated in a coordinate-independent way. We can, however, take advantage of the fact that we have a transformation $f_t : \Omega \to \mathcal{D}_t \subset \mathbb{R}^n$ such that the metric is Euclidean in $\mathcal{D}_t$. Also, since we are looking for a short time existence, our metric expressed in the y-coordinates in Ω $g_{ij}(t,y)$ is equivalent to the metric at $t = 0$, $g_{ij}(0,y)$. Similarly, the induced metric on $\partial\Omega$ $\gamma_{ij}(t,\overline{y})$ is equivalent to $\gamma_{ij}(0,\overline{y})$. Throughout this section, ∇ will refer to covariant differentiation with respect to the metric g_{ij} in Ω, and $\overline{\nabla}$ will refer to covariant differentiation on $\partial\Omega$ with respect to the induced metric γ_{ij} on $\partial\Omega$ as defined in the beginning of Section 3.

We will assume that the normal N to $\partial\Omega$ is extended to a vector field in the interior of Ω satisfying $g_{ij}N^iN^j \le 1$ there such that, in a neighborhood of $\partial\Omega$, N is the unit normal to the sets $\partial\Omega_\rho = \{y : \mathrm{dist}_g(y, \partial\Omega) = \rho\}$ and N has the regularity described by Lemmas 3.10 and 3.11. Then $\gamma_{ij} = g_{ij} - N_iN_j$ where $N_i = g_{ij}N^j$ is a positive, semidefinite, pseudo-Riemannian metric in Ω. Using the decomposition into normal and tangential components $g^{ij} = N^iN^j + \gamma^{ij}$, we can write

$$g^{ij}g^{kl}\nabla_i\beta_k\nabla_j\beta_l = \big(N^iN^jg^{kl} + g^{ij}N^kN^l + \gamma^{ik}\gamma^{jl} - N^iN^kN^jN^l + \gamma^{ij}\gamma^{kl} - \gamma^{ik}\gamma^{jl}\big)\nabla_i\beta_k\nabla_j\beta_l \tag{5.1}$$

$$g^{ij}g^{kl}\nabla_k\beta_i\nabla_l\beta_j = \big(g^{ij}\gamma^{kl} + \gamma^{ij}g^{kl} - (\gamma^{ik}\gamma^{jl} - N^iN^kN^jN^l) - (\gamma^{ij}\gamma^{kl} - \gamma^{ik}\gamma^{jl})\big)\nabla_i\beta_k\nabla_j\beta_l\,. \tag{5.2}$$

The terms $(\gamma^{ik}\gamma^{jl} - N^iN^kN^jN^l)\nabla_i\beta_k\nabla_j\beta_k$ and $(\gamma^{ij}\gamma^{kl} - \gamma^{ik}\gamma^{jl})\nabla_i\beta_k\nabla_j\beta_l$ are going to be lower order: the first one because it can be controlled by $\operatorname{div}\beta = g^{ik}\nabla_i\beta_k$, which we expect to be lower order, and the second one because the boundary term vanishes if we integrate by parts using Green's theorem. Hence (5.1) and (5.2) say that we essentially can control $|\nabla\beta|^2 = g^{ij}g^{kl}\nabla_i\beta_k\nabla_j\beta_l$ by the normal-tangential components $\gamma^{ij}N^kN^l\nabla_i\beta_k\nabla_j\beta_l$ and either the normal-normal components $N^iN^jN^kN^l\nabla_i\beta_k\nabla_j\beta_l$ or the tangential-tangential components $\gamma^{ij}\gamma^{kl}\nabla_i\beta_k\nabla_j\beta_l$.

DEFINITION 5.1 Let $\beta_k = \beta_{Ik} = \nabla_I^r u_k$ where $\nabla_I^r = \nabla_{i_1}\cdots\nabla_{i_r}$, u is a $(0,1)$ tensor, and $[\nabla_i, \nabla_j] = 0$. Let $\operatorname{div}\beta = \nabla_i\beta^i = \nabla^r \operatorname{div} u$, and let $\operatorname{curl}\beta_{ij} = \nabla_i\beta_j - \nabla_j\beta_i = \nabla^r \operatorname{curl} u_{ij}$.

LEMMA 5.2 *Let β be as in Definition 5.1, and let Q be a positive semidefinite quadratic form $Q(\nabla_i\beta_k, \nabla_j\beta_l) = q^{IJ}(\nabla_i\beta_{Ik})\nabla_j\beta_{Jl}$. Then*

$$g^{ij}g^{kl}Q(\nabla_i\beta_k, \nabla_j\beta_l) \le \big(2(N^iN^jg^{kl} + g^{ij}N^kN^l) + 2g^{ik}g^{jl} + (\gamma^{ij}\gamma^{kl} - \gamma^{ik}\gamma^{jl})\big)Q(\nabla_i\beta_k, \nabla_j\beta_l) \tag{5.3}$$

$$g^{ij}g^{kl}Q(\nabla_k\beta_i, \nabla_l\beta_j) \le \big(n(g^{ij}\gamma^{kl} + \gamma^{ij}g^{kl}) + 2g^{ik}g^{jl}\big)Q(\nabla_i\beta_k, \nabla_j\beta_l) \tag{5.4}$$

and

$$N^iN^j\gamma^{kl}Q(\nabla_i\beta_k, \nabla_j\beta_l) \le 2N^kN^l\gamma^{ij}Q(\nabla_i\beta_k, \nabla_j\beta_l) + N^kN^l\gamma^{ij}Q(\operatorname{curl}\beta_{ik}, \operatorname{curl}\beta_{jl})\,. \tag{5.5}$$

PROOF: Since $g^{ik} = \gamma^{ik} + N^iN^k$, we obtain

$$\gamma^{ik}\gamma^{jl}Q(\nabla_i\beta_k, \nabla_j\beta_l) \le \big(2g^{ik}g^{jl} + 2N^iN^kN^jN^l\big)Q(\nabla_i\beta_k, \nabla_j\beta_l)\,, \tag{5.6}$$

$$N^iN^kN^jN^lQ(\nabla_i\beta_k, \nabla_j\beta_l) \le \big(2g^{ik}g^{jl} + 2\gamma^{ik}\gamma^{jl}\big)Q(\nabla_i\beta_k, \nabla_j\beta_l)\,. \tag{5.7}$$

Equations (5.3) and (5.4) follow from (5.6) and (5.7) and

$$\gamma^{ik}\gamma^{jl}Q\big(\alpha_{ik},\alpha_{jl}\big) \le (n-1)\gamma^{ij}\gamma^{kl}Q(\alpha_{ik},\alpha_{jl})\,. \tag{5.8}$$

To prove (5.8), let $\mathrm{tr}_\gamma(\alpha) = \gamma^{ik}\alpha_{ik}$ and let $\hat{\alpha}_{ik} = \alpha_{ik} - \gamma_{ik}\gamma^{pq}\alpha_{pq}/(n-1)$ be the traceless part. Then

$$\mathrm{tr}_\gamma(\alpha)\,\mathrm{tr}_\gamma(\sigma) = (n-1)\big(\gamma^{ij}\gamma^{kl}\alpha_{ik}\sigma_{jl} - \gamma^{ij}\gamma^{kl}\hat{\alpha}_{ik}\hat{\sigma}_{jl}\big)\,.$$

□

Let us recall the Gauss formula for Ω and $\partial\Omega$:

$$\int_\Omega \nabla_m(\beta^m)d\mu_g = \int_{\partial\Omega} N_m\beta^m\,d\mu_\gamma \quad \text{and} \quad \int_{\partial\Omega} \overline{\nabla}_i\overline{f}^i\,d\mu_\gamma = 0 \tag{5.9}$$

if $\overline{f}$ is tangential to $\partial\Omega$ and N is the unit conormal to $\partial\Omega$. The last part of (5.9) follows since, by (3.8), $\overline{\nabla}_i\overline{f}^i = \not{\nabla}_i\overline{f}^i$ is the intrinsic divergence on $\partial\Omega$ if the coordinates are chosen so $\partial\Omega$ is given by $y^n = 0$.

LEMMA 5.3 *Let R^{ijklIJ} be any quadratic form q^{IJ} multiplied with $(N^kN^lg^{ij} - g^{ki}N^lN^j)$ or $(g^{kl}\gamma^{ij} - \gamma^{ik}g^{lj})$. Then*

$$\int_\Omega R^{ijklIJ}\nabla_k\alpha_{Ii}\nabla_j\beta_{Jl}\,d\mu = \int_{\partial\Omega} N^l\gamma^{ij}q^{IJ}\alpha_{Ii}\nabla_j\beta_{Jl}\,d\mu_\gamma - \int_\Omega(\nabla_k R^{ijklIJ})\alpha_{Ii}\nabla_j\beta_{Jl}\,d\mu\,, \tag{5.10}$$

$$\int_\Omega R^{ijklIJ}\nabla_k\alpha_{Ii}\nabla_j\beta_{Jl}\,d\mu = -\int_{\partial\Omega} N^l\gamma^{ik}q^{IJ}\nabla_k\alpha_{Ii}\beta_{Jl}\,d\mu_\gamma - \int_\Omega(\nabla_j R^{ijklIJ})\nabla_k\alpha_{Ii}\beta_{Jl}\,d\mu\,. \tag{5.11}$$

Moreover, if R^{ijklIJ} is any quadratic form q^{IJ} multiplied with $(\gamma^{kl}\gamma^{ij} - \gamma^{ik}\gamma^{lj})$, then

$$\int_\Omega R^{ijklIJ}\nabla_k\alpha_{iI}\nabla_j\beta_{lJ}\,d\mu = -\int_\Omega(\nabla_k R^{ijklIJ})\alpha_{iI}\nabla_j\beta_{lJ}\,d\mu\,. \tag{5.12}$$

PROOF: Note that we have the following identities:

$$R^{ijklIJ}\nabla_k\alpha_{Ii}\nabla_j\beta_{Jl} = \nabla_k(R^{ijklIJ}\alpha_{Ii}\nabla_j\beta_{Jl}) - (\nabla_k R^{ijklIJ})\alpha_{Ii}\nabla_j\beta_{lJ}\,, \tag{5.13}$$

$$R^{ijklIJ}\nabla_k\alpha_{Ii}\nabla_j\beta_{Jl} = \nabla_j(R^{ijklIJ}\nabla_k\alpha_{Ii}\beta_{Jl}) - (\nabla_j R^{ijklIJ})\nabla_k\alpha_{Ii}\beta_{Jl}\,. \tag{5.14}$$

Integrating (5.13) and (5.14) over Ω by using Gauss formula (5.7), we get a boundary term from the divergence. The lemma now follows from

$$\begin{aligned} &N_k(N^kN^lg^{ij} - g^{ki}N^lN^j) = N_k(g^{kl}\gamma^{ij} - \gamma^{ik}g^{lj}) = N^l\gamma^{ij}\,, \\ &N_j(N^kN^lg^{ij} - g^{ki}N^lN^j) = N_j(g^{kl}\gamma^{ij} - \gamma^{ik}g^{lj}) = -N^l\gamma^{ik}\,, \\ &N_k(\gamma^{kl}\gamma^{ij} - \gamma^{ik}\gamma^{lj}) = 0\,, \end{aligned} \tag{5.15}$$

□

DEFINITION 5.4 If $|I| = |J| = r$, set

$$g^{IJ} = g^{i_1j_1}\cdots g^{i_rj_r} \quad \text{and} \quad \gamma^{IJ} = \gamma^{i_1j_1}\cdots\gamma^{i_rj_r}\,.$$

If α and β are $(0, r)$ tensors, let $\langle\alpha, \beta\rangle = g^{IJ}\alpha_I\beta_J$ and $|\alpha|^2 = \langle\alpha, \alpha\rangle$. If $(\Pi\beta)_I = \gamma_I^J\beta_J$ is the projection, then $\langle\Pi\alpha, \Pi\beta\rangle = \gamma^{IJ}\alpha_I\beta_J$. Let

$$\|\beta\|_{L^2(\Omega)} = \left(\int_\Omega |\beta|^2\,d\mu_g\right)^{1/2}, \quad \|\beta\|_{L^2(\partial\Omega)} = \left(\int_{\partial\Omega} |\beta|^2\,d\mu_\gamma\right)^{1/2},$$

$$\|\Pi\beta\|_{L^2(\partial\Omega)} = \left(\int_{\partial\Omega} |\Pi\beta|^2\,d\mu_\gamma\right)^{1/2},$$

where $d\mu_g$ is the Riemannian volume element on Ω and $d\mu_\gamma$ is the induced surface measure on $\partial\Omega$.

LEMMA 5.5 *Let β be as in Definition* 5.1 *and ι_0 be as in Definition* 3.4*. If $|\theta| + 1/\iota_0 \le K$, then*

$$|\nabla\beta|^2 \le C\big(g^{ij}\gamma^{kl}\gamma^{IJ}\nabla_k\beta_{Ii}\nabla_l\beta_{Jj} + |\operatorname{div}\beta|^2 + |\operatorname{curl}\beta|^2\big)\,, \tag{5.16}$$

$$\int_\Omega |\nabla\beta|^2\,d\mu \le C\int_\Omega \big(N^iN^jg^{kl}\gamma^{IJ}\nabla_k\beta_{Ii}\nabla_l\beta_{Jj} + |\operatorname{div}\beta|^2 + |\operatorname{curl}\beta|^2 + K^2|\beta|^2\big)d\mu\,. \tag{5.17}$$

PROOF: The proof follows by induction from repeated use of Lemma 5.2. $|\beta|^2 = g^{IJ}\beta_I\beta_J$ can be written as a sum of terms of the form

$$N^{i_1}N^{j_1}\cdots N^{i_s}N^{j_s}\gamma^{i_{s+1}j_{s+1}}\cdots\gamma^{i_rj_r}\beta_{i_1\cdots i_r}\beta_{j_1\cdots j_r}\,. \tag{5.18}$$

If $s = 0, 1$, then (5.18) is bounded by the right-hand side of (5.16). If we inductively assume that we can bound the right-hand side of (5.18) for $s \le s_0$, then the bound for $s = s_0 + 1$ follows from (5.4)–(5.5) in Lemma 5.2. On the other hand, if we control the right-hand side of (5.17), then we have a bound for the integral of (5.18) for $s = 1, 2$. However, by (5.3) in Lemma 5.2 and (5.12) in Lemma 5.3, this gives us the integral of (5.18) also for $s = 0$, but then we can use (5.16) to obtain (5.17). □

LEMMA 5.6 *Let β be as in Definition* 5.1 *and ι_0 be as in Definition* 3.4. *If $|\theta| + 1/\iota_0 \le K$, then*

$$\|\beta\|^2_{L^2(\partial\Omega)} \le C\big(\|\nabla\beta\|_{L^2(\Omega)} + K\|\beta\|_{L^2(\Omega)}\big)\|\beta\|_{L^2(\Omega)}\,, \tag{5.19}$$

$$\|\beta\|^2_{L^2(\partial\Omega)} \le C\|\Pi\beta\|^2_{L^2(\partial\Omega)} + C\big(\|\operatorname{div}\beta\|_{L^2(\Omega)} + \|\operatorname{curl}\beta\|_{L^2(\Omega)} + K\|\beta\|_{L^2(\Omega)}\big)\|\beta\|_{L^2(\Omega)}\,. \tag{5.20}$$

and

$$\|\nabla\beta\|^2_{L^2(\Omega)} \le C\|\nabla\beta\|_{L^2(\partial\Omega)}\|\beta\|_{L^2(\partial\Omega)} + C\big(\|\operatorname{div}\beta\|_{L^2(\Omega)} + \|\operatorname{curl}\beta\|_{L^2(\Omega)}\big)^2\,. \tag{5.21}$$

Furthermore,

$$\|\nabla\beta\|^2_{L^2(\Omega)} \le C\|\Pi\nabla\beta\|_{L^2(\partial\Omega)}\|\Pi N\cdot\beta\|_{L^2(\partial\Omega)} + C\big(\|\operatorname{div}\beta\|_{L^2(\Omega)} + \|\operatorname{curl}\beta\|_{L^2(\Omega)} + K\|\beta\|_{L^2(\Omega)}\big)^2 \tag{5.22}$$

$$\|\nabla\beta\|^2_{L^2(\Omega)} \le C\|\Pi N\cdot\nabla\beta\|_{L^2(\partial\Omega)}\|\Pi\beta\|_{L^2(\partial\Omega)} + C\big(\|\operatorname{div}\beta\|_{L^2(\Omega)} + \|\operatorname{curl}\beta\|_{L^2(\Omega)} + K\|\beta\|_{L^2(\Omega)}\big)^2 \tag{5.23}$$

where $N\cdot\beta_I = N^i\beta_{iI}$ and $N\cdot\nabla\beta_{kI} = N^i\nabla_k\beta_{iI}$.

PROOF: Let N be the extension of the normal to the interior as in Lemmas 3.10 and 3.11. Then

$$\int_{\partial\Omega} |\beta|^2\,d\mu_\gamma = \int_\Omega \nabla_k\big(N^k|\beta|^2\big)d\mu\,,$$

and since $|\nabla N| \le K$, by Lemmas 3.10 and 3.11, (5.19) follows. (5.20) follows by induction as in the proof of Lemma 5.5 from

$$\Big|\int_{\partial\Omega} q^{IJ}(N^iN^j - \gamma^{ij})\beta_{Ii}\beta_{Jj}\,d\mu_\gamma\Big| \le C\big(\|\operatorname{div}\beta\|_{L^2(\Omega)} + \|\operatorname{curl}\beta\|_{L^2(\Omega)} + K\|\beta\|_{L^2(\Omega)}\big)\|\beta\|_{L^2(\Omega)}\,,$$

if q^{IJ} is any product of factors $q^{i_kj_k}$ of the form $g^{i_kj_k}$, $\gamma^{i_kj_k}$, or $N^{i_k}N^{j_k}$. The left-hand side is

$$\int_\Omega \nabla_k\big(N^kq^{IJ}(N^iN^j - \gamma^{ij})\beta_{Ii}\beta_{Jj}\big)d\mu$$
$$= 2\int_\Omega N^kq^{IJ}\big(N^iN^j - \gamma^{ij}\big)\beta_{Ii}\nabla_k\beta_{Jj}\,d\mu + \int_\Omega (\nabla_kN^k)q^{IJ}\big(N^iN^j - \gamma^{ij}\big)\beta_{Ii}\beta_{Jj}\,d\mu$$

$$= -2\int_\Omega N^k q^{IJ}\gamma^{ij}\beta_{Ii}\big(\nabla_k\beta_{Jj} - \nabla_j\beta_{Jk}\big)d\mu$$
$$+ 2\int_\Omega q^{IJ}(N^iN^j + \gamma^{ij})(\nabla_j\beta_{Ii})N^k\beta_{Jk}\,d\mu$$
$$+ 2\int_\Omega \nabla_j\big(q^{IJ}\gamma^{ij}N^k\big)\beta_{Ii}\beta_{Jk}\,d\mu + \int_\Omega (\nabla_k N^k)q^{IJ}\big(N^iN^j - \gamma^{ij}\big)\beta_{Ii}\beta_{Jj}\,d\mu\,.$$

(5.21) is just integration by parts twice. (5.22) and (5.23) follow from Lemmas 5.5 and 5.3. □

One can actually get away with a less regular boundary for some of the estimates:

LEMMA 5.7 *Let β be as in Definition* 5.1. *Then there is $\varepsilon_1(r) > 0$ such that if the condition in Definition* 3.5 *holds with $\varepsilon_1 \le \varepsilon_1(r)$, we have with $K_1 \ge 1/\iota_1$*

$$\|\beta\|^2_{L^2(\partial\Omega)} \le C\big(\|\nabla\beta\|_{L^2(\Omega)} + K_1\|\beta\|_{L^2(\Omega)}\big)\|\beta\|_{L^2(\Omega)}\,, \tag{5.24}$$

$$\|\beta\|^2_{L^2(\partial\Omega)} \le C\|\Pi\beta\|^2_{L^2(\partial\Omega)} + C\big(\|\mathrm{div}\,\beta\|_{L^2(\Omega)} + \|\mathrm{curl}\,\beta\|_{L^2(\Omega)} + K_1\|\beta\|_{L^2(\Omega)}\big)\|\beta\|_{L^2(\Omega)}\,. \tag{5.25}$$

PROOF: We will prove (5.24) and (5.25) in the x-coordinates

$$\Omega \ni y \to x(t,y) \in \mathcal{D}_t \subset \mathbb{R}^n\,.$$

Since the metric there is the induced metric from $\mathbb{R}^n$, we can then compare the normal $\mathcal{N}$ to $\partial\mathcal{D}_t$ at different points. Let χ_p be the partition of unity in Lemma 3.4, let $\mathcal{N}_p = \mathcal{N}(x_p)$ be the unit normal at some fixed point $x_p \in \mathrm{supp}(\chi_p) \cap \partial\mathcal{D}_t$, and let N be the unit normal to $\partial\mathcal{D}_t$. Then

$$\int_{\partial\mathcal{D}_t} \chi_p|\beta|^2\langle\mathcal{N}_p, N\rangle dS = \int_{\mathcal{D}_t} \mathcal{N}_p^k\partial_k\big(\chi_p|\beta|^2\big)dx\,,$$

where N is the unit normal to $\partial\mathcal{D}_t$ and $\langle\mathcal{N}_p, N\rangle = \delta_{ij}\mathcal{N}_p^i\mathcal{N}^j \ge \frac{1}{2}$. Since $|\partial\chi_p| \le CK_1$, (5.24) follows.

To prove (5.25), we will use a similar estimate to the one in the proof of (5.20), with $\mathcal{N}$ replaced by $\mathcal{N}_p$, $\gamma^{ij} = \delta^{ij} - \mathcal{N}^i\mathcal{N}^j$ replaced by ${\gamma_p}^{ij} = \delta^{ij} - \mathcal{N}_p^i\mathcal{N}_p^j$, and q^{IJ} replaced by q_p^{IJ}, a product of factors δ^{ij}, ${\gamma_p}^{ij}$, and $\mathcal{N}_p^i\mathcal{N}_p^j$. We will use the identity

$$\mathcal{N}_p^k\partial_k\big(\delta^{ij}{q_p}^{IJ}\chi_p\beta_{Ii}\beta_{Jj}\big) - 2\delta^{ij}\partial_i\big(\mathcal{N}_p^k{q_p}^{IJ}\chi_p\beta_{Ik}\beta_{Jj}\big)$$
$$= -2\mathcal{N}_p^k{q_p}^{IJ}\chi_p\beta_{Ik}\delta^{ij}\partial_i\beta_{Jj} + 2\delta^{ij}\mathcal{N}_p^k{q_p}^{IJ}\chi_p(\partial_i\beta_{Ik} - \partial_k\beta_{Ii})\beta_{Jj}$$
$$+ \mathcal{N}_p^k(\partial_k\chi_p)\big(\delta^{ij}{q_p}^{IJ}\beta_{Ii}\beta_{Jj}\big) - 2\delta^{ij}(\partial_i\chi_p)\big(\mathcal{N}_p^k{q_p}^{IJ}\beta_{Ik}\beta_{Jj}\big)\,.$$

Integrating this over $\mathcal{D}_t$ by using the Gauss theorem, we get

$$\left| \int_{\partial \mathcal{D}_t} \big(\langle \mathcal{N}_p, \mathcal{N}\rangle \delta^{ij} - 2\mathcal{N}^j \mathcal{N}_p^i\big) q_p{}^{IJ} \chi_p \beta_{Ii} \beta_{Jj}\, dS \right| \leq \int_{\mathcal{D}_t} \big(2\chi_p(|\operatorname{div}\beta| + |\operatorname{curl}\beta|) + 3|\partial\chi_p||\beta|\big)|\beta| dx\, .$$

We now assume that $|\mathcal{N} - \mathcal{N}_p| \leq \varepsilon_1$ in the support of χ_p, where $\varepsilon_1 = \varepsilon_1(r)$ is to be determined. Writing $\mathcal{N} = a\mathcal{N}_p + b\mathcal{T}_p$, where $a = \langle \mathcal{N}_p, \mathcal{N}\rangle$, $b = \sqrt{1-a^2} \leq \varepsilon_1$, $\langle \mathcal{T}_p, \mathcal{T}_p\rangle = 1$, and $\langle \mathcal{T}_p, \mathcal{N}_p\rangle = 0$, we get

$$\langle \mathcal{N}_p, \mathcal{N}\rangle \delta^{ij} - 2\mathcal{N}^j \mathcal{N}_p^i = a\big(\gamma_p^{ij} - \mathcal{N}_p^i \mathcal{N}_p^j\big) - 2b\mathcal{N}_p^i \mathcal{T}_p^j\, .$$

Let $Q_p(\beta_i, \beta_j) = q_p{}^{IJ}\chi_p \beta_{Ii}\beta_{Jj}$, and let $R_p(\beta,\beta) = \big(a(\gamma_p^{ij} - \mathcal{N}_p^i\mathcal{N}_p^j) - 2b\mathcal{N}_p^i\mathcal{T}_p^j\big)$ $Q_p(\beta_i, \beta_j)$. It follows that

$$\begin{aligned}
\mathcal{N}_p^i \mathcal{N}_p^j Q_p(\beta_i,\beta_j) &\leq \left(\gamma_p^{ij} - \frac{b}{a}\big(\mathcal{N}_p^i\mathcal{T}_p^j + \mathcal{T}_p^i\mathcal{N}_p^j\big)\right) Q_p(\beta_i,\beta_j) + \frac{1}{a} R_p(\beta,\beta) \\
&\leq \left(\gamma_p^{ij} - \frac{b}{a}\left(\frac{1+b}{a}\mathcal{T}_p^i\mathcal{T}_p^j + \frac{a}{1+b}\mathcal{N}_p^i\mathcal{N}_p^j\right)\right) Q_p(\beta_i,\beta_j) \\
&\quad + \frac{1}{a} R_p(\beta,\beta) \\
&\leq \left(\frac{1}{1-b}\gamma_p^{ij} + \frac{b}{1+b}\mathcal{N}_p^i\mathcal{N}_p^j\right) Q_p(\beta_i,\beta_j) + \frac{1}{a} R_p(\beta,\beta)\, ,
\end{aligned}$$

since $\mathcal{T}_p^i\mathcal{T}_p^j Q_p(\beta_i,\beta_j) \leq \gamma_p^{ij} Q_p(\beta_i,\beta_j)$ and $a^2 = 1 - b^2$. Moving the term with the normal component over to the other side, we obtain

$$\delta^{ij} Q_p(\beta_i,\beta_j) \leq \frac{2}{1-b}\gamma_p^{ij} Q_p(\beta_i,\beta_j) + \frac{1+b}{a} R_p(\beta,\beta)\, .$$

Integrating this gives

$$\begin{aligned}
\int_{\partial\mathcal{D}_t} \delta^{ij} q_p{}^{IJ}\chi_p\beta_{Ii}\beta_{Jj}\, dS &\leq \frac{2}{1-\varepsilon_1}\int_{\partial\mathcal{D}_t} \gamma_p^{ij} q_p{}^{IJ}\chi_p\beta_{Ii}\beta_{Jj}\, dS \\
&\quad + 4\int_{\mathcal{D}_t} \big(\chi_p(|\operatorname{div}\beta| + |\operatorname{curl}\beta|) + |\partial\chi_p||\beta|\big)|\beta| dx\, .
\end{aligned}$$

Repeated use of this gives

$$(5.26)\quad \int_{\partial\mathcal{D}_t} \delta^{ij}\delta^{IJ}\chi_p\beta_{Ii}\beta_{Jj}\,dS \le$$
$$A\int_{\partial\mathcal{D}_t} \gamma_p^{ij}\gamma_p{}^{IJ}\chi_p\beta_{Ii}\beta_{Jj}\,dS + B\int_{\mathcal{D}_t} \big(\chi_p(|\mathrm{div}\,\beta| + |\mathrm{curl}\,\beta|) + |\partial\chi_p||\beta|\big)|\beta|dx$$

for some constants A and B that depend only on the order r of the tensor β.

We now claim that if q^{IJ} is any positive definite quadratic form, then

$$(5.27)\qquad \gamma_p^{ij}q^{IJ}\chi_p\beta_{Ii}\beta_{Jj} \le \gamma^{ij}q^{IJ}\chi_p\beta_{Ii}\beta_{Jj} + b\delta^{ij}q^{IJ}\chi_p\beta_{Ii}\beta_{Jj}\,.$$

In fact, if $Q(\beta_i,\beta_j) = q^{IJ}\chi_p\beta_{Ii}\beta_{Jj}$,

$$\begin{aligned}
&\gamma_p^{ij}Q(\beta_i,\beta_j) - \gamma^{ij}Q(\beta_i,\beta_j)\\
&\quad = (\mathcal{N}^i\mathcal{N}^j - \mathcal{N}_p^i\mathcal{N}_p^j)Q(\beta_i,\beta_j)\\
&\quad = \big(b^2\mathcal{T}_p^i\mathcal{T}_p^j - b^2\mathcal{N}_p^i\mathcal{N}_p^j + ab(\mathcal{N}_p^i\mathcal{T}_p^j + \mathcal{T}_p^i\mathcal{N}_p^j)\big)Q(\beta_i,\beta_j)\\
&\quad \le \left(b^2\mathcal{T}_p^i\mathcal{T}_p^j - b^2\mathcal{N}_p^i\mathcal{N}_p^j + ab\left(\frac{1+b}{a}\mathcal{N}_p^i\mathcal{N}_p^j + \frac{a}{1+b}\mathcal{T}_p^i\mathcal{T}_p^j\right)\right)Q(\beta_i,\beta_j)\\
&\quad = b(\mathcal{N}_p^i\mathcal{N}_p^j + \mathcal{T}_p^i\mathcal{T}_p^j)Q(\beta_i,\beta_j)\\
&\quad \le b\delta^{ij}Q(\beta_i,\beta_j)\,,
\end{aligned}$$

since $a^2 = 1 - b^2$. Using (5.27) now, we can replace $\gamma_p^{ij}\gamma_p{}^{IJ}$ by $\gamma^{ij}\gamma^{IJ}$ in (5.26) with a small error that can be absorbed into the left-hand side if $b \le \varepsilon_1$ is sufficiently small. Finally, summing over p by using $\sum_p \chi_p = 1$, $\sum_p |\partial\chi_p| \le CK_1$, and Hölder's inequality gives (5.25). □

Lemma 5.6 applied to $\beta = \nabla q$, where q is a function, gives estimates for both the Dirichlet problem and the Neumann problem. In fact, if $q = 0$ on $\partial\Omega$, then $\Pi\nabla^2 q = \theta\nabla_N q$. Thus (5.22) and (5.20) give

$$\begin{aligned}
\|\nabla^2 q\|^2_{L^2(\Omega)} &\le CK\|\nabla_N q\|^2_{L^2(\partial\Omega)} + C\big(\|\triangle q\|_{L^2(\Omega)} + K\|\nabla q\|_{L^2(\Omega)}\big)^2\\
&\le C\big(\|\triangle q\|_{L^2(\Omega)} + K\|\nabla q\|_{L^2(\Omega)}\big)^2\,.
\end{aligned}$$

Similarly, if $\nabla_N q = 0$ on $\partial\Omega$, then $N^i\overline{\nabla}_j\nabla_i q = -\theta_j^i\overline{\nabla}_i q$, and by (5.23) and (5.20),

$$\begin{aligned}
\|\nabla^2 q\|^2_{L^2(\Omega)} &\le CK\|\overline{\nabla} q\|^2_{L^2(\partial\Omega)} + C\big(\|\triangle q\|_{L^2(\Omega)} + K\|\nabla q\|_{L^2(\Omega)}\big)^2\\
&\le C\big(\|\triangle q\|_{L^2(\Omega)} + K\|\nabla q\|_{L^2(\Omega)}\big)^2\,.
\end{aligned}$$

Similarly, we can get estimates for higher-order derivatives. More generally, we have the following:

PROPOSITION 5.8 *Let ι_0 and ι_1 be as in Definitions* 3.4 *and* 3.5*, and suppose that $|\theta| + 1/\iota_0 \leq K$ and $1/\iota_1 \leq K_1$. Then with $\tilde{K} = \min(K, K_1)$ we have, for any $r \geq 2$ and $\delta > 0$,*

$$\|\nabla^r q\|_{L^2(\partial\Omega)} + \|\nabla^r q\|_{L^2(\Omega)} \leq C\|\Pi\nabla^r q\|_{L^2(\partial\Omega)} + C(\tilde{K}, \mathrm{Vol}\,(\Omega)) \sum_{s \leq r-1} \|\nabla^s \triangle q\|_{L^2(\Omega)} \tag{5.28}$$

$$\|\nabla^r q\|_{L^2(\Omega)} + \|\nabla^{r-1} q\|_{L^2(\partial\Omega)} \leq \delta\|\Pi\nabla^r q\|_{L^2(\partial\Omega)} + C(1/\delta, K, \mathrm{Vol}\,(\Omega)) \sum_{s \leq r-2} \|\nabla^s \triangle q\|_{L^2(\Omega)}\,. \tag{5.29}$$

PROOF: (5.28) with an extra lower-order term $C(\tilde{K})\|\nabla q\|_{L^2(\Omega)}$ in the right follows from (5.20) or (5.25) together with repeated use of (5.21) and (5.19) or (5.24). The lower-order term can then be bounded by (5.17) in Lemma A.5. (5.29) with the same extra lower-order term follows from (5.22) together with repeated use of (5.19) and (5.21). □

Remark. One should be able to improve the results of Proposition 5.8 and replace the sum in the right-hand side of (5.28) by the sum over $s = 0, \frac{1}{2}$, at least when $|\nabla_N q| > \varepsilon > 0$ on $\partial\Omega$. However, then one has to make sense of fractional derivatives.

PROPOSITION 5.9 *Assume that $0 \leq r \leq 4$ or $r \geq (n-1)/2 + 2$. Suppose that $|\theta| \leq K$ and $\iota_1 \geq 1/K_1$, where ι_1 is as in Definition* 3.5*. If $q = 0$ on $\partial\Omega$, then for $m = 0, 1$,*

$$\begin{aligned}\|\Pi\nabla^r q\|_{L^2(\partial\Omega)} \leq{}& 2\|\overline{\nabla}^{r-2}\theta\|_{L^2(\partial\Omega)}\|\nabla_N q\|_{L^\infty(\partial\Omega)} + C\sum_{k=1}^{r-1} \|\theta\|^k_{L^\infty(\partial\Omega)}\|\nabla^{r-k} q\|_{L^2(\partial\Omega)} \\ &+ C(K, K_1)\Big(\|\theta\|_{L^\infty(\partial\Omega)} + \sum_{k \leq r-2-m} \|\overline{\nabla}^k\theta\|_{L^2(\partial\Omega)}\Big) \sum_{k \leq r-2+m} \|\nabla^k q\|_{L^2(\partial\Omega)}\,,\end{aligned} \tag{5.30}$$

and if $r > (n-1)/2 + 2$, then for any $\delta > 0$

$$\|\Pi\nabla^{r-1} q\|_{L^2(\partial\Omega)} \leq \delta\|\nabla^{r-1} q\|_{L^2(\partial\Omega)} + C_\delta\big(K, K_1, \|\theta\|_{L^2(\partial\Omega)}, \|\overline{\nabla}^{r-3}\theta\|_{L^2(\partial\Omega)}\big) \sum_{k=0}^{r-2} \|\nabla^k q\|_{L^2(\partial\Omega)}\,. \tag{5.31}$$

If, in addition, $|\nabla_N q| \geq \varepsilon > 0$ *and* $|\nabla_N q| \geq 2\varepsilon \|\nabla_N q\|_{L^\infty(\partial\Omega)}$*, then*

$$\|\overline{\nabla}^{r-2}\theta\|_{L^2(\partial\Omega)} \leq C\left(\frac{1}{\varepsilon}\right)\left(\|\Pi\nabla^r q\|_{L^2(\partial\Omega)} + \sum_{k=1}^{r-1} \|\theta\|^k_{L^\infty(\partial\Omega)} \|\nabla^{r-k} q\|_{L^2(\partial\Omega)}\right) + C\left(K, K_1, \frac{1}{\varepsilon}\right)\left(\|\theta\|_{L^\infty(\partial\Omega)} + \sum_{k\leq r-3} \|\overline{\nabla}^k\theta\|_{L^2(\partial\Omega)}\right) \sum_{k\leq r-1} \|\nabla^k q\|_{L^2(\partial\Omega)}\,. \tag{5.32}$$

Furthermore, if $r \leq 4$*, then the second line of* (5.30) *and* (5.32) *drop out.*

PROOF: (5.30) and (5.32) follow from Proposition 4.5. To prove (5.30) we can take $\varepsilon = 1$, and to prove (5.32) we take $m = 1$ in Proposition 4.5. (5.31) follows from (5.30) and Sobolev's lemma, (A.8). □

PROPOSITION 5.10 *Assume that* $0 \leq r \leq 4$ *or* $r \geq (n-1)/2 + 2$ *and that* $|\theta| + 1/\iota_0 \leq K$*. If* $q = 0$ *on* $\partial\Omega$*, then*

$$\|\nabla^{r-1} q\|_{L^2(\partial\Omega)} \leq C\big(\|\overline{\nabla}^{r-3}\theta\|_{L^2(\partial\Omega)} \|\nabla_N q\|_{L^\infty(\partial\Omega)} + \|\nabla^{r-2}\triangle q\|_{L^2(\Omega)}\big) + C\big(K, \mathrm{Vol}\,(\Omega), \|\theta\|_{L^2(\partial\Omega)}, \dots, \|\overline{\nabla}^{r-4}\theta\|_{L^2(\partial\Omega)}\big)\left(\|\nabla_N q\|_{L^\infty(\partial\Omega)} + \sum_{s\leq r-3} \|\nabla^s \triangle q\|_{L^2(\Omega)}\right). \tag{5.33}$$

If $r > (n-1)/2 + 2$*, then*

$$\|\nabla^{r-1} q\|_{L^2(\partial\Omega)} + \|\nabla q\|_{L^\infty(\partial\Omega)} \leq C\|\nabla^{r-2}\triangle q\|_{L^2(\Omega)} + C\big(K, \mathrm{Vol}\,(\Omega), \|\theta\|_{L^2(\partial\Omega)}, \dots, \|\overline{\nabla}^{r-3}\theta\|_{L^2(\partial\Omega)}\big) \sum_{s\leq r-3} \|\nabla^s \triangle q\|_{L^2(\Omega)}\,. \tag{5.34}$$

PROOF: (5.33) follows from (5.28) and (5.30) with $m = 1$ and r replaced by $r-1$. The estimate for $\|\nabla^{r-1} q\|_{L^2(\partial\Omega)}$ in (5.34) follows from (5.28), with r replaced by $r-1$, and (5.31). The estimate for $\|\nabla q\|_{L^\infty(\partial\Omega)}$ in (5.34) follows from the estimate for $\|\nabla^{r-1} q\|_{L^2(\partial\Omega)}$ and Sobolev's lemma, Lemma A.2. □

There are two possible energies, given in Proposition 5.11 and Proposition 5.12, respectively.

PROPOSITION 5.11 *Let* $Q(\alpha, \alpha) = \gamma^{IJ}\alpha_I\alpha_J$ *and* $h_{ij} = D_t g_{ij}/2$*, and set*

$$E(t) = \int_{\partial\Omega} \gamma^{ij} Q(\alpha_i, \alpha_j)\nu\, d\mu_\gamma + \int_{\Omega} g^{ij} N^k N^l Q(\nabla_i\beta_k, \nabla_j\beta_l) d\mu_g\,,$$

where $0 < \nu < \infty$. *Let* K *be a constant such that*

$$|h| \leq K \quad \text{in } [0, T] \times \Omega\,, \tag{5.35}$$

$$|\theta| + \frac{1}{\iota_0} + \left|\frac{\nu_t}{\nu}\right| \leq K \quad \text{on } [0, T] \times \partial\Omega\,. \tag{5.36}$$

Then

$$\begin{aligned}\frac{dE}{dt} \leq C\sqrt{E}\big(\big\|\Pi(D_t\alpha + \nu N^k\nabla\beta_k)\big\|_{L^2(\partial\Omega)} + \big\|D_t\nabla\beta - \nabla\alpha\big\|_{L^2(\Omega)}\big)\\ + CKE + C\big(\|\operatorname{div}\alpha\|_{L^2(\Omega)} + \|\operatorname{curl}\alpha\|_{L^2(\Omega)} + K\|\alpha\|_{L^2(\Omega)}\\ + \|\operatorname{div}\beta\|_{L^2(\Omega)} + \|\operatorname{curl}\beta\|_{L^2(\Omega)} + K\|\beta\|_{L^2(\Omega)}\big)^2\end{aligned} \tag{5.37}$$

PROOF: Since by Lemma 3.9 $D_t\, d\mu_\gamma = (\operatorname{tr} h - h_{NN})d\mu_\gamma$ and $D_t\, d\mu = \operatorname{tr} h\, d\mu$, we obtain

$$\begin{aligned}\frac{dE}{dt} = 2\int_{\partial\Omega} \gamma^{ij} Q(\alpha_i, D_t\alpha_j)\nu\, d\mu_\gamma\\ + 2\int_{\Omega} g^{ij}N^kN^l Q(\nabla_i\beta_k, D_t\nabla_j\beta_l)d\mu_g\\ + \int_{\partial\Omega} \Big(D_t(\gamma^{ij}\gamma^{IJ}) + \Big(\operatorname{tr} h - h_{NN} + \frac{\nu_t}{\nu}\Big)\gamma^{ij}\gamma^{IJ}\Big)\alpha_{Ii}\,\alpha_{Jj}\,\nu\, d\mu_\gamma\\ + \int_{\Omega} \big(D_t(g^{ij}N^kN^l\gamma^{IJ}) + \operatorname{tr} h\, g^{ij}N^kN^l\gamma^{IJ}\big)\nabla_i\beta_{Ik}\,\nabla_j\beta_{Jl}\, d\mu_g\,.\end{aligned} \tag{5.38}$$

Since $D_t\gamma^{ij} = -2\gamma^{im}\gamma^{jn}h_{mn}$, the second line is bounded by the boundary term in the energy E, and the third line is bounded by $\|\nabla\beta\|^2_{L^2(\Omega)}$. By Lemma 5.3

$$\begin{aligned}\int_{\Omega} g^{ij}N^kN^l\gamma^{IJ}\nabla_k\alpha_{Ii}\nabla_j\beta_{Jl}\, d\mu_g\\ = \int_{\partial\Omega} N^l\gamma^{ij}\gamma^{IJ}\alpha_{Ii}\nabla_j\beta_{Jl}\, d\mu_\gamma + \int_\Omega g^{ik}N^jN^l\gamma^{IJ}\nabla_k\alpha_{Ii}\nabla_j\beta_{Jl}\, d\mu_g\\ - \int_\Omega \nabla_k\big(g^{ij}N^kN^l\gamma^{IJ} - g^{ik}N^jN^l\gamma^{IJ}\big)\alpha_{Ii}\nabla_j\beta_{Jl}\, d\mu_g\,.\end{aligned}$$

The first term on the second line is bounded by $\|\operatorname{div}\alpha\|_{L^2(\Omega)}\|\nabla\beta\|_{L^2(\Omega)}$, and the second by $K\|\alpha\|_{L^2(\Omega)}\|\nabla\beta\|_{L^2(\Omega)}$. Recall now that by Lemma 5.5

$$\|\nabla\beta\|^2_{L^2(\Omega)} \leq CE + C\big(\|\operatorname{div}\beta\|_{L^2(\Omega)} + \|\operatorname{curl}\beta\|_{L^2(\Omega)} + K\|\beta\|_{L^2(\Omega)}\big)^2\,.$$

This proves Proposition 5.11. □

PROPOSITION 5.12 *Let* $Q(\alpha,\alpha)=\gamma^{IJ}\alpha_I\alpha_J$ *and* $h_{ij}=D_t g_{ij}/2$*, and set*

$$E(t)=\int_{\partial\Omega}\gamma^{ij}Q(\alpha_i,\alpha_j)\nu\,d\mu_\gamma+\int_{\Omega}g^{kl}\gamma^{ij}Q(\nabla_i\beta_k,\nabla_j\beta_l)d\mu_g \tag{5.39}$$

where $0<\nu<\infty$*. Let* K *be a constant such that*

$$|h|\le K\quad \text{in } [0,T]\times\Omega\,, \tag{5.40}$$

$$|\theta|+\frac{1}{\iota_0}+\left|\frac{\nu_t}{\nu}\right|\le K\quad \text{on } [0,T]\times\partial\Omega\,. \tag{5.41}$$

Then

$$\begin{aligned}\frac{dE}{dt}\le{}& C\sqrt{E}\big(\big\|\Pi(D_t\alpha+\nu N^k\nabla\beta_k)\big\|_{L^2(\partial\Omega)}+\|D_t\nabla\beta-\nabla\alpha\|_{L^2(\Omega)}\big)\\&+CKE+C\|\operatorname{curl}\alpha\|_{L^2(\Omega)}\sqrt{E}+C\|\alpha\|_{L^2(\Omega)}\|\nabla\operatorname{div}\beta\|_{L^2(\Omega)}\\&+\big(K\|\alpha\|_{L^2(\Omega)}+\|\operatorname{div}\beta\|_{L^2(\Omega)}+\|\operatorname{curl}\beta\|_{L^2(\Omega)}\big)^2\,.\end{aligned}\tag{5.42}$$

PROOF: Since by Lemma 3.9 $D_t\,d\mu_\gamma=(\operatorname{tr}h-h_{NN})d\mu_\gamma$ and $D_t\,d\mu=\operatorname{tr}h\,d\mu$, we obtain

$$\begin{aligned}\frac{dE}{dt}={}&2\int_{\partial\Omega}\gamma^{ij}Q(D_t\alpha_i,\alpha_j)\nu\,d\mu_\gamma+2\int_{\Omega}g^{kl}\gamma^{ij}Q(D_t\nabla_i\beta_k,\nabla_j\beta_l)d\mu_g\\&+\int_{\partial\Omega}\left(D_t(\gamma^{ij}\gamma^{IJ})+\left(\operatorname{tr}h-h_{NN}+\frac{\nu_t}{\nu}\right)\gamma^{ij}\gamma^{IJ}\right)\alpha_{Ii}\,\alpha_{Jj}\,\nu\,d\mu_\gamma\\&+\int_{\Omega}\big(D_t(g^{kl}\gamma^{ij}\gamma^{IJ})+\operatorname{tr}h\,g^{kl}\gamma^{ij}\gamma^{IJ}\big)\nabla_i\beta_{Ik}\,\nabla_j\beta_{Jl}\,d\mu_g\,.\end{aligned}$$

Since $D_t\gamma^{ij}=-2\gamma^{im}\gamma^{jn}h_{mn}$, the second line is bounded by the boundary term in the energy E, and the third line is bounded by $\|\nabla\beta\|^2_{L^2(\Omega)}$. The second term on the first line is bounded by $\|\operatorname{curl}\alpha\|_{L^2(\Omega)}\sqrt{E}$ plus

$$\begin{aligned}&\int_{\Omega}g^{kl}\gamma^{ij}\gamma^{IJ}\nabla_k\alpha_{Ii}\,\nabla_j\beta_{Jl}\,d\mu_g=\\&\qquad\int_{\partial\Omega}N^l\gamma^{ij}\gamma^{IJ}\alpha_{Ii}\,\nabla_j\beta_{Jl}\,d\mu_\gamma+\int_{\Omega}\gamma^{ik}g^{jl}\gamma^{IJ}\alpha_{Ii}\,\nabla_k\nabla_j\beta_{Jl}\,d\mu_g\\&\qquad\qquad-\int_{\Omega}\nabla_k\left(g^{kl}\gamma^{ij}\gamma^{IJ}\right)\alpha_{Ii}\,\nabla_j\beta_{Jl}\,d\mu_g\,,\end{aligned}$$

where we have used Lemma 5.3. The first term on the second line is bounded by $\|\alpha\|_{L^2(\Omega)}\|\nabla\operatorname{div}\beta\|_{L^2(\Omega)}$, and the second by $K\|\alpha\|_{L^2(\Omega)}\|\nabla\beta\|_{L^2(\Omega)}$. Recall now that by Lemma 5.5

$$\|\nabla\beta\|^2_{L^2(\Omega)}\le CE+C\big(\|\operatorname{div}\beta\|_{L^2(\Omega)}+\|\operatorname{curl}\beta\|_{L^2(\Omega)}\big)^2\,.$$

This proves Proposition 5.12. □

6 Euler's Equations and Higher-Order Derived Equations

Recall Euler's equations

$$D_t v_i + \partial_i p = 0\,, \qquad \partial_i v^i = 0\,, \tag{6.1}$$

where

$$D_t = \frac{d}{dt}\Big|_{y=\text{const}} = \frac{d}{dt}\Big|_{x=\text{const}} + v^k \partial_k \quad \text{and} \quad \partial_i = \frac{\partial}{\partial x^i} = \frac{\partial y^d}{\partial x^i}\frac{\partial}{\partial y^d}\,. \tag{6.2}$$

We now want to get higher-order versions of (6.1) in terms of higher-order tensors $\partial^r v_i$. By Lemma 2.3

$$D_t \partial^r v_i + \partial^r \partial_i p = -\sum_{s=0}^{r-1} \binom{r}{s+1} (\partial^{1+s} v) \cdot \partial^{r-s} v_i\,. \tag{6.3}$$

In particular, if $r = 1$,

$$D_t \partial_i v_j + \partial_i \partial_j p = -(\partial_i v^k)\partial_k v_j\,. \tag{6.4}$$

We now want to change coordinates and calculate $D_t \nabla^r u$. By Lemma 2.2,

$$\begin{aligned} &D_t \nabla_{a_1} \cdots \nabla_{a_r} u_a \\ &\quad= \frac{\partial x^{i_1}}{\partial y^{a_1}} \cdots \frac{\partial x^{i_r}}{\partial y^{a_r}} \frac{\partial x^i}{\partial y^a} \partial_{i_1} \cdots \partial_{i_r} v_i \\ &\quad= \frac{\partial x^{i_1}}{\partial y^{a_1}} \cdots \frac{\partial x^{i_r}}{\partial y^{a_r}} \frac{\partial x^i}{\partial y^a} \\ &\qquad \Big(\partial_t \partial_{i_1} \cdots \partial_{i_r} v_i + \frac{\partial v^l}{\partial x^{i_1}} \partial_l \cdots \partial_{i_r} v_i + \cdots + \frac{\partial v^l}{\partial x^{i_r}} \partial_{i_1} \cdots \partial_l v_i \\ &\qquad\quad + \frac{\partial v^l}{\partial x^i} \partial_{i_1} \cdots \partial_{i_r} v_l \Big)\,. \end{aligned} \tag{6.5}$$

It follows from (6.4) and (6.5) that

$$\begin{aligned} &D_t \nabla^r u_a + \nabla^r \nabla_a p \\ &\quad= -\sum_{s=1}^{r-1} \binom{r}{s+1} (\nabla^{1+s} u) \cdot \nabla^{r-s} u_a + (\nabla_a u^c) \nabla^r u_c \\ &\quad= (\nabla_a u_c - \nabla_c u_a) \nabla^r u^c - \sum_{s=1}^{r-2} \binom{r}{s+1} (\nabla^{1+s} u) \cdot \nabla^{r-s} u_a\,. \end{aligned} \tag{6.6}$$

In particular, if $r = 1$, we get

$$D_t \nabla_a u_b + \nabla_a \nabla_b p = (\nabla_a u^c) \nabla_b u_c\,, \tag{6.7}$$

so

$$D_t(\nabla_a u_b - \nabla_b u_a) = 0\,. \tag{6.8}$$

The higher-order Euler's equations (6.3) or (6.6) will be used in the interior together with the facts that

$$\operatorname{div} v = 0\,, \qquad D_t \operatorname{curl} v = O(\nabla v)\,. \tag{6.9}$$

On the boundary we will instead use an equation which has to do with the geometry of the boundary that depends only on Euler's equations indirectly through the change of coordinates. By Lemma 2.3,

$$\begin{aligned} D_t \partial_i p &= \partial_i D_t p - (\partial_i v^k)\partial_k p\,,\\ D_t \partial_i \partial_j p &= \partial_i \partial_j D_t p - (\partial_i v^k)\partial_k \partial_j p - (\partial_i v^k)\partial_k \partial_j p + (\partial_i \partial_j v^k)\partial_k p\,. \end{aligned} \tag{6.10}$$

It is, however, more convenient to formulate the higher-order version for $D_t \nabla^r p$. By Lemma 2.4

$$\begin{aligned} D_t \nabla^r p &= \nabla^r D_t p - \sum_{s=1}^{r-1} \binom{r}{s+1} (\nabla^{1+s} u)\cdot \nabla^{r-s} p\\ &= \nabla^r D_t p - (\nabla^r u)\cdot \nabla p - \sum_{s=1}^{r-2} \binom{r}{s+1} (\nabla^{1+s} u)\cdot \nabla^{r-s} p\,. \end{aligned} \tag{6.11}$$

We also want to calculate equations for p. By (6.1)

$$0 = D_t(\delta^{ij}\partial_i v_j) = \delta^{ij}\partial_i D_t v_j - \delta^{ij}(\partial_i v^k)\partial_k v_j$$

so

$$\triangle p = -(\partial_i v^k)\partial_k v^i\,. \tag{6.12}$$

Since $\triangle$ is invariant, we also have

$$\triangle p = -(\nabla_a u^b)\nabla_b u^a = -g^{ab}g^{cd}(\nabla_a u_d)\nabla_c u_b = -\operatorname{tr}\big((\nabla u)^2\big)\,, \tag{6.13}$$

where we used the notation $(\nabla u)^2_{ab} = \big((\nabla u)\cdot \nabla u\big)_{ab} = (\nabla_a u^c)\nabla_c u_b$ and the trace of a tensor is defined to be the trace over the first and last indices. It follows that

$$\nabla^r \triangle p = -\nabla^r\big(\operatorname{tr}(\nabla u)^2\big) = -\sum_{s=0}^{r} \binom{r}{s} (\nabla^{r-s}\nabla_a u)\cdot \nabla^{s+1} u^a\,. \tag{6.14}$$

By Lemma 2.4

$$\begin{aligned} \triangle D_t p &= -D_t\big(g^{ab}g^{cd}(\nabla_a u_d)\nabla_c u_b\big) + h^{ab}\nabla_a \nabla_b p + (\triangle u^e)\nabla_e p\\ &= 2g^{ab}h^{cd}(\nabla_a u_d)\nabla_c u_b + 2g^{ab}g^{cd}(\nabla_a u_d)\big(\nabla_c\nabla_b p - (\nabla_c u^e)\nabla_b u_e)\big)\\ &\quad + h^{ab}\nabla_a\nabla_b p - (\triangle u^e)\nabla_e p\\ &= 4g^{ab}g^{cd}(\nabla_a u_c)\nabla_b\nabla_d p + 2(\nabla_a u^d)(\nabla_d u^c)\nabla_c u^a - (\triangle u^e)\nabla_e p\,, \end{aligned}$$

since $D_t g^{ab} = -h^{ab}$, $h_{ab} = \nabla_a u_b + \nabla_b u_a$. To write things in a more appealing way, we will use the notation $(\nabla u)^3_{ab} = \big((\nabla u)\cdot(\nabla u)\cdot\nabla u\big)_{ab} = (\nabla_a u^d)(\nabla_d u^c)\nabla_c u_b$ and $\big((\nabla u)\cdot\nabla^2 p\big)_{ab} = (\nabla_a u^d)\nabla_d\nabla_b p$,

$$\triangle D_t p = 4\,\mathrm{tr}\big((\nabla u)\cdot\nabla^2 p\big) + 2\,\mathrm{tr}\big((\nabla u)^3\big) - (\triangle u)\cdot\nabla p\,, \tag{6.15}$$

and hence

$$\nabla^{r-2}\triangle D_t p = \nabla^{r-2}\big(4\,\mathrm{tr}\big((\nabla u)\cdot\nabla^2 p\big) + 2\,\mathrm{tr}\big((\nabla u)^3\big) - (\triangle u)\cdot\nabla p\big)\,. \tag{6.16}$$

The exact interpretations of what the dot product and trace mean are not so important since the right-hand side will be lower order and since ∇^{r-2} will be subject to Leibniz' rule. Summing up, we have the following:

LEMMA 6.1

$$\big|D_t\nabla^r u + \nabla^{r+1}p\big| + \big|D_t\nabla^{r-1}\,\mathrm{curl}\,u\big| + \big|\nabla^{r-1}\triangle p\big| \le C\sum_{s=0}^{r-1}\big|\nabla^{1+s}u\big|\,\big|\nabla^{r-s}u\big|, \tag{6.17}$$

$$\big|\Pi\big(D_t\nabla^r p + (\nabla^r u)\cdot\nabla p - \nabla^r D_t p\big)\big| \le C\sum_{s=1}^{r-2}\big|\Pi\big((\nabla^{1+s}u)\cdot\nabla^{r-s}p\big)\big|, \tag{6.18}$$

and

$$\big|\nabla^{r-2}\triangle D_t p - (\nabla^{r-2}\triangle u)\cdot\nabla p\big| \le C\sum_{s=0}^{r-2}\big|\nabla^{1+s}u\big|\big|\nabla^{r-s}p\big| + C\sum_{r_1+r_2+r_3=r-2}\big|\nabla^{1+r_1}u\big|\big|\nabla^{1+r_2}u\big|\big|\nabla^{1+r_3}u\big|\,. \tag{6.19}$$

7 Energy Estimates for Euler's Equations

Let

$$E_r(t) = \int_\Omega g^{mn}\gamma^{ij}\,Q\big(\nabla^{r-1}\nabla_i u_m, \nabla^{r-1}\nabla_j u_n\big)d\mu + \int_\Omega |\nabla^{r-1}\,\mathrm{curl}\,u|^2\,d\mu + \int_{\partial\Omega}\gamma^{ij}\,Q\big(\nabla^{r-1}\nabla_i p, \nabla^{r-1}\nabla_j p\big)\nu\,d\mu_\gamma \tag{7.1}$$

where $\nu = 1/(-\nabla_N p)$. We will prove that there are continuous functions C_r such that

$$\left|\frac{dE_r(t)}{dt}\right| \le C_r\left(K, \frac{1}{\varepsilon}, L, M, \mathrm{Vol}\,\Omega, \sum_{s=0}^{r-1}E_s(t)\right)\sum_{s=0}^{r}E_s(t) \tag{7.2}$$

if $0 \le r \le 4$ or $r \ge n/2 + 3/2$, provided that some a priori assumptions are true:

$$|\theta| + 1/\iota_0 \le K \quad \text{on } [0, T]\times\partial\Omega\,, \tag{7.3}$$

$$-\nabla_N p \geq \varepsilon > 0 \quad \text{on } [0,T]\times\partial\Omega\,, \tag{7.4}$$

$$|\nabla^2 p| + |\nabla_N p_t| \leq L \quad \text{on } [0,T]\times\partial\Omega\,. \tag{7.5}$$

Since $h_{ab} = \nabla_a u_b + \nabla_b u_a$, the bound for $|h|$ of course follows from the bound for $|\nabla u|$. We also assume

$$|\nabla p| \leq M\,, \quad |\nabla u| \leq M\,, \quad \text{in } [0,T]\times\Omega\,. \tag{7.6}$$

It is not clear to what extent we need the bound for $\nabla^2 p$, but it is natural to assume it, since $\triangle p = -\operatorname{tr}(\nabla u)^2$ and $\Pi\nabla^2 p = \theta\nabla_N p$. The bound for $\nabla^2 p$ together with (7.4) of course implies the bound for θ.

Remark. Instead of the energy (7.1) coming from Proposition 5.12, we could alternatively have used the energy coming from Proposition 5.11. The one we use gives a better control of $\|\nabla^r u\|_{L^2(\Omega)}$, which is needed to prove Theorem 7.2 below with minimal r_0, but it only works when $\operatorname{div} u = 0$.

Since $E_0(t) = \int_\Omega |v|^2\,d\mu = E_0(0)$ and $\operatorname{Vol}\Omega(t) = \operatorname{Vol}\Omega(0)$, we get the following recursively from (7.2):

THEOREM 7.1 *If $r \geq 0$ and $n \leq 7$, then there are continuous functions $\mathcal{F}_r$, with $\mathcal{F}_r|_{t=0} = 1$, such that for any smooth solution of Euler's equations* (1.1)–(1.5) *for $0 \leq t \leq T$ satisfying* (7.3)–(7.6), *we have*

$$\sum_{s=0}^{r} E_s(t) \leq \mathcal{F}_r\left(t, K, \frac{1}{\varepsilon}, L, M, E_0(0), \dots, E_{r-1}(0), \operatorname{Vol}\Omega\right)\sum_{s=0}^{r} E_s(0)\,, \quad 0 \leq t \leq T\,. \tag{7.7}$$

Let $\mathcal{K}(t)$ and $\varepsilon(t)$ be the maximum and minimum values, respectively, such that (7.3)–(7.4) hold at time t:

$$\mathcal{K}(t) = \max\left(\|\theta(t,\cdot)\|_{L^\infty(\partial\Omega)}, \frac{1}{\iota_0(t)}\right),$$
$$\mathcal{E}(t) = \|(\nabla_N p(t,\cdot))^{-1}\|_{L^\infty(\partial\Omega)} = \frac{1}{\varepsilon(t)}\,. \tag{7.8}$$

THEOREM 7.2 *Let $r \geq r_0 > n/2+3/2$. Then there is a continuous function $\mathcal{T}_r > 0$ such that if*

$$T \leq \mathcal{T}_r\big(\mathcal{K}(0), \mathcal{E}(0), E_0(0), \dots, E_{r_0}(0), \operatorname{Vol}\Omega\big)\,, \tag{7.9}$$

any smooth solution of the free boundary problem for Euler's equations (1.1)–(1.5) *for $0 \leq t \leq T$ satisfies*

$$\sum_{s=0}^{r} E_s(t) \leq 2\sum_{s=0}^{r} E_s(0)\,, \quad 0 \leq t \leq T\,. \tag{7.10}$$

7.1 Proof of Theorem 7.1

In the proof it is convenient to replace the a priori bound (7.3) by

$$|\theta| \le K', \quad \frac{1}{\iota_1} \le K_1; \tag{7.11}$$

see Definition 3.4 for ι_0 and Definition 3.5 for ι_1. However, by Lemma 3.6,

$$\frac{1}{\iota_0} \le \max\left(\frac{K_1}{2}, \|\theta\|_{L^\infty}\right) \quad \text{and} \quad \frac{1}{\iota_1} \le \max\left(\frac{\|\theta\|_{L^\infty}}{\varepsilon_1}, \frac{1}{2\iota_0}\right). \tag{7.12}$$

Now, to get the iteration started we need bounds for some low norms. For u, $E_0 = \|u\|^2_{L^2(\Omega)}$ is conserved, but we cannot control the low norms of p and p_t in terms of the energies only. Thus to control these we must use the fact that the $\operatorname{Vol}\Omega$ is conserved.

Before starting with the proof of (7.2), let us first see what a bound for the energy (7.1) implies.

LEMMA 7.3 *We have*

$$\|\nabla^r u\|^2_{L^2(\Omega)} \le C E_r, \quad \|\Pi\nabla^r p\|^2_{L^2(\partial\Omega)} \le \|\nabla p\|_{L^\infty(\partial\Omega)} E_r, \tag{7.13}$$

$$\|\nabla^r p\|^2_{L^2(\partial\Omega)} + \|\nabla^r p\|^2_{L^2(\Omega)} \le C(K_1, \operatorname{Vol}\Omega)\big(\|\nabla p\|_{L^\infty(\partial\Omega)} + \|\nabla u\|^2_{L^\infty(\Omega)}\big)\sum_{k=0}^{r} E_k. \tag{7.14}$$

PROOF OF LEMMA 7.3: That $\|\Pi\nabla^r p\|_{L^2(\partial\Omega)} \le \|\partial p\|_{L^\infty(\partial\Omega)} E_r$ follows from the definition of the projection, $\gamma^{ij} Q(\alpha_i, \alpha_j) = |\Pi\alpha|^2$ on $\partial\Omega$, and the fact that the measure in the energy is $(-\nabla_N p)^{-1} dS$. Since $\operatorname{div} u = 0$, the bound $\|\nabla^r u\|^2_{L^2(\Omega)} \le C E_r$ follows from Lemma 5.5. By Lemmas 6.1 and A.3

$$\|\nabla^{r-1} \triangle p\|_{L^2(\Omega)} \le C\|\nabla u\|_{L^\infty(\Omega)} \sum_{k=0}^{r} K_1^{r-\ell} \|\nabla^k u\|_{L^2(\Omega)}.$$

(7.14) follows from (5.28) in Proposition 5.8 and the second part of (A.17) in Lemma A.5. □

The most interesting observation is now that the bounds in particular of the boundary term in Lemma 7.3 actually imply a bound on the second fundamental form of the boundary:

LEMMA 7.4 *With $L^\infty = L^\infty(\partial\Omega)$ we have*

$$\|\overline{\nabla}^{r-2}\theta\|^2_{L^2} \le C\Bigg(K_1, \|\theta\|_{L^\infty}, \|(\nabla_N p)^{-1}\|_{L^\infty}, \|\nabla p\|_{L^\infty}, \|\nabla u\|_{L^\infty(\Omega)}, \operatorname{Vol}\Omega, \sum_{s=0}^{r-1} E_s(t)\Bigg) \sum_{s=0}^{r} E_s(t). \tag{7.15}$$

PROOF: Lemma 7.4 is of course just (5.32) in Proposition 5.9 and (7.14) in Lemma 7.3, the crucial point being a lower bound $-\nabla_N p > \varepsilon > 0$. □

Lemma 7.3 suffices to control the interior terms, as we shall see. To control the boundary terms, it turns out that the crucial point is to estimate

$$\|\Pi \nabla^r D_t p\|_{L^2(\partial\Omega)} ,$$

which uses the bound in Lemma 7.4 to estimate $\|\nabla^{r-2} \triangle D_t p\|_{L^2(\Omega)}$. We have the following:

LEMMA 7.5 *Let $p_t = D_t p$ and $L^\infty = L^\infty(\partial\Omega)$. We have*

$$\|\Pi \nabla^r p_t\|^2_{L^2(\partial\Omega)} + \|\nabla^{r-1} p_t\|^2_{L^2(\partial\Omega)} + \|\nabla^r p_t\|^2_{L^2(\Omega)} \le C\Bigg(K_1, \|\theta\|_{L^\infty}, \|(\nabla_N p)^{-1}\|_{L^\infty}, \|\nabla p\|_{L^\infty}, \|\nabla u\|_{L^\infty(\Omega)}, \|\nabla_N p_t\|_{L^\infty}, \operatorname{Vol}\Omega, \sum_{s=0}^{r-1} E_s(t)\Bigg) \sum_{s=0}^{r} E_s(t) . \tag{7.16}$$

PROOF: By Lemmas 6.1 and A.3

$$\|\nabla^{r-2} \triangle D_t p\|_{L^2(\Omega)} \le C(K_1)\big(\|\nabla p\|_{L^\infty(\Omega)} + \|\nabla u\|_{L^\infty(\Omega)}\big) \Bigg(\sum_{k=0}^{r} \big(\|\nabla^k u\|_{L^2(\Omega)} + \|\nabla^k p\|_{L^2(\Omega)}\big)\Bigg) + C(K_1)\|\nabla u\|^2_{L^\infty(\Omega)} \sum_{k=0}^{r-1} \|\nabla^k u\|_{L^2(\Omega)} . \tag{7.17}$$

The bound in (7.16) for $\|\nabla^{r-1} p_t\|_{L^2(\partial\Omega)}$ is just (5.33) in Proposition 5.10 together with (7.17) and Lemmas 7.3 and 7.4. The bound for $\|\Pi \nabla^r p_t\|_{L^2(\partial\Omega)}$ follows (5.30) in Proposition 5.9 and the bound just obtained for $\|\nabla^s p_t\|_{L^2(\partial\Omega)}$ for $s \le r - 1$. Finally, the bound for $\|\nabla^r p_t\|_{L^2(\Omega)}$ follows from (5.29) in Proposition 5.8 and the bounds for $\|\nabla^{r-1} p_t\|_{L^2(\partial\Omega)}$ and $\|\Pi \nabla^r p_t\|_{L^2(\partial\Omega)}$ just obtained. □

After having seen what a bound for the energy implies, we now want to prove (7.2). The main ingredient is Proposition 5.12 applied to $\alpha = -\nabla^r p$, $\beta = \nabla^{r-1} u$, and $\nu = 1/(-\nabla_N p)$. Then $\operatorname{div} \beta = 0$ and $\operatorname{curl} \alpha = 0$, so we get from Proposition 5.12 and Lemma 7.3

$$\frac{dE_r}{dt} \le C\big(K_1, \|\theta\|_{L^\infty}, \|(\nabla_N p)^{-1}\|_{L^\infty}, \|\nabla p\|_{L^\infty}, \|\nabla u\|_{L^\infty(\Omega)}\big) E_r + C\sqrt{E_r}\big(\|\Pi(-D_t \nabla^r p + \nu N^k \nabla^r u_k)\|_{L^2(\partial\Omega)} + \|D_t \nabla^r u + \nabla^{r+1} p\|_{L^2(\Omega)} + \|D_t \nabla^{r-1} \operatorname{curl} u\|_{L^2(\Omega)}\big) . \tag{7.18}$$

Using Lemmas 6.1 and A.3, we can directly control the interior terms in (7.18):

$$\|D_t\nabla^r u + \nabla^{r+1} p\|_{L^2(\Omega)} + \|D_t\nabla^{r-1}\operatorname{curl} u\|_{L^2(\Omega)} \leq C\|\nabla u\|_{L^\infty(\Omega)}\sum_{k=0}^{r} K_1^{r-k}\|\nabla^k u\|_{L^2(\Omega)}\,. \tag{7.19}$$

Hence it only remains to control the boundary term in (7.18). By Lemma 6.1,

$$\|\Pi\big(D_t\nabla^r p + (\nabla^r u)\cdot\nabla p\big)\|_{L^2(\partial\Omega)} \leq \|\Pi\nabla^r D_t p\|_{L^2(\partial\Omega)} + C\sum_{s=1}^{r-2}\big\|\Pi\big((\nabla^{1+s}u)\cdot\nabla^{r-s}p\big)\big\|_{L^2(\partial\Omega)}\,. \tag{7.20}$$

Since the first term in the right-hand side of (7.20) is controlled by Lemma 7.5, it only remains to estimate

$$\big\|\Pi\big((\nabla^{1+s}u)\cdot\nabla^{r-s}p\big)\big\|_{L^2(\partial\Omega)} \quad \text{for } 1\leq s\leq r-2\,. \tag{7.21}$$

Clearly these terms are lower order, so there is no problem in estimating them, say, using Sobolev's lemma to bound them with interior norms. However, in order to get a bound that is linear in the highest-order derivative provided the a priori assumptions (7.3)–(7.6) hold, we must work a bit harder. Let us therefore look at the endpoints. If $s = r-2$, this can be estimated by

$$\|\nabla^2 p\|_{L^\infty(\partial\Omega)}\|\nabla^{r-1}u\|_{L^2(\partial\Omega)} \leq CL\left(\sum_{k=0}^{r} E_k\right)^{1/2} \tag{7.22}$$

where we used the a priori assumption (7.5) and Sobolev's lemma (Lemma A.2),

$$\|\nabla^{r-1}u\|_{L^2(\partial\Omega)} \leq C\|\nabla^{r-1}u\|_{L^{2(n-1)/(n-2)}(\partial\Omega)} \leq C(K_1)\sum_{k=0}^{r}\|\nabla^k u\|_{L^2(\Omega)}\,. \tag{7.23}$$

If $s = 0$ (which actually is excluded), we could estimate it with

$$\|\nabla u\|_{L^\infty(\partial\Omega)}\|\nabla^r p\|_{L^2(\partial\Omega)} \leq C(K_1)M\left(\sum_{k=0}^{r} E_k\right)^{1/2} \tag{7.24}$$

by Lemma 7.3. Hence, we must now somehow control the intermediate terms. If the derivatives were tangential, we could do this with the interpolation inequality Lemma A.1. But because of the projection to the tangential components in (7.21), the highest-order derivatives will be mostly tangential. By (4.48)

$$\begin{aligned}\big\|\Pi\big((\nabla^{1+s}u)\cdot\nabla^{r-s}p\big)\big\|_{L^2(\partial\Omega)} &\leq \big\|\big|\Pi(\nabla^{1+s}u)\big|\,\big|\Pi\nabla^{r-s}p\big|\big\|_{L^2(\partial\Omega)}\\ &\quad + \big\|\big|\Pi(N^k\nabla^{1+s}u_k)\big|\,\big|\Pi N^k\nabla^{r-1-s}\nabla_k p\big|\big\|_{L^2(\partial\Omega)}\end{aligned} \tag{7.25}$$

$$\le \|\Pi(\nabla^{1+s}u)\|_{L^{2(r-2)/s}(\partial\Omega)}\,\|\Pi\nabla^{r-s}p\|_{L^{2(r-2)/(r-2-s)}(\partial\Omega)}$$
$$+\|\Pi\big(N^k\nabla^{1+s}u_k\big)\|_{L^{2(r-2)/s}(\partial\Omega)}\,\|\Pi N^k\nabla^{r-1-s}\nabla_k p\|_{L^{2(r-2)/(r-2-s)}(\partial\Omega)}\,.$$

These terms can now be estimated by (4.46) in Proposition 4.11 with $\alpha = \nabla u$ and $\beta = \nabla^2 p$. This concludes the proof of Theorem 7.1.

7.2 Proof of Theorem 7.2

Let us now show how Theorem 7.2 follows. We will be using Sobolev's lemma (Lemmas A.2–A.4). But then we must first make sure that we can control the Sobolev constants. By the results in the appendix, these depend on the constant $K_1 = 1/\iota_1$ in Definition 3.5. Alternatively, the change of the Sobolev constants in time are controlled by a bound for the time derivative of the metric in the y-coordinates; see the appendix. We also need to have control of the constant $1/\varepsilon$. We have the following:

LEMMA 7.6 *Let K_1 be as in Definition* 3.5*, $\mathcal{E}(t)$ as in* (7.8)*, and $r_0 > n/2 + 3/2$. Then there are continuous functions G_{r_0}, H_{r_0}, I_{r_0} and J_{r_0} such that*

$$\|\nabla u\|_{L^\infty(\Omega)} \le G_{r_0}(K_1, E_0, \dots, E_{r_0})\,, \tag{7.26}$$

$$\|\nabla p\|_{L^\infty(\Omega)} + \|\nabla^2 p\|_{L^\infty(\partial\Omega)} \le H_{r_0}(K_1, E_0, \dots, E_{r_0}, \operatorname{Vol}\Omega)\,, \tag{7.27}$$

$$\|\theta\|_{L^\infty(\partial\Omega)} \le I_{r_0}(K_1, \mathcal{E}, E_0, \dots, E_{r_0}, \operatorname{Vol}\Omega)\,, \tag{7.28}$$

$$\|\nabla p_t\|_{L^\infty(\partial\Omega)} \le J_{r_0}(K_1, \mathcal{E}, E_0, \cdots, E_{r_0}, \operatorname{Vol}\Omega)\,. \tag{7.29}$$

PROOF: By Sobolev's lemma

$$\|\nabla u\|_{L^\infty(\Omega)} \le C(K_1)\sum_{s\le r}\|\nabla^s u\|_{L^2(\Omega)}\,, \qquad r-1 > \frac{n}{2}\,, \tag{7.30}$$

$$\|\nabla p\|_{L^\infty(\Omega)} \le C(K_1)\sum_{s\le r}\|\nabla^s p\|_{L^2(\Omega)}\,, \qquad r-1 > \frac{n}{2}\,, \tag{7.31}$$

$$\|\nabla^2 p\|_{L^\infty(\partial\Omega)} \le C(K_1)\sum_{s\le r}\|\nabla^s p\|_{L^2(\partial\Omega)}\,, \qquad r-2 > \frac{n-1}{2}\,. \tag{7.32}$$

(7.26) follows from (7.30) and (7.13) in Lemma 7.3, and (7.27) follows from (7.31), (7.14), and (7.26). (Note that p enters quadratic in the left-hand side of (7.14) but only linear in the right-hand side.) (7.32) follows in the same way. The bounds for $\|\theta\|_{L^\infty}$ and $\|\nabla p_t\|_{L^\infty}$ cannot be obtained directly by Sobolev's lemma since the right-hand side of (7.15) depends on $\|\theta\|_{L^\infty}$ and the right-hand side of (7.16) depends on $\|\nabla p_t\|_{L^\infty}$. However,

$$|\nabla^2 p| \ge |\Pi\nabla^2 p| = |\nabla_N p|\,|\theta| \ge \mathcal{E}^{-1}|\theta|\,, \tag{7.33}$$

so (7.28) follows from (7.27). (7.29) follows from (5.34) in Proposition 5.10. □

LEMMA 7.7 *Let* $K_1 \geq 1/\iota_1$ *and* $\varepsilon_1 = \varepsilon_1(r)$ *be as in Definition* 3.5 *and Lemma* 5.7. *Then if* $r_0 > n/2 + 3/2$,

$$\left|\frac{d}{dt}E_r\right| \leq C_r(K_1, \mathcal{E}, E_0, \dots, E_{r_0}, \operatorname{Vol}\Omega) \sum_{s=0}^{r} E_s \tag{7.34}$$

and

$$\left|\frac{d}{dt}\mathcal{E}\right| \leq C_r(K_1, \mathcal{E}, E_0, \dots, E_{r_0}, \operatorname{Vol}\Omega)\,. \tag{7.35}$$

PROOF: (7.34) is a consequence of Lemma 7.6 and the estimates in the proof of Theorem 7.1. (7.35) follows from

$$\left|\frac{d}{dt}\|(-\nabla_N p(t,\cdot))^{-1}\|_{L^\infty(\partial\Omega)}\right| \leq C\left\|\left(-\nabla_N p(t,\cdot)\right)^{-1}\right\|^2_{L^\infty(\partial\Omega)}\|\nabla_N p_t(t,\cdot)\|_{L^\infty(\partial\Omega)}$$

and (7.29). □

As a result of Lemma 7.7, we get the following:

LEMMA 7.8 *If* $r \geq r_0$, *there is continuous function* $\mathcal{T}_r(K_1, \mathcal{E}(0), E_0(0), \dots, E_r(0), \operatorname{Vol}\Omega) > 0$ *such that for*

$$0 \leq t \leq \mathcal{T}_r\big(K_1, \mathcal{E}(0), E_0(0), \dots, E_r(0), \operatorname{Vol}\Omega\big) \tag{7.36}$$

the following statements hold: *We have*

$$E_s(t) \leq 2E_s(0)\,, \quad 0 \leq s \leq r\,, \qquad \mathcal{E}(t) \leq 2\mathcal{E}(0)\,. \tag{7.37}$$

Furthermore,

$$\frac{g_{ij}(0,y)X^iX^j}{2} \leq g_{ij}(t,y)X^iX^j \leq 2g_{ij}(0,y)X^iX^j\,, \tag{7.38}$$

and with $\varepsilon_1(r) > 0$ *as in Lemma* 5.7,

$$|\mathcal{N}(x(t,\overline{y})) - \mathcal{N}(x(0,\overline{y})| \leq \frac{\varepsilon_1(r)}{1}6\,, \quad \overline{y} \in \partial\Omega\,, \tag{7.39}$$

$$|x(t,y) - x(0,y)| \leq \frac{\iota_1}{16}\,, \qquad y \in \Omega\,, \tag{7.40}$$

$$\left|\frac{\partial x(t,\overline{y})}{\partial y} - \frac{\partial x(0,\overline{y})}{\partial y}\right| \leq \frac{\varepsilon_1(r)}{16}\,, \qquad \overline{y} \in \partial\Omega\,. \tag{7.41}$$

PROOF: We get (7.37) from Lemma 7.7 if $\mathcal{T}_r(K_1, \mathcal{E}(0), E_0(0), \dots, E_r(0), \operatorname{Vol}\Omega) > 0$ is sufficiently small. We have

$$\|\nabla u\|_{L^\infty(\Omega)} + \|\nabla p\|_{L^\infty(\Omega)} \leq C(K_1, \mathcal{E}(0), E_0(0), \dots, E_{r_0}(0))\,, \tag{7.42}$$

$$\|\nabla^2 p\|_{L^\infty(\partial\Omega)} + \|\theta\|_{L^\infty(\partial\Omega)} \leq C(K_1, \mathcal{E}(0), E_0(0), \dots, E_{r_0}(0), \operatorname{Vol}\Omega)\,, \tag{7.43}$$

$$\|\nabla p_t\|_{L^\infty(\Omega)} \leq D(K_1, \mathcal{E}(0), E_0(0), \cdots, E_{r_0}(0), \operatorname{Vol}\Omega)\,. \tag{7.44}$$

In fact, (7.42)–(7.44) follows from (7.37) and Lemma 7.6. It follows from this that

$$\|\nabla u(t,\cdot)\|_{L^\infty(\partial\Omega)} \leq 2\|\nabla u(0,\cdot)\|_{L^\infty(\partial\Omega)}\,, \tag{7.45}$$

$$\|\nabla p(t,\cdot)\|_{L^\infty(\Omega)} \le 2\|\nabla p(0,\cdot)\|_{L^\infty(\Omega)}\,, \tag{7.46}$$

$$\|v(t,\cdot)\|_{L^\infty(\Omega)} \le 2\|v(0,\cdot)\|_{L^\infty(\Omega)}\,. \tag{7.47}$$

In fact, by (6.7) we have

$$|D_t\nabla u| \le |\nabla^2 p| + |\nabla u|^2\,, \quad |D_t\partial v| \le |\partial^2 p| + |\partial v|^2\,. \tag{7.48}$$

Using (7.42)–(7.44) we get that

$$\int_0^T \|\nabla^2 p(t,\cdot)\|_{L^\infty} + \|\nabla u(t,\cdot)\|_{L^\infty}^2\, dt \le \|\nabla u(0,\cdot)\|_{L^\infty} \tag{7.49}$$

if T is sufficiently small, so (7.45) follows after possibly making $\mathcal{T} > 0$ smaller. (7.46) and (7.47) follow in a similar manner from $|D_t\nabla p| = |\nabla p_t|$ and $|D_t v| = |\partial p|$, respectively.

Also, (7.38) follows from the same argument since

$$D_t g_{ab} = h_{ab} = \nabla_a u_b + \nabla_b u_a \tag{7.50}$$

and by (7.44)

$$2\int_0^T \|\nabla_a u_b\|_{L^\infty(\Omega)}\, dt X^a X^b \le \frac{g_{ab}X^a X^b}{2} \tag{7.51}$$

if T is sufficiently small. Now the estimate for $\mathcal{N}$ follows from

$$D_t n_a = h_{NN} n_a\,, \tag{7.52}$$

and the estimates for x and $\partial x/\partial y$ from

$$\begin{aligned} &D_t x(t,y) = v(t,x(t,y))\,,\\ &D_t\frac{\partial x}{\partial y} = \frac{\partial v(t,x(t,y))}{\partial y} = \frac{\partial v(t,x)}{\partial x}\frac{\partial x}{\partial y}\,, \end{aligned} \tag{7.53}$$

and (7.47) and (7.45), respectively. □

The idea is now to use (7.38)–(7.41) to pick a K_1, i.e., ι_1 (see Definition 3.5), which depends only on its value at $t = 0$,

$$\iota_1(t) \ge \frac{\iota_1(0)}{2}\,. \tag{7.54}$$

LEMMA 7.9 *Suppose that* $\varepsilon_1(r)/2 \le \varepsilon_1 \le \varepsilon_1(r)$, *and let* $\mathcal{T}$ *be as in Lemma* 7.7. *Pick* $\iota_1 > 0$ *such that*

$$|\mathcal{N}(x(0,y_1)) - \mathcal{N}(x(0,y_2))| \le \frac{\varepsilon_1}{2} \quad \textit{whenever } |x_1(0,y_1) - x(0,y_2)| \le 2\iota_1\,. \tag{7.55}$$

Then if $t \le \mathcal{T}$ *we have*

$$|\mathcal{N}(x(t,y_1)) - \mathcal{N}(x(t,y_2))| \le \varepsilon_1 \quad \textit{whenever } |x_1(t,y_1) - x(t,y_2)| \le \iota_1\,. \tag{7.56}$$

PROOF: (7.56) follows from (7.55) and (7.39)–(7.40). □

Theorem 7.2 now follows directly from Lemmas 7.9 and 7.8. Lemma 7.9 allows us to pick a K_1 depending only on initial conditions, while Lemma 7.8 gives us $\mathcal{T} > 0$ that depends only on the initial conditions and K_1 such that, by Lemma 7.9, $1/\iota_1 \le K_1$ for $t \le \mathcal{T}$.

Note that there is also an evolution equation for θ, but using it would require control of one more derivative of u:

$$D_t\theta_{ij} = - \gamma_i^\ell \gamma_j^a N^d \nabla_\ell \nabla_a u_d + N^a N^b \nabla_a u_b \theta_{ij} + 2(\theta_{ia} N_j + \theta_{ja} N_i) g^{ab} N^c \nabla_b u_c \,. \tag{7.57}$$

We can control the size of θ through (7.43), but we cannot control it in terms of initial data without going to energies with one more derivative. This is why we need to estimate all the Sobolev constants in terms of K_1 instead of K, since (7.38)–(7.41) will allow us to control the time evolution of K_1.

Appendix: Sobolev Lemmas and Interpolation Inequalities

Let us now state some Sobolev lemmas and interpolation inequalities. Most of the results here are standard in $\mathbb{R}^n$, but we must control how it depends on the metric. There are two convenient ways to do this. The first is to use the fact that our set expressed in the x-coordinates $\mathcal{D}_t \subset \mathbb{R}^n$ inherits the metric in $\mathbb{R}^n$, and the surface $\partial\mathcal{D}_t$ can be expressed locally as a graph over $\mathbb{R}^{n-1}$.

Let $\mathcal{N}(\overline{x})$ be the unit normal at $\overline{x} \in \partial\mathcal{D}_t$, and suppose that

$$|\mathcal{N}(\overline{x}_1) - \mathcal{N}(\overline{x}_2)| \le \varepsilon_1 \quad \text{whenever } |\overline{x}_1 - \overline{x}_2| \le \iota_1\,, \quad \overline{x}_1\,, \overline{x}_2 \in \partial\mathcal{D}_t\,. \tag{A.1}$$

By (A.1) we can write the surface as a graph within a ball of radius $\iota_1 = 1/K_1$, and for functions supported in such a ball we can thus use Sobolev's lemma in $\mathbb{R}^{n-1}$ or $\mathbb{R}^n$. In general, we make a partition of unity into functions supported in such balls, and the Sobolev constant will thus depend only on K_1.

When controlling how the metric changes with time, we can use that our metrics γ on $\partial\Omega$ and g in Ω are equivalent to the same metrics at $t = 0$ in the y-coordinates:

$$C_0^{-1}\gamma_{ij}^0(y) Z^i Z^j \le \gamma_{ij}(t,y) Z^i Z^j \le C_0 \gamma_{ij}^0(y) Z^i Z^j \quad \text{if } Z \in T(\Omega)\,, \tag{A.2}$$

$$C_0^{-1} g_{ij}^0(y) Z^i Z^j \le g_{ij}(t,y) Z^i Z^j \le C_0 g_{ij}^0(y) Z^i Z^j \quad \text{if } Z \in T(\Omega)\,, \tag{A.3}$$

and use Sobolev's lemma for the metrics γ_{ij}^0 and g_{ij}^0, respectively. In this case, the Sobolev constants depend only on $\gamma_{ij}^0(y) = \gamma_{ij}(0,y)$ and $g_{ij}^0(y) = g_{ij}(0,y)$, respectively, and on C_0.

LEMMA A.1 *If α is a $(0,r)$ tensor, then with $a = k/m$ and a constant C that only depends on m and n,*

$$\|\overline{\nabla}^k \alpha\|_{L^s(\partial\Omega)} \le C \|\alpha\|_{L^q(\partial\Omega)}^{1-a} \|\overline{\nabla}^m \alpha\|_{L^p(\partial\Omega)}^a \quad \textit{if } \frac{m}{s} = \frac{k}{p} + \frac{m-k}{q}, \;\; 2 \le p \le s \le q \le \infty\,. \tag{A.4}$$

PROOF: Let us first prove (A.4) in the case $m = 2$ and $k = 1$. We claim that

$$\|\overline{\nabla}\alpha\|_{L^s}^2 \le C_s \big\| |\alpha| \, |\overline{\nabla}^2\alpha| \big\|_{L^{s/2}} \quad \text{if } s \ge 2 \text{ and } C_s = s - 2 + \sqrt{n-1}\,, \tag{A.5}$$

from which (A.4) follows in the case $m = 2$ and $k = 1$. Then, the norm in the left of (A.4) to the power r is the limit as $\varepsilon \to 0$ of

$$\begin{aligned}\int_{\partial\Omega} &\big(\langle\overline{\nabla}\alpha, \overline{\nabla}\alpha\rangle + \varepsilon\big)^{s/2-1} \langle\overline{\nabla}\alpha, \overline{\nabla}\alpha\rangle d\mu_\gamma \\ &= -\int_{\partial\Omega} \big(\langle\overline{\nabla}\alpha, \overline{\nabla}\alpha\rangle + \varepsilon\big)^{s/2-1} \langle\alpha, \overline{\triangle}\alpha\rangle d\mu_\gamma \\ &\quad - \int_{\partial\Omega} 2\left(\frac{s}{2} - 1\right)\big(\langle\overline{\nabla}\alpha, \overline{\nabla}\alpha\rangle + \varepsilon\big)^{s/2-2} \langle\overline{\nabla}\alpha, \overline{\nabla}^2\alpha\rangle \cdot \langle\alpha, \overline{\nabla}\alpha\rangle d\mu_\gamma\end{aligned}$$

where we have integrated by parts. As $\varepsilon \to 0$ we see that

$$\begin{aligned}\|\overline{\nabla}\alpha\|_{L^s}^s &\le C_s \int \langle\overline{\nabla}\alpha, \overline{\nabla}\alpha\rangle^{s/2-1} |\alpha| |\overline{\nabla}^2\alpha| d\mu_\gamma \\ &\le C_s \|\overline{\nabla}\alpha\|_{L^s}^{s-2} \big\| |\alpha| \, |\overline{\nabla}^2\alpha| \big\|_{L^{s/2}} \,.\end{aligned} \tag{A.6}$$

Dividing both sides by $\|\overline{\nabla}\alpha\|_{L^s}^{s-2}$ gives the desired inequality (A.4).

For fixed m, p, and q, let $s = s(k)$ be defined by (A.4) and set $M_k = \|\nabla^k\alpha\|_{L^{s(k)}}$. Then we have just proven that $M_k^2 \le C_m M_{k-1} M_{k+1}$ for $1 \le k \le m-1$. Hence $N_k = C_m^{k^2} M_k$ satisfies $N_k^2 \le N_{k-1}N_{k+1}$, and this logarithmic convexity implies that $N_k \le N_0^{(m-k)/m} N_m^{k/m}$, which proves (A.4) in general. □

LEMMA A.2 *Suppose that* (A.1) *and* (A.2) *hold with* $\iota_1 \ge 1/K_1$. *Then if* α *is a* $(0, r)$ *tensor,*

$$\|\alpha\|_{L^{(n-1)p/(n-1-kp)}(\partial\Omega)} \le C(K_1) \sum_{\ell=0}^{k} \|\nabla^\ell\alpha\|_{L^p(\partial\Omega)}\,, \qquad 1 \le p < \frac{n-1}{k}\,, \tag{A.7}$$

$$\begin{aligned}&\|\alpha\|_{L^\infty(\partial\Omega)} \le \\ &\qquad \delta\|\nabla^k\alpha\|_{L^p(\partial\Omega)} + C_\delta(K_1) \sum_{0\le\ell\le k-1} \|\nabla^\ell\alpha\|_{L^p(\partial\Omega)}\,, \quad k > \frac{n-1}{p}\,,\end{aligned} \tag{A.8}$$

for any $\delta > 0$.

Remark. For the boundary there are two possible interpretations of (A.7) and (A.8). One is to let the norm be given by the inner product $\langle\alpha, \alpha\rangle = \gamma^{IJ}\alpha_I\alpha_J$ and the covariant differentiation given by $\overline{\nabla}$, which corresponds to covariant differentiation on the boundary. The other interpretation is to let the inner product on the boundary be that of the interior $\langle\alpha, \alpha\rangle = g^{IJ}\alpha_I\alpha_J$ and the covariant differentiation be that of the interior ∇. In fact, in both cases the proof reduces to $k = 1$ as before. If ϕ is a function, then the lemma for ϕ follows from using covariant differentiation on the boundary. Applying this result to a norm gives

$$|\gamma_i^{\ j}\nabla_j\langle\alpha,\alpha\rangle| = 2|\langle\alpha,\gamma_i^{\ j}\nabla_j\alpha\rangle| \le |\alpha|\,|\gamma_i^{\ j}\nabla_j\alpha|\,, \tag{A.9}$$

which is bounded by $|\alpha|\,|\overline{\nabla}\alpha|$ and $|\alpha|\,|\nabla\alpha|$, respectively.

PROOF OF LEMMA A.2: We may assume that $p > n$ and hence $k \le 1$ in (A.8) and $k = 1$ in (A.7). In fact, the general case follows from first using (A.8) and (A.7), respectively, in this case and then repeatedly using (A.7). Second, the case $r > 0$ can be reduced to the case of functions $r = 0$ by applying it to the norms $\phi = |\alpha|$. Hence we may assume that α is a function and $k = 1$.

Using the partition of unity $\{\chi_i\}$ in Lemma 3.8, we write $\phi = \sum_i \phi_i$ where $\phi_i = \chi_i\phi$. The support of each ϕ_i is then contained in a set S_i where the surface can be written as a graph $x_n = f_i(x')$ with $|\partial f_i| \le \varepsilon_1 \le 1$ as in (3.20). Then $dx' \le dS \le C dx'$ and $|\partial_{x'}\phi|/C \le |\overline{\nabla}\phi| \le |\partial_{x'}\phi|$ where $C = (1+\varepsilon_1)^{1/2} \le 2$; thus, apart from a constant factor, Sobolev's lemma on S_i reduces to Sobolev's lemma in $\mathbb{R}^{n-1}$. By using Minkowski's inequality, Sobolev's lemma in $\mathbb{R}^{n-1}$, and Minkowski's inequality again, we get

$$\begin{aligned}\int_{\partial\Omega}\Big(\sum|\phi_i|\Big)^q\,dS &\le 2\sum\int_{B(4r_0,x_i)}|\phi_i|^q\,dx' \\ &\le 2C\sum\Bigg(\int_{B(4r_0,x_i)}|\nabla\phi_i|^p dx'\Bigg)^{q/p} \\ &\le 8C\Bigg(\int_{\partial\Omega}\Big(\sum|\nabla\phi_i|^q\Big)^{p/q}dS\Bigg)^{q/p}\end{aligned} \tag{A.10}$$

since $q > p$. Here

$$\begin{aligned}\Big(\sum|\nabla\phi_i|^q\Big)^{p/q} &= \Big(\sum(|\nabla\chi_i||\phi| + |\chi_i||\nabla\phi|)^q\Big)^{p/q} \\ &\le CK_1^p(32)^{(n-1)p/q}(|\phi|r_0^{-1} + |\nabla\phi|)^p\,,\end{aligned} \tag{A.11}$$

which proves (A.7). (A.8) with δ replaced by a constant follows in the same way. Finally, we get (A.8) by considering (A.8) with δ replaced by a constant and $k = 1$ applied to α replaced by $|\alpha|^2$. In fact, we then get $\|\alpha\|_{L^\infty}^2 \le C\||\alpha|\,|\nabla\alpha|\|_{L^q} + C\||\alpha|^2\|_{L^q}$ for some $(n-1)/k < q < p$. Using Hölder's inequality, we can estimate the first term by $C\|\alpha\|_{L^{pq/(p-q)}}\|\nabla\alpha\|_{L^p} \le \delta\|\nabla\alpha\|_{L^p}^2 + C^2\delta^{-1}\|\alpha\|_{L^{pq/(p-q)}}^2$, where the last term is bounded by $C_\delta\|\alpha\|_{L^\infty}^{1-(p-q)/q}\|\alpha\|_{L^p}^{(p-q)/q}$. □

LEMMA A.3 *With notation as in Lemmas* A.1 *and* A.2, *we have*

$$\sum_{j=0}^{k}\|\nabla^j\alpha\|_{L^s(\Omega)} \le C\|\alpha\|_{L^q(\Omega)}^{1-a}\Bigg(\sum_{i=0}^{m}\|\nabla^i\alpha\|_{L^p(\Omega)}K_1^{m-i}\Bigg)^a\,. \tag{A.12}$$

PROOF: As in the proof of (A.5), the general case of (A.12) will follow from the special case $m = 2$ and $k = 1$. If we integrate by parts as in the proof of (A.4), we also get a boundary term

$$\int_\Omega |\nabla\alpha|^s \, d\mu \le C \int_\Omega |\nabla\alpha|^{s-2}|\alpha| \, |\nabla^2\alpha| d\mu + C \int_{\partial\Omega} |\nabla\alpha|^{s-1}|\alpha| d\mu_\gamma \, .$$

If α has compact support in Ω, then the boundary term cancels. Then by the proof of (A.4)

$$\|\nabla\alpha\|^2_{L^s(\Omega)} \le C\|\alpha\|_{L^q(\Omega)}\|\nabla^2\alpha\|_{L^p(\Omega)} \, . \tag{A.13}$$

We will prove that (A.13) is also true if α has compact support in a neighborhood of the boundary $\iota_1 < \operatorname{dist}(y, \partial\Omega) \le 0$. We have

$$\begin{aligned}
&\int_{\partial\Omega} |\nabla\alpha|^{s-1}|\alpha| d\mu_\gamma \\
&\le \left(\int_{\partial\Omega} |\nabla\alpha|^{(s-1)t} \, d\mu_\gamma\right)^{1/t} \left(\int_{\partial\Omega} |\alpha|^{t/(t-1)} \, d\mu_\gamma\right)^{(t-1)/t} \\
&\le C \left(\int_\Omega \left|\nabla_N|\nabla\alpha|^{(s-1)t}\right| d\mu\right)^{1/t} \left(\int_\Omega \left|\nabla_N|\alpha|^{t/(t-1)}\right| d\mu\right)^{(t-1)/t} \\
&\le C \left(\int_\Omega |\nabla\alpha|^{(s-1)t-1}|\nabla^2\alpha| d\mu\right)^{1/t} \left(\int_\Omega |\alpha|^{1/(t-1)}|\nabla\alpha| d\mu\right)^{(t-1)/t} .
\end{aligned}$$

Now we want to use Hölder's inequality again on each factor with $\|\nabla^2\alpha\|_{L^p}$, $\|\alpha\|_{L^q}$, and $\|\nabla\alpha\|_{L^s}$ where $1/q + 1/p = 2/s$. Let $1/q' = 1 - /q$, $1/p' = 1 - 1/p$, and $1/s' = 1 - 1/s$. We will show that we can pick t so that $s = p'((s-1)t - 1)$ and $s' = (t-1)q$. We need to show that the two expressions for t are the same, i.e., that $(s - s/p + 1)/(s-1) = t = (s - 1 + s/q)/(s-1)$, which is equivalent to $1/p + 1/q = 2/s$.

The boundary term can hence be bounded by $\|\nabla\alpha\|^{s-2/t}_{L^s(\Omega)}\|\nabla^2\alpha\|^{1/t}_{L^p(\Omega)}\|\alpha\|^{1/t}_{L^q(\Omega)}$. On the other hand, the interior term can be estimated as in the proof of (A.5) so we get

$$\begin{aligned}
&\|\nabla\alpha\|^s_{L^s(\Omega)} \le \\
&\qquad C\|\nabla\alpha\|^{s-2}_{L^s(\Omega)}\|\alpha\|_{L^q(\Omega)}\|\nabla^2\alpha\|_{L^p(\Omega)} + C\|\nabla\alpha\|^{s-2/t}_{L^s(\Omega)}\|\nabla^2\alpha\|^{1/t}_{L^p(\Omega)}\|\alpha\|^{1/t}_{L^q(\Omega)} \, ,
\end{aligned}$$

from which (A.13) follows also in the case where α is supported in the neighborhood $\iota_1 < \operatorname{dist}(y, \partial\Omega) \le 0$. Let $\{\chi_i\}$ be the partition of unity in Lemma 3.8. Now, since $|\nabla^\ell\chi_i| \le C{\iota_1}^{-\ell}$, it follows that $\|\nabla^2(\chi_i\alpha)\|_{L^p}$ is bounded by the sum in the right-hand side of (A.12) if $m = 2$ and $k = 1$. Since $\|\alpha\|^2_{L^s} \le \||\alpha|^2\|^2_{L^{s/2}} \le$

$\|\alpha\|_{L^q}\|\alpha\|_{L^p}$ by Hölder's inequality, (A.12) follows in the case where $m = 2$ and $k = 1$.

The general case of (A.12) follows from the special case as in the proof of (A.5) with the only exception being that now $M_k = \sum_{i=0}^k \|\nabla^i\alpha\|_{L^{s(k)}}$. So far we have only proven that $M_1 \le CM_0M_2$, but the general case of $M_k^2 \le CM_{k-1}M_{k+1}$ follows by induction from the previous case applied to $M_k' = \sum_{i=0}^k \|\nabla^i\nabla\alpha\|_{L^{s(k)}}$, $(M_{k-1}')^2 \le CM_{k-2}'M_k'$, and Hölder's inequality $\|\alpha\|_{L^s} \le \||\alpha|^{1-a}|\alpha|^a\|_{L^s} \le \|\alpha\|_{L^q}^{1-a}\|\alpha\|_{L^p}^a$ again. □

LEMMA A.4 *Suppose that* $\iota_1 \ge 1/K_1$ *and* α *is a* $(0,r)$ *tensor. Then*

$$\|\alpha\|_{L^{np/(n-kp)}(\Omega)} \le C\sum_{\ell=0}^{k} K_1^{k-\ell}\|\nabla^\ell\alpha\|_{L^p(\Omega)}, \qquad 1 \le p < \frac{n}{k}, \tag{A.14}$$

$$\|\alpha\|_{L^\infty(\Omega)} \le C\sum_{0\le\ell\le k} K_1^{n/p-\ell}\|\nabla^k\alpha\|_{L^p(\Omega)}, \quad k > \frac{n}{p}. \tag{A.15}$$

PROOF: As in the proof of Lemma A.2, we may assume that α is a function and $k = 1$. We now want to extend the functions to outside Ω and then use Sobolev's lemma in $\mathbb{R}^n$. We can extend the function by writing the surface as a graph $x_n = f(x')$, $(x', x_n) \in \mathbb{R}^n$, as in the proof of Lemma A.2. Let $\{\chi_i\}$ be the partition of unity in Lemma 3.8 and set $\phi_i = \chi_i\phi$. In a neighborhood of $\operatorname{supp}(\chi_i)$, we can then write $\partial\mathcal{D}_t$ as a graph after a rotation:

$$x^n = f(x'), \quad (x', x^n) \in \mathbb{R}^n, \quad |\partial f| \le 1.$$

We now define

$$\hat\phi_i(x) = \begin{cases} \phi_i(x) & \text{when } x \in \Omega \\ \phi(\hat x) & \text{when } x \notin \Omega \end{cases} \tag{A.16}$$

where $\hat x = (\hat x', \hat x^n) = (x', x^n - 2(x^n - f(x')))$.

In proving estimates (A.14) and (A.15), we may assume that $\phi \in C^\infty(\overline{\Omega})$ since this is dense in $W^{1,p}(\Omega)$; see [10]. Then by Sobolev's lemma in $\mathbb{R}^n$:

$$\|\hat\phi_i\|_{L^q(\mathbb{R}^n)} \le C\|\nabla\hat\phi_i\|_{L^p(\mathbb{R}^n)} \le C\|\nabla\phi_i\|_{L^p(\Omega)} + C\|\nabla\hat\phi_i\|_{L^p(\complement\Omega)} \le C'\|\nabla\phi_i\|_{L^p(\Omega)}$$

since $|\partial\hat x^i/\partial x^j| \le C$. Since $|\nabla\chi_i| \le CK_1$, this proves (A.14); (A.15) follows in a similar manner. □

LEMMA A.5 *Suppose that* $q = 0$ *on* $\partial\Omega$. *Then*

$$\begin{aligned} \|q\|_{L^2(\Omega)} &\le C(\operatorname{Vol}\Omega)^{1/n}\|\nabla q\|_{L^2(\Omega)}, \\ \|\nabla q\|_{L^2(\Omega)} &\le C(\operatorname{Vol}\Omega)^{1/2n}\|\triangle q\|_{L^2(\Omega)}. \end{aligned} \tag{A.17}$$

PROOF: The first inequality is Faber-Krahns theorem. Its proof uses a symmetrization argument; see [14]. The second follows from the first and integration by parts. □

We state two more lemmas.

LEMMA A.6 *If the metric satisfies*

$$C_0^{-1} g^0_{ij}(y) Z^i Z^j \le g_{ij}(t,y) Z^i Z^j \le C_0 g^0_{ij}(y) Z^i Z^j \quad \text{if } Z \in T(\Omega)\,, \tag{A.18}$$

where g^0 is a positive definite metric, then with a constant depending only on g^0 and c_0,

$$\|\partial_t^k \alpha\|_{L^s(\Omega\times[0,T])} \le C \|\alpha\|^{1-a}_{L^q(\Omega\times[0,T])} \|\partial_t^m \alpha\|^a_{L^p(\Omega\times[0,T])}\,, \tag{A.19}$$

provided that $\partial_t^j \alpha(0,\cdot) = 0$ for $j = 0, \dots, m-1$.

PROOF: It remains to prove (A.19), which is done similarly to the proof of (A.12). Suppose now that $\alpha(0,\cdot) = \partial_t \alpha(0,\cdot) = 0$. By (A.18) we can bound the norm and the measure from above and below by a measure that is independent of t. Thus, as before, it follows that

$$\int_0^T \int_\Omega |\partial_t \alpha|^s \, d\mu\, dt \le C \int_0^T \int_\Omega |\partial_t \alpha|^{s-2} |\alpha| |\partial_t^2 \alpha| d\mu\, dt + C \int_\Omega |\partial_t \alpha|^{s-1} |\alpha| d\mu(T)$$

$$\int_\Omega |\partial_t \alpha|^{s-1} |\alpha|\, d\mu(T) \le \left(\int_0^T \int_\Omega |\partial_t \alpha|^s \, d\mu\, dt \right)^{1-2/ts} \left(\int_0^T \int_\Omega |\partial_t^2 \alpha|^p \, d\mu\, dt \right)^{1/tp} \left(\int_0^T \int_\Omega |\alpha|^q \, d\mu\, dt \right)^{1/tq},$$

from which (A.19) follows as before. □

Using Lemma A.2 and the proof of Lemma 5.6, we can get a slightly improved version of Lemma 5.6:

LEMMA A.7 *Let α be $(0,r)$ tensor, and assume that $|\theta|_{L^\infty(\partial\Omega)} + 1/\iota_0 \le K$ and $\operatorname{Vol}(\Omega) \le V$. Then there is $C = C(K, V, r, n)$ such that*

$$\|\alpha\|_{L^{(n-1)p/(n-p)}(\partial\Omega)} \le C \|\nabla \alpha\|_{L^p(\Omega)} + C \|\alpha\|_{L^p(\Omega)}\,, \quad 1 \le p < n\,, \tag{A.20}$$

$$\|\nabla^2 \alpha\|_{L^2(\Omega)} \le C \big(\|\Pi \nabla^2 \alpha\|_{L^{2(n-1)/n}(\partial\Omega)} + \|\triangle \alpha\|_{L^2(\Omega)} + \|\nabla \alpha\|_{L^2(\Omega)} \big)\,. \tag{A.21}$$

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