The Formation of Vortices from a Surface of Discontinuity

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Proceedings of the Royal Society of London. Series A, Containing Papers of a Mathematical and Physical Character is currently published by The Royal Society.
and that in general the deviation from classical scattering increased rapidly with the velocity of the α-particle.

A general explanation of the phenomenon has been given from the point of view of wave mechanics, and the experimental data have been used to obtain a rough idea of the size of the scattering nuclei.

I wish to express my thanks to Dr. J. Chadwick for having suggested the problem and for his continuous interest in the work, and to Dr. Guido Beck for many interesting discussions about the theoretical points of the problem. My thanks are due to Mr. Crowe for his help in preparing the radio-active sources.

The Formation of Vortices from a Surface of Discontinuity.

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(Communicated by H. Jeffreys, F.R.S.—Received July 28, 1931.)

1. Introduction.

Helmholtz* was the first to remark on the instability of those “liquid surfaces” which separate portions of fluid moving with different velocities, and Kelvin,† in investigating the influence of wind on waves in water, supposed frictionless, has discussed the conditions under which a plane surface of water becomes unstable. Adopting Kelvin’s method, Rayleigh‡ investigated the instability of a surface of discontinuity. A clear and easily accessible rendering of the discussion is given by Lamb.§

The above investigations are conducted upon the well-known principle of “small oscillations”—there is a basic steady motion, upon which is superposed a flow, the squares of whose components of velocity can be neglected. This method has the advantage of making the equations of motion linear. If by this method the flow is found to be stable, the equations of motion give the subsequent history of the system, for the small oscillations about the steady state always remain “small.” If, however, the method indicates that the

* Helmholtz. ‘Phil. Mag.,’ vol. 36 (1868).
† Kelvin. ‘Phil. Mag.,’ vol. 42 (1871), or ‘Collected Works,’ vol. 4.
system is unstable, that is, if the deviations from the steady state increase exponentially with the time, the assumption of small motions cannot, after an appropriate interval of time, be applied to the case under consideration, and the equations of motion, in their approximate form, no longer give a picture of the flow. For this reason, which is well known, the investigations of Rayleigh only prove the existence of instability during the initial stages of the motion. It is the object of this note to investigate the form assumed by the surface of discontinuity when the displacements and velocities are no longer small.

In a recent paper on "The Wake in Fluid Flow Past a Solid,"* Dr. Jeffreys stated "... wherever the surface (of discontinuity) projects into the upper fluid the velocity of the upper fluid is increased and that of the lower diminished, by considerations of continuity; hence the changes in pressure resulting tend to push the lower fluid further into the upper. But if each wave is symmetrical about its crest, the distribution of pressure is also symmetrical. Therefore each wave must grow symmetrically, whatever its amplitude. This is not what is observed. Surfaces of discontinuity of velocity do not give symmetrical waves of large amplitude; they give series of vortices." When, however, it was pointed out to Dr. Jeffreys that a surface of discontinuity does not tend to grow symmetrically, as shown later in this paper, he detected an error in the above argument by referring to a work on "The Formation of Water Waves by Wind."† In equation (11) of his paper, with \( \rho' = \rho, \quad U' = -U, \quad g = 0, \quad T = 0, \) in the general solution, we get

\[
\sigma^2 + U^2\rho^2 = 0.
\]

If then the surface displacement \( \zeta \) is given by \( \zeta \propto e^{i\nu} \cos kx \), we have

\[
\gamma^2 = k^2U^2,
\]

and by inserting this in equation (8) of the same paper we see

\[
\frac{P}{\rho} = -\left(\frac{\sigma + Up}{r}\right)^2 \zeta
\]

\[
= -\frac{2\sigma Up}{r} \zeta \propto 2\gamma U \sin kx,
\]

where \( P \) is the pressure, \( \sigma \equiv \partial/\partial t, \quad p \equiv \partial/\partial x \), and \( r \) is here some constant. We see, therefore, that the pressure is antisymmetrical instead of symmetrical with respect to the crest of a wave.

In section 3 of the present investigation a second approximation to the

original Rayleigh theory is obtained. It is shown that the surface of dis-
continuity, instead of growing according to the formula
\[ y = a_1 e^{kx} \cos kx, \]
obeys the law
\[ y = a_1 e^{kx} \cos kx + \frac{1}{2} U k^2 a_1^2 e^{2kx} \sin 2kx, \]
correct to the second order of small quantities. The tendency to become
unsymmetrical is shown by the second term in the equation.

The method of the Rayleigh theory, that of expressing the displacement of
the surface as a Fourier series in \( x \), whose coefficients are functions of the time,
cannot show the rolling up of a surface of discontinuity, and in investigating
the ultimate shape of the surface an approximate numerical method has been
used. The principle of this method is that, instead of assuming a continuous
distribution of vorticity over the surface, we approximate to it by assuming a
distribution of finite "elemental" vortices, and we follow up the paths of these
vortices by a numerical step-by-step method. The line joining these elemental
vortices at any instant of time is an approximation to the actual shape of the
surface at that time. A very similar idea appears to have been suggested in a
letter* to 'Nature' which has recently been brought to my notice. In it
A. R. Low says that in order better to understand the motion of fluids, "A
mass of perfect fluid consisting of several distinct parts, each with its own
velocity potential, and separated by thin sheets of transition (in the math-
ematical limit, vortex sheets) may be effectively redistributed . . . . . and
new mean elements with an effective finite distribution of vorticity become the
objects of physical observation and measurement through the whole or part
of the joint mass."

In the following paper the author is greatly indebted to Professor Prandtl
and to Dr. Jeffreys, whose suggestions and criticisms have greatly increased
the scope of the work. Various sections of the following discussion are capable
of extension and it is hoped that their results will shortly be published.

2. Summary.

The flow of a stream of density \( \rho \) and velocity \( U \) in the direction of the axis
of \( x \), above a stream of the same density but velocity \( U \) in the opposite direction,
is investigated by the method of small oscillations and by an approximate
numerical method. The motion is two-dimensional and the common surface,
when undisturbed, is the axis of \( x \). The flow on both sides of the surface of

discontinuity is continuous and irrotational. Initially the disturbed surface of separation is supposed to be a sine curve of small amplitude.

The solution of the problem by the method of small oscillations is composed of two terms—a first and a second order term. The first-order term, which corresponds to the original Rayleigh solution, represents a sine curve of continuously increasing amplitude. The second-order term, which ultimately dominates, introduces a disturbance which is antisymmetrical with respect to a crest—and hence the surface of discontinuity does not grow symmetrically. The Rayleigh method can only be applied during the first stages of flow, in which the displacement can be represented by a Fourier series in $x$.

The motion is further investigated by means of an approximate numerical method. The surface of discontinuity, which is a vortex sheet, is replaced by a distribution of finite elemental vortices along its trace, and the paths of these vortices are determined by a numerical step-by-step method. The line joining these vortices at any instant is assumed to be an approximation to the actual shape of the surface at that time. It is shown that the effect of instability upon a surface of discontinuity of sine form is to produce concentrations of vorticity at equal intervals along the surface, and it is also shown that the surface of discontinuity tends to roll up round these points of concentration with an accompanying increase in the amplitude of the displacement. For comparison, the effect of putting 2, 4, 8, 12 elemental vortices of equal strength, initially equally spaced along the surface, is investigated, but the results are of the same nature.

The effect of diffusion, which is just mentioned qualitatively, is to retard the rolling up process. The results of the investigation are purely qualitative, but they account for the formation of vortices from a surface of discontinuity. No attempt is made to discover that value of the wave-length which ultimately becomes dominant in the disturbance, and so fixes the distance between successive vortices.

3. The Method of Small Oscillations.

We assume that we are dealing with the flow of a stream of density $\rho$ and velocity $U$ in the direction of the axis of $x$, above a stream of the same density but velocity $U$ in exactly the opposite direction. The motion is two-dimensional and the common surface, when undisturbed, is the axis of $x$. The flow on both sides of the surface of discontinuity is continuous and irrotational. The initial form of the disturbed surface is $y = a \cos kx$, where $a$ is of the first order of small quantities.
Following Rayleigh, we put
\[
\begin{align*}
\phi &= Ux + \phi_1 + \phi_2 \\
\phi' &= -Ux + \phi_1' + \phi_2'
\end{align*}
\]
where \(\phi\) is the velocity potential of the upper stream and \(\phi'\) that of the lower one. From considerations of continuity we must have that \(\nabla^2 \phi = 0 = \nabla^2 \phi'\). We assume also that \(\phi_1, \phi_1'\) are of the first order, and that \(\phi_2, \phi_2'\) are of the second order. The form of the surface at time \(t\) is assumed to be
\[
y = a e^{i(\sigma t - kx)} + y_2,
\]
where \(\sigma\) is to be determined, and where \(y_2\) is a function of \(x\) and \(t\) only, and is of the second order. As in other problems of a similar nature, it is appropriate to put
\[
\begin{align*}
\phi_1 &= Ae^{-ky + i(\sigma t - kx)} \\
\phi_1' &= A'e^{ky + i(\sigma t - kx)}
\end{align*}
\]
the streams being assumed to be of infinite depth. If \(u, v\) represent the component velocities in the upper stream, \(u', v'\) those in the lower stream, the kinematical conditions to be satisfied are \(DF/\partial t = 0\) at the boundary, that is
\[
\begin{align*}
\frac{\partial F}{\partial t} + u \frac{\partial F}{\partial x} + v \frac{\partial F}{\partial y} &= 0 \\
\frac{\partial F}{\partial t} + u' \frac{\partial F}{\partial x} + v' \frac{\partial F}{\partial y} &= 0
\end{align*}
\]
at the boundary. The expression represented by \(F\) is
\[
F \equiv y - a e^{i(\sigma t - kx)} - y_2 \equiv 0.
\]
Also, the expression for the pressure in the upper stream is
\[
-\frac{p}{\rho} = \frac{\partial \phi}{\partial t} + \frac{1}{2} (u^2 + v^2) + gy,
\]
and in the lower stream
\[
-\frac{p'}{\rho} = \frac{\partial \phi'}{\partial t} + \frac{1}{2} (u'^2 + v'^2) + gy,
\]
so that the equation representing the continuity of pressure at the common surface is
\[
\frac{\partial \phi}{\partial t} + \frac{1}{2} (u^2 + v^2) = \frac{\partial \phi'}{\partial t} + \frac{1}{2} (u'^2 + v'^2),
\] (5)
to be satisfied at the boundary.

If we only require accuracy to the first order of small quantities, equations (4) and (5) become
\[
\begin{align*}
A k &= -ua (\sigma - kU) \\
A'k &= u(a (\sigma + kU) \\
A (\sigma - kU) &= A' (\sigma + kU)
\end{align*}
\] (6)
from which
\[
(\sigma - kU)^2 + (\sigma + kU)^2 = 0,
\]
or
\[
\frac{\sigma}{k} = \pm iU.
\] (7)
This is the Rayleigh solution. The most general form, therefore, correct to the first order, is
\[
y = a_1 e^{kU t} \cos kx + a_1' e^{-kU t} \cos kx,
\]
\[
\phi_1 = a_1 U e^{kU t - ky} \cos kx + a_1' U e^{-kU t - ky} \cos kx,
\]
\[
\phi_1' = a_1 U e^{kU t + ky} \cos kx + a_1' U e^{-kU t + ky} \cos kx,
\]
where \(a_1\) and \(a_1'\) are of the first order and satisfy the relation
\[
a_1 + a_1' = a.
\]
The above values are incorporated in the general equations, and the following eight relations give the value at the common surface, correct to the second order, of the relevant expressions. (For simplicity we here introduce the abbreviations \(s \equiv \sin kx, c \equiv \cos kx.\))
\[
\frac{\partial \phi}{\partial t} = kU^2 [a_1 e^{kU t} (s - c) - a_1' e^{-kU t} (s + c)] + \frac{\partial \phi_2}{\partial t}
\]
- \(k^2 U^2 [a_1^2 e^{2kU t} (sc - c^2) - a_1'^2 e^{-2kU t} (sc + c^2) - 2a_1 a_1' c^2],
\]
\[
\frac{\partial \phi}{\partial x} = U + kU [a_1 e^{kU t} (s + c) - a_1' e^{-kU t} (s - c)] + \frac{\partial \phi_2}{\partial x}
\]
- \(k^2 U [a_1^2 e^{2kU t} (sc + c^2) - a_1'^2 e^{-2kU t} (sc - c^2) + 2a_1 a_1' c^2],
\]
\[
\frac{\partial \phi}{\partial y} = -kU [a_1 e^{kU t} (s - c) + a_1' e^{-kU t} (s + c)] + \frac{\partial \phi_2}{\partial y}
\]
+ \(k^2 U [a_1^2 e^{2kU t} (sc - c^2) + a_1'^2 e^{-2kU t} (sc + c^2) + 2a_1 a_1' sc],
\]
\[
\frac{1}{2} (u^2 + v^2) = \frac{1}{2} U^2 + kU^2 \left[ a_1 e^{kU} (s + c) - a_1' e^{-kU} (s - c) \right] + U \frac{\partial \phi_2}{\partial x} \\
+ k^2 U^2 \left[ a_1 e^{2kU} (s^2 - sc) + a_1' e^{-2kU} (s^2 + sc) - a_1 a_1' c^2 \right],
\]

\[
\frac{\partial \phi'}{\partial t} = kU \left[ a_1 e^{kU} (s + c) - a_1' e^{-kU} (s - c) \right] + \frac{\partial \phi_2'}{\partial x} \\
+ k^2 U \left[ a_1 e^{2kU} (sc + c^2) - a_1' e^{-2kU} (sc - c^2) + 2a_1 a_1' c^2 \right],
\]

\[
\frac{\partial \phi'}{\partial x} = - U - kU \left[ a_1 e^{kU} (s - c) - a_1' e^{-kU} (s + c) \right] + \frac{\partial \phi_2'}{\partial y} \\
- k^2 U \left[ a_1 e^{2kU} (sc - c^2) - a_1' e^{-2kU} (sc + c^2) + 2a_1 a_1' sc \right],
\]

\[
\frac{\partial \phi'}{\partial y} = kU \left[ a_1 e^{kU} (s + c) + a_1' e^{-kU} (s - c) \right] + \frac{\partial \phi_2'}{\partial y} \\
+ k^2 U \left[ a_1 e^{2kU} (sc + c^2) + a_1' e^{-2kU} (sc - c^2) + 2a_1 a_1' sc \right],
\]

With these values the kinematical conditions become

\[
\frac{\partial y_2}{\partial t} + U \frac{\partial y_2}{\partial x} - \frac{\partial \phi_2}{\partial y} = k^2 U \left[ a_1 e^{2kU} (\sin 2kx - \cos 2kx) \right. \\
\left. + a_1' e^{-2kU} (\sin 2kx + \cos 2kx) + 2a_1 a_1' \sin 2kx \right], \tag{8}
\]

\[
\frac{\partial y_2}{\partial t} - U \frac{\partial y_2}{\partial x} - \frac{\partial \phi_2'}{\partial y} = k^2 U \left[ a_1 e^{2kU} (\sin 2kx + \cos 2kx) \right. \\
\left. + a_1' e^{-2kU} (\sin 2kx - \cos 2kx) + 2a_1 a_1' \sin 2kx \right],
\]

and the equation for the continuity of pressure is

\[
\frac{\partial \phi_2}{\partial t} + U \frac{\partial \phi_2}{\partial x} - k^2 U^2 \left[ a_1 e^{2kU} - a_1' e^{-2kU} \right] \sin 2kx \\
= \frac{\partial \phi_2'}{\partial t} - U \frac{\partial \phi_2'}{\partial x} + k^2 U^2 \left[ a_1 e^{2kU} - a_1' e^{-2kU} \right] \sin 2kx, \tag{9}
\]

to be satisfied at \( y = 0 \). Equations (8) and (9) can be re-arranged as follows:

\[
\frac{\partial \phi_2}{\partial y} = \frac{\partial y_2}{\partial t} + U \frac{\partial y_2}{\partial x} - k^2 U [\sin 2kx (a_1 e^{kU} + a_1' e^{-kU})]^2 \\
- \cos 2kx (a_1 e^{2kU} - a_1' e^{-2kU})], \tag{10}
\]

\[
\frac{\partial \phi_2'}{\partial y} = \frac{\partial y_2}{\partial t} - U \frac{\partial y_2}{\partial x} - k^2 U [\sin 2kx (a_1 e^{kU} + a_1' e^{-kU})]^2 \\
+ \cos 2kx (a_1 e^{2kU} - a_1' e^{-2kU})],
\]

\[
\frac{\partial}{\partial t} (\phi_2 - \phi_2') + U \frac{\partial}{\partial x} (\phi_2 + \phi_2') = 2k^2 U^2 \sin 2kx [a_1 e^{2kU} - a_1' e^{-2kU}]. \tag{11}
\]
Formation of Vortices from Surface of Discontinuity.

From these equations it is evident that \( y_2 \) can be expressed in the form \( P \cos 2kx + \phi \sin 2kx \). (A rigorous proof of this can easily be obtained by a simple refinement of the following argument.) It follows therefore that the values of \( \phi_2 \) and \( \phi_2' \) when \( y \) is not zero, are

\[
\begin{align*}
\phi_2 &= -\frac{e^{-2k}\nu}{2k} \left\{ \frac{\partial y_2}{\partial t} + U \frac{\partial y_2}{\partial x} - k^2 U \left[ \sin 2kx (a_1 e^{kU} + a_1' e^{-kU})^2 
- \cos 2kx (a_1^2 e^{2kU} - a_1' a_1 e^{-2kU}) \right] \right\}, \\
\phi_2' &= \frac{e^{2k}\nu}{2k} \left\{ \frac{\partial y_2}{\partial t} - U \frac{\partial y_2}{\partial x} - k^2 U \left[ \sin 2kx (a_1 e^{kU} + a_1' e^{-kU})^2 
+ \cos 2kx (a_1^2 e^{2kU} - a_1' a_1 e^{-2kU}) \right] \right\},
\end{align*}
\]  

(12)

and hence

\[
\frac{\partial}{\partial t} (\phi_2 - \phi_2') = -\frac{1}{k} \frac{\partial^2 y_2}{\partial t^2} + 2k^2 U^2 \left( a_1^2 e^{2kU} - a_1' a_1 e^{-2kU} \right),
\]

\[
U \frac{\partial}{\partial x} (\phi_2 + \phi_2') = -\frac{U^2}{k} \frac{\partial^2 y_2}{\partial x^2} + 2k^2 U^2 \left( a_1^2 e^{2kU} - a_1' a_1 e^{-2kU} \right),
\]

so that on substituting in (11) we get

\[
\frac{\partial^2 y_2}{\partial t^2} + U^2 \frac{\partial^2 y_2}{\partial x^2} = 2k^2 U^2 \left( a_1^2 e^{2kU} - a_1' a_1 e^{-2kU} \right) \sin 2kx.
\]

(13)

We require only the "particular" solution of this equation, whence it follows that

\[
y_2 = \frac{1}{2} U k^2 \left( a_1^2 e^{2kU} + a_1' a_1 e^{-2kU} \right) \sin 2kx.
\]

(14)

The amplitude of the disturbance represented by \( y_2 \) grows much more rapidly than that of \( (a_1 e^{kU} + a_1' e^{-kU}) \cos kx \), and ultimately dominates it. The expression represented by \( y_2 \) is also antisymmetrical with respect to \( x = 0 \), that is with respect to a crest of the original wave, and its effect, as \( t \) increases, is to give the original cosine curve a shape somewhat like that of a breaker in the first stages of its growth, instead of the symmetrical shape suggested by the Rayleigh solution. Ultimately, of course, when \( t \) is very big, the surface, according to this theory, is virtually a sine curve of period \( \pi/k \) instead of a cosine curve of period \( 2\pi/k \). The theory, however, is not applicable when \( t \) is large, for by that time the third and higher order terms can no longer be neglected. In addition, a picture of the actual ultimate state of affairs cannot be obtained without introducing diffusion and viscosity. It is evident, too, that the method of representing the displacement \( y \) by a Fourier series in \( x \) cannot show the rolling up of the surface, and is, as a matter of fact, only valid...
during the first stages of the motion. The following sections describe a method which gives the form of the surface without being subject to this restriction.

4. An Approximate Solution.

An approximation to the surface of discontinuity, which is a vortex sheet, can be obtained by replacing the continuous distribution of vorticity by isolated vortices of appropriate strength along its trace. We shall not investigate the effect of a general disturbance involving all wave-lengths, but shall restrict ourselves to one particular wave-length. The final result cannot be obtained by superposing the effects of the separate wave-lengths, but the general nature of the flow is the same for each component wave. We can imagine that the wave-length considered is the one that ultimately dominates as a result of instability, and so fixes the distance between successive vortices of the resulting system. The actual determination of this wave-length is not here attempted, and it seems probable that this can only be done by taking diffusion and viscosity into account.

4.1. Two Vortices to a Wave-length.—As a preliminary step we investigate the case of two vortices to a wave-length, and the system reduces to a single row of vortices of strength $-\kappa$ equally spaced along the $x$ axis. This system is in a steady state, but, as is well known,* it is unstable. If we assume, following Lamb, that the distance between consecutive vortices is $\lambda/2$, and that the $m$th vortex, initially at $(m\lambda/2, 0)$, suffers a displacement

$$x_m = \alpha e^{m\phi}, \quad y_m = \beta e^{m\phi},$$

where $\phi$ lies between 0 and $2\pi$, then we get

$$\frac{d\alpha}{dt} = \mu\beta, \quad \frac{d\beta}{dt} = \mu\alpha,$$

where

$$\mu = \frac{4\kappa}{\pi\lambda^2} \left( \frac{1 - \cos \phi}{1^2} + \frac{1 - \cos 2\phi}{2^2} + \frac{1 - \cos 3\phi}{3^2} \ldots \right) = \frac{\kappa}{\pi \lambda^2} \phi (2\pi - \phi). \quad (1)$$

Since the velocity of the stream above the vortices is $U$ and that below is $-U$, then $\kappa = \lambda U$, and

$$\mu = \frac{U}{\pi \lambda} \phi (2\pi - \phi).$$

We see that the component of the disturbance that grows most quickly is that

---

for which \( \phi = \pi \), and if we replace the whole disturbance by the most rapidly growing component, we have

\[
x_m = (Ae^{\mu_1 t} + Be^{-\mu_1 t}) (-)^m, \quad y_m = (-Ae^{\mu_1 t} + Be^{-\mu_1 t}) (-)^m,
\]

where \( \mu_1 = \pi U/\lambda \).

In this type of disturbance consecutive vortices are displaced to the same extent but in opposite directions. In this case the vortices depart from their initial position along arms of rectangular hyperbolas given by the formula

\[
x_m^2 - y_m^2 = 4AB.
\]

These paths are shown by the broken curves in fig. 1A.

The preceding theory is, however, only a first approximation, and if we can assume that the whole disturbance can be represented effectively by the component that ultimately becomes dominant, an exact theory can be applied as follows. The vortices are in the positions \( \left( m \frac{\lambda}{2} + x_m, y_m \right) \) where \( x_m = (-)^m x_0 \), \( y_m = (-)^m y_0 \), and can therefore be separated into two rows

\[
\left( 2m \frac{\lambda}{2} + x, y_0 \right) \quad \text{and} \quad \left( 2m + 1 \frac{\lambda}{2} - x, y_0 \right).
\]

In the subsequent motion, by symmetry, the vortices will be at

\[
(m \lambda + x, y), \quad \left( 2m + 1 \frac{\lambda}{2} - x, -y \right)
\]

where \( x \) and \( y \) vary. The exact velocity components of the vortex at \( (x, y) \) are

\[
u = \frac{\kappa}{2\lambda} \frac{\sinh \frac{4\pi y}{\lambda}}{\cosh \frac{4\pi y}{\lambda} + \cos \frac{4\pi x}{\lambda}},
\]

\[
v = \frac{\kappa}{2\lambda} \frac{\sin \frac{4\pi x}{\lambda}}{\cosh \frac{4\pi y}{\lambda} + \cos \frac{4\pi x}{\lambda}}.
\]

and the paths of the vortices are

\[
u \frac{dx}{v} = \frac{d}{dy} \frac{\sinh \frac{4\pi y}{\lambda}}{\sin \frac{4\pi x}{\lambda}},
\]

or

\[
cosh \frac{4\pi}{\lambda} y + \cos \frac{4\pi}{\lambda} x = \cosh \frac{4\pi}{\lambda} y_0 + \cos \frac{4\pi}{\lambda} x_0.
\]
This curve is an undulating one repeating itself at intervals of $\lambda/2$, as shown by the unbroken lines in fig. 1A, and to a first approximation is identical with that of equation (2).

![Fig. 1.](image)

This method suggests then that, owing to instability, a single row of vortices splits itself into two rows travelling in opposite directions, the vortices themselves moving in an undulatory path. Further, the paths of the vortices suggest to some extent the turbulent nature of the flow in the wake behind a solid. The distribution of vortices according to this method is a good approximation to the surface of discontinuity only in the first stages of the flow, but even here the rolling up of the surface is suggested, as in fig. 1B. A more accurate approximation is obtained, as in the later sections, by putting more vortices into each wave-length.

It was suggested to me, however, by Dr. Jeffreys that it would be interesting to see what happens to a single undulation on a surface of discontinuity as the elemental vortices might immediately roll up into a vortex somewhere near the undulation. Section 4.2 discusses this case.

4.2. A Single Undulation on a Surface of Discontinuity.—As an approximation the system is replaced by a row of vortices of equal strength $-\kappa$, the distance between consecutive vortices being $a$. The undulation is represented by a displacement of one of the vortices through a distance $h$ perpendicular to the row, as in fig. 2. The system is not a steady one, and if $u$, $v$, are the velocity components of the $n$th vortex, the one initially at the origin having the number 0 attached to it, we have that the initial velocities of the vortices are

\[
\begin{align*}
\frac{u_0}{2a} &= \frac{\kappa}{2a} \left( \frac{\sinh \frac{2\pi h}{a}}{\cosh \frac{2\pi h}{a} - 1} \right) = \frac{\kappa}{2a} \left( \frac{\cosh \pi \frac{h}{a} - \frac{1}{\pi h/a}}{\cosh \pi \frac{h}{a} - 1} \right) > 0, \\
v_0 &= 0,
\end{align*}
\]

\[\text{(4)}\]
\[ u_n = -\frac{\kappa}{2\pi} \frac{h}{h^2 + n^2 a^2} \]
\[ v_n = -\frac{\kappa}{2\pi} \frac{na}{h^2 + n^2 a^2} + \frac{\kappa}{2\pi} \frac{1}{na} = \frac{\kappa}{2\pi} \frac{h^2}{na (h^2 + n^2 a^2)} \]

and so \( v_n/u_n = -h/na \).

Vortex 0 therefore moves downstream. Vortices 1 to \( \infty \) move towards vortex 0, and numbers \(-1\) to \(-\infty\) move away from it, as shown by the arrows in fig. 2.

Without going more deeply into the matter it seems plausible to suggest that

\[ \rightarrow U \]

\[ \leftarrow U \]

Fig. 2.

vortices \( 0, 1, 2, \ldots, \infty \), are going to combine to form one vortex that will travel in the direction of the positive \( x \)-axis, while vortices \(-1, -2, \ldots, -\infty\), will unite to form a vortex more slowly in the direction of the negative \( x \)-axis, and that there will be a rupture of the surface of discontinuity between vortices 0 and \(-1\).

4.3. *Four Vortices to a Wave-length.*—Initially the vortices are at \( \left( m \frac{\lambda}{2}, 0 \right) \); \( (m + \frac{1}{2})\lambda, h \); \( (m + \frac{3}{2})\lambda, -h \), as an approximation to a sine curve of amplitude \( h \). By symmetry, the vortices at \( \left( m \frac{\lambda}{2}, 0 \right) \) will always be at rest, and the other vortices will form two rows, representative members of which will be the points \( (m + \frac{1}{2})\lambda + x_1, y_1 \); \( (m + \frac{3}{2})\lambda - x_1, -y_1 \). The velocity components of the vortex at \( (x_1, y_1) \) is

\[ u = \frac{3\kappa}{2\lambda} \frac{\sinh 4\pi \frac{y_1}{\lambda}}{\cosh 4\pi \frac{y_1}{\lambda} + \cos 4\pi \frac{x_1}{\lambda}} \]
\[ v = \frac{3\kappa}{2\lambda} \frac{\sin 4\pi \frac{x_1}{\lambda}}{\cosh 4\pi \frac{y_1}{\lambda} + \cos 4\pi \frac{x_1}{\lambda}} \]

and as in 4.1, the path of the vortex is

\[ \cosh 4\pi \frac{y_1}{\lambda} + \cos 4\pi \frac{x_1}{\lambda} = \cosh 4\pi \frac{h}{\lambda} + 1. \]
Here again the vortices split themselves into two sections which move in opposite directions along sinuous curves.

4.4. *Eight and Twelve Vortices to a Wave-length.*—The cases of eight and twelve vortices to a wave-length will now be discussed in some detail, as they are better approximations than those considered above, to the continuous distribution of vorticity. A comparison of the various cases considered is of importance because the solution here given is a numerical one, and at first glance it is not evident whether an increase in the number of vortices to a wave-length slows down, or accelerates, the characteristic motion of the surface of discontinuity. We have seen, however, that the cases corresponding to \( n = 2 \) or 4 show, during the first stages of the motion, a tendency to roll up like a succession of breakers, but these cases cannot be accepted as showing the formation of vortices, for the elemental vortices show no tendency to concentrate themselves at intervals along the surface. In view of the drastic nature of the approximation it is not surprising that this is so, but it is rather remarkable that with \( n = 8 \) the phenomenon of the rolling up is quite evident, and certain of the vortices rotate, and continue to rotate, round the points of concentration and show no tendency to move away. This is much more evident with \( n = 12 \), and so we are led to believe that an increase in the number of elemental vortices to the wave-length accelerates the rolling up process. This can be seen more explicitly by a comparison of Tables I and II, which correspond to \( n = 8 \) and \( n = 12 \) respectively. The "concentration" process is much more in evidence in the latter of these cases. The numerical solution has not been continued far enough to show whether all the vortices maintain their connection with the surface or whether some of them travel away. It is very likely that those furthest away (in each wave-length) from the points of concentration move away from the system.

For the sake of simplicity therefore we shall assume that the surface
of separation is a sine curve of wave-length \( \lambda \) whose amplitude increases according to the Rayleigh theory until it is equal to \( \alpha \lambda \) (say). The curve 
\[ y = a e^{\nu t} \cos kx \] 
gives a picture of the motion, until \( \left( \frac{a}{\lambda} e^{\nu t} \right)^2 \) can no longer be neglected, and if we are working to an accuracy of 1 per cent. the Rayleigh theory can be applied until
\[ t = \frac{\lambda}{2\pi U} \log (\lambda/10a). \]

If \( a/\lambda \) should be equal to 1/100 (say) at \( t = 0 \), then the form given above is valid during the interval \( t = 0 \) to \( t = 0.35 \lambda/U \) second. In other words the ratio \( a e^{\nu t}/\lambda \) increases from 1/100 to 1/10 in the time 0.35 \( \lambda/U \) second.

For the purposes of the following work the origin of time is taken at the instant where \( \alpha = 1/10 \). The equation of the surface of separation is then
\[ y = \alpha \lambda \sin 2\pi x/\lambda, \]
or putting \( y/\lambda = \eta, \ x/\lambda = \xi, \)
\[ \eta = \alpha \sin 2\pi \xi. \] (7)

The normal velocity on both sides of the surface of separation is continuous, but the tangential component has an abrupt discontinuity at the surface. In other words, we are dealing with a vortex sheet. If \( ds_0 \) be an element of length of the sheet in the \( x, y \) plane, \( dx_0 \) its projection on the \( x \)-axis, then the strength of vorticity at a point \( x \) on the sheet is given by the relation*
\[ \kappa ds_0 = Ud\xi_0 - (-U) dx_0 = 0, \]
or
\[ \kappa = -2U dx_0/ds. \] (8)

Since the surface of separation repeats itself at intervals of \( \lambda \) we may consider our system as being formed by the juxtaposition of rows of vortices of strength \( \kappa ds \) \((=-2U dx)\), the interval between members of the row being \( \lambda \), as in fig. 4, \( a \), if we imagine the whole of the surface of discontinuity filled with elemental vortices. The complex potential, however, for a row of vortices of strength \( \kappa \) and interval \( \lambda \) is known, and from it, by integration, we can build

* This relation is obtained by taking the circulation round the contour bounded by the lines \( x = x_0, x = x_0 + 8x_0, y = R, y = -R \), with \( R \) tending to infinity. Equation (8) is obtained on the assumption that the contributions to the circulation from the lines \( x = x_0, x = x_0 + 8x_0 \) cancel, but Professor Prandtl pointed out that these lines may yield a small first-order term. The value of \( \kappa \) given by (8) can, however, be accepted as a very good approximation.
up the stream function for a surface of separation of any form. The complex potential for a row of vortices of strength $\kappa_0$ and interval $\lambda$ is

$$\omega = -\frac{\kappa_0}{2\pi} \log \sin \pi (z - z_0)/\lambda,$$

where one of the vortices is at $z_0$. The components of velocity at a point $(x_1, y_1)$ are

$$u = -\frac{\kappa_0}{2\lambda} \frac{\sinh k (y_1 - y_0)}{\cosh k (y_1 - y_0) - \cos k (x_1 - x_0)},$$

$$v = \frac{\kappa_0}{2\lambda} \frac{\sin k (x_1 - x_0)}{\cosh k (y_1 - y_0) - \cos k (x_1 - x_0)}.$$

The components of velocity at a point $(x_1, y_1)$ due to the vorticity of the surface of separation are therefore

$$u = -\left\{ \int_x^{x+\lambda} \frac{\kappa ds}{2\lambda} \frac{\sinh k (y_1 - y)}{\cosh k (y_1 - y) - \cos k (x_1 - x)} \right\},$$

$$v = \int_x^{x+\lambda} \frac{\kappa ds}{2\lambda} \frac{\sin k (x_1 - x)}{\cosh k (y_1 - y) - \cos k (x_1 - x)} \right\}.$$ (9)

where the integrals are line integrals taken over the surface of separation between any two limits of $x$ differing by a wave-length. We may therefore put

$$u = \frac{U}{\lambda} \int_0^\lambda \frac{\sinh k (y_1 - y)}{\cosh k (y_1 - y) - \cos k (x_1 - x)} \, dx,$$

$$v = -\frac{U}{\lambda} \int_0^\lambda \frac{\sin k (x_1 - x)}{\cosh k (y_1 - y) - \cos k (x_1 - x)} \, dx.$$ (10)

If the point $(x_1, y_1)$ lies above or below the surface of separation, the integrals (10) are line integrals with no singularities on the contour of integration. If, however, we wish to find the velocity of a point on the surface of separation, the integrals have a simple infinity at the point whose velocity we desire to find, and in the usual way the "principal values" of the integrals give the velocity components. We are not dealing with a problem in steady motion for the form of the surface of separation alters with time; $y$ is a function of $x$ and $t$.

If the equation of the surface of separation is given by $F(x, y, t) = 0$, then the equation must satisfy the condition

$$\frac{DF}{Dt} = \frac{\partial F}{\partial t} + u \frac{\partial F}{\partial x} + v \frac{\partial F}{\partial y} = 0,$$ (11)

where $u$ and $v$ are given by (6). This is the condition that a point on the surface of separation at time $t$ is on the new position of the surface at time $t + \delta t$. 
The exact solution of (10) and (11) is a matter of some difficulty, and in attempting to obtain an approximate solution we introduce the following simplification. Instead of having a continuous distribution of vorticity over the surface of separation, we assume that the surface is broken up into many parts by lines at equal intervals of $x$, and the vorticity in each of these sections is imagined concentrated into an elemental vortex at its "centre of gravity"—so that our approximate picture is exactly as shown in fig. 4, a. We now investigate the motion of these vortices, and at any time the line joining them gives an approximation to the form of the surface of discontinuity. If we insert $n$ vortices at equal intervals between $x = 0$ and $x = \lambda$, we put, as an approximation

$$n\kappa = -2U\lambda,$$

(12)

where $\kappa$ is the strength of each of these elemental vortices.

The velocity of a vortex at a point $(x_1, y_1)$ is thus given by

$$u = \frac{U}{n} \sum \frac{\sinh k(y_1 - y)}{\cosh k(y_1 - y) - \cos k(x_1 - x)}$$
$$v = -\frac{U}{n} \sum \frac{\sin k(x_1 - x)}{\cosh k(y_1 - y) - \cos k(x_1 - x)}$$

(13)

or

$$u = \frac{1}{U} \sum \frac{\sinh 2\pi (\eta_1 - \eta)}{\cosh 2\pi (\eta_1 - \eta) - \cos 2\pi (\xi_1 - \xi)}$$
$$v = -\frac{1}{U} \sum \frac{\sin 2\pi (\xi_1 - \xi)}{\cosh 2\pi (\eta_1 - \eta) - \cos 2\pi (\xi_1 - \xi)}$$

(13a)

the summation being over all values of $(x, y)$ which are the co-ordinates of the other vortices. The exact solution of this problem is also difficult, for if we assume that the vortices lie on the curve $F_1(x, y, t) = 0$, then the differential equation to be satisfied by $F_1$ is

$$\frac{\partial F_1}{\partial t} + u \frac{\partial F_1}{\partial x} + v \frac{\partial F_1}{\partial y} = 0,$$

where $u$ and $v$ are given by (13). The exact solution of this problem is probably more complicated than that of the previous one, for here we have summations over a finite number of points instead of integrals over a range.

We can, however, adopt a method which is virtually a numerical solution of the differential equation. We assume that the surface of separation increases in amplitude uniformly until it is given by the equation $\eta = \alpha \sin 2\pi \xi$. We approximate to the subsequent shape of this surface by assuming in every
wave-length, instead of a continuous distribution of vorticity, \( n \) elemental vortices each of strength \(-2U \lambda/n\) initially uniformly spaced along the curve at intervals whose projections on the \( x \)-axis are \( \lambda/n \). We now investigate the motion of the vortices numerically. The instantaneous velocities of the vortices in their initial positions, \((u_1, \ v_1)\) are calculated by (13), and as an approximation the position of a vortex, initially at \((x_1, \ y_1)\), at time \( \delta t_1 \) is 
\[
(x_1 + u_1 \delta t_1, \ y_1 + v_1 \delta t_1) = (x_2, \ y_2).
\]
The instantaneous velocities at \((x_2, \ y_2)\) are calculated from (13), and the positions after another interval of time \( \delta t_2 \) are 
\[
(x_2 + u_2 \delta t_2, \ y_2 + v_2 \delta t_2) = (x_3, \ y_3),
\]
and so on for any number of times. The line joining all the positions \((x_1, \ y_1)\) gives the form of the surface from which we commence our approximation. The line joining the points \((x_2, \ y_2)\) gives the form at time \( \delta t_1 \) later, and the points \((x_3, \ y_3)\) give the curve after a further interval \( \delta t_2 \), and so on. The errors introduced at each step are of the order
\[
\left| \frac{u}{\delta t} \right|^2.
\]
The order of the error can be reduced by increasing the amount of calculation and using the Kutta-Runge method, or some such process of numerical solution, but after the rather drastic approximation of imagining the surface split into elemental vortices, there is very little point in introducing a higher order of accuracy into the numerical calculations at the cost of much labour. Care, however, is taken in choosing the intervals \( \delta t \) in such a way that the directions of the motion of a vortex before and after any interval of time do not differ greatly.

The above method of approximation is useful only if the curvature of the paths travelled by the vortices is small, but it is useless when the curvature is large as in the case of vortices 4, 5, 7, 8 of fig. 4, d. In these cases the angular velocity round vortex 6, and not the linear velocity, remains approximately constant during the interval of time from \(0.35\lambda/U\) second to \(0.40\lambda/U\) second. Also this method cannot be used for a large number of steps, for the accumulation of the errors introduced at each step may become large enough to destroy the value of the approximation.

In actual practice, however, diffusion of vorticity plays an important rôle, so that if \( U \) is small the surface of separation may “disappear” before the rolling-up becomes evident. It should also be remembered that we are only considering the motion of one of the components of the general disturbance, and that all these components tend to act in the same way.

4.5. Numerical Evaluation.—The equations of motion of the surface have
been put into a non-dimensional form and the only quantities to which numerical values have to be given are \( n \), the number of elemental vortices per wave-length, and \( z \), the ratio of the initial amplitude of the surface to its wave-length.

Let us compare the initial distributions of velocity along surfaces of sine form but of different amplitudes. When the amplitude is small we see that

\[
\frac{u}{U} = (\text{amplitude}) \times g(x),
\]

\[
\frac{v}{U} = (\text{amplitude})^2 \times h(x),
\]

where \( g(x) \) and \( h(x) \) are functions that vary along the surface. Hence a change in the amplitude, when it is small, will produce a proportionately larger change in the \( x \)-velocity than in the \( y \)-velocity. Further, if one considers the surface made up of rectilinear vortices, it can easily be seen that the crests of the wave move along the positive \( x \)-axis, and the troughs along the negative \( x \)-axis, corresponding to the directions of the upper and lower streams. At the first instant of motion also, vortices at equal distances on either side of a crest (or trough) have their \( x \)-velocities equal, but their \( y \)-velocities equal and opposite. Hence at first the surface tends to roll over itself like a breaker, and the bigger the initial amplitude the more marked is this tendency, but the general nature

\[\text{Fig. 4.}\]
of the flow is the same in all cases. In order to avoid undue calculation we therefore take $\alpha$ to be rather large, say 1/10, as this should show the change in the form of the surface sooner than surfaces of smaller amplitude (see fig. 4). A more exact picture of the flow would be obtained if the initial amplitude were taken much smaller, say 1/100, but with such small values the Rayleigh theory can be applied to the system, with the modification that the surface does not maintain its sine-form as its amplitude grows, but it becomes distorted to a shape very similar to that of fig. 4, b. After this the small oscillations method cannot be applied and the subsequent forms are given by the approximate theory sketched above.

We therefore take the surface of discontinuity to be

$$y = 0.1 \lambda \sin \frac{2\pi x}{\lambda},$$

or

$$\eta = 0.1 \sin \frac{2\pi \xi}{\lambda},$$

at time $t = 0$, and investigate the motion. We put $n = 8$, and divide the surface in such a way that there are elemental vortices at the points $r\lambda/2$, where $r$ assumes all integral values from $-\infty$ to $+\infty$. The remaining ones are uniformly spaced between these points. Owing to the symmetry that is bound to exist during the subsequent motion, the vortices at $x = r\lambda/2$ have zero velocity for all time for they are points of antisymmetry with respect to the system. The shapes are calculated from Table I, which is self-explanatory. An increased accuracy in determining the shape of the surface would be obtained by taking $n$ larger than 8, and such a case is given by Table II which corresponds to $n = 12$.

The general features of the flow are well marked. The various quantities in Tables I and II are given in non-dimensional form and can be applied to any other case in which $\alpha$ is equal to 1/10 at $t = 0$ second.

The diagrams, which are only shown for the more accurate case $n = 12$, show several well-known features. The elemental vortices appear to concentrate at intervals along the surface equal to the wave-length of the disturbance—which may be interpreted as demonstrating a periodic concentration of vorticity when a surface of discontinuity breaks up. The process of the concentration of vorticity is accompanied by a rolling up of the surface of discontinuity, which is a characteristic that has been noted by many investigators. The numerical calculation is ended at time $t = 0.40 \lambda/U$ second for the case $n = 12$, but the rolling up process proceeds with increased rapidity for the elemental vortices in the neighbourhood of vortex No. 6 have a large
### Table I: \( \eta = 8 \)

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Formation of Vortices from Surface of Discontinuity. 189
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<th>Co-ordinates at ( t = 0.45 \lambda / U ) sec.</th>
<th>Velocity components at ( t = 0.45 \lambda / U ) sec.</th>
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Vortex No. 2

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Formation of Vortices from Surface of Discontinuity.
angular velocity round it. As the surface rolls up, the amplitude of the disturbance increases—in the case considered it is approximately doubled in the time $0.40 \lambda/U$ second.

The whole investigation accounts in an approximate theoretical manner for the formation of vortices along the surface, when the surface of discontinuity breaks up owing to instability.

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On Dirichlet’s Divisor Problem.

By J. R. Wilton, University of Adelaide.

(Communicated by G. H. Hardy, F.R.S.—Received June 11, 1931.)

1. Let $d(n)$ denote the number of divisors of the positive integer $n$, so that,

$$ n = p_1^{a_1} \ldots p_r^{a_r} $$

is the canonical expression of $n$ in prime factors,

$$ d(n) = (1+a_1) \ldots (1 + a_r), $$

and let $d(x) = 0$ if $x$ is not an integer; then, if

$$ D(x) = \sum_{n \leq x} d(n) = \sum_{n \leq x} d(n) - \frac{1}{2} d(x), $$

and

$$ \Delta(x) = D(x) - x \log x - (2C - 1) x - \frac{1}{6}, $$

where $C$ is Euler’s constant, it was proved by Dirichlet* in 1849 that

$$ \Delta(x) = O(\sqrt{x}), $$

and more than half-a-century later this result was improved by Voronoï† to

$$ \Delta(x) = O(x^{3/5} \log x); $$

the most that has yet been proved in this connection is van der Corput’s theorem‡:

$$ \Delta(x) = O(x^{27/83} \log^{11/41} x), $$